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Uniform in time mean field limits for 1D Riesz gases

Pierre Le Bris*

Abstract

This note is based on a talk given at the *Séminaire Laurent Schwartz* in February 2024. The goal is to review a recent joint work with Arnaud Guillin (Université Clermont-Auvergne) and Pierre Monmarché (Sorbonne Université) published in *Journal de l'École polytechnique - Mathématiques* (2023) [GLM23].

Consider a one dimensional N-particle system in singular repulsive mean field interaction. The main motivating example is the (generalized) Dyson Brownian motion which holds importance in Random Matrix Theory. We wish to show the convergence, as N goes to infinity, of the empirical distribution of the system towards the solution of a non linear PDE.

We describe a method that relies only on the well posedness of the system of particles and which provides a quantitative (and in some cases uniform in time) result. We make full use of the fact that in dimension one the particles will stay ordered, and that as a consequence the interaction we consider will be convex. Using a coupling method, we prove that by taking any independent sequence of empirical measures, it is a Cauchy sequence. Then, independence ensures the fact that the limit is an almost surely constant random variable which we then identify. This method requires in particular no study of the non linear limit.

Contents

1 Introduction

1.1 On propagation of chaos

Consider a system of N interacting particles described by the following system of Stochastic Differential Equations (SDEs)

$$dX_t^{i,N} = \frac{1}{N}\sum_{j=1}^N K(X_t^{i,N} - X_t^{j,N})dt + \sqrt{2\sigma_N}dB_t^i, \quad i \in \{1,\ldots,N\}. \tag{1.1}$$

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Each particle is represented by a quantity $X \in \mathbb{R}^d$, most likely its position, where $X_t^{i,N}$ denotes the position at time t of the i-th particle. K is a function (which we will call an interaction kernel), σ_N is a diffusion coefficient that may or may not depend on the total number of particles, and $(B^i)_i$ are independent Brownian motions. We assume that the system is *exchangeable*, meaning that the joint law of the particles is invariant by permutation of indices.

The main question we try and address is: what happens when N, the total number of particles, goes to infinity?

This is a classical problem in Statistical Physics which concerns itself with understanding how one can go from a *microscopic* description of a system of particles, i.e. write the dynamics of each individual particle such as in (1.1), to a *mesoscopic* description, i.e. an evolution equation for the statistical distribution of particles. The aim is to reduce the study of a (very) large system to the study of a single object.

In doing so, we lose some information. In particular, the mesoscopic description gives the probability distribution of one particle in the system, but we do not know *a priori* the joint law of two or more particles, i.e. the correlations in the system.

The answer to this problem lies in a phenomenon named *propagation of chaos* by Kac [Kac56] (and then generalized by McKean [McK66]) which states that, as N grows, two given particles become "more and more" statistically independent. This can be intuitively understood by noticing that two given particles only act on each other via a force of order $\frac{1}{N}$. In fact, let's rewrite the SDE system (1.1) in a way that emphasizes the fact that particles only see each other through their common law. Denoting $\mu_t^N := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}$ the empirical measure, we may see that (1.1) is equivalent to

$$dX_t^{i,N} = K * \mu_t^N\big(X_t^{i,N}\big)dt + \sqrt{2\sigma_N}dB_t^i,$$

where $*$ denotes the convolution operation: $K * \mu(x) = \int K(x-y)\mu(dy)$. If the particles are indeed expected to become independent in the limit $N \to \infty$, as well as identically distributed (since they are exchangeable), we can guess that via some form of law of large numbers μ_t^N will converge to a measure $\bar{\rho}_t$, the law of a typical particle in the limit. Hence, very formally, a natural candidate for limit SDE

$$\begin{cases} d\bar{X}_t = K * \bar{\rho}_t(\bar{X}_t)dt + \sqrt{2\sigma}dB_t, \\ \bar{\rho}_t = \text{Law}(\bar{X}_t), \end{cases} \tag{1.2}$$

where $\sigma = \lim \sigma_N$. This non linear SDE is said to be of *McKean-Vlasov* type because of the non linearity induced by the interaction with its own law. We may also consider its related Fokker-Planck equation

$$\partial_t\bar{\rho}_t(x) = -\nabla\cdot((K * \bar{\rho}_t(x))\bar{\rho}_t(x)) + \sigma\Delta\bar{\rho}_t(x). \tag{1.3}$$

To come back to the initial motivation, (1.1) would thus be the microscopic point of view, while (1.3) would be the mesoscopic point of view.

Let us denote $\rho_t^{k,N} = \text{Law}\big(X_t^{1,N}, \dots, X_t^{k,N}\big)$ the joint law of the subset of the first k particles of the N particle system (with the convention $\rho_t^N = \rho_t^{N,N}$) and $\bar{\rho}_t^{\otimes k} = \bar{\rho}_t \otimes \cdots \otimes \bar{\rho}_t$ the non linear limit law $\bar{\rho}_t$ tensorized k times.

Goal. Our main objective is to show a result of the form:

Propagation of chaos:

$$\forall k \in \mathbb{N}, \qquad \lim_{N\to\infty} \rho_0^{k,N} = \bar{\rho}_0^{\otimes k} \implies \forall t \geq 0, \lim_{N\to\infty} \rho_t^{k,N} = \bar{\rho}_t^{\otimes k},$$

Notice that this property yields "independence at the limit", as the joint law converges towards a tensorized law. The *chaos* aspect of this property obviously refers to the independence, while the *propagation* alludes to the fact that it will be sufficient to prove this convergence as N goes to infinity at time $t = 0$ for it to also hold at later times t.

Equivalently, for an exchangeable system of particles, we may show
Mean field limit:

$$\lim_{N\to\infty} \mu_0^N = \bar{\rho}_0 \implies \forall t \geq 0, \lim_{N\to\infty} \mu_t^N = \bar{\rho}_t.$$

We do not at this stage specify the types of convergence.

Focuses. Given these objectives, we also focus on a few sub-goals.

First, we wish to obtain a *quantitative* convergence. Indeed, we always see one model as an approximation of the other. Either, as it was the historical motivation, we want to boil down (1.1) to the study of (1.3), or, on the contrary, we already have a non linear equation similar to (1.3) which we want, for instance, to numerically simulate and thus use (1.1) as a model which can be easily simulated via an Euler-Maruyama scheme. In both cases, we wish to quantify the error made in the approximation.

Secondly, notice how we only mention a convergence as $N \to \infty$ for each fixed time $t > 0$. If one wishes to study the stationary distribution of one model using the other, we need to be able to exchange the limits in t and in N. Hence we look for *uniform in time* propagation of chaos, i.e convergence in N independent of t.

Finally, the case of singular kernel K holds importance because of the number of applications and links with other research areas. Consider in dimensions d the kernel $K(x) = \mathbb{M}\nabla g(x)$ where $\mathbb{M}$ is a constant $d\times d$ matrix either antisymmetric (in which case we speak of a conservative system) or $\mathbb{M} = \pm\mathbb{I}$ (dissipative, and resp. attractive or repulsive interactions), and assume there is $s \in [0, d[$ such that

$$g(x) = \begin{cases} -\log|x|, & \text{if } s = 0, \\ |x|^{-s}, & \text{if } s > 0. \end{cases}$$

Here, we speak of singularity since $|K(x)| \xrightarrow[|x|\to 0]{} \infty$ These type of interactions are often referred to as *Riesz interactions* and have attracted a lot of attention recently. The specific case of $s = d-2$ is known as the *Coulomb interaction*. We refer to [CD22a, JW17, BJW19], and more specifically to the recent review [Lew22] and references therein, but to name a few cases of interest :

- $d \geq 2, s = d-2, \mathbb{M} = \mathbb{I}$ yields an approximation of the second-order Newtonian dynamics,
- $d = 1, s = 0, \mathbb{M} = -\mathbb{I}$ has links to random matrix theory (we refer to the classical book [AGZ10]). This model will be the main concern of this note,
- $d = 2, s = 0, \mathbb{M} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ corresponds to the 2D vortex model (see [JW18]),
- $d = 2, s = 0, \mathbb{M} = -\mathbb{I}$ for the Ginzburg-Landau vortices (see for instance [SS12, Ser15]),
- $d = 2, s = 0, \mathbb{M} = \mathbb{I}$ corresponds to the Patlak-Keller-Segel model for chemotaxis in biology (see for instance [BJW19]).

1.2 The Dyson Brownian Motion

We simplify the problem and now wonder what we can do in dimension 1. As mentioned above, there exists a case of interest for 1D Riesz gases known as the (generalized) Dyson Brownian motion

$$dX_t^{i,N} = \sqrt{\frac{2\sigma}{N}} dB_t^i - \lambda X_t^{i,N} dt + \frac{1}{N} \sum_{j\neq i} \frac{1}{X_t^{i,N} - X_t^{j,N}} dt. \tag{1.4}$$

The dynamics above are satisfied, for $\lambda = 0$, by the eigenvalues of an $N \times N$ symmetric matrix valued Brownian motion, as observed by Dyson in 1962 [Dys62]. For $\lambda > 0$, it correspond to the eigenvalues of an $N \times N$ symmetric matrix valued Ornstein-Uhlenbeck process [Cha92].

Let us in fact give a short and informal justification of this claim which comes from the course at *Collège de France* by Lions [Lio22], the rigorous proof can be found in [AGZ10].

Consider the matrix

$$D_t^N = \frac{1}{\sqrt{N}} \begin{pmatrix} \sqrt{2}W_t^{1,1} & W_t^{1,2} & \dots & W_t^{1,N} \\ W_t^{2,1} & \sqrt{2}W_t^{1,2} & \dots & W_t^{1,N} \\ & & \dots & \\ W_t^{N,1} & W_t^{N,2} & \dots & \sqrt{2}W_t^{N,N} \end{pmatrix}$$

where the $W^{i,j}$ are one dimensional Brownian motions, $W^{i,j} = W^{j,i}$ for all $j \neq i$ and all Brownian motions are independent (up to the symmetricity assumption). To compute the time evolution of its eigenvalues, we do a Taylor expansion and write for a time increment h

$$D_{t+h}^N = D_t^N + \frac{1}{\sqrt{N}} \left(\Delta W^{i,j}\right)_{i,j},$$

where $\Delta W^{i,j} = W_{t+h}^{i,j} - W_t^{i,j}$ (with a multiplicative constant if $i = j$). We diagonalize D_t^N thanks to an orthogonal matrix O, denoting $\lambda_t^1, \dots, \lambda_t^N$ its eigenvalues. One can check that the law of the matrix $\left(\Delta W^{i,j}\right)_{i,j}$ is invariant by multiplication by an orthogonal matrix, so there exists a matrix $\left(\Delta \tilde{W}^{i,j}\right)_{i,j}$ with the same law as $\left(\Delta W^{i,j}\right)_{i,j}$ such that

$$O^T D_{t+h}^N O = \begin{pmatrix} \lambda_t^1 & 0 & \dots & 0 \\ 0 & \lambda_t^2 & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & \lambda_t^N \end{pmatrix} + \frac{1}{\sqrt{N}} \left(\Delta \tilde{W}^{i,j}\right)_{i,j}.$$

Because $O^T D_{t+h}^N O$ has the same eigenvalues as D_{t+h}^N, we solve for X

$$\det \left(\begin{pmatrix} X - \lambda_t^1 & 0 & \dots & 0 \\ 0 & X - \lambda_t^2 & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & X - \lambda_t^N \end{pmatrix} - \frac{1}{\sqrt{N}} \left(\Delta \tilde{W}^{i,j}\right)_{i,j} \right) = 0$$

We expand the determinant and, just as one would do when proving Itô's formula based on a Taylor expansion, we keep the terms of order $\Delta \tilde{W}^{i,j}$ as Brownian increment, replace those of order $(\Delta \tilde{W}^{i,j})^2$ by the time increment h, and discard the next order terms since they are negligible. The expansion of the determinant yields

$$\prod_{i=1}^N \left(X - \lambda_t^i - \sqrt{\frac{2}{N}} \Delta \tilde{W}^{i,i} \right) - \sum_{k,l=1,k<l}^N \frac{1}{\sqrt{N}} \Delta \tilde{W}^{k,l} \frac{1}{\sqrt{N}} \Delta \tilde{W}^{l,k} \prod_{j \neq k,l} \left(X - \lambda_t^j - \sqrt{\frac{2}{N}} \Delta \tilde{W}^{j,j} \right) \\ + O\left(h^{3/2}\right) = 0,$$

and thus, using the symmetry assumption and discarding the higher order terms,

$$\prod_{i=1}^N \left(X - \lambda_t^i - \sqrt{\frac{2}{N}} \Delta \tilde{W}^{i,i} \right) - \frac{h}{N} \sum_{k,l=1,k<l}^N \prod_{j \neq k,l} \left(X - \lambda_t^j - \sqrt{\frac{2}{N}} \Delta \tilde{W}^{j,j} \right) = 0.$$

Let us now look for X the first eigenvalue of D^N_{t+h}, which we denote λ^1_{t+h}, and write $\lambda^1_{t+h} = \lambda^1_t + \mu^1 + \sqrt{2/N}\Delta\tilde{W}^{1,1}$, for some μ^1 which we can see is of order h. Then, separating the case $k = 1$ from the rest, we get

$$\begin{aligned}\mu^1 \prod_{j=2}^N &\left(\mu^1 + \lambda^1_t - \lambda^j_t + \sqrt{\frac{2}{N}}(\Delta\tilde{W}^{1,1} - \Delta\tilde{W}^{j,j})\right) \\ &= \frac{h}{N}\sum_{\substack{k,l=2 \\ k<l}}^N \mu^1 \prod_{j\neq 1,k,l}\left(\mu^1 + \lambda^1_t - \lambda^j_t - \sqrt{\frac{2}{N}}(\Delta\tilde{W}^{1,1} - \Delta\tilde{W}^{j,j})\right) \\ &\quad + \frac{h}{N}\sum_{l=2}^N \prod_{j\neq 1,l}\left(\mu^1 + \lambda^1_t - \lambda^j_t - \sqrt{\frac{2}{N}}(\Delta\tilde{W}^{1,1} - \Delta\tilde{W}^{j,j})\right),\end{aligned}$$

and, again simplifying the term of order greater than h (observe that μ^1 is of order h), we get

$$\mu^1 \prod_{j=2}^N \left(\lambda^1_t - \lambda^j_t\right) = \frac{h}{N}\sum_{l=2}^N \prod_{j\neq 1,l}\left(\lambda^1_t - \lambda^j_t\right),$$

i.e $\mu^1 = \frac{h}{N}\sum_{j=2}^N \frac{1}{\lambda^1_t - \lambda^j_t}$. We thus obtain

$$\lambda^1_{t+h} = \lambda^1_t + \frac{h}{N}\sum_{j=2}^N \frac{1}{\lambda^1_t - \lambda^j_t} + \sqrt{\frac{2}{N}}(\tilde{W}^{1,1}_{t+h} - \tilde{W}^{1,1}_t),$$

hence the aforementioned dynamics.

1.3 Short review of methods

We thus now try and look for a method which would allow us to deal with a singular kernel and a vanishing diffusion. We start by mentioning some classical methods of proof. The goal is not to do an exhaustive review and the choice of methods we mention in what follows is motivated by the various comparisons we might wish to do, and many results from this very active field of research are unfortunately left out. We instead refer to the classical courses [Szn91, Mél96], and to the more recent reviews [JW17] and [CD22a, CD22b].

Coupling methods. Historically, one of the first tool used to show propagation of chaos is a probabilistic tool, as used by McKean (see for instance [McK66]) and then popularized by Sznitman [Szn91], known as a coupling method. It comes from the idea that a natural distance between probability measures is the Wasserstein distance, strongly linked to the theory of optimal transport, and is based on Dobrushin's inequality in the deterministic case [Dob79]. The goal is to simultaneously construct a solution of (1.1) and N independent copies of (1.2) (i.e constructing the Brownian motions). The distance between these two solutions will yield an upper bound on the Wasserstein distance. Recently, one of the biggest advance consists in the reflection coupling [Ebe16].

To the best of our knowledge, such methods still fail in dealing with singular kernels K. So far, they may only deal with Lipschitz continuous interactions, or possibly discontinuous but bounded interactions.

Furthermore, in the case of the Dyson Brownian motion, there should not be any Brownian motion in the limit (1.2) (adapted to (1.4), i.e $K(x) = 1/x$ and $\sigma = 0$).

Hence, we do not hope to be able to use such a method.

PDE approach. Using tools from PDE analysis, and functional inequalities, in order to show convergence of ρ_t^N towards $\bar{\rho}_t^{\otimes N}$, recent progress have been made using a modulated energy [Ser18, Ser20, RS23], by considering the relative entropy of ρ_t^N with respect to $\bar{\rho}_t^{\otimes N}$ [JW18] or by combining these two quantities into a modulated free energy [BJW19]. These quantities have proven useful in showing propagation of chaos for systems of particles in singular interaction by making full use of the regularity of (1.2). Notice however that, in our case, there would not be any Laplacian term (1.3), which would typically be the main regularizing term for the solution. The study of the non linear limit (1.3) can however still be done, see [BDLL22], but we want to avoid this direction.

Compactness/Tightness. Another method, that lies somewhere in between the fields of probability and PDE analysis, consists in proving the tightness or compactness of the set of empirical measure, showing that the limit of any convergent subsequence satisfies (1.2) (again, adapted to (1.4)), and proving the uniqueness of the solution of (1.2). This has been for instance done for singular interaction kernels, in the specific case of (1.4) [RS93, CL97]. This method, however, does not provide quantitative convergence rates.

Large deviations principles. Finally, let us briefly mention some results on large deviations (LDP), which are particularly sought-after, especially in the case of large random matrices. Pathwise propagation of chaos and LDP for the empirical measures in a regular case are obtained in the pioneering article [BAB90] based on a Girsanov's transform and Varadhan's lemma. Then, large deviations have been obtained in the specific case of the Dyson Brownian motion (see for instance [Fon04, Gui04]). To the best of our knowledge, apart from these results, establishing LDPs for general singular drifts remains an open problem. Let us however mention the recent [HHMT24] which deals with a general class of interaction (without, however, reaching the same singularity as the Dyson Brownian motion).

2 The approach

We consider, in a slightly more general setting, the 1D N-particle system in mean field interaction

$$dX_t^{i,N} = \sqrt{2\sigma_N} dB_t^i - U'(X_t^{i,N})dt - \frac{1}{N}\sum_{j\neq i} V'(X_t^{i,N} - X_t^{j,N})dt, \tag{2.1}$$

where

- the $(B^i)_i$ are independent Brownian motion,
- $\sigma_N \xrightarrow[N\to\infty]{} 0$,
- U is "nice" (think of U' lipschitz continuous, or U convex, or even $U'(x) = \lambda x$),
- there exists $\alpha \in [1, 2[$, such that for all $x \in \mathbb{R}^*$, $V'(x) = -x/|x|^{\alpha+1}$.

We wish to prove the following result in Wasserstein-2 distance (denoted $\mathcal{W}_2$).

Theorem 1. *Consider a sequence of initial empirical measures $(\mu_0^N)_{N\geq 1}$ such that there exists $\bar{\rho}_0 \in \mathcal{P}_2(\mathbb{R})$ such that $\lim_{N\to 0} \mathbb{E}\left(\mathcal{W}_2(\mu_0^N, \bar{\rho}_0)^2\right) = 0$. Assuming $U'(x) = \lambda x$, $\alpha \in [1, 2[$ and $\sigma_N \leq 1/N$ if $\alpha = 1$, there exist a deterministic family of measures $(\bar{\rho}_t)_{t\geq 0} \in \mathcal{C}(\mathbb{R}^+, \mathcal{P}_2(\mathbb{R}))$, as well as universal constants $C_1, C_2 > 0$ and a quantity $C_0^N > 0$ that depends on the initial condition and such that $C_0^N \to 0$ as $N \to \infty$, such that for all $N \geq 1$ and all $t \geq 0$*

$$\mathbb{E}\left(\mathcal{W}_2(\mu_t^N, \bar{\rho}_t)^2\right) \leq e^{-2\lambda t} C_0^N + \frac{C_1}{N^{(2-\alpha)/\alpha}} + C_2 \sigma_N.$$

In particular, notice that we require the diffusion coefficient σ_N to go to 0 in order to obtain the convergence of the empirical measure.

Remark 2.1. *This is a quantitative result and the constants C_1, C_2 above are explicit and independent of N and t.*

Furthermore, this is a uniform in time control thanks to the nonpositive power of the exponential $e^{-2\lambda t}$. In fact, we obtain a slightly better result: we obtain generation of chaos. Indeed we do not require the convergence of C_0^N (which is the term which depends on the initial condition) since, for t large enough, it becomes negligible.

Remark 2.2. *At this stage, we do not require a priori any property on $\bar{\rho}_t$. We in fact prove its existence as the (deterministic) limit of the empirical measures, and only then obtain the PDE is should satisfy.*

2.1 Well posedness of the particle system

We have the following result

Lemma 2.1. *Consider $N \geq 2$, and $-\infty < x_1 < \cdots < x_N < \infty$.*

- *If $\alpha > 1$, for any $\sigma_N \geq 0$, there exists a unique strong solution $X = (X^1, \ldots, X^N)$ to the stochastic differential equation* (2.1) *with initial condition $X_0^1 = x_1, \ldots, X_0^N = x_N$, which furthermore satisfies $-\infty < X_t^1 < \cdots < X_t^N < \infty$ for all $t \geq 0$, $\mathbb{P}$-a.s.*
- *The same result holds for $\alpha = 1$ and $\sigma_N \leq 1/N$.*

Notice that the fact that the problem is one-dimensional and that the interaction is "sufficiently" repulsive implies that the particle stay ordered. This will be an important property we rely on.

Sketch of the proof. The idea is to consider the "energy" $H(x_1, \ldots, x_N) = \frac{1}{2N} \sum_{i \neq j} V(x_i - x_j)$ and show that this quantity does not (on average) increase along the flow of $(X_t^1, \ldots, X_t^N)_t$. This will ensure that there are a.s. no collisions between particles. Using Itô's formula to compute the time evolution of $H(X_t^1, \ldots, X_t^N)$, we see that there is a competition between two terms:

- the term $-\sum_i \left(\frac{1}{N} \sum_{j \neq i} V'(X_t^i - X_t^j) \right)^2$, coming from the drift (2.1) which tends to decrease the energy,
- and $\frac{\sigma_N}{N} \sum_i \sum_{j \neq i} V''(X_t^i - X_t^j)$, coming from the Brownian motion which, in this case, may cause the particles to collide.

Hence, when formally comparing the order of magnitude of these two terms, we end up trying to control $\frac{\sigma_N}{N} V''(x) \sim \frac{\sigma_N}{N} \frac{1}{|x|^{\alpha+1}}$ by $-\frac{1}{N^2}(V'(x))^2 \sim \frac{1}{N^2} \frac{1}{|x|^{2\alpha}}$. This can always be done if $2\alpha > \alpha + 1$ i.e $\alpha > 1$. If $\alpha = 1$ we need to be careful with the constants, which is what yields the condition $\sigma_N \leq 1/N$. □

2.2 Cauchy-type estimate and conclusion

The proof of mean-field limit relies on a Cauchy-type control that we state below. We only write the simplified case $\alpha = 1$ for the sake of conciseness.

Lemma 2.2. *Let $(\mu^N)_{N \in \mathbb{N}}$ be any sequence of independent empirical measures, such that μ_t^N is the empirical measure of the N particle system at time t. For $\alpha = 1$, $U'(x) = \lambda x$ and $\sigma_N = \sigma/N$, there exists C such that we have for all $t \geq 0$ and all $N, M \geq 1$*

$$\mathbb{E}\left(\mathcal{W}_2 \left(\mu_t^N, \mu_t^M \right)^2 \right) \leq e^{-2\lambda t} \mathbb{E}\left(\mathcal{W}_2 \left(\mu_0^N, \mu_0^M \right)^2 \right) + \frac{C}{N \wedge M}. \tag{2.2}$$

Remark 2.3. *The same result* (2.2) *holds:*

- *for* $\alpha \in [1,2[$, *but with rate* $C/(N \wedge M)^{(2-\alpha)/\alpha}$,
- *for* $U'(x) = 0$ *or* U' *only Lipschitz continuous, though no longer uniform in time (i.e., the constant* C *may depend on* t*),*
- *for* $\mathbb{E}\big(\sup_{s\in[0,t]} \mathcal{W}_2\left(\mu_s^N, \mu_s^M\right)^2\big)$*, though, again, no longer uniform in time.*

A priori, Lemma 2.2 only ensures the convergence of the law of the empirical measure and not of the empirical measure itself. However, using the independence, Lemma 2.2 in fact yields the existence of a (deterministic) $\bar{\rho}_t$ such that

$$\mathbb{E}\left(\mathcal{W}_2(\mu_t^N, \bar{\rho}_t)^2\right) \to 0,$$

with the same rate of convergence, which satisfies, for all f sufficiently regular and all $t \geqslant 0$,

$$\begin{aligned}\int_{\mathbb{R}} f(x)\bar{\rho}_t(dx) = \int_{\mathbb{R}} f(x)\bar{\rho}_0(dx) - \int_0^t \int_{\mathbb{R}} f'(x)U'(x)\bar{\rho}_s(dx)ds \\ + \frac{1}{2}\int_0^t \int\int_{\{x\neq y\}} \frac{(f'(x)-f'(y))(x-y)}{|x-y|^{\alpha+1}}\bar{\rho}_s(dx)\bar{\rho}_s(dy)ds.\end{aligned}$$

In particular, we do not rely on the study of the limit PDE, and only identify it after the proof of convergence. Hence how we obtain uniform in time mean-field limit for 1D Riesz gases.

2.3 Proof of the Cauchy-type estimate

We now, in this last section, present the proof of Lemma 2.2.

The starting point is that, in dimension one, the optimal coupling in Wasserstein-2 distance between two sets of points is known. Indeed, let $x_1 \leq \cdots \leq x_N$ and $y_1 \leq \cdots \leq y_N$ be two (ordered) sets of points which define the empirical measures $\mu^X = \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ and $\mu^Y = \frac{1}{N}\sum_{i=1}^N \delta_{y_i}$ then

$$\mathcal{W}_2(\mu^X, \mu^Y)^2 = \frac{1}{N}\sum_{i=1}^N |x_i - y_i|^2$$

In our case, for $N, M \geq 1$, to define μ_t^M and μ_t^N, let $(\tilde{B}^i)_{i\in\{1,\dots,M\}}$ and $(\tilde{B}'^j)_{j\in\{1,\dots,N\}}$ be two independent families of Brownian motions, and consider $x_1 < \cdots < x_M$ and $y_1 < \cdots < y_N$ two sets of initial conditions. Denote by $(\tilde{X}^{i,M})_{i\in\{1,\dots,M\}}$ (resp. $(\tilde{Y}^{j,N})_{j\in\{1,\dots,N\}}$) the unique strong solution of (2.1) with initial conditions $x_1 < \cdots < x_M$ and Brownian motions $(\tilde{B}^i)_{i\in\{1,\dots,M\}}$ (resp. initial conditions $y_1 < \cdots < y_N$ and Brownian motions $(\tilde{B'}^j)_{j\in\{1,\dots,N\}}$).

In order to compare the two sets $(\tilde{X}^{i,M})_{i\in\{1,\dots,M\}}$ and $(\tilde{Y}^{j,N})_{j\in\{1,\dots,M\}}$ despite the difference in the number of particles, we consider N exact copies of the system $(\tilde{X}^{i,M})_{i\in\{1,\dots,M\}}$, and M exact copies of $(\tilde{Y}^{j,N})_{j\in\{1,\dots,N\}}$. We denote $(X^i)_{i\in\{1,\dots,NM\}}$ and $(Y^i)_{i\in\{1,\dots,NM\}}$ the resulting processes, numbered such that for all $t \geq 0$

$$\begin{aligned}-\infty < X_t^1 = \cdots = X_t^N < \cdots < X_t^{N(M-1)+1} = \cdots = X_t^{NM} < \infty \\ -\infty < Y_t^1 = \cdots = Y_t^M < \cdots < Y_t^{M(N-1)+1} = \cdots = Y_t^{NM} < \infty.\end{aligned}$$

Thus

$$\mu_t^M = \frac{1}{M}\sum_{i=1}^M \delta_{\tilde{X}_t^{i,M}} = \frac{1}{NM}\sum_{i=1}^{NM} \delta_{X_t^i}, \quad \mu_t^N = \frac{1}{N}\sum_{i=1}^N \delta_{\tilde{Y}_t^{i,N}} = \frac{1}{NM}\sum_{i=1}^{NM} \delta_{Y_t^i},$$

and

$$\mathcal{W}_2\left(\mu_t^N,\mu_t^M\right)^2=\frac{1}{NM}\sum_{i=1}^{NM}\left|X_t^i-Y_t^i\right|^2.$$

By convention, and for the sake of clarity, we assume $V'(0)=0$.

We can now, using Itô's formula, explicitly compute the time evolution of $\mathcal{W}_2\left(\mu_t^N,\mu_t^M\right)^2$. Direct calculations yield

$$\begin{aligned} d\left(\mathcal{W}_2\left(\mu_t^N,\mu_t^M\right)^2\right)=&-2\lambda\mathcal{W}_2\left(\mu_t^N,\mu_t^M\right)^2dt+2\sigma\left(\frac{1}{N}+\frac{1}{M}\right)dt+dM_t\\ &-\frac{2}{(NM)^2}\sum_i\left(X_t^i-Y_t^i\right)\sum_j\left(V'(X_t^i-X_t^j)-V'(Y_t^i-Y_t^j)\right)dt, \end{aligned}\tag{2.3}$$

where M_t is a martingale which will disappear when considering the expectation. The first term is the one which will yield the exponential term with negative exponent when using Gronwall's lemma, and the second comes from the diffusion via Itô's term (and it is to deal with this term that we require the diffusion coefficient σ_N to vanish to 0). Finally, let us consider this last term. We have

$$\begin{aligned} &\sum_i\left(X_t^i-Y_t^i\right)\sum_j\left(V'(X_t^i-X_t^j)-V'(Y_t^i-Y_t^j)\right)\\ &\qquad=\sum_{i>j}\left(V'(X_t^i-X_t^j)-V'(Y_t^i-Y_t^j)\right)\left(\left(X_t^i-Y_t^i\right)-\left(X_t^j-Y_t^j\right)\right)\\ &\qquad=\sum_{i>j}\left(V'(X_t^i-X_t^j)-V'(Y_t^i-Y_t^j)\right)\left(\left(X_t^i-X_t^j\right)-\left(Y_t^i-Y_t^j\right)\right). \end{aligned}$$

Here, notice that we obtain summands of the form $(V'(x)-V'(y))(x-y)$. Since $V'(x)=-1/x$ is an increasing function for positive x, it turns out that whenever $X_t^i>X_t^j$ and $Y_t^i>Y_t^j$, the summand is actually positive. So it only remains to control the cases in which $X_t^i=X_t^j$ or $Y_t^i=Y_t^j$

$$\begin{aligned} &\sum_i\left(X_t^i-Y_t^i\right)\sum_j\left(V'(X_t^i-X_t^j)-V'(Y_t^i-Y_t^j)\right)\\ &\qquad\geq\sum_{i>j\text{ s.t }Y_t^i=Y_t^j}V'(X_t^i-X_t^j)\left(X_t^i-X_t^j\right)+\sum_{i>j\text{ s.t }X_t^i=X_t^j}V'(Y_t^i-Y_t^j)\left(Y_t^i-Y_t^j\right)\\ &\qquad\geq\sum_{i>j\text{ s.t }Y_t^i=Y_t^j}-1+\sum_{i>j\text{ s.t }X_t^i=X_t^j}-1\\ &\qquad=-\frac{M(M-1)}{2}N-\frac{N(N-1)}{2}M. \end{aligned}$$

In summary, since we consider ordered sets of points, most summands disappear and we only have to count the number of repetitions. Hence, plugging this back into (2.3), taking the expectation and using Gronwall's lemma, we obtain the desired result

$$\mathbb{E}\left(\mathcal{W}_2\left(\mu_t^N,\mu_t^M\right)^2\right)\leq e^{-2\lambda t}\mathbb{E}\left(\mathcal{W}_2\left(\mu_0^N,\mu_0^M\right)^2\right)+\frac{C}{N\wedge M}.$$

Remark 2.4. *For $\alpha\in[1,2[$, we need to control quantities of the form*

$$\mathbb{E}\left(\int_0^t\sum_{i>j}\frac{1}{\left|X_s^{i,N}-X_s^{j,N}\right|^{\alpha-1}}ds\right),$$

which can be done separately and are typically the type of quantities that are considered when proving tightness of the empirical measures.

Remark 2.5. *This is thus quite a quick proof. However, we do not expect this method to work in other settings. It relies heavily on the geometry of the problem (the dimension one implies repulsive convex interaction, the particle stay in order, etc) and the fact that the noise disappears.*

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