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SELF-SIMILARITY IN THE SINGULARITY FORMATION FOR THE UNSTEADY PRANDTL'S EQUATIONS AND RELATED PROBLEMS

CHARLES COLLOT

Abstract. The main issue of this text is the singularity formation problem for the two dimensional Prandtl's system on the upper half plane, as well as for related models. The scaling invariance of the equation is partly responsible for the appearance of a self-similar phenomenon. It involves the formation of a shock along the tangential direction, together with an expansion along the transversal component due to incompressibility.

1. Introduction

We consider Prandtl's equations in two dimensions on the upper half-plane:

$$\begin{cases} u_t - u_{yy} + uu_x + vu_y = -p_x^E \qquad (t,x,y) \in [0,T) \times \mathbb{R} \times \mathbb{R}_+, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0, \qquad u|_{y\to\infty} = u^E, \end{cases} \tag{1.1}$$

where u is the tangential (with respect to the boundary $\mathbb{R}\times\{0\}$) velocity and v the normal velocity. The trace at the boundary of the tangential component of the outer inviscid flow and the pressure are denoted by u^E and p^E. This equation is used to describe the behaviour of a fluid close to a boundary at large Reynolds numbers, as a formal limit of the Navier-Stokes equation. For more on the derivation of these equations, we refer to [45, 46, 37, 48].

In [7, 8], we investigated the singularity formation problem for (1.1) and related problems. Our analysis brings a new perspective on (1.1), and also on the usual Burgers equation, the inviscid Prandtl's equations, and Burgers equation with transverse viscosity. The aim of the present document is to sum up these results, and present them in a way that explains the role of each term in (1.1). The spirit behind the study is that of the obtention of blow-up profiles, which describe the solution to leading order near the singularity.

Early reliable numerical evidences for the possibility of blow-ups for (1.1) were obtained by Van Dommelen and Shen [50]. Additional numerical and formal results have been obtained since then, see [10, 19, 27, 29, 44] and references therein. However, there is still no precise description of the singularity formation. E and Engquist [14] proved the existence of finite time blow-up for certain solutions of (1.1), and Theorems 4.3 and 4.4 here can be viewed as a refinement of their result. Their approach by contradiction does unfortunately not provide a description of the singularity formation. For a more general class of outer inviscid flows, the existence of singularity formation was proved in [35], where also no precise details about the singularity are available.

Many other interesting issues arise in the study of Prandtl's equations. Local existence of solutions is obtained within the analytical setting [45, 36, 34, 13]. If the solution satisfies in addition some monotonicity properties, the regularity requirement can be lowered and local well-posedness is established in Sobolev regularity [43, 38, 1] and global weak solutions were constructed in [53]. The singular solutions we consider here do not possess this monotonicity property. In this case, (1.1) can be ill-posed in the Sobolev setting [20, 24]. Related instabilities can prevent the Prandtl's equations from being a good approximation of the Navier-Stokes equations in the inviscid limit [23]. Indeed, either the monotonicity property and/or Gevrey regularity in the tangential variable x are necessary to insure that approximation, see [45, 21]. Finally, in the steady case solutions might also be singular, which is the so-called Goldstein singularity, and which has been rigorously established recently in [11]. We believe this singularity is different than the singularities of the unsteady case.

The document is organised as follows. In Section 2 we describe the singularity formation for Burgers equation, which encodes the tangential compression in (1.1). Then we investigate how singularities of the unsteady Prandtl's equation might form in Section 3. The main result of this document concerning (1.1) is then stated in Theorems 4.3 and 4.4 of Section 4, and a sketch of proof is given. Finally, a full description of the singularity formation is given for Burgers equation with transverse viscosity in Section 6.

2. Compression along the tangential variable: Burgers equation

Taking only in account the horizontal convection in (1.1), Prandtl's equations reduce to the inviscid Burgers equation, sometimes also called Hopf equation. This is the simplest model for nonlinear one-dimensional wave propagation:

$$U_t + UU_x = 0, \quad U(0,x) = U_0(x) \tag{2.1}$$

for $x \in \mathbb{R}$ and $U : I \times \mathbb{R} \to \mathbb{R}$. This equation, as well as other nonlinear hyperbolic systems, has been extensively studied, see [12, 39, 47]. The discussion here borrows mostly from [8], and from [15]. We would like to emphasise the roles played in singularity formation by the scaling invariances and by the strong "locality" of the equation. There should be some similarities with the singularity formation for the unsteady Prandtl's system (1.1), as we will point out in Section 4.

The invariances are the following. If U is a solution to (2.1) then so is

$$\frac{\mu}{\lambda} U\left(\frac{t-t_0}{\lambda}, \frac{x-x_0-ct}{\mu}\right) + c,$$

for any time and space scales $\lambda, \mu > 0$, time and space shifts parameters t_0, x_0 and Galilean transformation of speed $c \in \mathbb{R}$. The strong locality lies in the transport aspect of the equation. Namely, the equation can be solved using the method of characteristics, which are straight lines along which the solution u is conserved. There indeed holds the following formula for solutions to (2.1):

$$U(t,x) = U_0(\Phi_t^{-1}(x)), \quad \Phi_t(X) = X + tU_0(X), \tag{2.2}$$

Φ_t being the diffeomorphism from Lagrangian coordinates X towards Eulerian coordinates x. From the above formula, one easily sees that finite time singularity happens under the condition $\inf_x \partial_x U_0 < 0$, as Φ_t will cease to be a diffeomorphism.

If we assume that U_0 is C^1 and compactly supported for example, then the blow-up time is $T = (-\min(\partial_x U_0))^{-1}$ and one has wave steepening, $\partial_x U \to +\infty$ as $t \to T$, announcing a shock formation.

Shocks in one-dimensional hyperbolic equations are mostly studied so that unique global weak solutions can be constructed. The formation of shocks is more difficult to study for higher order hyperbolic systems or in higher dimensions. Two points of view on shock formation exist. The first one, starting from the seminal work [30] is to study how the solution is compressed by differentiating the gradient of the solution. Here, along the characteristics:

$$\frac{d}{dt}\partial_x U(t, \Phi_t(X)) = -|\partial_x U(t, \Phi_t(X))|^2 \tag{2.3}$$

which makes the Ricatti equation $b_t = -b^2$ appear. This equation has indeed finite time blow-up solutions of the form $b = -(T-t)^{-1}$ and shows how the gradient of the solution becomes unbounded. Another point of view is the study of the geometry of the characteristics, and how they degenerate, see [2, 6]. Here we propose another one, in which the focus is put on the existence and stability of self-similar blow-up profiles. Note that these profiles were sometimes implicitly used in some works on the vanishing viscosity limit of Burgers equation [28].

Among all finite time blow-up solutions, the backward self-similar ones are those enjoying extra symmetries. Thanks to the invariances of the equation, one can always choose the blow-up time to be zero, and the characteristic on which the shock is forming to be stationary at the origin, which is summed up in the following hypothesis (HP):

$$(HP) \quad T = 0, \quad \min U_x(-1, x) = U_x(-1, 0) = -1 \quad \text{and} \quad U(-1, 0) = 0$$

(we choose -1 as a reference negative time here but it can be any other negative time without loss of generality). Self-similarity refers to such solutions that are left invariant by one scaling transformation, i.e., with a nontrivial stabilizer:

$$\mathcal{G}_s := \left\{ (\lambda, \mu) \in (0, +\infty)^2, \ \ \frac{\mu}{\lambda} U\left(\frac{t}{\lambda}, \frac{x}{\mu}\right) = U(t, x) \ \forall t, x \right\} \neq \{(1, 1)\}.$$

For Burgers equation (2.1), these solutions can be classified.

Proposition 2.1 ([8]). *Consider $U \in \mathcal{C}^1$ a solution of* (2.1) *which satisfies* (HP), *such that $U \neq x/t$. Then $\mathcal{G}_s$ is non-trivial if and only if one the following holds.*

(i) *Self-similarity (SS): There exists $i > 0$ with*

$$U(t, x) = \mu^{-1}(-t)^{1/2i}\Psi_i\left(\mu\frac{x}{(-t)^{1+1/2i}}\right), \ \ \mu > 0,$$

where Ψ_i is a profile which is analytic if $i \in \mathbb{N}^*$, else C^{1+2i}.

(ii) *Discrete self-similarity (DSS): There exists $i > 0$, $\lambda > 1$ such that*

$$U(t, x) = \lambda^{k/2i}U\left(\frac{t}{\lambda^k}, \frac{x}{\lambda^{k(1+1/2i)}}\right), \quad \forall k \in \mathbb{Z},$$

and $U \notin C^{1+2i}$.

Remark 2.2.

- It is remarkable to see on such a simple equation already a vast range of behaviours. There are several possible scaling laws, which has to do with the fact that the scaling group is two-dimensional. The self-similarity is then said to be of the second type [3].
- The scaling laws are clearly linked to the regularity of the profiles. This is an issue for nonlinear wave equations in higher dimensions [33, 32]. Regularity brings rigidity, since there is only a countable number of analytic profiles.
- For discrete self-similarity, U must have some kind of fractal behaviour near the point where the shock will form, which is highly instable. For other equations, discrete self-similar blow-up is unfortunately still poorly understood.

Proof. The proof involves direct computations using the characteristics. We refer to [8] for further developments regarding Burgers equation. □

We now restrict ourselves to the case of smooth solutions, and focus on the role played by the family of analytic profiles.

Lemma 2.3. *The profile $\Psi_i : \mathbb{R} \to \mathbb{R}$ for $i \in \mathbb{N}$ involved in Proposition 2.1 is odd, analytic and solves the implicit equation:*

$$X = -\Psi_i - \Psi_i^{2i+1}. \tag{2.4}$$

It is decreasing, positive on $(-\infty, 0)$ and admits the development near the origin $\Psi_i(X) \approx -X + X^{2i+1}$ and $\Psi_i(X) \sim \mp |X|^{1/(2i+1)}$ as $X \to \pm\infty$.

These profiles are important thanks to the fact that they are the attractors of the singularity formation. It is worth noting that they appear as profiles for compactly supported solutions even though they are unbounded at infinity.

Proposition 2.4. *Let $U_0 \in C^\infty(\mathbb{R})$ be such that $\partial_x U_0$ is minimal at x_0 with*

$$U_0(x_0) = c, \quad \partial_x U_0(x_0) < 0, \quad \partial_x^j U_0(x_0) = 0 \ \text{ for } j = 2, ..., 2i, \quad \text{and } \ \partial_x^{2i+1} U_0(x_0) > 0$$

for some $i \in \mathbb{N}$. Then the associated solution of (2.1) U blows up at time $T = -1/U_x(x_0)$ at the point $x_\infty = x_0 + cT$ with:

$$U(t,x) = \mu^{-1}(T-t)^{\frac{1}{2i}} \Psi_i \left(\mu \frac{x - x_0 - ct}{(T-t)^{1+1/2i}} \right) + c + w(t,x) \tag{2.5}$$

for some $\mu, c \in \mathbb{R}$, where the remainder is negligible

$$\frac{w}{(T-t)^{1/2i} \Psi_i \left(\mu \frac{x - x_0 - ct}{(T-t)^{1+1/2i}} \right)} \to 0 \quad \text{as } (x,t) \to (x_\infty, T).$$

Proof. Again, the proof involves direct computations and we refer to [8]. □

To sum up, the picture for Burgers equation is the following. Smooth solutions may become singular in finite time. When they do so, a profile appears near the singularity, which belongs to a universal countable family of backward self-similar profiles. The structure of these profiles is intimately linked to the scaling invariance of the equation, and to the limited development near the characteristics where the shock will form.

3. On singularity formation for the inviscid Prandtl's system

Having in mind the singularity formation scenarios described in the previous Section 2 for Burgers equation, we now move to what happens for the inviscid two-dimensional Prandtl's system. It corresponds to (1.1) without the normal viscosity:

$$\begin{cases} u_t + uu_x + vu_y = 0, \\ u(t,x,0) = 0 = v(t,x,0), \quad u_x + v_y = 0 \end{cases} \tag{3.1}$$

and with a vanishing outer flow for simplification. This is a nonlinear hyperbolic transport equation, and we claim that the singularity formation here is somehow similar to the case of Burgers equation (2.1). This is ongoing work and will be the subject of an upcoming article, so that we just give partial results here. The locality is reflected in the fact that the equation can be solved using characteristics, see for example [27]. We now explain the characteristics of (3.1). In (3.1), the tangential velocity is u, and u is transporting itself along the tangential direction. Hence, given a particle with initial position (X,Y) (in Lagrangian variables), the tangential coordinate in Eulerian variables at time t of this particule will be $x = X + tu_0(X,Y)$. The tangential displacements are thus those of Burgers equation (2.2). One then retrieves the normal displacements by using the fact that the characteristics must preserve volume since (u,v) is divergence free. The characteristics $x(t,X,Y)$ and $y(t,X,Y)$ are then given by the following formula:

$$x = \xi[t,Y](X) = X + tu_0(X,Y), \quad y = \int_0^\eta \frac{dZ}{1 + tu_{0x}(\xi^{-1}[t,Z](x(t,X,Y)),Z)}.$$

Above, $\xi[t,Y] : X \mapsto X$ is thought of as a mapping where t and Y are fixed parameters, which to X associates the value $x(t,X,Y)$. $\xi^{-1}[t,Y](x)$ is then its inverse as a function of X alone. We see from the above formula that the solution becomes singular under the condition $\inf_x \partial_x U_0 < 0$, and that the singularity occurs at time $T = (-\inf_x \partial_x U_0)^{-1}$. This is the same as for Burgers equation (2.1). We claim that there exist backward self-similar solutions for (3.1). Here, we only state the existence of a particular profile which is involved in our main Theorem 4.3 concerning the original Prandtl's system (1.1).

Proposition 3.1. *There exists a function* $\Theta \in C^1(\mathbb{R} \times [0,+\infty), \mathbb{R})$*, such that*

$$u(t,x,y) = (T-t)^{1/2} \Theta\left(\frac{x}{(T-t)^{3/2}}, \frac{y}{(T-t)^{-1/2}}\right) \tag{3.2}$$

is a solution of (3.1) *for any* $T \in \mathbb{R}$*. The profile* Θ *is odd in* X*, with* $\Theta(X,Y) \geq 0$ *for* $X \leq 0$ *and* $\Theta(X,Y) \leq 0$ *for* $X \geq 0$*. In addition,* $\inf_X \partial_X \Theta$ *being attained only at* $(0,\pi)$ *with:*

$$\Theta(X, \pi + Y) = -X + X^3 + \frac{Y^2}{4} X + O(|X|^5 + |Y|^4 X) \quad \text{as } (X,Y) \to (0,0).$$

Moreover, the restriction of the tangential derivative on the vertical axis satisfies:

$$\partial_X \Theta(0,Y) = -\sin^2(Y/2)\, 1_{0 \leq Y \leq 2\pi}. \tag{3.3}$$

Proof. For the proof of the above Proposition and additional properties of the profile Θ, we refer to the upcoming article. □

For the solution (3.2), what happens is the following. The flow compresses particles onto the vertical axis $X = 0$, with a typical tangential scale of $x \sim (T-t)^{3/2}$ for the singularity which is the same as for the stable blow-up of Burgers equation ((2.5) with $i = 1$). This results in an expansion along the normal direction thanks to incompressibility, with a typical normal scale of $y \sim (T-t)^{-1/2}$. The blow-up point's position is $(0, \pi/\sqrt{T-t})$, so this point is ejected toward infinity in finite time.

Note that the profile Θ in (3.2) does not correspond to the so-called terminal profiles of [16, 49], and the normal scale here of $y \sim (T-t)^{-1/2}$ is different than there which is $y \sim (T-t)^{-1/4}$. This has to do with the fact that Θ is not the generic profile due to the particular symmetry of oddness in X.

The existence of other backward self-similar profiles for (3.1), as well as the proof of stability and universality for the induced blow-up dynamics, will be treated in a forthcoming article. We end this section here by raising the following question: can the inviscid blow-up dynamics induced by the profile Θ and described above appear for the full original Prandtl's system (1.1)? Section 4 gives a partial positive answer to that question, and Section 6 gives a complete answer for the simplified model of Burgers equation with transverse viscosity.

4. Main result for the unsteady Prandtl's system

We describe in this section our main result for the system (1.1). We show precisely for a particular class of solutions how tangential compression happens. This can be interpreted as a stability result for the inviscid blow-up dynamics induced by the self-similar profile Θ described in Proposition 3.1, even in the presence of transverse viscosity. Countably many other unstable dynamics are also found. Let us assume that there is no outer flow for simplicity, $u^E = 0$ and $p^E = 0$ so that (1.1) can be rewritten as:

$$\begin{cases} u_t - u_{yy} + uu_x + vu_y = 0 \qquad (t,x,y) \in [0,T) \times \mathbb{R} \times \mathbb{R}_+, \\ u_x + v_y = 0, \\ u|_{y=0} = v|_{y=0} = 0. \end{cases} \tag{4.1}$$

Our result is also hold true for a class of outer flows preserving oddness in x for the solution, as will be mentioned in the remarks after the main Theorems 4.3 and 4.4. indeed, the presence of a nontrivial outer flow gives rise to forcing terms. As it is the case in many blow-up dynamics, though such forcing terms might play a role in the onset of the singularity formation, near the blow-up time a universal blow-up mechanism takes place and they cease to play a role, being of a lower order.

We now want to study tangential compression via looking at the gradient of a solution, having the identity (2.3) for Burgers equation in mind. If u solves (4.1) and is odd in x, the trace of the derivative $\xi(t,y) := -u_x(t,0,y)$ solves on the vertical ray $y \in [0,+\infty)$:

$$\begin{cases} \xi_t - \xi^2 + \left(\int_0^y \xi(t,\tilde{y})d\tilde{y}\right)\xi_y - \xi_{yy} = 0, \\ \xi(t,0) = 0. \end{cases} \tag{4.2}$$

E and Engquist [14] showed that some solutions to (4.2) blow up in finite time using a convexity argument, without describing the singularity on the vertical axis, and also outside the axis. Note that in (4.2), the singular dynamics is due to the quadratic term ξ^2, while the solution also undergoes dissipation and a nonlinear and nonlocal transport.

Without the nonlocal nonlinear transport term and the boundary condition, (4.2) reduces to the nonlinear heat equation:

$$\xi_t - \xi^2 - \xi_{yy} = 0. \tag{4.3}$$

The blow-up dynamics for the above equation is now completely understood. We state the following result in the setting of even function to avoid taking into account the position of the blow-up point.

Theorem 4.1 ([4, 22, 26, 40, 51])**.** *There exists an open set of even functions in $L^\infty(\mathbb{R})$ of solutions to* (4.3) *that blow-up at some time $T>0$ with as $t\to T$:*

$$\xi(t,y)=\frac{1}{T-t}\,\frac{1}{1+\frac{y^2}{8(T-t)|\log(T-t)|}}+O_{L^\infty(\mathbb{R})}\left(\frac{1}{(T-t)|\log(T-t)|^\eta}\right),\quad \eta>0,$$

and instable solutions for each $k\in\mathbb{N}$, $k\geq 2$ such that as $(t,y)\to(T,0)$:

$$\xi(t,y)=\frac{1}{T-t}\,\frac{1}{1+ay^{2k}/(T-t)}+O_{L^\infty(\mathbb{R})}\left(\frac{\left((T-t)^{1/2k}+|y|\right)^{1/2}}{T-t+y^{2k}}\right),\quad a>0.$$

We will show that the blow-up dynamics of (4.2) is deeply affected by the addition of nonlinear transport, when compared to that of (4.3) described in the Theorem above. But first, let us give a local well-posedness result for (4.2). If ξ is a solution, then so is $\lambda^2\xi(\lambda^2 t,\lambda y)$. The problem is then $L^{\frac12}$ critical. The problem might then be well-posed in supercritical Lebesgue spaces, and this is indeed the case for L^1.

Proposition 4.2. *Let $\xi_0\in L^1([0,+\infty))$. There exists $T(\|\xi_0\|_{L^1})>0$ and a unique solution to* (4.2) *in Duhamel formulation such that $\xi\in C([0,T],L^1([0,+\infty)))$, $\xi(0,\cdot)=\xi_0(\cdot)$ and $\|\partial_y\xi(t)\|_{L^1}\lesssim t^{-1/2}$. Moreover, there holds the regularisation $\xi\in C^\infty((0,T]\times[0,+\infty))$ and for each $k\in\mathbb{R}$, $\partial_y^k\xi\in C((0,T],L^1([0,+\infty)))$. For any $k\in\mathbb{N}$ and $0<T_1\leq T$, the flow is locally uniformly continuous from L^1 into $C([T_1,T],W^{k,1}[0,+\infty))$. Finally, there holds the blow-up criterion for the maximal time of existence T:*

$$T<+\infty \quad \textit{iff} \quad \limsup_{t\uparrow T}\|u(t)\|_{L^\infty([0,+\infty))}=+\infty.$$

Proof. The above proposition can be proved by adapting the ideas of [52]. □

We now turn to our main result. First, note that the reduced one dimensional problem (4.2) with a different domain and different boundary conditions was also studied in [18], appearing for a particular class of infinite energy solutions to the three dimensional Navier-Stokes equations. In [18], the authors investigate the existence of a similar stable blow-up dynamics as described below, and for a particular class of solutions. The approach however is based on different techniques, involving parabolic regularity, maximum principles and comparison principles. It allows for a non-perturbative argument, but also requires numerous additional assumptions. As a consequence, their argument cannot be applied to the problem that we consider here. Moreover, our approach, which is based on energy methods, is more robust, and allows us to prove the stability of the fundamental profile, as well as to construct unstable blow-ups and to derive weighted estimates. The first result, for the stable dynamics (within the symmetry class), is the following.

Theorem 4.3 (Stable blow-up for Equation (4.2) [7])**.** *There exists $\lambda_0^*\gg 1$ such that for all $\lambda_0\geq\lambda_0^*$, an $\epsilon(\lambda_0)>0$ exists such that if initially:*

$$\xi_0(y)=\lambda_0^2\cos^2\left(\frac{y-\lambda_0\pi}{2\lambda_0}\right)1_{0\leq y\leq 2\lambda_0\pi}+\tilde{\xi}_0(y),\quad \textit{with}\quad \|\tilde{\xi}_0\|_{L^1([0,+\infty))}\leq\epsilon(\lambda_0),$$

then the solution to (4.2) *blows up at some time $T>0$, with:*

$$\xi(t,y)=\lambda^2\cos^2\left(\frac{y-y^*(t)}{2\lambda(t)\mu(t)}\right)1_{-\pi\leq\frac{y-y^*}{\lambda\mu}\leq\pi}+\tilde{\xi},$$

where, for some $\mu_\infty > 0$*:*

$$\lambda(t) = \frac{1 + O((T-t)^{3/2})}{\sqrt{T-t}}, \quad \mu(t) = \mu_\infty + O((T-t)), \quad y^*(t) = \frac{\mu_\infty \pi + O((T-t)^{1/4})}{\sqrt{T-t}},$$

and

$$\|\tilde{\xi}\|_{L^\infty} \leq (T-t)^{-1+1/8}.$$

Moreover, on any compact set, the solution remains uniformly regular up to time T*, so that for any* $y \in [0, +\infty)$*, the limit at blow-up time* $\lim_{t\uparrow T} \xi(t,y) = \xi^*(y)$ *exists and satisfies:*

$$\xi^*(y) \sim \frac{y^2}{4\mu_\infty^2} \quad as \quad y \to +\infty.$$

Other blow-up profiles also exist, but the corresponding dynamics is linearly unstable. The result is the following.

Theorem 4.4 (Instable blow-ups for Equation (4.2) [7])**.** *For any* $k \in \mathbb{N}$*, with* $k \geq 2$*, there exists a solution to* (4.2) *blowing up at time* $T > 0$*, with:*

$$\xi(t,y) = \lambda^{2k/(2k-1)} G_k\left(\frac{y - y^*(t)}{\lambda(t)\mu(t)}\right) \mathbb{1}_{-a_k \leq \frac{y-y^*}{\lambda\mu} \leq a_k} + \tilde{\xi},$$

where $a_k = \pi/(2k\sin(\pi/2k))$*. Above,* G_k *is an even nonnegative* C^1 *function supported on* $[-a_k, a_k]$ *increasing on* $[-a_k, 0]$ *with* $G_k(Z) \approx 1 - Z^{2k}$ *near* 0*. The parameters satisfy for some* $\mu_\infty, \nu > 0$*:*

$$\lambda(t) = \frac{1 + O((T-t)^\nu}{(T-t)^{1-1/2k}}, \quad \mu(t) = \mu_\infty + O((T-t)^\nu),$$

$$y^*(t) = \frac{\mu_\infty a_k}{(T-t)^{1-1/2k}}(1 + O((T-t)^\nu), \quad \|\tilde{\xi}\|_{L^\infty} \leq (T-t)^{-1+\nu}.$$

Moreover, on any compact set, the solution remains uniformly regular up to time T*, so that for any* $y \in [0, +\infty)$*, the limit* $\lim_{t\uparrow T} \xi(t,y) = \xi^*(y)$ *exists and satisfies:*

$$\xi^*(y) \sim \left(\frac{2k-1}{\mu_\infty}\right)^{2k/(2k-1)} y^{1+1/(2k-1)} \quad as \quad y \to +\infty.$$

Comments on the results. *1. Implications for the boundary layer.* One does not know a priori that if, given u_0 an initial condition of Prandtl's equations (1.1), the maximal time of existence for u the corresponding solution of (1.1), and ξ the associated solution of (4.2), are the same. It could be the case that another singularity happens before outside the vertical axis. However, if the singularity of u happens indeed on the vertical axis, our result shows that the point where the shock will form is ejected to infinity. The typical scale for the normal variable being $y \sim (T-t)^{-1/2}$, the transverse viscosity is asymptotically negligible in the blow-up dynamics near T. This indicates that the blow-up mechanism could be that of the inviscid Prandtl's equations (3.1). In light of Proposition 3.1 and (3.3), this conclusion is supported by the interpretation of (4.3) as a partial stability result for the self-similar dynamics of the inviscid equations (3.1) driven by the profile Θ.

2. On more general outer flows. Our results above can be extended to other nontrivial outer flows satisfying suitable symmetry assumptions (e.g. u^E odd and p^E even in x). Indeed, this will just induce the presence of new terms that are of lower order asymptotically during singularity formation, and will not perturb the blow-up mechanism. Hence the statements of Theorems 4.3 and 4.4 remain true. This is the case, in particular, of the impulsively started cylinder [50] $u^E = \kappa \sin x$ and $p^E = (\kappa^2/4)\cos(2x)$, for which the reduced equation (4.2) becomes:

$$\begin{cases} \xi_t - \xi_{yy} - \xi^2 + \left(\int_0^y \xi\right) \xi_y = -\kappa^2, \\ \xi(t,0) = 0, \quad \xi(t,y) \underset{y\to+\infty}{\longrightarrow} -\kappa. \end{cases}$$

5. Sketch of proof of Theorem 4.3

We give in this document a sketch of the proof of Theorem 4.3, with a focus on the main ideas. The full proof of Theorem 4.3 and its adaptation to prove Theorem 4.4 can be found in [7].

5.1. **Formal matched asymptotics to find the scaling exponents.** Assume $\xi(t,y)$ solves (4.2), that its maximum is at $y^*(t)$, with its speed is given by the transport part of the equation: $y^*_t = \int_0^{y^*} \phi$. In the parabolic self-similar variables (which amounts to zoom near the maximum y^* of the solution at scale $\sqrt{T-t}$):

$$Y = \frac{y - y^*}{\sqrt{T-t}}, \quad s = -\log(T-t), \quad f(s,Y) = (T-t)\xi(t,y), \tag{5.1}$$

the renormalised function f solves, neglecting the boundary conditions:

$$f_s + f + \frac{Y}{2}\partial_Y f - f^2 + \partial_Y^{-1} f \partial_Y f - \partial_{YY} f = 0.$$

An obvious static solution to the above equation is $f = 1$. It however does not satisfy the boundary conditions, so that there is a "free boundary" encoding where the function f starts to deviate from 1 and goes to zero. To find this "free boundary", we write the evolution of the correction $\varepsilon = f - 1$:

$$\varepsilon_s + \mathcal{L}\varepsilon = NL, \quad \mathcal{L}\varepsilon := -\varepsilon + \frac{3}{2} Y \partial_Y \varepsilon - \varepsilon_{yy}. \tag{5.2}$$

Above, the operator $\mathcal{L}$ is well known (linked to hypergeometric functions), self-adjoint on $L^2(e^{-3Y^2/4})$ with spectrum $\{3i/2 - 1, \ i \in \mathbb{N}\}$. The corresponding eigenfunctions h_i are Hermite polynomials of order i. One then looks for the dominant mode as $t \to T$ (equivalently as $s \to \infty$). The first eigenvalue is -1 associated to the constant in space instable eigenfunction 1, but this mode is in fact not excited (this instability being linked to the invariance of the equation by time translation, it is under control by choosing suitably the time T in (5.1)). The second mode, $h_1 = y$, cannot be excited since it would violate the fact that the maximum is at $y = y^*$ ($Y = 0$) (this is linked to the invariance by translation of the equation). Assuming that the $i = 2$ mode dominates gives, with $h_2 = 3y^2 - 2$:

$$\varepsilon(s,Y) \approx Ce^{-2s}(3Y^2 - 2), \quad i \geq 1.$$

f then starts to deviate from 1 where the correction ε starts to be of the same size as the leading order term 1, which is in the zone

$$Y \sim e^s, \quad \text{i.e.} \quad y - y^* \sim (T-t)^{-\frac{1}{2}}.$$

This introduces the new variable (and the guess $y \sim (T-t)^{-1/2}$ for the scaling in original variable):

$$Z := \frac{Y}{e^s} = \frac{y - y^*}{(T-t)^{-1/2}}, \quad F(s,Z) := f(s,Y) \tag{5.3}$$

and F solves

$$F_s+F-F^2+\left(-\frac{1}{2}Z+\int_0^Z F(s,\tilde{Z})d\tilde{Z}\right)\partial_Z F-e^{-s}\partial_{ZZ}F=0. \tag{5.4}$$

5.2. **The profiles.** In (5.4), the viscosity is negligible as $s\to\infty$. Assuming that the renormalisation done in the previous subsection is correct, F should converge towards some steady state, solution to

$$F-F^2+\left(-\frac{1}{2}Z+\int_0^Z F(\tilde{Z})d\tilde{Z}\right)\partial_Z F=0 \tag{5.5}$$

Lemma 5.1. *Equation* (5.5) *admits the one-parameter family of solutions*

$$F_1\left(Z/\mu\right),\quad \mu>0,\quad F_1(Z)=\cos^2\left(Z/2\right)1_{-\pi\leq Z\leq \pi}.$$

Remark 5.2. Assuming that higher order modes in (5.2) dominate yields other scaling exponents for the free boundary (5.3), which in turn gives another equation than (5.5) for the profiles. Such a case is possible, but instable due to the linear structure of $\mathcal{L}$, and corresponds to Theorem 4.4.

5.3. **Bootstrap argument: the setup.** We prove Theorem 4.3 by considering a solution close to the concentrating blow-up profile, stabilising the resulting dynamics by choosing appropriately the corresponding parameters: scaling parameter λ (in direct correspondance with the blow-up time T since $\lim_{t\to T}\lambda=\infty$), position of the blow-up point y^*, spatial scale μ. Such an argument is standard in blow-up problems, and such a solution is said to be in the "bootstrap" regime. The first step is to decompose in a suitable way the solution. We write

$$\xi(t,y)=\lambda^2(t)F_1\left(\frac{y-y^*(t)}{\mu(t)\lambda(t)}\right)+\tilde{\xi}(t,y)$$

such that in the variables:

$$s=\int_0^t\lambda^2(\tilde{t})d\tilde{t},\quad Y=\lambda(y-y^*),\quad Z=\frac{y-y^*}{\lambda\mu},\quad f(s,Y)=F(s,Z)=\lambda^{-2}\xi(t,y), \tag{5.6}$$

one has

$$f(s,Y)=F_1\left(Y/\lambda^2\mu\right)+\varepsilon,\quad \varepsilon\perp h_0,h_1,h_2 \text{ in } L^2(e^{-3Y^2/4}) \tag{5.7}$$

where the h_i are the eigenmodes of $\mathcal{L}$. Such a decomposition can be proved to be unique in the vicinity of the set $(\lambda^2(F_1(\cdot/(\lambda\mu))))_{\lambda,\mu>0}$. We also introduce a suitable decomposition for the blow-up point:

$$y^*=\lambda\mu(\pi+a).$$

5.4. **Modulation equations and interior Lyapunov functional.** The most sensitive part of the dynamics happens close to the maximum of the function y^*. The variables (5.6) are suitable to show local stability of F_1 near the maximum (that is to say the decay of ε given by (5.7)), and to compute the evolution of the modulation parameters. It is possible to show the following estimates (where $L^2_{\tilde{\rho}}$ is a Lebesgue space with measure $e^{-3Y^2/4}dY$ with the appropriate renormalised boundary):

Lemma 5.3 (Modulation equations). *In the bootstrap regime there holds:*

$$\begin{aligned} \left|\frac{\mu_s}{\mu}-\frac{1}{2\lambda^4\mu^2}\right| &\lesssim \lambda^{-8}+\|\varepsilon\|_{L^2_{\tilde\rho}}+hot,\\ \left|\frac{\lambda_s}{\lambda}-\frac{1}{2}+\frac{1}{4\lambda^4\mu^2}\right| &\lesssim \lambda^{-12}+\lambda^{-4}\|\varepsilon\|_{L^2_{\tilde\rho}}+hot,\\ \left|a_s+\frac{a}{2}-\int_{-\pi-a}^{-\pi}F_1-\frac{1}{\lambda^2\mu}\int_{-\lambda y^*}^{0}\varepsilon\right| &\lesssim \lambda^{-4}+\|\varepsilon\|_{L^2_{\tilde\rho}}+hot. \end{aligned}$$

Remark 5.4. The lemma above says roughly that $\mu_s = 0$, $\lambda_s = \lambda/2$ and $a_s = -a/2$ to leading order. This explains why μ will stabilise to a finite limit, λ will tend to ∞ at the desired rate $e^{s/2}$ and a will tend to 0.

Lemma 5.5 (Interior Lyapunov functional). *In the bootstrap regime there holds:*

$$\frac{d}{ds}\Big(\frac12\|\varepsilon\|^2_{L^2_{\tilde\rho}}\Big)+e^{-\eta s}\|\partial_Y\varepsilon\|^2_{L^2_{\tilde\rho}}\le-\Big(\frac72-Ce^{-\eta s}\Big)\|\varepsilon\|^2_{L^2_{\tilde\rho}}+C\|\varepsilon\|_{L^2_{\tilde\rho}}\lambda^{-12}+Ce^{-e^s}.$$

Proof of Lemmas 5.3 and 5.5. The key idea is that the dynamics is driven by the operator $\mathcal{L}$ defined in (5.2) for Y close to the origin, and that what happens further away is not important, which is reflected by the fact that the measure $L^2(e^{-\frac34 Y^2})$ decays quickly. The proof then uses the orthogonality conditions (5.7) linked to a decoupling between the parameters and the remainder, the spectral structure for the linearised operator $\mathcal{L}$, the Taylor expansion of F_1 near the origin and the Poincare inequality $\int_{\mathbb{R}}(1+Y^2)u^2(Y)e^{-Y^2}\lesssim\int_{\mathbb{R}}(u^2+(\partial_Y u)^2)e^{-Y^2}$. The control of the nonlinear terms is done through direct L^∞ estimates. This argument originates from [4, 17, 40, 51]. □

5.5. **Exterior Lyapunov functional.** What we did in the previous subsection gives only the control of the remainder ε near the maximum y^*, in the variable Y (i.e., at a scale of order $\sqrt{T-t}$ in original variable). To control it on a scale of order $(T-t)^{-1/2}$ in original variables (which corresponds to the scale of the blow-up profile), we need to perform a different analysis. We change variables using (5.6) for the remainder: $\varepsilon(s,Y)=u(s,Z)$. It then solves, where hot denotes higher order error and nonlinear terms:

$$u_s+T\partial_Z u+Vu+\partial_Z^{-1}u\partial_Z F_1-\frac{1}{\lambda^4\mu^2}\partial_{ZZ}u=hot, \tag{5.8}$$

with the transport and the potential term being defined by

$$T(Z):=\begin{cases}-\left(\frac Z2+\frac\pi2\right) & \text{if } Z\le\pi,\\ \frac12\sin Z & \text{if } -\pi\le Z\le\pi,\\ -\left(\frac Z2-\frac\pi2\right) & \text{if } \pi\le Z,\end{cases}\quad V(Z):=\begin{cases}1 & \text{if } Z\le-\pi,\\ -\cos Z & \text{if } -\pi\le Z\le\pi,\\ 1 & \text{if } \pi\le Z,\end{cases}$$

and the integral term being $\partial_Z^{-1}u=\int_0^Z u dZ$. We adopt a kinetic point of view to look at (5.8). The transport term $T\partial_Z$ pushes away from the origin towards the points $-\pi$ and π. It however vanishes at the origin so that particles may take an infinite amount of time to escape from a neighbourhood of the origin. The potential term V is negative close to the origin and positive away. The problem for the dynamics generated by $T\partial_Z+V$ lies therefore at the origin, where particles may spend a lot of time while the remainder u is being amplified. The integral term $\partial_Z^{-1}u\partial_Z F_1$ will not generate problems: the integral is computed from the origin, and $T\partial_Z$ pushes away from the origin, hence in appropriate weighted spaces $\partial_Z^{-1}u\partial_Z F_1$ is "slaved" by $\partial_Z^{-1}u\partial_Z F_1$. Finally, the viscosity $\frac{1}{\lambda^4\mu^2}\partial_{ZZ}$ only plays a role at the scale $Z\sim e^{-s}$, which corresponds to the zone $Y\sim1$ studied in the previous subsection.

We thus control u in a zone outside the maximum of ξ (which is located at $Z=0$), treating the information coming from the origin as a forcing term controlled in Y variable in the previous subsection, and in a suitable weighted space adapted to $T\partial_Z+V$, where $\partial_Z^{-1}u\partial_Z F_1$ and $\frac{1}{\lambda^4\mu^2}\partial_{ZZ}$ are lower order. The weight is the following, for an even function q, $q(0)=0$, $q(1)=1$, $q'>0$ on $(0,\pi)$ with some additional properties:

$$w(s,Z):=\begin{cases} \dfrac{1+\cos Z}{(1-\cos Z)\sin^4 Z}\dfrac{1}{\sin(-Z)}4(\pi+Z)^3\dfrac{1}{s^{q(Z)}} & \text{if} \quad Z\in(-\pi,0), \\ \dfrac{1+\cos Z}{(1-\cos Z)\sin^4 Z}\dfrac{1}{\sin Z}4(\pi-Z)^3\dfrac{1}{s^{q(Z)}} & \text{if} \quad Z\in(0,\pi), \\ \dfrac{1}{s}, & \text{if} \quad |Z|\geq \pi. \end{cases}$$

We only state the corresponding result on the left $Z=0$, the result being exactly the same on the right.

Lemma 5.6 (Exterior Lyapunov Functional on the left). *Let $M\gg 1$. Assume that u solves* (5.8)*, then:*

$$\begin{aligned} \frac{d}{ds}\left(\frac{1}{2}\int_{-\pi-a}^{-Me^{-s}}u^2 w dZ\right)+\frac{1}{\lambda^2\mu}\int_{-\pi-a}^{-Me^{-s}}|\partial_Z u|^2 w dZ \\ \leq \left(-\frac{1}{2}+\frac{C}{M^2}+\frac{C}{\ln s}\right)\int_{-\pi-a}^{-Me^{-s}}u^2 w dZ \\ +Ce^{6s}u^2(-Me^{-s})+e^{-8s}(\partial_Z u)^2(-Me^{-s})+hot \end{aligned}$$

Remark 5.7. In the energy estimate of the lemma above, one notices a dissipative term which has a correct sign and makes the quantity decrease. Then, for M and s large enough, $-1/2+C/M^2+C/\ln s\approx -1/2$ which shows a linear exponential decay of exponent $1/2$ for the exterior Lyapunov functional. Formally, for M large enough we are far away from the origin and the dissipation is lower order, and for s large enough the integral term $\partial_Z^{-1}u$ is of lower order in the $L^2(wdZ)$ space as w depends on s. The terms $Ce^{6s}u^2(-Me^{-s})+e^{-8s}(\partial_Z u)^2(-Me^{-s})$ are boundary terms corresponding to the information coming from the origin, and are estimated using the information obtained in Lemma 5.5.

Proof. Having in mind that the leading order terms are $T\partial_Z+V$, the above Lyapunov estimate is an approximation of a corresponding weighted L^∞ estimate on the characteristics, that is put in an L^2 form via a duality method. The penalisation of the nonlocal term is obtained via a suitable time dependent modification of the weight. The nonlinear terms are estimated by L^∞ bounds. □

We end this subsection by mentioning that in order to obtain L^∞ estimates to control the nonlinear terms, we also control derivatives of u with similar energy estimates. To take derivatives we use the vector field $A\partial_Z u$ where A

$$A(Z):=\begin{cases} -1 \ \text{ for } Z\leq -\pi/2, \\ \sin Z \ \text{ for } -\pi/2\leq Z\leq \pi/2, \\ 1 \ \text{ for } \pi/2\leq Z. \end{cases}$$

commutes well with the linear operator $T\partial_Z+V$.

5.6. **Behaviour near the origin.** After having controlled the solution near the maximum, and away from the maximum at the scale of the profile, one finally needs to control the solution near the boundary $y = 0$ to control the various boundary terms. A refinement of a no blow-up argument from [22], also inspired by [25] to obtain the trace of the solution at the blow-up time, yields:

Lemma 5.8 (No blow-up near the origin)**.** *Let $N, L, L' \geq 1$, $q \in 2\mathbb{N}$, $b \in \mathbb{R}$. Assume $\xi \in C^3([0, t(s_1)] \times [0, 2N])$, with:*

$$\xi(t(s_0)) = by^2 + \tilde{\xi}(t(s_0)), \quad \|\tilde{\xi}(t(s_0))\|_{L^\infty([0,2N])} \leq L, \quad \|\partial_y \tilde{\xi}(t(s_0))\|_{L^2([0,2N])} \leq L'.$$

and for all $t \geq t(s_0)$:

$$\|\xi\|_{L^\infty([0,2N])} \leq e^{(1-1/8)s}, \quad \|\partial_y \xi(t(s_0))\|_{L^2([0,2N])} \leq e^s,$$

then, writing $\xi = by^2 + \tilde{\xi}$, for all $t \geq t(s_0)$:

$$\|\tilde{\xi}\|_{L^q([0,N])} \lesssim LN^{1/q} + N^{2+1/q} e^{-s_0/16}, \quad \|\partial_y \tilde{\xi}\|_{L^2([0,N])} \lesssim L' + N^{3/2} e^{-s_0/8q}.$$

Remark 5.9. The above lemma states that if a solution is close to by^2 on some compact set of y going up to the boundary, then it will remain close to by^2 up to the blow-up time. In particular, by parabolic regularisation, the solution remains smooth up to the blow-up time on any compact set.

Proof. The proof relies on a standard parabolic bootstrap argument using parabolic regularisation, with a careful control of the forcing and nonlinear terms. □

6. On Burgers equation with transverse viscosity

An interesting question linked to Theorems 4.3 and 4.4 is wether or not the information obtained on the vertical axis there can be enough to understand the shock formation for the full solution, i.e., outside the vertical axis. We believe the answer is yes in some cases, and solved this problem for a simplification of (1.1) in [8] and now describe this result. According to Theorem 4.3, some blow-ups of (1.1) happen away from the boundary and the boundary does not have effects on the singularity. If one removes the vertical transport term in (1.1) and the boundary, one obtains Burgers equation with transverse viscosity:

$$u_t + uu_x = u_{yy}. \tag{6.1}$$

We look again for a reduced equation encoding tangential compression, similar to (2.3) for Burgers and (4.2) for Prandtl. For a solution u to (6.1) that is odd in x, the behaviour on the transverse axis $\{x = 0\}$ is encoded by a closed system, which is the motivation for this symmetry assumption. It admits solutions blowing up simultaneously with a precise behaviour. Indeed, assume $\partial_x^j u_0(0, y) = 0$ for all $y \in \mathbb{R}$ for $2 \leq j \leq 2i$ for some integer $i \in \mathbb{N}$. This remains true for later times and the trace of the derivatives

$$\xi(t, y) := -\partial_x u(t, 0, y) \quad \text{and} \quad \zeta(t, y) = \partial_x^{2i+1} u(t, 0, y) \tag{6.2}$$

solve the parabolic system

$$\begin{cases} (NLH) & \xi_t - \xi^2 - \partial_{yy}\xi = 0, \\ (LFH) & \zeta_t - (2i+2)\xi\zeta - \partial_{yy}\zeta = 0. \end{cases} \tag{6.3}$$

Solutions to the nonlinear heat equation (NLH) might blow up in finite time, see Theorem 4.1. For the singular solutions ξ to (NLH) of Theorem 4.1, the solution to the linearly forced heat equation (LFH) may also blow-up in finite time with a related precise asymptotic. From precise information on ξ and ζ on the vertical axis, we are able to infer precise information on u the solution of (6.1), i.e., to gain information outside the vertical axis. The main result of [8] is the construction and precise description of finite time blow-up solutions for which the gradient become unbounded. At the main order, a solution is given by a backward self-similar solution of (2.1) (the inviscid Burgers equation) along the tangential x variable, whose associated scaling parameters depend on the vertical variable y and are directly related to the solutions of (6.3).

Theorem 6.1. *For any $i \in \mathbb{N}^*$ and $b > 0$, there exists solution to* (6.1) *blowing up at time T with*

$$u(t,x,y) = b^{-1}\lambda^{-1/2i}(t,y)\Psi_i\left(b\lambda^{1+1/2i}(t,y)x\right) + \tilde{u}(t,x,y)$$

where Ψ_i is defined by (2.4)*, the transverse scale satisfies*

$$\lambda(t,y) = \frac{1}{T-t}\,\frac{1}{1+y^2/8(T-t)|\log(T-t)|},$$

one has the convergence in self-similar variables (X,Z)

$$(T-t)^{-1/2i}u\left((T-t)^{1+1/2i}X, \sqrt{(T-t)|\log(T-t)|}Z\right) \longrightarrow b^{-1}(1+Z^2/8)^{1/2i}\Psi_i\left(\frac{bX}{(1+Z^2/8)^{1+1/2i}}\right) \quad (6.4)$$

in C^1 on compacts sets, and for some constants $C > 0$ the remainder satisfies

$$\|\partial_x \tilde{u}\|_{L^\infty} \le C(T-t)^{-1}|\log(T-t)|^{-1/4}. \quad (6.5)$$

Remark 6.2.

- Theorem 6.1 states that there exist solutions to (6.1) which concentrate any backward self-similar profile of Burgers equation (2.4) along the horizontal variable, at a scale related to the stable blow-up of the semilinear heat equation (4.3) along the transverse variable (see Theorem 4.1). Let us mention that it is also possible to concentrate any profile of Burgers equation at a scale related to any instable blow-up of (4.3), see [8].
- There are only few results concerning anisotropic singularity formation, despite its relevance for fluid mechanics. In [9, 41], anisotropic blow-ups were constructed for the supercritical semi-linear heat equation in large dimensions.

Proof. We do not detail the proof of Theorem 6.1 here, and refer to [8]. Let us mention that there, a new framework to deal with the mixed hyperbolic and parabolic features is developed and that we revisit the construction of flat blow-up profiles for the semi-linear heat equation. □

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