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FRACTIONAL KELLER-SEGEL EQUATIONS

SAMIR SALEM AND LAURENT LAFLECHE

ABSTRACT. This note summarizes some results provided in the papers [14, 17], concerning the study of the fractional Keller-Segel model. This diffusion aggregation equation arises in the modeling of the chemotaxis motion of bacteria. The diffusion part consists in a fractional Laplacian, and the aggregation kernel is up to the Newtonian one. In the case where the aggregation and diffusion are well balanced, we present how this model can be obtained from an interacting particle system. Then we present some results about well-posedness of the model when the diffusion is not overtaken by the aggregation, and finite time blow-up in the opposite case.

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1. INTRODUCTION

1.1. Chemotaxis model. Biologists observed the following phenomenon. In an environment where *Dictyostelium discoideum* bacteria are cultivated, not long after the experimenter makes the bacteria colony starve by stopping the supply of the chemical substance they feed with, the colony starts to shrink into a sort of slug or mushroom. This formation is meant to help in the survival of a sufficient fraction of the colony, so that it can start colonizing a more suitable environment. The motion of the bacteria is based on chemotaxis, that is the motion of microorganisms toward an increasing or decreasing gradient of a chemical substance to which they are either attracted or repulsed. And this phenomenon is essential in our understanding of the formation of multicellular life beings.

The parabolic-elliptic Keller Segel equation is a simple mathematical model which relates this biological phenomenon. More precisely the evolution of the density of bacteria ρ_t and the concentration of chemoattractant c_t is given by the equation

$$\begin{aligned} \partial_t \rho_t + \chi \nabla \cdot (\rho_t \nabla c_t) &= \Delta \rho_t, \\ - \Delta c_t &= \rho_t, \end{aligned} \tag{1.1}$$

where $\chi > 0$ is a sensitivity parameter encoding the intensity of the attractiveness of the chemoattractant. We refer to [3] for a proper biological and mathematical motivation. This model has been extensively studied, especially in dimension 2 which is the best understood and which makes particular biological sense in the context of bacteria motion. Some blow up phenomena are known to arise if the initial mass is too large [3, Corollary 2.2], and global well posedness holds when the mass is small enough [7].

1.2. α -stable processes. Let $\alpha \in (0, 2)$ and M be a Poisson random measure (see for instance [6, Definition 2.3, Chapter V]) on $\mathbb{R}^+ \times \mathbb{R}^d$ of intensity $ds \times c_{d,\alpha} |x|^{-d-\alpha} dx$, where $c_{d,\alpha}$ is some normalization constant. Denote \bar{M} its compensated measure, i.e. $\bar{M}(ds, dx) = M(ds, dx) - ds \times c_{d,\alpha} |x|^{-d-\alpha} dx$, and denote $(Z_t)_{t \geq 0}$ the following Lévy process

$$Z_t = \int_{[0,t] \times \mathbb{R}^d} x \bar{M}(ds, dx). \tag{1.2}$$

Due to Ito's rule for jump processes [1, Theorem 4.4.7, p 226] we have for a test function ϕ smooth enough

$$\begin{aligned} \phi(Z_t) &= \phi(Z_s) + \int_{[s,t] \times \mathbb{R}^d} (\phi(Z_{u-} + x) - \phi(Z_{u-})) \bar{M}(du, dx) \\ &\quad + c_{d,\alpha} \int_s^t \int_{\mathbb{R}^d} \frac{\phi(Z_u + x) - \phi(Z_u) - x \cdot \nabla \phi(Z_u)}{|x|^{d+\alpha}} dx du. \end{aligned} \tag{1.3}$$

The process $(Z_t)_{t \geq 0}$ defined in (1.2) is an α -stable Lévy process, i.e. $(Z_t)_{t \geq 0}$ has the same law as $(u^{-1/\alpha} Z_{ut})_{t \geq 0}$ for any $u > 0$. Necessarily, such a process can only exist for $\alpha \in [0, 2]$ [6, Exercice 2.34, Chapter VI], the case $\alpha = 0$ corresponding to the null process, and the case $\alpha = 2$, to the standard Brownian motion. It is well known, but we also see from (1.3), that the infinitesimal generator of some α -stable Lévy process is the fractional Laplacian $-(-\Delta)^{\alpha/2}$ of exponent $\alpha/2 \in (0, 1)$, defined for smooth function $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ as

$$-(-\Delta)^{\alpha/2} \phi(z) = c_{d,\alpha} \int_{\mathbb{R}^d} \frac{\phi(z+x) - \phi(z) - x \cdot \nabla \phi(z)}{|x|^{d+\alpha}} dx, \tag{1.4}$$

(see [13] for equivalent definitions of the fractional Laplacian).

Some bacteria are known for their "run and tumble" motion, therefore their trajectories are better described by Lévy flights than Brownian motion (see for instance [4]). This inclines to replace the classical diffusion in the evolution equation of the density of bacteria with a fractional diffusion.

Therefore not only for the purpose of modeling, but also because of the recent popularity of fractional diffusion equation, the problem

$$\begin{aligned} \partial_t \rho_t + \chi \nabla \cdot (\rho_t \nabla c_t) + (-\Delta)^{\alpha/2} \rho_t &= 0, \\ (-\Delta)^{(4-a)/2} c_t &= \rho_t, \end{aligned} \tag{1.5}$$

has been studied under various perspectives by different authors in the case $a = 2$ (see for instance [12],[8],[4],[2]). The aim of this paper is to describe several results about equation (1.5) for general values of the couple (a, α)

2. DERIVATION OF THE MODEL FROM MANY PARTICLES SYSTEM

This section summarizes some results of [17]. We discuss here the derivation of the model (1.5) from an interacting particle system. Define K_a on \mathbb{R}^d as

$$K_a(x) = -\frac{x}{|x|^a},$$

so that the fractional Poisson equation in (1.5) can be rewritten

$$(-\Delta)^{(4-a)/2} c_t = \rho_t \iff \nabla c_t(x) = K_a * \rho_t(x)$$

For $(a, \alpha) \in (0, 2) \times (0, 2)$ and $N \geq 1$ let $(\mathcal{Z}_t^i)_{i=1, \dots, N, t \geq 0}$ be N independent α -stable Lévy flights on \mathbb{R}^d , (X_0^1, \dots, X_0^N) a random variable on \mathbb{R}^{dN} independent of the N Lévy flights and consider the particle system evolving on the plane defined as

$$X_t^{i,N} = X_0^i + \frac{\chi}{N} \int_0^t \sum_{j \neq i}^N K_a(X_s^{i,N} - X_s^{j,N}) ds + \mathcal{Z}_t^i, \quad i = 1, \dots, N. \tag{2.1}$$

In this note, we will deal only with the case $a = \alpha$, as is is shorter to describe, but more interesting results concerning the case $a < \alpha$ can be found in .

2.1. Chaos. For the sake of completeness we recall some basic notions on the topic of molecular chaos, and refer to [18] for some further explanations. We begin with the

Proposition 2.1 (Proposition 2.2 of [18]). *Let be $(u_N)_{N \geq 1}$ be a sequence of symmetric probabilities on E^N (E a polish space), $(X^1, \dots, X^N)_{N \geq 1}$ a sequence of random vector of law u_N , and $\mu_N = \frac{1}{N} \sum_{i=1}^N \delta_{X^i}$ the empirical measure associated to this vector. Then*

- (i) u_N is u chaotic if and only if $(\mu_N)_{N \geq 1}$ converges in law (weakly in $\mathcal{P}(E)$) toward $u \in \mathcal{P}(E)$.
- (ii) The sequence of random variables $(\mu_N)_{N \geq 1}$ is tight if and only if the sequence of law of X^1 under u_N is tight.

Our aim is to prove that the dynamic (2.1) propagates chaos i.e. that if one starts this dynamic from some initial condition which law is ρ_0 -chaotic, the law of the solution at time $t > 0$ to (2.1) is ρ_t -chaotic, with ρ_t the solution at time $t > 0$ to (1.5) starting from ρ_0 . Or equivalently, due to the above Proposition, to prove that

$$\mu_0^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_0^{i,N}} \xrightarrow[N \rightarrow +\infty]{*, (\mathcal{L})} \rho_0 \implies \mu_t^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,N}} \xrightarrow[N \rightarrow +\infty]{*, (\mathcal{L})} \rho_t.$$

If the second convergence holds only up to the extraction of a subsequence, i.e. we are interested in weaker result of the type

$$\mu_0^N = \frac{1}{N} \sum_{i=1}^N \delta_{X_0^{i,N}} \xrightarrow[N \rightarrow +\infty]{*,(\mathcal{L})} \rho_0 \implies \exists (N_k)_{k \geq 1} \quad \mu_t^{N_k} = \frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,N_k}} \xrightarrow[k \rightarrow +\infty]{*,(\mathcal{L})} \rho_t,$$

with ρ_t some solution at time $t > 0$ to (1.5) starting from ρ_0 , then we talk in this case, we talk of convergence/consistency rather than propagation of chaos.

2.2. Convergence/consistency of particle system (2.1). In the rest of this section, we aim at sketching the proof of a convergence/consistency result in the case $a = \alpha$. The key point in order to prove such a result for the particle system (2.1) is to get an estimation of the expectation of some singular function of the distance between the first and second particle (by exchangeability). The idea is to give a bound from below of the Ito's correction of the process $(|Z_t|^\varepsilon)_{t \geq 0}$ with $\varepsilon \in (0, 1)$ and $(Z_t)_{t \geq 0}$ some 2 dimensional and α -stable Lévy process.

In view of Proposition 2.1, it is enough in order to prove the desired convergence/consistency result, to show the tightness of the trajectory of the first particle of the system, namely

$$X_t^{1,N} = X_0^1 + \int_0^t \frac{\chi}{N} \sum_{j>1}^N K_a(X_s^{1,N} - X_s^{j,N}) ds + \mathcal{Z}_t^1 := X_0^1 + J_t^{N,1} + \mathcal{Z}_t^1.$$

The only sequential part of this process is $((J_t^{N,1})_{t \in [0,T]})_{N \geq 1}$, so that it is enough to show its tightness, to deduce the tightness of the law of $((\frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,N}})_{t \in [0,T]})_{N \geq 1}$, due to point (ii) of Proposition 2.1, i.e. there is a subsequence of $((\frac{1}{N} \sum_{i=1}^N \delta_{X_t^{i,N}})_{t \in [0,T]})_{N \geq 1}$ converging in law.

Then let be $0 < s < t < T$ and note that for any $p \in (1, \alpha/(\alpha - 1))$

$$\begin{aligned} |J_t^{N,1} - J_s^{N,1}| &\leq \frac{1}{N} \sum_{j>1}^N \int_s^t |X_u^{1,N} - X_u^{j,N}|^{1-a} du \\ &\leq |t - s|^{(p-1)/p} \frac{1}{N} \sum_{j>1}^N \left(\int_0^T |X_u^{1,N} - X_u^{j,N}|^{(1-\alpha)p} \right)^{1/p} \\ &\leq |t - s|^{(p-1)/p} \left(1 + \frac{1}{N} \sum_{j>1}^N \int_0^T |X_u^{1,N} - X_u^{j,N}|^{(1-\alpha)p} du \right) := |t - s|^\beta Z_{N,p}^T. \end{aligned} \tag{2.2}$$

The tightness of the $(J_t^{N,1})_{t \in [0,T]}$, easily follows from a bound of the expectation of $Z_{N,p}^T$ uniformly in N (see for instance [11, 10, 9]). Indeed for any $R > 0$, by Ascoli-Azerla's Theorem the set

$$\mathcal{K}_R := \left\{ f \in \mathcal{C}([0, T], \mathbb{R}^d), f(0) = 0, \sup_{0 \leq s < t \leq T} \frac{|f(s) - f(t)|}{|s - t|^\beta} \leq R \right\},$$

is compact. Then using Markov's inequality yields

$$\begin{aligned} \sup_{N \geq 1} \mathbb{P} \left((J_t^{N,1})_{t \in [0, T]} \notin \mathcal{K}_R \right) &= \sup_{N \geq 1} \mathbb{P} (Z_{N,p}^T \geq R) \\ &\leq R^{-1} \sup_{N \geq 1} \mathbb{E} [Z_{N,p}^T], \end{aligned}$$

and the sequence $((J_t^{N,1})_{t \in [0, T]})_{N \geq 1}$ would be tight by Prokhorov's Theorem, should the desired bound hold. In this purpose we have the

Proposition 2.2. *Let be $1 < \alpha = a < 2$ and $(X_t^{i,N})_{i=1, \dots, N, t \geq 0}$ a solution to equation (2.1) for an initial condition with law $(F_0^N)_{N \geq 1} \in \mathcal{P}_\kappa(\mathbb{R}^{dN})$ for some $\kappa \in (1, a)$. There exists $\chi_a > 0$ such that if $\chi \in (0, \chi_a)$ then it holds for any $T > 0$ and some $\varepsilon \in (0, 1)$*

$$\sup_{N \geq 1} \mathbb{E} \left[Z_{N, \frac{\alpha-1}{\alpha-\varepsilon}}^T \right] \leq \sup_{N \geq 1} \int_0^T \sup_{1 \leq i \neq j \leq N} \mathbb{E} [|X_u^{i,N} - X_u^{j,N}|^{\varepsilon-a}] du < \infty.$$

The proof of this proposition is based on [10] itself inspired by [16]. We sketch it below

Proof. Denote $Z_s^{i,j} := X_s^{i,N} - X_s^{j,N}$ note that it solves

$$\begin{aligned} Z_t^{i,j} &= Z_0^{i,j} - \frac{\chi}{N} \int_0^t \sum_{k \neq i,j}^N \left(\frac{Z_s^{i,k}}{|Z_s^{i,k}|^a} - \frac{Z_s^{j,k}}{|Z_s^{j,k}|^a} \right) ds \\ &\quad - \frac{2\chi}{N} \int_0^t \frac{Z_s^{i,j}}{|Z_s^{i,j}|^a} ds + \int_{[0,t] \times \mathbb{R}^d} x (\bar{M}_i - \bar{M}_j) (ds, dx). \end{aligned}$$

Since M_i and M_j are independent, the process $(\int_{[0,t] \times \mathbb{R}^d} x (\bar{M}_i - \bar{M}_j) (ds, dx))_{t \geq 0}$ is equal in law to the process $(\int_{[0,t] \times \mathbb{R}^d} 2^{1/a} x \bar{M}_i (ds, dx))_{t \geq 0}$. Then applying Ito's rule to $Z_s^{i,j}$ with $\phi(x) = |x|^\varepsilon$ for some $\varepsilon \in (0, 1)$ is not possible, since the ϕ defined so is not \mathcal{C}^2 (not even \mathcal{C}^a), but let us perform the computations for the sake of the sketch. We then have

$$\begin{aligned} \phi(Z_t^{i,j}) &= \phi(Z_0^{i,j}) - \int_0^t \frac{\chi \varepsilon}{N} |Z_s^{i,j}|^{\varepsilon-2} Z_s^{i,j} \cdot \left(\sum_{k \neq i,j}^N \left(\frac{Z_s^{i,k}}{|Z_s^{i,k}|^a} - \frac{Z_s^{j,k}}{|Z_s^{j,k}|^a} \right) \right) ds \\ &\quad - \int_0^t \frac{2\chi \varepsilon}{N} |Z_s^{i,j}|^{\varepsilon-a} ds \\ &\quad + \int_{[0,t] \times \mathbb{R}^d} (\phi(Z_{s-}^{i,j} + 2^{1/a} x) - \phi(Z_{s-}^{i,j}) - 2^{1/a} x \cdot \nabla \phi(Z_{s-}^{i,j})) M_i(ds, dx) \\ &\quad + \int_{[0,t] \times \mathbb{R}^d} 2^{1/a} x \cdot \nabla \phi(Z_{s-}^{i,j}) \bar{M}_i(ds, dx). \end{aligned}$$

Taking the expectation kills the last martingale term, replaces the Poisson measure $M_i(ds, dx)$ with its intensity $dsc_{d,a}|x|^{-d-\alpha} dx$ so that with the change of variables $2^{1/a} x \rightarrow x$

and definition (1.4) it holds

$$\begin{aligned} \mathbb{E}[|Z_t^{i,j}|^\varepsilon] &= \mathbb{E}[|Z_t^{i,j}|^\varepsilon] - \int_0^t \frac{\varepsilon\chi}{N} \mathbb{E}\left[|Z_s^{i,j}|^{\varepsilon-2} Z_s^{i,j} \cdot \left(\sum_{k \neq i,j}^N \left(\frac{Z_s^{i,k}}{|Z_s^{i,k}|^a} - \frac{Z_s^{j,k}}{|Z_s^{j,k}|^a}\right)\right)\right] ds \\ &\quad - \int_0^t \frac{2\chi\varepsilon}{N} \mathbb{E}[|Z_s^{i,j}|^{\varepsilon-a}] ds + 2 \int_0^t \mathbb{E}[-(-\Delta)^{a/2} \phi(Z_s^{i,j})] ds. \end{aligned} \quad (2.3)$$

Then using some Fourier's analysis yields for some constant $c_{\varepsilon,a} > 0$

$$-(-\Delta)^{a/2} \phi(x) = \varepsilon c_{\varepsilon,a} |x|^{\varepsilon-a}.$$

Then the symmetry of the roles played by the particles yields

$$\mathbb{E}\left[|Z_s^{i,j}|^{\varepsilon-2} Z_s^{i,j} \cdot \left(\sum_{k \neq i,j}^N \left(\frac{Z_s^{i,k}}{|Z_s^{i,k}|^a} - \frac{Z_s^{j,k}}{|Z_s^{j,k}|^a}\right)\right)\right] \leq 2(N-2) \mathbb{E}[|Z_s^{i,j}|^{\varepsilon-a}],$$

so that in the end we obtain

$$\begin{aligned} \mathbb{E}[|Z_0^{i,j}|^\varepsilon] + 2\varepsilon(c_{\varepsilon,a} - \chi) \int_0^t \mathbb{E}[|Z_s^{i,j}|^{\varepsilon-a}] ds &\leq \mathbb{E}[|Z_t^{i,j}|^\varepsilon] \\ &\leq \mathbb{E}[|X_t^{i,N}|^\kappa] + \mathbb{E}[|X_t^{j,N}|^\kappa], \end{aligned}$$

and the result is proved thanks to the moment estimate, provided that

$$\chi < \chi_a = \sup_{\varepsilon \in (0,1)} c_{\varepsilon,a}. \quad \square$$

3. STUDY OF THE CONTINUUM MODEL

This section summarizes some results of [14]. We discuss here some aspects of equation (1.5) depending on the set of parameters $(\alpha, a) \in [0, 2) \times [0, d)$. We shall distinguish three different regimes:

- if $a < \alpha$, we are in the diffusion dominated case,
- if $a = \alpha$, we are in the fair competition case,
- if $a > \alpha$, we are in the aggregation dominated case.

These appellations are inspired from [5], where a nonlinear diffusion is considered.

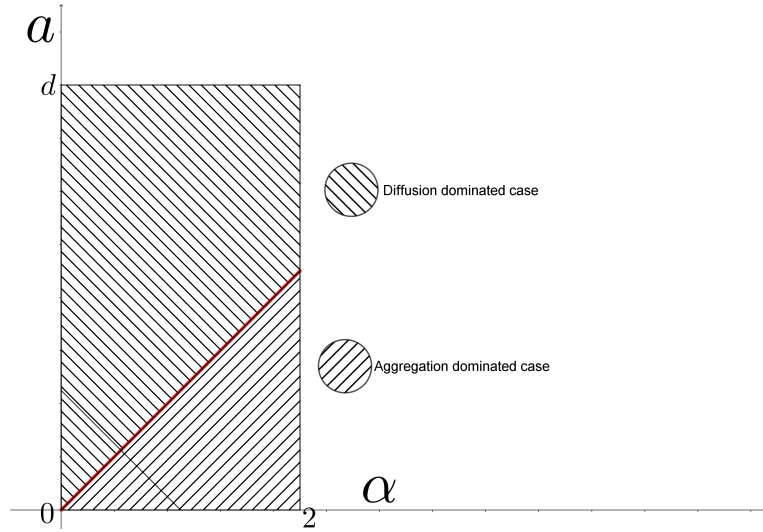


FIGURE 1. The three different regimes in the plane (α, a) . The set $\alpha + a < 1$ has to be excluded, see Proposition (3.1) below.

For the rest of the section we will denote the fractional Laplacian operator of exponent $\alpha/2$ defined by 1.4 by $\Delta^{\frac{\alpha}{2}}$, without loss of generality. We define

$$M_k = \int_{\mathbb{R}^d} \rho(x) \langle x \rangle^k dx,$$

L_k^1 the set of the $\rho \in L^1$ such that $M_k < \infty$ and $L \ln L$ the set of the $\rho \in L^1$ such that

$$\int_{\mathbb{R}^d} \rho \ln(\rho) < \infty.$$

3.1. Propagation of moments. The fractional Laplacian induces restrictions on the behaviour at infinity. In particular, the integration by parts fails when one tries to compute the derivative of moments of order greater than the order of the fractional Laplacian. However, we can still prove the propagation of moments of low order, provided that $a + \alpha > 1$, as stated in the next proposition.

Proposition 3.1. *Assume $a < 2$ if $\alpha < 1$, $k \in [(1 - a)_+, \alpha)$ and the initial condition verifies $\rho^{\text{in}} \in L_k^1$. Then*

$$\rho \in L_{\text{loc}}^\infty(\mathbb{R}_+, L_k^1).$$

The proof consists in differentiating moments of order k of the form and using Gronwall's inequality. It uses the fact that for such weight functions it holds

$$\Delta^{\frac{\alpha}{2}}(\langle x \rangle^k) \leq C_{d,k,\alpha} \langle x \rangle^{k-\alpha},$$

for $k \in (0, \alpha)$.

3.2. Lebesgue norms estimates. In the diffusion dominated case $\alpha > a$, we have an immediate regularization in stronger Lebesgue spaces even if the initial data is only in L^1 initially.

In the aggregation dominated case, it seems no more possible to get such gains of regularity, and two distinct behaviours appear. In the case of an initial condition with a small initial particular Lebesgue norm, the solution will remain in this Lebesgue space and spread out in the sense that this Lebesgue norm will converge to 0. If the solution is in this space but is not small enough, blow-up can occur in certain cases and one can only prove the boundedness in Lebesgue spaces on a given finite time interval depending on initial conditions.

The critical case $\alpha = a$ is a sort of mix of the two above behaviours, since we obtain both a regularization property from the L^1 space to some L^p space for $p > 1$, and a smallness condition on the mass that induces two behaviours. In the case of small initial mass, we again find the convergence to 0 in the L^p norm, and in the converse case, we could only prove a local in time propagation of the L^p norm. However, it remains an open problem to know whether or not a blow-up can occur or if the L^p norm remains bounded for larger times in this situation. These results can be summarized as follows (see Proposition 3.3 in [14]):

Proposition 3.2.

- When $a < \alpha$ and $p = q' \in (1, p_a)$, it holds

$$\|\rho\|_{L^p} \leq CM_0 \max(t^{-d/\alpha q}, M_0^{d/q(\alpha-a)}), \quad (3.1)$$

where $C > 0$ is a constant depending on d, a, α, p and χ .

- When $a > \alpha$, then for any $p \in (p_{a,\alpha}, p_a)$, there exists two constants $C = C_{a,\alpha,p}M_0(\chi M_0)^{-d/(a-\alpha)q}$ and $C^{\text{in}} = C_{a,\alpha,p}(\|\rho^{\text{in}}\|_{L^p})$ such that

$$\|\rho^{\text{in}}\|_{L^p} < C \implies \|\rho\|_{L^p} \leq C^{\text{in}}M_0t^{-d/\alpha q} \quad (3.2)$$

$$\|\rho^{\text{in}}\|_{L^p} > C \implies \rho \in L^\infty((0, T), L^p) \quad (3.3)$$

$$\|\rho^{\text{in}}\|_{L^p} = C \implies \rho \in L^\infty(\mathbb{R}_+, L^p), \quad (3.4)$$

where $T < C_{a,\alpha,p}(\chi, M_0)\|\rho^{\text{in}}\|_{L^p}^{-pb}$ with $b := \alpha/(p(\alpha - a) + d(p - 1))$.

- When $a = \alpha$, then there exists a constant $C_{a,d,p} > 0$, such that for any $p \in (1, p_a)$,

$$\chi M_0 \leq C_{a,d,p} \implies \|\rho\|_{L^p} \leq M_0(C^{\text{in}}b)^{-1/b}t^{-d/\alpha q} \quad (3.5)$$

$$\chi M_0 \geq C_{a,d,p} \implies \rho \in L^\infty((0, T), L^p), \quad (3.6)$$

where C^{in} is a nonnegative constant depending on the initial data and

$$T > \frac{1}{bC^{\text{in}}} \left(\frac{M_0}{\|\rho^{\text{in}}\|_{L^p}} \right)^{\alpha q/d}.$$

Again, the proof relies on a Gronwall's estimate. It consists in trying to use the dissipation of L^p norm coming from the fractional Laplacian in the spirit of Nash's inequality to control the nonlinear term, the latter being controlled by Lebesgue norms thanks to Hardy-Littlewood-Sobolev's inequality.

3.3. Well-posedness. Our results on well-posedness depends as for the Lebesgue estimates on the relative position of α and a . In all the cases we assume some initial moments by taking $\rho^{\text{in}} \in L_k^1$ with $k \in (0, \alpha)$. We obtain the following results.

Theorem 3.1.

- *When $a < \alpha$, then there exists a unique and global solution.*
- *When $a = \alpha$ then there exists a unique global solution if $\rho^{\text{in}} \in L \ln L$ and $\chi M_0 < C_{a,d}$ for a universal constant $C_{a,d} > 0$ and there exists a unique solution on $[0, T)$ for a time $T > 0$ depending on the initial condition if $\rho^{\text{in}} \in L^p$ for a given $p > 1$.*
- *When $a > \alpha$ and $\rho^{\text{in}} \in L^p$ with $p \in (p_{a,\alpha}, p_a)$, then again there exists a unique solution on $[0, T)$ for a time $T = T_{\rho^{\text{in}}} > 0$ depending on the initial condition and there is a constant $C_{\chi,p}(M_0)$ such that if*

$$\|\rho^{\text{in}}\|_{L^p} \leq C_{\chi,p}(M_0), \quad (3.7)$$

then the solution is global.

The proof of the existence and the uniqueness relies on an approximation argument and a stability estimate in the Wasserstein distance W_2 , reminiscent of the proof of uniqueness for the Vlasov equation by G.Loeper [15]. This estimate requires an a priori estimate on the $L^1((0, T), L^{p_a})$ norm of the solution, which, in the sub- and super-critical cases can be proved rather straightforwardly by using the above Proposition 3.2, coming back to the equation and integrating in time the variation of the L^p norm.

In the critical case $a = \alpha$, a good condition of regularity for the initial space density is to assume finite entropy. In this case, in the same spirit as for the estimates in Lebesgue norms, we get the following bound on the entropy

$$\int_{\mathbb{R}^d} \rho \ln(\rho) + 4C_{a,d}^{-1} (\chi M_0 - C_{a,d}) \int_0^t |\sqrt{\rho}|_{H^{\frac{a}{2}}}^2 \leq \int_{\mathbb{R}^d} \rho^{\text{in}} \ln(\rho^{\text{in}}),$$

where $C_{a,d}$ can be evaluated in terms of the optimal constants for the Gagliardo-Nirenberg's and Hardy-Littlewood-Sobolev's inequalities.

From this inequality and Sobolev's embeddings, we obtain a bound on the $L^1((0, T), L^{p_a})$ norm in case of small mass also in the critical case. All these cases can be summarized by writing that the solution to (1.5) satisfies

$$\rho \in L^1((0, T), L^{p_a}), \quad (3.8)$$

as soon as the initial condition ρ^{in} verifies

- $\rho^{\text{in}} \in L^1$ if $a < \alpha$
- $\rho^{\text{in}} \in L_k^1 \cap L \ln L$ and M_0 is small if $a = \alpha$
- $\rho^{\text{in}} \in L^p$ if $a > \alpha$.

Remark that we can take T as large as we want when the L^p estimates are global in time.

3.4. Blow-up. In the aggregation case $a > \alpha$, under fixed mass and large L^p norm, only a local in time existence can be proved, and this is optimal since we can prove that a

blow-up occurs (i.e. the solution cannot remain in L^1) if the initial data ρ^{in} is even and concentrated enough in the sense that

$$\int_{\mathbb{R}^d} \rho^{\text{in}} \langle x \rangle^k \leq C^* \chi^{k/2(a-k)} M_0^{(2a-k)/2(a-k)}, \quad (3.9)$$

for a given constant $C^* = C_{d,a,\alpha,k}$. The proof consists in introducing a particular kind of weight

$$m(x) := 1 + \varphi(|x|)|x|^a + \varphi^c(|x|)|x|^k.$$

where $\varphi \in C_c^\infty$ is an even nonincreasing cut-off function, and showing that if a solution ρ exists for a sufficiently large time T , then

$$\int_{\mathbb{R}^d} \rho m \xrightarrow{t \rightarrow T^*} 0,$$

for a finite time $T^* < T$, which is a contradiction. One of the ingredients of the proof is the fact that for $a > \alpha$, $|\Delta^{\frac{\alpha}{2}}(|x|^a \varphi)| \leq C \langle x \rangle^{-(d+\alpha)}$ and $|\Delta^{\frac{\alpha}{2}}(|x|^k \varphi^c)| \leq C \langle x \rangle^{k-\alpha}$ which implies that

$$\Delta^{\frac{\alpha}{2}}(m) \leq C \langle x \rangle^{k-\alpha}.$$

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