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## An explicit computation of $p$ -stabilized vectors

par MICHITAKA MIYAUCHI et TAKUYA YAMAUCHI

RÉSUMÉ. Nous donnons une méthode concrète pour calculer les vecteurs  $p$ -stables dans l'espace des éléments fixés par un sous-groupe parahorique d'un groupe réductif  $p$ -adique. Nous discutons d'une application globale et, en particulier, nous donnons un exemple explicite d'un relèvement de Saito-Kurokawa  $p$ -stable.

ABSTRACT. In this paper, we give a concrete method to compute  $p$ -stabilized vectors in the space of parahori-fixed vectors for connected reductive groups over  $p$ -adic fields. An application to the global setting is also discussed. In particular, we give an explicit  $p$ -stabilized form of a Saito-Kurokawa lift.

### 1. Introduction

Let  $F$  be a non-archimedean local field,  $\mathfrak{o}$  the ring of integers of  $F$ ,  $\mathfrak{p}$  the maximal ideal of  $\mathfrak{o}$ ,  $\varpi$  a uniformizer of  $F$ , and  $\mathbb{F} = \mathfrak{o}/\mathfrak{p}$  the residue field of  $F$ . We normalize the valuation  $|\cdot|$  of  $F$  so that  $|\varpi| = q^{-1}$ , where  $q$  is the cardinality of  $\mathbb{F}$ . Let  $G$  be a connected reductive group defined over  $F$ ,  $B$  its standard Borel subgroup and  $K = G(\mathfrak{o})$  a maximal compact subgroup of  $G(F)$  whenever it is defined. Let  $P$  be a parabolic subgroup of  $G$  which contains  $B$  and  $K_P = \{g \in K \mid g \bmod \mathfrak{p} \in P(\mathbb{F})\}$  the parahoric subgroup of  $K$  associated to  $P$ .

Let  $\pi$  be an irreducible smooth representation of  $G(F)$  such that the space  $\pi^{K_P}$  of  $K_P$ -fixed vectors in  $\pi$  is non-trivial. Then the Hecke algebra  $\mathcal{H}_{K_P}$  of  $G(F)$  associated to  $K_P$  acts on  $\pi^{K_P}$ . In this paper, we first give a method to compute the eigenvalues for the special elements of  $\mathcal{H}_{K_P}$  on  $\pi^{K_P}$ , which are called " $U_p$ -operators". We next give an explicit construction of simultaneous eigenvectors for these  $U_p$ -operators, which are called " $p$ -stabilized vectors".

An idea to compute eigenvalues of  $U_p$ -operators is to consider the Jacquet module of  $\pi$  associated to  $P$ . When  $P$  is the standard Borel subgroup of  $G(F)$ , this has been a well-known method for experts. We extend a result of Casselman [6] which is proved only for the standard Borel subgroup to any parabolic subgroup  $P$  and any compact open subgroup of  $G(F)$  which has an Iwahori factorization relative to  $P$  in the sense of [6]. Then it can

be reduced a calculation of the eigenvalues for  $U_p$ -operators to the same problem for the actions of the elements corresponding to those operators on the Jacquet modules (Proposition 2.3).

Our results might give a potential tool to study an arithmetic investigation of automorphic forms in Iwasawa theory, Hida theory, or deformation theory of Galois representations [14],[21],[20], though we do not discuss about this in this paper. In particular, we will know that what kind of  $p$ -stabilized forms can be embedded into a Hida family with respect a specific parabolic subgroup of  $G$ .

This paper is organized as follows. In Section 2, we study the action of Hecke algebras on the space of parahori-fixed vectors by using Jacquet modules. In Section 3, we introduce the notion of  $U_p$ -operators and  $p$ -stabilized vectors. In Section 4, we outline a method to construct  $p$ -stabilized vectors when  $\pi$  has a non-zero  $K$ -fixed vector. Without this assumption on  $\pi$ , it seems to be difficult to check the non-triviality of the vector which we will construct. In Section 5, 6, and 7, we make up a list of all  $U_p$ -eigenvalues and  $p$ -stabilized vectors for  $GL_2, U(2, 1)$ , and  $GSp_4$ . A relation to the global setting is discussed in Section 8 and then the global  $p$ -stabilized forms are given in the final section in cases of  $GL_2$  and  $GSp_4$ . In particular, we will give them for Saito-Kurokawa lifts where the existence has been already discussed in Proposition 4.2.2, p.688 of [20] (see (9.2) and (9.3)).

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## 2. The action of Hecke operators via Jacquet modules

We keep the notation in Section 1. In this section, we study parahori-fixed vectors of smooth representations of  $G(F)$ . We fix a maximal torus  $T$  of  $G$  and a minimal parabolic subgroup  $B$  of  $G$  which contains  $T$ . Then we have the Levi decomposition  $B = TU$ , where  $U$  is the unipotent radical of  $B$ . Let  $P$  be a parabolic subgroup of  $G$  containing  $B$  with Levi decomposition  $P = MN$ . We denote by  $\bar{N}$  the unipotent radical of the parabolic subgroup opposite to  $P$ .

Henceforth, for any algebraic group  $H$ , we sometimes denote by  $H$  the group of  $F$ -valued points of  $H$  for the sake of simplicity. This should cause no confusion in the remainder of the paper.

For any smooth representation  $(\pi, V)$  of  $G(F)$ , we define its Jacquet module  $(\pi_N, V_N)$  as follows (cf. Section 3 of [6]): Set  $V(N) = \langle \pi(n)v - v \mid n \in N, v \in V \rangle$  and  $V_N = V/V(N)$ . We define a representation  $(\pi_N, V_N)$  of  $M$  by

$$\pi_N(m)r_N(v) = \delta_P^{-\frac{1}{2}}(m)r_N(\pi(m)v), \quad m \in M, \quad v \in V,$$

where  $r_N$  is the natural projection from  $V$  to  $V_N$  and  $\delta_P$  is the modulus character of  $P(F)$ .

Let  $I := \{g \in K \mid g \bmod \mathfrak{p} \in B(\mathfrak{o}/\mathfrak{p})\}$  be the standard Iwahori subgroup of  $G(F)$ . If a smooth representation  $(\pi, V)$  of  $G(F)$  is admissible, then by p.7, Theorem of [8], the canonical projection  $r_U : V \rightarrow V_U$  induces a  $\mathbb{C}$ -linear isomorphism

$$(2.1) \quad V^I \xrightarrow{\sim} (V_U)^{I \cap T(F)}.$$

The following theorem generalizes this isomorphism to any parabolic subgroups.

**Theorem 2.1.** *Let  $(\pi, V)$  be an admissible representation of  $G(F)$ . Suppose that a compact subgroup  $J$  of  $G(F)$  contains  $I$  and it has an Iwahori factorization with respect to  $P$  in the sense of [6] (see before Proposition 1.4.4 in loc.cit.). Then the canonical projection  $r_N : V \rightarrow V_N$  induces a linear isomorphism*

$$V^J \xrightarrow{\sim} (V_N)^{J \cap M}.$$

*Proof.* By Theorem 3.3.3 of [6], the map  $r_N : V^J \rightarrow (V_N)^{J \cap M}$  is surjective. We now prove the injectivity of this map. By (2.1), we get  $V^I \cap V(U) = \{0\}$ . Since  $I \subset J$  and  $U \supset N$ , we have  $V^I \supset V^J$  and  $V(U) \supset V(N)$ . This gives us that  $V^J \cap V(N) \subset V^I \cap V(U) = \{0\}$ . This implies that  $r_N : V^J \rightarrow (V_N)^{J \cap M}$  is injective.  $\square$

As in Theorem 2.1, let  $J$  be a compact subgroup of  $G(F)$  which contains  $I$ . Assume that  $J$  has an Iwahori factorization with respect to  $P$ .

**Definition 2.2.** ([5] Definition 6.5) We say that an element  $m \in M$  is *positive relative to  $(P, J)$*  if the following conditions are fulfilled:

$$m(J \cap N)m^{-1} \subset J \cap N, \quad m^{-1}(J \cap \overline{N})m \subset J \cap \overline{N}.$$

We denote by  $M^+$  the set of all positive elements in  $M$ . We say that an element  $m$  in  $M$  is *negative relative to  $(P, J)$*  if  $m^{-1}$  is positive. We write  $M^-$  for the set of all negative elements in  $M$ .

Given a compact open subgroup  $J$  of  $G(F)$ , we define the Hecke algebra  $\mathcal{H}_J := \mathcal{H}[G(F)//J]$  of  $G(F)$  associated to  $J$  to be the space of all compactly supported functions  $f : G(F) \rightarrow \mathbb{C}$  which satisfy  $f(j_1 g j_2) = f(g)$ , for  $j_1, j_2 \in J$  and  $g \in G(F)$ . Then  $\mathcal{H}_J$  becomes an algebra under the convolution with respect to the Haar measure on  $G(F)$  normalized so that the volume of  $J$  is one. For any  $g \in G(F)$ , we denote by  $f_g = [JgJ] \in \mathcal{H}_J$  the characteristic function of  $JgJ$ . Since the algebra  $\mathcal{H}_J$  is generated by  $f_g$ ,  $g \in G(F)$ , if we define  $\mathcal{H}_{J, \mathbb{Q}} := \mathbb{Q}[f_g \mid g \in G(F)]$ , then we have  $\mathcal{H}_J = \mathcal{H}_{J, \mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C}$ . For any  $\mathbb{Q}$ -algebra  $A$ , we put  $\mathcal{H}_{J, A} := \mathcal{H}_{J, \mathbb{Q}} \otimes_{\mathbb{Q}} A$ .

If  $(\pi, V)$  is a smooth representation of  $G(F)$ , then the Hecke algebra  $\mathcal{H}_J$  acts on  $V^J$  (cf. [6]). We denote by  $Z_M$  the center of  $M$ . We consider Hecke operators associated to positive elements in  $Z_M$ .

**Proposition 2.3.** *Let  $J$  be as in Definition 2.2 and  $(\pi, V)$  an admissible representation of  $G(F)$ . Then for any  $\zeta \in Z_M \cap M^+$ , we have*

$$r_N(\pi(f_\zeta)v) = \delta_P^{-\frac{1}{2}}(\zeta)\pi_N(\zeta)r_N(v), \quad v \in V^J,$$

where  $r_N : V^J \xrightarrow{\sim} (V_N)^{J \cap M}$  is the isomorphism given in Theorem 2.1.

*Proof.* For  $\zeta \in Z_M \cap M^+$  and  $v \in V^J$ , we have

$$\pi(f_\zeta)v = \int_{J\zeta J} \pi(g)v dg = \sum_{k \in J/J \cap \zeta J\zeta^{-1}} \pi(k\zeta)v.$$

Since we assume that  $J$  has an Iwahori factorization, we get  $J = (J \cap \overline{N})(J \cap M)(J \cap N)$ . Because  $\zeta$  is positive and it belongs to  $Z_M$ , we have that  $J \cap \zeta J\zeta^{-1} = (J \cap \overline{N})(J \cap M)\zeta(J \cap N)\zeta^{-1}$ , so that  $J/J \cap \zeta J\zeta^{-1} = (J \cap N)/\zeta(J \cap N)\zeta^{-1}$ . Therefore we obtain

$$\pi(f_\zeta)v = \sum_{k \in (J \cap N)/\zeta(J \cap N)\zeta^{-1}} \pi(k\zeta)v.$$

Since  $r_N(\pi(k\zeta)v) = r_N(\pi(\zeta)v) = \delta_P^{\frac{1}{2}}(\zeta)\pi_N(\zeta)r_N(v)$ ,  $k \in J \cap N$  and  $[J \cap N : \zeta(J \cap N)\zeta^{-1}] = \delta_P^{-1}(\zeta)$ , we have

$$r_N(\pi(f_\zeta)v) = \delta_P^{-\frac{1}{2}}(\zeta)\pi_N(\zeta)r_N(v),$$

as required. □

**Proposition 2.4.** *Suppose that a compact open subgroup  $J$  of  $G(F)$  has an Iwahori factorization relative to  $P$ . Then  $f_{\zeta_1}$  and  $f_{\zeta_2}$  are commutative, for any  $\zeta_1, \zeta_2 \in Z_M \cap M^+$ .*

*Proof.* We shall claim that  $f_{\zeta_1} * f_{\zeta_2} = f_{\zeta_1\zeta_2}$ . Then we obtain

$$f_{\zeta_1} * f_{\zeta_2} = f_{\zeta_1\zeta_2} = f_{\zeta_2\zeta_1} = f_{\zeta_2} * f_{\zeta_1}$$

because  $\zeta_1\zeta_2 = \zeta_2\zeta_1$ . It follows from [5] (6.6) that

$$[J\zeta_1 J : J] = [J \cap \overline{N} : \zeta_1^{-1}(J \cap \overline{N})\zeta_1][J \cap M : \zeta_1^{-1}(J \cap M)\zeta_1 \cap (J \cap M)].$$

Hence we get

$$[J\zeta_1 J : J] = [J \cap \overline{N} : \zeta_1^{-1}(J \cap \overline{N})\zeta_1]$$

since  $\zeta_1$  lies in the center of  $M$ . Similarly, we obtain  $[J\zeta_2 J : J] = [J \cap \overline{N} : \zeta_2^{-1}(J \cap \overline{N})\zeta_2]$  and  $[J\zeta_1\zeta_2 J : J] = [J \cap \overline{N} : (\zeta_1\zeta_2)^{-1}(J \cap \overline{N})\zeta_1\zeta_2]$ . Since  $\zeta_1$  and  $\zeta_2$  are both positive, we have

$$J \cap \overline{N} \supset \zeta_2^{-1}(J \cap \overline{N})\zeta_2 \supset (\zeta_1\zeta_2)^{-1}(J \cap \overline{N})\zeta_1\zeta_2.$$

So we obtain

$$[J\zeta_1\zeta_2J : J] = [J\zeta_1J : J][J\zeta_2J : J],$$

and hence  $f_{\zeta_1} * f_{\zeta_2} = f_{\zeta_1\zeta_2}$  by Proposition 2.2 in Chapter 3 of [10]. □

### 3. $p$ -stabilized vectors

For simplicity, we assume that the dimension of the center of  $G$  is at most one. Let  $\pi$  be an irreducible smooth representation of  $G(F)$ . Then  $\pi$  is admissible by [12]. In this section, we give a notion of  $p$ -stabilized vectors (or of  $p$ -stabilization) for parahori-fixed vectors in  $\pi$ .

Let  $\Delta$  be the set of all simple roots of  $(G, T)$  which is a subset of the character group  $X^*(T) := \text{Hom}_{\text{alg}}(T, GL_1)$ . Let  $P$  be a parabolic subgroup of  $G$  containing  $B$ ,  $P = MN$  its Levi decomposition, and  $K_P$  the parahoric subgroup which corresponds to  $P$ . Let  $\Delta_P$  be the subset of  $\Delta$  corresponding to  $P$ . We define  $T_P^-$  to be the semi-group consisting of the elements  $t$  in  $T(F)/T(\mathfrak{o})$  such that

$$(3.1) \quad |\alpha(t)| \leq 1 \text{ for all } \alpha \in \Delta \text{ and } t(K_P \cap N)t^{-1} \subset K_P \cap N.$$

We can choose a complete system of representatives for  $T_P^-$  as elements in  $Z_M \cap M^+$ . Put  $m_P = \#\!(\Delta \setminus \Delta_P)$ . Note that  $\Delta_B = \emptyset$ . For each  $\alpha \in \Delta$ , there exists  $t_\alpha \in T_P^-$  such that  $\text{ord}_\varpi \alpha(t_\alpha) = 1$  and  $\text{ord}_\varpi \alpha(t_\beta) = 0$  for all  $\beta \in \Delta \setminus \{\alpha\}$ . Put

$$t_{m_P+1} := \begin{cases} \varpi^{-1}\text{Id}, & \text{if } Z_M \supset F^\times \\ \text{Id}, & \text{otherwise,} \end{cases}$$

where  $\text{Id}$  is the identity element of  $G(F)$ . We write  $\Delta \setminus \Delta_P = \{\alpha_1, \dots, \alpha_{m_P}\}$  and  $t_i = t_{\alpha_i}$  for  $i \in \{1, \dots, m_P\}$ . Then the semi-group  $T_P^-$  is generated by  $t_1, \dots, t_{m_P}, t_{m_P+1}$ . For any  $\mathbb{Q}$ -algebra  $A$  which is contained in  $\mathbb{C}$ , we consider the subalgebra

$$\mathcal{U}_{P,A} := A[[K_P t K_P \mid t \in T_P^-]]$$

of the Hecke algebra  $\mathcal{H}_{K_P,A}$  over  $A$ .

**Lemma 3.1.** *Put  $U_{\varpi,i}^P := [K_P t_i K_P] \in \mathcal{H}_{K_P}$ , for  $i \in \{1, \dots, m_P + 1\}$ . Then the ring  $\mathcal{U}_{P,A}$  is a commutative  $A$ -algebra generated by  $U_{\varpi,1}^P, \dots, U_{\varpi,m_P+1}^P$ .*

*Proof.* Recall that we take a complete system of representatives for  $T_P^-$  as elements in  $Z_M \cap M^+$ . Then the commutativity follows from Proposition 2.4. The later claim follows from the fact that  $T_P^-$  is generated by  $t_1, \dots, t_{m_P}, t_{m_P+1}$ . □

**Definition 3.2.** Let  $(\pi, V)$  be an irreducible smooth representation of  $G(F)$  such that  $V^{K_P} \neq \{0\}$ . We say that a non-zero vector  $v$  in  $V^{K_P}$  is a  $p$ -stabilized vector with respect to  $\widehat{P}$  if it is a simultaneous eigenvector for all  $U_{\varpi,1}^P, \dots, U_{\varpi,m_P}^P$ . Here  $\widehat{P}$  is the Langlands dual of  $P$  (cf. [3]).

**Remark 3.3.** *The condition (3.1) on  $T_P^-$  is crucial to get the commutativity of  $\mathcal{U}_{P,A}$ . In general, this property does not hold for  $\mathcal{H}_{K_P,A}$ .*

### 4. Construction of $p$ -stabilized vectors

Let  $(\pi, V)$  be an irreducible smooth representation of  $G(F)$  which has a non-zero  $K$ -fixed vector. In this section, we give a method to produce  $p$ -stabilized vectors for  $\pi$ . Let  $P = MN$  be a parabolic subgroup of  $G(F)$  containing  $B$ . Then by Theorem 2.1, the Jacquet functor  $r_N$  induces an isomorphism  $r_N : V^{K_P} \simeq (V_N)^{K_P \cap M}$ . We set  $W = (V_N)^{K_P \cap M}$ . Let  $H$  denote the subgroup of  $Z_M$  generated by  $t_1, \dots, t_{m_P} \in T_P^-$ . As we have seen before, we may assume  $H \subset Z_M \cap M^+$ . For a quasi-character  $\chi$  of  $H$  and  $n \in \mathbb{N}$ , we define

$$W_{\chi,n} = \{w \in W \mid (\pi_N(t) - \chi(t))^n w = 0 \text{ for any } t \in H\}$$

and put  $W_{\chi,\infty} = \bigcup_{n \in \mathbb{N}} W_{\chi,n}$ . Similarly, we define  $(V_N)_{\chi,\infty}$  for  $V_N$ . Let  $\mathcal{S}$  denote the set of quasi-characters  $\chi$  of  $H$  such that  $W_{\chi,\infty} \neq \{0\}$ . Since  $W$  is a finite-dimensional  $H$ -module, we have

$$W = \bigoplus_{\chi \in \mathcal{S}} W_{\chi,\infty}.$$

For an element  $w$  in  $W$ ,  $w$  is a simultaneous eigenvector for  $t_1, \dots, t_{m_P}$  if and only if  $w$  lies in  $W_{\chi,1}$ , for some  $\chi \in \mathcal{S}$ .

Let  $\phi_K$  be a non-zero  $K$ -fixed vector in  $V$ . By the Iwasawa decomposition  $G = PK$ , the element  $v = r_N(\phi_K)$  generates  $V_N$  as an  $M$ -module, so does  $W$ . Since  $H$  is contained in the center of  $M$ , we have

$$V_N = \bigoplus_{\chi \in \mathcal{S}} (V_N)_{\chi,\infty}$$

as an  $M$ -module. We claim that the  $W_{\chi,\infty}$ -component of  $v$  is not zero, for any  $\chi \in \mathcal{S}$ . If the  $W_{\chi,\infty}$ -component of  $v$  is zero, then  $v$  lies in the proper  $M$ -submodule  $\bigoplus_{\chi' \neq \chi} (V_N)_{\chi',\infty}$  of  $V_N$ . This contradicts the fact that  $v$  generates  $V_N$  as an  $M$ -module. So the claim follows.

We fix a character  $\chi$  of  $H$  in  $\mathcal{S}$ . For any  $\chi' \in \mathcal{S}$  which is different from  $\chi$ , there exists an integer  $1 \leq i(\chi') \leq m_P$  such that  $\chi(t_{i(\chi')}) \neq \chi'(t_{i(\chi')})$ . Put  $n(\chi') = \dim W_{\chi',\infty}$ . Then

$$v' = \prod_{\chi' \neq \chi} (\pi_N(t_{i(\chi')}) - \chi'(t_{i(\chi')}))^{n(\chi')} v$$

is a non-zero vector in  $W_{\chi,\infty}$ . Therefore, there exist non-negative integers  $n(\chi, i)$  for  $1 \leq i \leq m_P$  such that  $v'' := \prod_{1 \leq i \leq m_P} (\pi_N(t_i) - \chi(t_i))^{n(\chi,i)} v'$  is a

non-zero vector in  $W_{\chi,1}$ . By Proposition 2.3,

$$\begin{aligned} \phi &= \prod_{1 \leq i \leq m_P} (\delta_P^{\frac{1}{2}}(t_i)\pi(U_{\varpi,i}^P) - \chi(t_i))^{n(\chi,i)} \\ &\quad \times \prod_{\chi' \neq \chi} (\delta_P^{\frac{1}{2}}(t_{i(\chi')})\pi(U_{\varpi,i(\chi')}^P) - \chi'(t_{i(\chi')}))^{n(\chi')} \phi_K \end{aligned}$$

is a  $p$ -stabilized vector with respect to  $\widehat{P}$ , which satisfies

$$\pi(U_{\varpi,i}^P)\phi = \delta_P(t_i)^{-\frac{1}{2}}\chi(t_i)\phi,$$

for all  $i \in \{1, \dots, m_P\}$ .

In the following series of sections, we give examples of  $p$ -stabilized vectors in various settings.

### 5. $GL_2$ -case

Let  $\alpha$  be the simple root of  $GL_2$  such that  $\alpha : T \rightarrow F^\times$ ,  $\text{diag}(a, b) \mapsto ab^{-1}$ . We have a  $U_p$ -operator  $U_{\varpi,1}^B = [It_1I]$ , where  $t_1 = \text{diag}(1, \varpi^{-1})$ . Let  $\pi = \pi(\chi)$  be an unramified principal series representation of  $GL_2(F)$  where  $\chi = \chi_1 \otimes \chi_2$  and  $\chi_1, \chi_2$  are unramified quasi-characters of  $F^\times$ . Then  $\pi$  has a non-zero  $K$ -fixed vector  $\phi_0$ , where  $K = GL_2(\mathfrak{o})$ . We shall give an explicit  $p$ -stabilized vector for  $\pi$ . The semisimplification of  $\pi_U$  is

$$\chi_1 \otimes \chi_2 + \chi_2 \otimes \chi_1.$$

The element  $t_1$  acts on each irreducible component of  $\pi_U$  by  $\chi_2(\varpi^{-1})$  and  $\chi_1(\varpi^{-1})$  respectively. If  $\chi_1(\varpi^{-1}) \neq \chi_2(\varpi^{-1})$ , then we have  $\pi_U = (\chi_1 \otimes \chi_2) \oplus (\chi_2 \otimes \chi_1)$ . It follows from Proposition 2.3 and the results in Section 4 that

$$f_1 := (\delta_B^{\frac{1}{2}}(t_1)U_{\varpi,1}^B - \chi_2(\varpi^{-1}))\phi_0, \quad f_2 := (\delta_B^{\frac{1}{2}}(t_1)U_{\varpi,1}^B - \chi_1(\varpi^{-1}))\phi_0$$

are  $p$ -stabilized vectors with respect to  $B$  with the eigenvalues  $q^{\frac{1}{2}}\chi_1(\varpi^{-1})$ ,  $q^{\frac{1}{2}}\chi_2(\varpi^{-1})$  respectively.

If  $\chi_1(\varpi^{-1}) = \chi_2(\varpi^{-1})$ , then we have  $\chi_1 = \chi_2$  since  $\chi_1$  and  $\chi_2$  are unramified. In this case, any irreducible component of  $\pi_U$  is isomorphic to  $\chi_1 \otimes \chi_1$ , but  $\pi_U$  is not decomposed as  $(\chi_1 \otimes \chi_1)^{\oplus 2}$ . This follows from the fact that  $\mathbb{C} \simeq \text{Hom}_{GL_2(F)}(\pi, \pi) = \text{Hom}_T(\pi_U, \chi_1 \otimes \chi_1)$  (Schur's lemma and Frobenius reciprocity). In this case,  $f_3 := (\delta_B^{\frac{1}{2}}(t_1)U_{\varpi,1}^B - \chi_1(\varpi^{-1}))\phi_0$  is a  $p$ -stabilized vector with respect to  $B$  with the eigenvalue  $q^{\frac{1}{2}}\chi_1(\varpi^{-1})$ .

We can express  $f_i$  in terms of Iwahori fixed vectors as follows. Let  $\phi$  be a generator of  $\pi^K$ . Choose a basis  $\{\phi, \phi' = \pi(t_1^{-1})\phi\}$  of  $\pi^I$ . Then we have

$$U_{\varpi,1}^B(\phi, \phi') = (\phi, \phi') \begin{pmatrix} a(\phi) & q \\ -\chi_1(\varpi^{-1})\chi_2(\varpi^{-1}) & 0 \end{pmatrix}$$



where  $a(\phi) = \chi_1(\varpi^{-1}) + \chi_2(\varpi^{-1})$ . Therefore we have

$$\chi_1(\varpi)f_1 = \phi - q^{-\frac{1}{2}}\chi_2(\varpi^{-1})\phi', \quad \chi_2(\varpi)f_2 = \phi - q^{-\frac{1}{2}}\chi_1(\varpi^{-1})\phi'$$

(see Section 9.1 for the relation to the global setting). It is the same for  $f_3$ .

### 6. $U(2, 1)$ -case

Let  $U(2, 1)$  be the quasi-split unitary group in three variables associated to an quadratic extension  $E/F$ . Put

$$\Phi = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We denote by  $\bar{\phantom{x}}$  the conjugate for the non-trivial element in  $\text{Gal}(E/F)$ . We realize  $U(2, 1)(F)$  as the subgroup of  $\text{GL}_3(E)$  consisting of all  $g$  satisfying  ${}^t\bar{g}\Phi g = \Phi$ . Let  $B$  be the upper triangular Borel subgroup of  $U(2, 1)$ ,  $T$  the diagonal subgroup of  $B$  and  $K = U(2, 1)(F) \cap \text{GL}_3(\mathfrak{o}_E)$ , where  $\mathfrak{o}_E$  is the ring of integers in  $E$ . We write  $E^1$  for the norm-one subgroup of  $E/F$ . Then  $T$  is isomorphic to  $E^\times \times E^1$ . Let  $\chi$  be an unramified quasi-character of  $E^\times$  and let  $\mathbf{1}_{E^1}$  denote the trivial character of  $E^1$ . Due to [13], the corresponding parabolically induced representation  $\text{Ind}_{B(F)}^{U(2,1)(F)}(\chi \otimes \mathbf{1}_{E^1})$  is irreducible except for the following cases:

- (i)  $\chi = |\cdot|_E^\pm$ , where  $|\cdot|_E$  denotes the normalized absolute value of  $E$ ;
- (ii)  $\chi|_{F^\times} = \omega_{E/F}|\cdot|^\pm$ , where  $\omega_{E/F}$  is the non-trivial character of  $F^\times$  which is trivial on  $N_{E/F}(E^\times)$ ;
- (iii)  $\chi|_{F^\times}$  is trivial and  $\chi$  is not trivial.

Suppose that  $\pi = \text{Ind}_{B(F)}^{U(2,1)(F)}(\chi \otimes \mathbf{1}_{E^1})$  is irreducible. Then  $\pi$  has a non-zero  $K$ -fixed vector  $\phi_0$ . We shall produce an explicit  $p$ -stabilized vector for  $\pi$ .

We fix a uniformizer  $\varpi_E$  of  $E$  and set

$$t_1 = \begin{pmatrix} \varpi_E & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \overline{\varpi_E}^{-1} \end{pmatrix}.$$

Let  $U$  be the unipotent radical of  $B$ . Then  $t_1$  is positive relative to  $(B, U)$ . The semisimplification of  $\pi_U$  is  $\chi \otimes \mathbf{1}_{E^1} + \bar{\chi}^{-1} \otimes \mathbf{1}_{E^1}$ , where  $\bar{\chi}$  denotes the quasi-character of  $E^\times$  defined by  $\bar{\chi}(x) = \chi(\bar{x})$ , for  $x \in E^\times$ . The element  $t_1$  acts on  $\chi \otimes \mathbf{1}_{E^1}$  and  $\bar{\chi}^{-1} \otimes \mathbf{1}_{E^1}$  by  $\chi(\varpi_E)$  and  $\bar{\chi}^{-1}(\varpi_E)$  respectively. As in the  $GL_2$ -case, by Proposition 2.3 and the results in Section 4,

$$(\delta_B^{\frac{1}{2}}(t_1)U_{\varpi,1}^B - \bar{\chi}^{-1}(\varpi_E))\phi_0, \quad (\delta_B^{\frac{1}{2}}(t_1)U_{\varpi,1}^B - \chi(\varpi_E))\phi_0$$

are  $p$ -stabilized vectors with respect to  $B$  with the eigenvalues  $q_E\chi(\varpi_E)$  and  $q_E\bar{\chi}^{-1}(\varpi_E)$  respectively.

### 7. $GSp_4$ -case

Hereafter we follow the notations in [16]. Put

$$J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

We realize  $GSp_4(F)$  as the subgroup of  $GL_4(F)$  consisting of all  $g$  such that  ${}^t g J g = \lambda J$ , for some  $\lambda \in F^\times$ . Let  $B$  be the Borel subgroup of  $GSp_4$  consisting of the upper triangular elements,  $T$  the diagonal subgroup of  $B$  and  $U$  the unipotent radical of  $B$ . Let  $P$  (respectively  $Q$ ) be the Siegel (respectively Klingen) parabolic subgroup of  $GSp_4$  containing  $B$ . Let  $P = M_P N_P$  and  $Q = M_Q N_Q$  be Levi decompositions. Let  $\alpha_1, \alpha_2$  be simple roots defined by  $\alpha_1(t) = ab^{-1}$  and  $\alpha_2(t) = b^2c^{-1}$  for  $t = \text{diag}(a, b, cb^{-1}, ca^{-1})$ ,  $a, b, c \in F^\times$ . Note that  $\Delta_P = \{\alpha_1\}$  and  $\Delta_Q = \{\alpha_2\}$ . Thus there exist two  $U_p$ -operators  $U_{\varpi, i}^B = [It_i I]$ ,  $i = 1, 2$  where  $t_1 = \text{diag}(1, 1, \varpi^{-1}, \varpi^{-1})$  and  $t_2 = \text{diag}(1, \varpi^{-1}, \varpi^{-1}, \varpi^{-2})$ . Note that  $t_1, t_2$  are positive elements in  $T_B^-$  relative to  $(B, I)$ , and  $t_1$  (respectively  $t_2$ ) is a positive element in  $T_P^-$  (respectively  $T_Q^-$ ) relative to  $(P, K_P)$  (respectively  $(Q, K_Q)$ ). In this section, we give  $p$ -stabilized vectors according to the classification of the parahorospherical representations of  $GSp_4(F)$  by Roberts and Schmidt [16].

**7.1. Iwahori case.** In this subsection, we study  $p$ -stabilized vectors with respect to  $\hat{B} = B$ . The strategy taking here is as follows:

- (i) Let  $(\pi, V)$  be an irreducible admissible representation of  $GSp_4(F)$  admitting a non-zero Iwahori-fixed vector. Then  $\pi$  is an irreducible constituent of some unramified principal series representation  $\chi_1 \times \chi_2 \times \sigma$ . We use the classification of such representations in Table A.15 of [16].
- (ii) We make up a list of the simultaneous eigenvalues for  $U_p$ -operators  $U_{\varpi, 1}^B, U_{\varpi, 2}^B$  in terms of the following Satake parameters of  $\chi_1 \times \chi_2 \times \sigma$ :

$$(7.1) \quad \alpha = \chi_1 \chi_2 \sigma(\varpi^{-1}), \quad \beta = \chi_1 \sigma(\varpi^{-1}), \quad \gamma = \chi_2 \sigma(\varpi^{-1}), \quad \delta = \sigma(\varpi^{-1}).$$

By Proposition 2.3, the problem is reduced to the computation of the simultaneous eigenvalues for  $t_1, t_2$  on  $(V_U)^{I \cap T}$ . We note that  $V_U = (V_U)^{I \cap T}$ . Table A.3 of [16] gives the semisimplification of the Jacquet module  $\pi_{N_P}$  of  $\pi$  associated to the Siegel parabolic subgroup  $P$ . So we can easily get the semisimplification  $\pi_U^{ss}$  of  $\pi_U$  by using the transitivity of Jacquet functors. The elements  $t_1, t_2$  act on each irreducible component of  $\pi_U^{ss}$  by its central character. Thus we can obtain the set  $S'$  of pairs of simultaneous eigenvalues for  $(t_1, t_2)$  on  $V_U = (V_U)^{I \cap T}$ . It will turn out that  $S'$  is contained

in  $S$ , where  $S$  is the set of pairs of simultaneous eigenvalues for  $(t_1, t_2)$  on  $(\chi_1 \times \chi_2 \rtimes \sigma)_U = (\chi_1 \times \chi_2 \rtimes \sigma)_U^{I \cap T}$ , that is,

$$S = \{(\delta, \delta\gamma), (\delta, \delta\beta), (\alpha, \alpha\beta), (\alpha, \alpha\gamma), (\gamma, \gamma\delta), (\gamma, \gamma\alpha), (\beta, \beta\delta), (\beta, \beta\alpha)\}.$$

(iii) Suppose that  $S'$  contains just  $\dim V_U$ -elements. Since  $t_1, t_2$  generates  $T/Z_G$ , this implies that  $(\pi_U)^{I \cap T}$  is a semisimple and multiplicity-free  $T$ -module. We further assume that  $\pi$  has a non-zero  $K$ -fixed vector  $\phi_K$ . In this case, given an element  $(s, t)$  in  $S'$ , the vector

$$\phi_{s,t} := \prod_{\substack{(s',t') \in S' \\ s' \neq s}} (\delta_B^{\frac{1}{2}}(t_1)U_{\varpi,1}^B - s') \prod_{\substack{(s',t') \in S' \\ t' \neq t}} (\delta_B^{\frac{1}{2}}(t_2)U_{\varpi,2}^B - t') \phi_K$$

is a  $p$ -stabilized vector with respect to  $\widehat{B} = B$  with the eigenvalues  $(\delta_B^{-\frac{1}{2}}(t_1)s, \delta_B^{-\frac{1}{2}}(t_2)t)$  because of Proposition 2.3 and the results in Section 4. We note that Table A. 15 of [16] gives a list of the  $K$ -spherical representations of  $\mathrm{GSp}_4(F)$ .

**7.1.1. Case I.** Let  $\chi_1, \chi_2, \sigma$  be unramified quasi-characters of  $F^\times$ . Then the corresponding parabolically induced representation  $\pi = \chi_1 \times \chi_2 \rtimes \sigma$  is irreducible if and only if  $\chi_1 \neq \nu^{\pm 1}$ ,  $\chi_2 \neq \nu^{\pm 1}$  and  $\chi_1 \neq \nu^{\pm 1} \chi_2^{\pm 1}$ . Here let us put  $\nu = |\cdot|$ . Due to Table A. 15 of [16],  $\pi$  has a non-zero  $K$ -fixed vector. The semisimplification of  $\pi_{N_P}$  is given in Table A.3 of [16]. We use the notation in section A.3 of [16]. Let  $r_{T, M_P}$  denote the Jacquet functor from the category of the smooth representations of  $M_P$  to that of  $B$ . Note that the semisimplification of  $r_{T, M_P}((\chi_1 \times \chi_2) \otimes \sigma)$  is  $(\chi_1 \otimes \chi_2 \otimes \sigma) + (\chi_2 \otimes \chi_1 \otimes \sigma)$ . Since  $r_U = r_{T, M_P} \circ r_{N_P}$ , the semisimplification of  $\pi_U$  is

$$\begin{aligned} & (\chi_1 \otimes \chi_2 \otimes \sigma) + (\chi_2 \otimes \chi_1 \otimes \sigma) + (\chi_1^{-1} \otimes \chi_2^{-1} \otimes \chi_1 \chi_2 \sigma) \\ & + (\chi_2^{-1} \otimes \chi_1^{-1} \otimes \chi_1 \chi_2 \sigma) + (\chi_1 \otimes \chi_2^{-1} \otimes \chi_2 \sigma) + (\chi_2^{-1} \otimes \chi_1 \otimes \chi_2 \sigma) \\ & + (\chi_2 \otimes \chi_1^{-1} \otimes \chi_1 \sigma) + (\chi_1^{-1} \otimes \chi_2 \otimes \chi_1 \sigma). \end{aligned}$$

The elements  $t_1, t_2$  act on each component of the semisimplification of  $\pi_U$  by the following pairs of scalars:

$$(\delta, \delta\gamma), (\delta, \delta\beta), (\alpha, \alpha\beta), (\alpha, \alpha\gamma), (\gamma, \gamma\delta), (\gamma, \gamma\alpha), (\beta, \beta\delta), (\beta, \beta\alpha).$$

The pairs above are pairwise distinct if and only if  $\chi_1 \neq 1$ ,  $\chi_2 \neq 1$  and  $\chi_1 \neq \chi_2^{\pm 1}$ .

**7.1.2. Case II.** Let  $\chi, \sigma$  be unramified quasi-characters of  $F^\times$  such that  $\chi^2 \neq \nu^{\pm 1}$  and  $\chi \neq \nu^{\pm \frac{3}{2}}$ . Put  $\chi_1 = \nu^{\frac{1}{2}} \chi$  and  $\chi_2 = \nu^{-\frac{1}{2}} \chi$ . In what follows, we consider the irreducible constituents  $\pi$  of  $\chi_1 \times \chi_2 \rtimes \sigma$ .

**Case IIa.** Put  $\pi = \chi \text{St}_{\text{GL}(2)} \rtimes \sigma$ . Noting that  $r_{T, M_P}(\chi \text{St}_{\text{GL}(2)} \otimes \sigma) = \nu^{\frac{1}{2}} \chi \otimes \nu^{-\frac{1}{2}} \chi \otimes \sigma$ . It follows from Table A.3 of [16] that the semi-simplification of  $\pi_U$  is

$$(\chi_1 \otimes \chi_2 \otimes \sigma) + (\chi_2^{-1} \otimes \chi_1^{-1} \otimes \chi_1 \chi_2 \sigma) + (\chi_1 \otimes \chi_2^{-1} \otimes \chi_2 \sigma) + (\chi_2^{-1} \otimes \chi_1 \otimes \chi_2 \sigma).$$

Hence the elements  $t_1, t_2$  act on each irreducible component of  $\pi_U$  by

$$(\delta, \delta\gamma), (\alpha, \alpha\gamma), (\gamma, \gamma\delta), (\gamma, \gamma\alpha)$$

respectively. The pairs above are pairwise distinct if and only if  $\chi^2 \neq 1$ .

**Case IIb.** If  $\pi = \chi \mathbf{1}_{\text{GL}(2)} \rtimes \sigma$ , then  $\pi$  has a non-zero  $K$ -fixed vector. Since  $r_{T, M_P}(\chi \mathbf{1}_{\text{GL}(2)} \otimes \sigma) = \nu^{-\frac{1}{2}} \chi \otimes \nu^{\frac{1}{2}} \chi \otimes \sigma$ , it follows from Table A.3 of [16] that the semisimplification of  $\pi_U$  is

$$(\chi_2 \otimes \chi_1 \otimes \sigma) + (\chi_1^{-1} \otimes \chi_2^{-1} \otimes \chi_1 \chi_2 \sigma) + (\chi_2 \otimes \chi_1^{-1} \otimes \chi_1 \sigma) + (\chi_1^{-1} \otimes \chi_2 \otimes \chi_1 \sigma).$$

The elements  $t_1, t_2$  act on each component by

$$(\delta, \delta\beta), (\alpha, \alpha\beta), (\beta, \beta\delta), (\beta, \beta\alpha)$$

respectively. The pairs above are pairwise distinct if and only if  $\chi^2 \neq 1$ .

**7.1.3. Case III.** Let  $\chi$  and  $\rho$  be unramified quasi-characters of  $F^\times$ . We assume that  $\chi \neq 1$  and  $\chi \neq \nu^{\pm 2}$ . Put  $\chi_1 = \chi$ ,  $\chi_2 = \nu$ , and  $\sigma = \nu^{-\frac{1}{2}} \rho$ . Next, we consider the irreducible constituents  $\pi$  of  $\chi_1 \times \chi_2 \rtimes \sigma$ .

**Case IIIa.** If  $\pi = \chi \rtimes \rho \text{St}_{\text{GSp}(2)}$ , then the semisimplification of  $\pi_U$  is

$$(\chi_1 \otimes \chi_2 \otimes \sigma) + (\chi_2 \otimes \chi_1 \otimes \sigma) + (\chi_2 \otimes \chi_1^{-1} \otimes \chi_1 \sigma) + (\chi_1^{-1} \otimes \chi_2 \otimes \chi_1 \sigma).$$

The actions of  $t_i, i = 1, 2$  on each irreducible component of  $\pi_U$  are just

$$(\delta, \delta\gamma), (\delta, \delta\beta), (\beta, \beta\delta), (\beta, \beta\alpha)$$

respectively. The pairs above are pairwise distinct if and only if  $\chi \neq \nu^{\pm 1}$ .

**Case IIIb.** Suppose that  $\pi = \chi \rtimes \rho \mathbf{1}_{\text{GSp}(2)}$ . Then  $\pi$  admits a non-zero  $K$ -fixed vector and the semisimplification of  $\pi_U$  is

$$(\chi_1^{-1} \otimes \chi_2^{-1} \otimes \chi_1 \chi_2 \sigma) + (\chi_2^{-1} \otimes \chi_1^{-1} \otimes \chi_1 \chi_2 \sigma) + (\chi_1 \otimes \chi_2^{-1} \otimes \chi_2 \sigma) + (\chi_2^{-1} \otimes \chi_1 \otimes \chi_2 \sigma).$$

Hence the actions of  $t_i, i = 1, 2$  on each component are

$$(\alpha, \alpha\beta), (\alpha, \alpha\gamma), (\gamma, \gamma\delta), (\gamma, \gamma\alpha)$$

respectively. The pairs above are pairwise distinct if and only if  $\chi \neq \nu^{\pm 1}$ .

**7.1.4. Case IV.** Let  $\rho$  be an unramified quasi-character of  $F^\times$ . Put  $\chi_1 = \nu^2$ ,  $\chi_2 = \nu$ , and  $\sigma = \nu^{-\frac{3}{2}}\rho$ . We will consider the irreducible constituents  $\pi$  of  $\chi_1 \times \chi_2 \rtimes \sigma$ . In this case,  $\alpha, \beta, \gamma, \delta$  are different from each other. So  $\pi_U$  is a semisimple and multiplicity-free  $T$ -module.

**Case IVa.** Suppose that  $\pi = \rho \text{St}_{\text{GSp}(4)}$ . Then we have  $\pi_U = \chi_1 \otimes \chi_2 \otimes \sigma$ , and the elements  $t_1, t_2$  act on it by  $(\delta, \delta\gamma)$ . Since  $\dim \pi^I = 1$ , any non-zero  $I$ -fixed vector is itself a  $p$ -stabilized vector with the eigenvalues  $(\delta_B^{-\frac{1}{2}}(t_1)\delta, \delta_B^{-\frac{1}{2}}(t_2)\delta\gamma)$ .

**Case IVb.** If  $\pi = L(\nu^2, \nu^{-1}\rho \text{St}_{\text{GSp}(2)})$ , then  $\pi_U$  is isomorphic to

$$(\chi_2 \otimes \chi_1 \otimes \sigma) \oplus (\chi_2 \otimes \chi_1^{-1} \otimes \chi_1\sigma) \oplus (\chi_1^{-1} \otimes \chi_2 \otimes \chi_1\sigma).$$

Hence  $t_i, i = 1, 2$  act on each component by

$$(\delta, \delta\beta), (\beta, \beta\delta), (\beta, \beta\alpha)$$

respectively.

**Case IVc.** Suppose that  $\pi = L(\nu^{\frac{3}{2}}\text{St}_{\text{GL}(2)}, \nu^{-\frac{3}{2}}\rho)$ . Then  $\pi_U$  is isomorphic to

$$(\chi_2^{-1} \otimes \chi_1^{-1} \otimes \chi_1\chi_2\sigma) \oplus (\chi_1 \otimes \chi_2^{-1} \otimes \chi_2\sigma) \oplus (\chi_2^{-1} \otimes \chi_1 \otimes \chi_2\sigma),$$

and the elements  $t_1, t_2$  act on each irreducible component by

$$(\alpha, \alpha\gamma), (\gamma, \gamma\delta), (\gamma, \gamma\alpha)$$

respectively.

**Case IVd.** If  $\pi = \rho \mathbf{1}_{\text{GSp}(4)}$ , then we have  $\dim \pi^I = \dim \pi^K = 1$ , and hence  $\pi^I = \pi^K$ . We also get  $\pi_U = \chi_1^{-1} \otimes \chi_2^{-1} \otimes \chi_1\chi_2\sigma$ , and the elements  $t_1, t_2$  act on  $\pi_U$  by  $(\alpha, \alpha\beta)$ . Thus any non-zero  $K$ -fixed vector is a  $p$ -stabilized vector with the eigenvalues  $(\delta_B^{-\frac{1}{2}}(t_1)\alpha, \delta_B^{-\frac{1}{2}}(t_2)\alpha\beta)$ .

**7.1.5. Case V.** Let  $\xi$  and  $\rho$  be unramified quasi-characters of  $F^\times$ . We assume that  $\xi^2 = 1$  and  $\xi \neq 1$ . Put  $\chi_1 = \nu\xi$ ,  $\chi_2 = \xi$ , and  $\sigma = \nu^{-\frac{1}{2}}\rho$ . We consider the irreducible constituents  $\pi$  of  $\chi_1 \times \chi_2 \rtimes \sigma$ . In this case,  $\alpha, \beta, \gamma, \delta$  are different from each other. This means that  $\pi_U$  is a semisimple and multiplicity-free  $T$ -module.

**Case Va.** If  $\pi = \delta([\xi, \nu\xi], \nu^{-\frac{1}{2}}\rho)$ , then we have

$$\pi_U = (\chi_1 \otimes \chi_2 \otimes \sigma) \oplus (\chi_1 \otimes \chi_2^{-1} \otimes \chi_2\sigma).$$

The elements  $t_1, t_2$  act on each component by

$$(\delta, \delta\gamma), (\gamma, \gamma\delta)$$

respectively.

**Case Vb.** Suppose that  $\pi = L(\nu^{\frac{1}{2}}\xi\text{St}_{\text{GL}(2)}, \nu^{-\frac{1}{2}}\rho)$ . Then we have

$$\pi_U = (\chi_2^{-1} \otimes \chi_1^{-1} \otimes \chi_1\chi_2\sigma) \oplus (\chi_2^{-1} \otimes \chi_1 \otimes \chi_2\sigma),$$

and hence the elements  $t_1, t_2$  act on each irreducible component by

$$(\alpha, \alpha\gamma), (\gamma, \gamma\alpha)$$

respectively.

**Case Vc.** In the case when  $\pi = L(\nu^{\frac{1}{2}}\xi\text{St}_{\text{GL}(2)}, \xi\nu^{-\frac{1}{2}}\rho)$ , we have

$$\pi_U = (\chi_2 \otimes \chi_1 \otimes \sigma) \oplus (\chi_2 \otimes \chi_1^{-1} \otimes \chi_1\sigma).$$

The elements  $t_1, t_2$  act on each component by

$$(\delta, \delta\beta), (\beta, \beta\delta)$$

respectively.

**Case Vd.** If  $\pi = L(\nu\xi, \xi \rtimes \nu^{-\frac{1}{2}}\rho)$ , then  $\pi$  has a non-zero  $K$ -fixed vector. We have

$$\pi_U = (\chi_1^{-1} \otimes \chi_2^{-1} \otimes \chi_1\chi_2\sigma) \oplus (\chi_1^{-1} \otimes \chi_2 \otimes \chi_1\sigma).$$

Hence the elements  $t_1, t_2$  act on each component by

$$(\alpha, \alpha\beta), (\beta, \beta\alpha)$$

respectively.

**7.1.6. Case VI.** Let  $\rho$  be an unramified quasi-character of  $F^\times$ . Put  $\chi_1 = \nu$ ,  $\chi_2 = \mathbf{1}_{F^\times}$  and  $\sigma = \nu^{-1/2}\rho$ . Finally, we consider the irreducible constituents  $\pi$  of  $\chi_1 \times \chi_2 \rtimes \sigma$ . In this case, we have  $\alpha = \beta$ ,  $\gamma = \delta$  and  $\alpha \neq \gamma$ .

**Case VIa.** Suppose that  $\pi = \tau(S, \nu^{-\frac{1}{2}}\rho)$ . Then the semisimplification of  $\pi_U$  is

$$(\chi_1\chi_2 \otimes \chi_1 \otimes \sigma)^{\oplus 2} + (\chi_2 \otimes \chi_1 \otimes \sigma).$$

Hence the actions of  $t_i, i = 1, 2$  on each component are

$$(\gamma, \gamma^2), (\gamma, \gamma^2), (\gamma, \gamma\alpha)$$

respectively.

**Case VIb.** If  $\pi = \tau(T, \nu^{-\frac{1}{2}}\rho)$ , then we have  $\dim \pi^{K_P} = \dim \pi^I = 1$  by Table A. 15 in [16], and hence  $\pi^I = \pi^{K_P}$ . We also get  $\pi_U = \chi_2 \otimes \chi_1 \otimes \sigma$ , and the elements  $t_1, t_2$  act on  $\pi_U$  by  $(\gamma, \gamma\alpha)$ . Therefore any non-zero  $K_P$ -spherical vector is a  $p$ -stabilized vector with the eigenvalues  $(\delta_B^{-\frac{1}{2}}(t_1)\gamma, \delta_B^{-\frac{1}{2}}(t_2)\gamma\alpha)$ .

**Case VIc.** Suppose that  $\pi = L(\nu^{\frac{1}{2}}\text{St}_{\text{GL}(2)}, \nu^{-\frac{1}{2}}\rho)$ . Then we get  $\pi_U = \chi_2 \otimes \chi_1^{-1} \otimes \chi_1\sigma$ , and  $t_i, i = 1, 2$  acts on it by  $(\alpha, \gamma\alpha)$ . By Table A. 15 in [16], the space  $\pi^{K_Q}$  is one-dimensional. Therefore a non-zero  $K_Q$ -fixed vector is a  $p$ -stabilized vector with the eigenvalues  $(\delta_B^{-\frac{1}{2}}(t_1)\alpha, \delta_B^{-\frac{1}{2}}(t_2)\gamma\alpha)$ .

**Case VIId.** If  $\pi = L(\nu, \mathbf{1}_{F^\times} \rtimes \nu^{-1/2}\rho)$ , then  $\pi$  has a non-zero  $K$ -fixed vector. The semisimplification of  $\pi_U$  is

$$(\chi_1^{-1} \otimes \chi_2 \otimes \chi_1\sigma)^{\oplus 2} + (\chi_2 \otimes \chi_1^{-1} \otimes \chi_1\sigma).$$

Hence the actions of  $t_i, i = 1, 2$  on each component are just

$$(\alpha, \alpha^2), (\alpha, \alpha^2), (\alpha, \gamma\alpha)$$

respectively.

In Table 7.1, we list the simultaneous eigenvalues for  $t_1, t_2$  on  $\pi_U$ , where  $\pi$  is a representation in groups I-V. For a representation in group VI, we always have  $\alpha = \beta, \gamma = \delta$  and  $\alpha \neq \gamma$ . So for such representations, we list the multiplicity of the simultaneous eigenvalues for  $t_1, t_2$  on  $\pi_U$  in Table 7.2.

**7.2. Siegel parahoric case.** In this subsection, we compute the eigenvalues of the  $U_P$ -operator  $U_{\varpi,1}^P$ , where  $P$  denotes the Siegel parahoric subgroup of  $\mathrm{GSp}_4(F)$ . The strategy is as follows:

- (i) We use the classification of the irreducible smooth representations  $(\pi, V)$  of  $\mathrm{GSp}_4(F)$  admitting  $K_P$ -fixed vectors in Table A.15 of [16]. As in the previous subsection, we realize  $\pi$  as an irreducible constituent of some unramified principal series representation  $\chi_1 \times \chi_2 \rtimes \sigma$ . We use the same notation as in subsection 7.1.
- (ii) We compute the set  $S'$  of eigenvalues of  $t_1$  on  $(V_{N_P})^{M_P \cap K_P}$  as follows: The semisimplification  $\pi_{N_P}^{\mathrm{ss}}$  of the Jacquet module  $\pi_{N_P}$  of  $\pi$  associated to  $P$  is given in Table A.3 of [16]. We denote by  $\pi_{N_P, M_P \cap K_P}$  the  $M_P$ -submodule of  $\pi_{N_P}^{\mathrm{ss}}$  spanned by the  $M_P \cap K_P$ -fixed vectors in  $\pi_{N_P}^{\mathrm{ss}}$ . Note that for any irreducible admissible representation  $\tau$  of  $M_P \simeq \mathrm{GL}_2(F) \times F^\times$ , we have  $\dim \tau^{M_P \cap K_P} \leq 1$ . Thus the length of  $\pi_{N_P, M_P \cap K_P}$  is equal to  $\dim(V_{N_P})^{M_P \cap K_P}$ . Since the element  $t_1$  lies in the center of  $M_P$ , the eigenvalues of  $t_1$  on  $(V_{N_P})^{M_P \cap K_P}$  is just those of  $t_1$  on  $\pi_{N_P, M_P \cap K_P}$ . So we can easily compute the eigenvalues of  $t_1$  on  $(V_{N_P})^{M_P \cap K_P}$  because  $t_1$  acts on each irreducible component of  $\pi_{N_P, M_P \cap K_P}$  by the central character. It will turn out that  $S'$  is contained in  $S = \{\alpha, \beta, \gamma, \delta\}$ , where  $\alpha, \beta, \gamma, \delta$  are the Satake parameters of  $\chi_1 \times \chi_2 \rtimes \sigma$  defined in (7.1).
- (iii) We assume that  $S'$  contains just  $\dim(\pi_{N_P})^{K_P \cap M_P}$ -elements. Then  $(\pi_{N_P})^{K_P \cap M_P}$  is a semisimple and multiplicity-free  $Z_{M_P}$ -module because  $t_1$  generates  $Z_{M_P}$  modulo  $Z_{M_P}(\mathfrak{o})$ . We further assume that  $\pi$  has a non-zero  $K$ -fixed vector  $\phi_K$ . Then, given an element  $s$  in  $S'$ , the vector

$$\phi_s := \prod_{\substack{s' \in S' \\ s' \neq s}} (\delta_P^{\frac{1}{2}}(t_1)U_{\varpi,1}^P - s')\phi_K$$

rep.	$(\alpha, \alpha\beta)$	$(\alpha, \alpha\gamma)$	$(\beta, \beta\alpha)$	$(\beta, \beta\delta)$	$(\gamma, \gamma\alpha)$	$(\gamma, \gamma\delta)$	$(\delta, \delta\beta)$	$(\delta, \delta\gamma)$	$\dim \pi^I$	$\dim \pi^K$
I	○	○	○	○	○	○	○	○	8	1
IIa	-	○	-	-	○	○	-	○	4	0
IIb	○	-	○	○	-	-	○	-	4	1
IIIa	-	-	○	○	-	-	○	○	4	0
IIIb	○	○	-	-	○	○	-	-	4	1
IVa	-	-	-	-	-	-	-	○	1	0
IVb	-	-	○	○	-	-	○	-	3	0
IVc	-	○	-	-	○	○	-	-	3	0
IVd	○	-	-	-	-	-	-	-	1	1
Va	-	-	-	-	-	○	-	○	2	0
Vb	-	○	-	-	○	-	-	-	2	0
Vc	-	-	-	○	-	-	○	-	2	0
Vd	○	-	-	-	-	-	-	-	2	1

TABLE 7.1

representation	$(\alpha, \alpha^2)$	$(\alpha, \alpha\gamma)$	$(\gamma, \gamma\alpha)$	$(\gamma, \gamma^2)$	$\dim \pi^I$	$\dim \pi^K$
VIa	0	0	1	2	3	0
VIb	0	0	1	0	1	0
VIc	0	1	0	0	1	0
VId	2	1	0	0	3	1

TABLE 7.2



is a  $p$ -stabilized vector with respect to  $\widehat{P} = Q$  with the eigenvalue  $\delta_P^{-\frac{1}{2}}(t_1)s$  because of Proposition 2.3 and the results in Section 4.

**7.2.1. Case I.** In this case,  $\pi_{N_P, M_P \cap K_P}$  is

$$(\chi_1 \times \chi_2) \otimes \sigma + (\chi_1^{-1} \times \chi_2^{-1}) \otimes \chi_1 \chi_2 \sigma + (\chi_1 \times \chi_2^{-1}) \otimes \chi_2 \sigma + (\chi_2 \times \chi_1^{-1}) \otimes \chi_1 \sigma.$$

The element  $t_1$  acts on each component of  $\pi_{N_P, M_P \cap K_P}$  by

$$\delta, \alpha, \gamma, \beta$$

respectively.

**7.2.2. Case II. Case IIa.** In this case, we have  $\pi_{N_P, M_P \cap K_P} = (\chi_1 \times \chi_2^{-1}) \otimes \chi_2 \sigma$ . The element  $t_1$  acts on it by  $\gamma$ . Since  $\dim V^{K_P} = 1$ , any non-zero  $K_P$ -fixed vector is a  $p$ -stabilized vector with respect to  $\widehat{P}$ .

**Case IIb.** In this case, we have

$$\pi_{N_P, M_P \cap K_P} = \chi \mathbf{1}_{\mathrm{GL}(2)} \otimes \sigma + \chi^{-1} \mathbf{1}_{\mathrm{GL}(2)} \otimes \chi^2 \sigma + (\chi_2 \times \chi_1^{-1}) \otimes \chi_1 \sigma,$$

and  $t_1$  acts on each component by  $\delta, \alpha, \beta$  respectively.

**7.2.3. Case III. Case IIIa.** In this case, we obtain

$$\pi_{N_P, M_P \cap K_P} = (\chi_1 \times \chi_2) \otimes \sigma + (\chi_2 \times \chi_1^{-1}) \otimes \chi_1 \sigma,$$

and  $t_1$  acts on each component by  $\delta$  and  $\beta$  respectively.

**Case IIIb.** We get

$$\pi_{N_P, M_P \cap K_P} = (\chi_2^{-1} \times \chi_1^{-1}) \otimes \chi_1 \chi_2 \sigma + (\chi_1 \times \chi_2^{-1}) \otimes \chi_2 \sigma.$$

The element  $t_1$  acts on each component by  $\alpha$  and  $\gamma$  respectively.

**7.2.4. Case IV. Case IVa.** The representations in case IVa have no  $K_P$ -fixed vectors.

**Case IVb.** We obtain

$$\pi_{N_P, M_P \cap K_P} = \nu^{3/2} \mathbf{1}_{\mathrm{GL}(2)} \otimes \sigma + (\chi_2 \times \chi_1^{-1}) \otimes \chi_1 \sigma.$$

The element  $t_1$  acts on each component by  $\delta$  and  $\beta$  respectively.

**Case IVc.** In this case,  $t_1$  acts on  $\pi_{N_P, M_P \cap K_P} = (\chi_1 \times \chi_2^{-1}) \otimes \chi_2 \sigma$  by  $\gamma$ .

**Case IVd.** The element  $t_1$  acts on  $\pi_{N_P, M_P \cap K_P} = \nu^{-3/2} \mathbf{1}_{\mathrm{GL}(2)} \otimes \chi_1 \chi_2 \sigma$  by  $\alpha$ .

**7.2.5. Case V. Case Va.** In this case,  $\pi$  admits no  $K_P$ -fixed vectors.

**Case Vb.** We have  $\pi_{N_P, M_P \cap K_P} = \nu^{1/2} \xi \mathbf{1}_{\text{GL}(2)} \otimes \chi_2 \sigma$ , and  $t_1$  acts on it by  $\gamma$ .

**Case Vc.** The element  $t_1$  acts on  $\pi_{N_P, M_P \cap K_P} = \nu^{1/2} \xi \mathbf{1}_{\text{GL}(2)} \otimes \sigma$  by  $\delta$ .

**Case Vd.** In this case, we obtain

$$\pi_{N_P, M_P \cap K_P} = \nu^{-1/2} \xi \mathbf{1}_{\text{GL}(2)} \otimes \chi_1 \sigma \oplus \nu^{-1/2} \xi \mathbf{1}_{\text{GL}(2)} \otimes \chi_1 \chi_2 \sigma.$$

So  $t_1$  acts on each component by  $\beta$  and  $\alpha$  respectively.

**7.2.6. Case VI. Case VIa.** In this case, the element  $t_1$  acts on the space  $\pi_{N_P, M_P \cap K_P} = \nu^{1/2} \mathbf{1}_{\text{GL}(2)} \otimes \sigma$  by  $\gamma$ .

**Case VIb.** We have  $\pi_{N_P, M_P \cap K_P} = \nu^{1/2} \mathbf{1}_{\text{GL}(2)} \otimes \sigma$ . The element  $t_1$  acts on it by  $\gamma$ .

**Case VIc.** In this case,  $\pi$  has no  $K_P$ -fixed vectors.

**Case VId.** We get  $\pi_{N_P, M_P \cap K_P} = (\nu^{-1/2} \mathbf{1}_{\text{GL}(2)} \otimes \chi_1 \sigma)^{\oplus 2}$  and  $t_1$  acts on each component by  $\alpha$ .

In Table 7.3, we list the eigenvalues for  $t_1$  on  $(\pi_{N_P})^{M_P \cap K_P}$ , where  $\pi$  is a representation in groups I-V. For representations in group VI, we list the multiplicity of the eigenvalues for  $t_1$  on  $(\pi_{N_P})^{M_P \cap K_P}$  in Table 7.4.

representation	$\alpha$	$\beta$	$\gamma$	$\delta$	$\dim \pi^{K_P}$	$\dim \pi^K$
I	○	○	○	○	4	1
IIa	-	-	○	-	1	0
IIb	○	○	-	○	3	1
IIIa	-	○	-	○	2	0
IIIb	○	-	○	-	2	1
IVa	-	-	-	-	0	0
IVb	-	○	-	○	2	0
IVc	-	-	○	-	1	0
IVd	○	-	-	-	1	1
Va	-	-	-	-	0	0
Vb	-	-	○	-	1	0
Vc	-	-	-	○	1	0
Vd	○	○	-	-	2	1

TABLE 7.3

representation	$\alpha$	$\gamma$	$\dim \pi^{K_P}$	$\dim \pi^K$
VIa	0	1	1	0
VIb	0	1	1	0
VIc	0	0	0	0
VIId	2	0	2	1

TABLE 7.4

**7.3. Klingen parahoric case.** In this subsection, we compute the eigenvalues of the  $U_p$ -operator  $U_{\varpi,1}^Q = [K_Q t_2 K_Q]$ . The strategy is exactly same to that for the Siegel parahoric case. So we shall be brief here.

- (i) According to the classification in Table A.15 of [16], we realize the representations  $(\pi, V)$  of  $\mathrm{GSp}_4(F)$  which admit  $K_Q$ -fixed vectors as an irreducible constituent of some unramified principal series representation  $\chi_1 \times \chi_2 \rtimes \sigma$ .
- (ii) We compute the set  $S'$  of eigenvalues of  $t_2$  on  $(V_{N_Q})^{M_Q \cap K_Q}$  as follows: The semisimplification  $\pi_{N_Q}^{\mathrm{ss}}$  of  $\pi_{N_Q}$  is given in Table A.3 of [16]. We denote by  $\pi_{N_Q, M_Q \cap K_Q}$  the  $M_Q$ -submodule of  $\pi_{N_Q}^{\mathrm{ss}}$  generated by the  $M_Q \cap K_Q$ -fixed vectors. Note that for any irreducible admissible representation  $\tau$  of  $M_Q \simeq F^\times \times \mathrm{GSp}_2(F)$ , we have  $\dim \tau^{M_Q \cap K_Q} \leq 1$ . So the length of  $\pi_{N_Q, M_Q \cap K_Q}$  is equal to  $\dim(V_{N_Q})^{M_Q \cap K_Q}$ . Since  $t_2 \in Z_{M_Q}$ , the eigenvalues of  $t_2$  on  $(V_{N_Q})^{M_Q \cap K_Q}$  is just those of  $t_2$  on  $\pi_{N_Q, M_Q \cap K_Q}$ . So we can compute the eigenvalues of  $t_2$  on  $(V_{N_Q})^{M_Q \cap K_Q}$  because  $t_2$  acts on each irreducible component of  $\pi_{N_Q, M_Q \cap K_Q}$  by the central character. It will turns out that  $S'$  is contained in  $S = \{\alpha\beta, \alpha\gamma, \delta\beta, \delta\gamma\}$ , where  $\alpha, \beta, \gamma, \delta$  are the Satake parameters of  $\chi_1 \times \chi_2 \rtimes \sigma$  defined in (7.1).
- (iii) If  $S'$  contains just  $\dim(\pi_{N_Q})^{K_Q \cap M_Q}$ -elements, then  $(\pi_{N_Q})^{K_Q \cap M_Q}$  is a semisimple and multiplicity-free  $Z_{M_Q}$ -module. We further assume that  $\pi$  has a non-zero  $K$ -fixed vector  $\phi_K$ . Then, given an element  $s$  in  $S'$ , the vector

$$\phi_s := \prod_{\substack{s' \in S' \\ s' \neq s}} (\delta_Q^{\frac{1}{2}}(t_2) U_{\varpi,1}^Q - s') \phi_K$$

is a  $p$ -stabilized vector with respect to  $\widehat{Q} = P$  with the eigenvalue  $\delta_Q^{-\frac{1}{2}}(t_2)s$  because of Proposition 2.3 and the result in Section 4.

**7.3.1. Case I.** In this case, we have

$$\begin{aligned} \pi_{N_Q, M_Q \cap K_Q} = & \chi_1 \otimes (\chi_2 \rtimes \sigma) + \chi_2 \otimes (\chi_1 \rtimes \sigma) + \chi_2^{-1} \otimes (\chi_1 \rtimes \chi_2 \sigma) \\ & + \chi_1^{-1} \otimes (\chi_2 \rtimes \chi_1 \sigma). \end{aligned}$$

The element  $t_2$  acts on each irreducible component of  $\pi_{N_Q, M_Q \cap K_Q}$  by

$$\delta\gamma, \beta\delta, \alpha\gamma, \alpha\beta$$

respectively.

**7.3.2. Case II. Case IIa.** In this case, we get  $\pi_{N_Q, M_Q \cap K_Q} = \chi_1 \otimes (\chi_2 \rtimes \sigma) + \chi_2^{-1} \otimes (\chi_1 \rtimes \chi_2 \sigma)$ . The element  $t_2$  acts on each component by  $\delta\gamma$  and  $\alpha\gamma$  respectively.

**Case IIb.** The element  $t_2$  acts on each component of  $\pi_{N_Q, M_Q \cap K_Q} = \chi_2 \otimes (\chi_1 \rtimes \sigma) + \chi_1^{-1} \otimes (\chi_2 \rtimes \chi_1 \sigma)$  by  $\beta\delta$  and  $\alpha\beta$  respectively.

**7.3.3. Case III. Case IIIa.** In this case,  $t_2$  acts on  $\pi_{N_Q, M_Q \cap K_Q} = \chi_2 \otimes (\chi_1 \rtimes \sigma)$  by  $\beta\delta$ .

**Case IIIb.** We have  $\pi_{N_Q, M_Q \cap K_Q} = \chi \otimes \rho \mathbf{1}_{\mathrm{GSp}(2)} + \chi^{-1} \otimes \chi \rho \mathbf{1}_{\mathrm{GSp}(2)} + \chi_2^{-1} \otimes (\chi_1 \rtimes \chi_2 \sigma)$ , and  $t_2$  acts on each component of it by  $\delta\gamma, \alpha\beta$  and  $\alpha\gamma$  respectively.

**7.3.4. Case IV. Case IVa.** In this case,  $\pi$  has no  $K_Q$ -fixed vectors.

**Case IVb.** The element  $t_2$  acts on  $\pi_{N_Q, M_Q \cap K_Q} = \chi_2 \otimes (\chi_1 \rtimes \sigma)$  by  $\beta\delta$ .

**Case IVc.** We have  $\pi_{N_Q, M_Q \cap K_Q} = \nu^2 \otimes \nu^{-1} \rho \mathbf{1}_{\mathrm{GSp}(2)} + \chi_2^{-1} \otimes (\chi_1 \rtimes \chi_2 \sigma)$ , and  $t_2$  acts on each component by  $\delta\gamma$  and  $\alpha\gamma$  respectively.

**Case IVd.** The element  $t_2$  acts on  $\pi_{N_Q, M_Q \cap K_Q} = \nu^{-2} \otimes \nu \rho \mathbf{1}_{\mathrm{GSp}(2)}$  by  $\alpha\beta$ .

**7.3.5. Case V. Case Va.** In this case,  $t_2$  acts on  $\pi_{N_Q, M_Q \cap K_Q} = \chi_1 \otimes (\chi_2 \rtimes \sigma)$  by  $\delta\gamma$ .

**Case Vb.** In this case,  $t_2$  acts on  $\pi_{N_Q, M_Q \cap K_Q} = \chi_2^{-1} \otimes (\chi_1 \rtimes \chi_2 \sigma)$  by  $\alpha\gamma$ .

**Case Vc.** The element  $t_2$  acts on  $\pi_{N_Q, M_Q \cap K_Q} = \chi_2 \otimes (\chi_1 \rtimes \sigma)$  by  $\beta\delta$ .

**Case Vd.** In this case,  $t_2$  acts on  $\pi_{N_Q, M_Q \cap K_Q} = \nu^{-1/2} \xi \otimes (\xi \rtimes \nu^{1/2} \rho)$  by  $\alpha\beta$ .

**7.3.6. Case VI. Case VIa.** We have  $\pi_{N_Q, M_Q \cap K_Q} = \chi_1 \otimes (\chi_2 \rtimes \sigma)$ , and hence  $t_2$  acts on it by  $\gamma^2$ .

**Case VIb.** In this case,  $\pi$  has no  $K_Q$ -fixed vectors.

**Case VIc.** The element  $t_2$  acts on  $\pi_{N_Q, M_Q \cap K_Q} = \mathbf{1}_{F^\times} \otimes \rho \mathbf{1}_{\mathrm{GSp}(2)}$  by  $\alpha\gamma$ .

**Case VId.** In this case,  $t_2$  acts on each component of  $\pi_{N_Q, M_Q \cap K_Q} = \mathbf{1}_{F^\times} \otimes \rho \mathbf{1}_{\mathrm{GSp}(2)} + \chi_1^{-1} \otimes (\chi_2 \rtimes \chi_1 \sigma)$  by  $\alpha\gamma$  and  $\alpha^2$  respectively.

In Table 7.5, we list the eigenvalues for  $t_2$  on  $(\pi_{N_Q})^{M_Q \cap K_Q}$ , where  $\pi$  is a representation in groups I-V. For representations in group VI, we list the multiplicity of the eigenvalues for  $t_2$  on  $(\pi_{N_Q})^{M_Q \cap K_Q}$  in Table 7.6.

representation	$\alpha\beta$	$\alpha\gamma$	$\delta\beta$	$\delta\gamma$	$\dim \pi^{K_Q}$	$\dim \pi^K$
I	○	○	○	○	4	1
IIa	-	○	-	○	2	0
IIb	○	-	○	-	2	1
IIIa	-	-	○	-	1	0
IIIb	○	○	-	○	3	1
IVa	-	-	-	-	0	0
IVb	-	-	○	-	1	0
IVc	-	○	-	○	2	0
IVd	○	-	-	-	1	1
Va	-	-	-	○	1	0
Vb	-	○	-	-	1	0
Vc	-	-	○	-	1	0
Vd	○	-	-	-	1	1

TABLE 7.5

representation	$\alpha^2$	$\alpha\gamma$	$\gamma^2$	$\dim \pi^{K_Q}$	$\dim \pi^K$
VIa	0	0	1	1	0
VIb	0	0	0	0	0
VIc	0	1	0	1	0
VId	1	1	0	2	1

TABLE 7.6

## 8. A relation to global objects

In this section, we will be concerned with global objects and give an answer why we consider the actions of the positive elements in local settings. Basic references of this section are [15] and [9] (see also Section 5 of II in [18]).

Let  $G$  be a non-compact connected semisimple algebraic group over  $\mathbb{Q}$  and  $K$  the maximal compact subgroup of  $G(\mathbb{R})$ . Let  $K_{\mathbb{C}}$  be the complexification of  $K$ . Put  $D = G(\mathbb{R})/K$ . Let  $\lambda : K_{\mathbb{C}} \rightarrow \text{Aut}(E_{\lambda})$  be an algebraic representation on a complex vector space  $E_{\lambda}$ . Then we obtain a homogeneous vector bundle  $\mathbb{E}_{\lambda} = G(\mathbb{R}) \times_{K, \lambda|_K} E_{\lambda} = \text{Ind}_K^{G(\mathbb{R})} \lambda|_K$  on  $D$ . Since  $D$  is simply connected, there is a (smooth) trivialization

$$\mathbb{E}_{\lambda} \xrightarrow{\sim} D \times E_{\lambda}.$$

With respect to this trivialization, the action of  $\gamma \in G(\mathbb{R})$  on  $D \times E_\lambda$  is given by

$$\gamma(z, v) = (\gamma z, J_\lambda(\gamma, z)v)$$

for some smooth mapping  $J_\lambda : G \times D \rightarrow \text{Aut}(E_\lambda)$  which is so called an automorphic factor. The action of  $G(\mathbb{R})$  induces the cocycle condition  $J_\lambda(\gamma_1\gamma_2, z) = J_\lambda(\gamma_1, \gamma_2 z)J_\lambda(\gamma_2, z)$  for all  $\gamma_1, \gamma_2 \in G(\mathbb{R})$ .

For a holomorphic  $E_\lambda$ -valued function  $f$  on  $D$ , the action of  $g \in G(\mathbb{R})$  is defined by

$$(g^{-1}f)(z) = J_\lambda^{-1}(g, z)f(gz).$$

For a discrete subgroup  $\Gamma$  of  $G(\mathbb{Q})$  and a finite character  $\chi : \Gamma \rightarrow \mathbb{C}^\times$ , we say that a holomorphic function  $f : D \rightarrow E_\lambda$  is an automorphic form of weight  $\lambda$  and level  $\Gamma$  with the character  $\chi$  if it satisfies  $\gamma f = \chi(\gamma)f$  for any  $\gamma \in \Gamma$ . We denote by  $M_\lambda(\Gamma, \chi)$  the space of automorphic forms of weight  $\lambda$  and level  $\Gamma$  with the character  $\chi$ .

Next we construct adelic forms from classical automorphic forms. Let  $G$  be a connected reductive group over  $\mathbb{Q}$  and  $Z_G$  the center of  $G$ . Then its derived group  $G^{\text{der}}$  is a semisimple connected algebraic group. Consider the following exact sequence

$$1 \rightarrow G^{\text{der}} \rightarrow G \xrightarrow{\nu} T := G/G^{\text{der}} \rightarrow 1$$

where  $T$  is a torus (cf. p.303 of [15]). If  $G = GL_2$  (respectively  $GSp_{2n}$ ), then  $T = \mathbb{G}_m$  and  $\nu$  is the determinant map (respectively the similitude character). Let  $G(\mathbb{R})^+$  be the connected component of the identity element in  $G(\mathbb{R})$  with respect to the real topology. For simplicity, in what follows we assume that  $G(\mathbb{R})^+ = Z_G(\mathbb{R})^+G^{\text{der}}(\mathbb{R})^+$  (for example,  $G = GSp_{2n}, GL_2, U(p, q), p, q > 0$  satisfy this condition).

Let  $\mathbb{A}$  be the adèle ring of  $\mathbb{Q}$  and  $\mathbb{A}_f$  the finite adèle of  $\mathbb{Q}$ . Let  $K$  be a compact open subgroup of  $G(\mathbb{A}_f)$ . Assume that  $\nu(K) \supset T(\hat{\mathbb{Z}})$ . Then it follows from the strong approximation theorem for  $G^{\text{der}}$  that

$$(8.1) \quad G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+K = G(\mathbb{Q})Z_G(\mathbb{R})^+G^{\text{der}}(\mathbb{R})^+K.$$

Set  $K_\infty = Z(\mathbb{R})^+K_\infty^{(1)}$ , where  $K_\infty^{(1)}$  is the maximal compact subgroup of  $G^{\text{der}}$ . Then  $D := G(\mathbb{R})^+/K_\infty = G^{\text{der}}(\mathbb{R})/K_\infty^{(1)}$  is the bounded symmetric domain endowed with an involution  $\iota$ . Let  $I \in D$  be the fixed point of  $\iota$ . From the description of  $D$  as above,  $Z(\mathbb{R})^+$  acts on  $I$  trivially. Put  $\Gamma = G^{\text{der}}(\mathbb{Q}) \cap K$ . For an automorphic form  $f \in M_\lambda(\Gamma, \chi)$ , we define the function  $F_f : G(\mathbb{A}) \rightarrow E_\lambda$  as follows. By (8.1), it is possible to write a given  $g \in G(\mathbb{A})$  as  $g = az_\infty g_\infty k$  with  $a \in G(\mathbb{Q})$ ,  $z_\infty \in Z_G(\mathbb{R})^+$ ,  $g_\infty \in G^{\text{der}}(\mathbb{R})^+$ , and  $k \in K$ . Then we put

$$F_f(g) = J_\lambda^{-1}(g_\infty, I)f(g_\infty I).$$

We can check that  $F_f$  is an automorphic form on  $G(\mathbb{A})$  in the sense of [4]. Let  $\pi_f$  be the maximal irreducible subquotient of the representation of  $G(\mathbb{A})$  generated by  $F_f$ .

We now consider Hecke operators and then compare them with those in local settings. For  $\alpha \in G(\mathbb{Q})^+ = G(\mathbb{Q}) \cap G(\mathbb{R})^+$ , consider the double coset

$$T(\alpha) := \Gamma\alpha\Gamma = \bigcup_i \Gamma\alpha_i.$$

We write  $\alpha_i = \alpha_{i,0}\alpha_{i,1}$ , where  $\alpha_{i,0} \in Z(\mathbb{R})^+$  and  $\alpha_{i,1} \in G^{\text{der}}(\mathbb{R})^+$ . Then we define the action of  $T(\alpha)$  on  $f$  by

$$T(\alpha)f(z) = n(\alpha)_\lambda \sum_i (\alpha_{i,1}f)(z)$$

where  $n(\alpha)_\lambda \in \mathbb{Q}^\times$  is a normalized factor depending on  $\lambda$  (and also on  $G$ ). Assume that  $G$  is unramified at a rational  $p$  and the  $p$ -component  $K_p$  of  $K$  is a compact open subgroup of  $G(\mathbb{Z}_p)$  which contains an Iwahori subgroup. There exists a compact open subgroup  $K^p$  of  $G(\mathbb{A}_f^{(p)})$  such that  $K = K^p \times K_p$ . Note that  $F_{f,p}$  is an Iwahori fixed vector of  $\pi_{f,p}$ . For  $\alpha \in G(\mathbb{Q}) \cap T(\mathbb{Q}_p)/T(\mathbb{Z}_p)$ , thus we obtain

$$\begin{aligned} T(\alpha)F_f(g) &= T(\alpha)F_f(g_\infty) = T(\alpha)J_\lambda^{-1}(g_\infty, I)F(g_\infty I) \\ &= n(\alpha)_\lambda \sum_i J_\lambda^{-1}(g_\infty, I)J_\lambda^{-1}(\alpha_{i,1}, g_\infty I)F(\alpha_{i,1}g_\infty I) \\ &= n(\alpha)_\lambda \sum_i J_\lambda^{-1}(\alpha_{i,1}g_\infty, I)F(\alpha_{i,1}g_\infty I) \\ &= n(\alpha)_\lambda \sum_i F_f(\alpha_{i,1}g_\infty). \end{aligned}$$

Note that

$$\begin{aligned} G(\mathbb{Q})Z_G(\mathbb{R})^+\alpha_{i,1}g_\infty K &= G(\mathbb{Q})Z_G(\mathbb{R})^+\alpha_i g_\infty K \\ &= G(\mathbb{Q})(g_\infty Z_G(\mathbb{R})^+ \times \alpha_i^{-1}K). \end{aligned}$$

Hence we have

$$\begin{aligned} T(\alpha)F_f(g) &= n(\alpha)_\lambda \sum_i F_f(g_\infty\alpha_i^{-1}) = n(\alpha)_\lambda \sum_{h \in K\alpha^{-1}K/K} F_f(g_\infty h) \\ &= n(\alpha)_\lambda \int_K F_f(g_\infty h g_f) dg_f = n(\alpha)_\lambda \int_{K\alpha^{-1}K} F_f(g_\infty g_f) dg_f \\ &= n(\alpha)_\lambda \int_{G(\mathbb{A}_f)} T(\alpha^{-1})_K F_f(g_\infty g_f) dg_f = n(\alpha)_\lambda [K_p\alpha^{-1}K_p]F_{f,p} \end{aligned}$$

where  $dg_f$  is the Haar measure on  $G(\mathbb{A}_f)$  normalized so that  $\text{vol}(K) = 1$  and  $T(\alpha^{-1})_K := [K_p\alpha^{-1}K_p] \otimes 1_{K^p}$ .

We can naturally define the notion of positive or negative elements in the global setting. In the global setting, we usually consider Hecke operators represented by negative elements. By the arguments above, the computation of such global Hecke operators is reduced to that of Hecke operators associated to positive elements in the local setting. In the next section, we will give examples of  $p$ -stabilized forms in the global setting.

### 9. $p$ -stabilized forms

**9.1.  $GL_2$ -case.** In this subsection we will refer [19] as a basic reference. Put  $G = GL_2$ . In this case, we have  $G^{\text{der}} = SL_2$ ,  $K = SO(2)(\mathbb{R})$ ,  $K_{\mathbb{C}} \simeq \mathbb{C}^{\times}$  and the corresponding Hermitian symmetric space is the upper half-plane  $\mathbb{H} = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ . The automorphic factor is given by  $J(\gamma, z) = cz + d$  for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$  and for an integer  $k \geq 1$  we define the algebraic representation  $\sigma_k : K_{\mathbb{C}} \rightarrow \mathbb{C}^{\times}, z \mapsto z^k$ . For an integer  $N \geq 1$ , we define the congruence subgroup  $\Gamma_0(N)$  (respectively  $\Gamma_1(N)$ ) to be the group consisting of the elements  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$  such that  $c \equiv 0 \pmod N$  (respectively  $a - 1 \equiv c \equiv 0 \pmod N$ ). For an integer  $N \geq 1$  and a Dirichlet character  $\chi : \Gamma_0(N) \rightarrow \mathbb{C}^{\times}$  so that  $\chi|_{\Gamma_1(N)} = 1$ , we define  $M_k(\Gamma_0(N), \chi) := M_{\sigma_k}(\Gamma_0(N), \chi)$ . For a prime  $p \nmid N$ , we define the action of Hecke operator  $T_p = [\Gamma_0(N)\text{diag}(1, p)\Gamma_0(N)]$  on  $M_k(\Gamma_0(N), \chi)$  by

$$\begin{aligned} T_p f(z) &:= p^{\frac{k}{2}-1} \sum_{\alpha \in \Gamma_0(N) \backslash \Gamma_0(N)\text{diag}(1, p)\Gamma_0(N)} ((p^{-\frac{1}{2}}\alpha)^{-1} f)(z) \\ &= p^{\frac{k}{2}-1} \sum_{\alpha \in \Gamma_0(N) \backslash \Gamma_0(N)\text{diag}(1, p)\Gamma_0(N)} j(p^{-\frac{1}{2}}\alpha, z)^{-k} f(p^{-\frac{1}{2}}\alpha z). \end{aligned}$$

Note that  $p^{-\frac{1}{2}}\alpha \in SL_2(\mathbb{R})$ . If  $p \nmid N$ , then we define the action of  $U_p = [\Gamma_0(pN)\text{diag}(1, p)\Gamma_0(pN)]$  on  $M_k(\Gamma_0(pN), \chi)$  by

$$U_p f(z) := p^{\frac{k}{2}-1} \sum_{\alpha \in \Gamma_0(pN) \backslash \Gamma_0(pN)\text{diag}(1, p)\Gamma_0(pN)} ((p^{-\frac{1}{2}}\alpha)^{-1} f)(z).$$

Let  $f \in M_k(\Gamma_0(N), \chi)$  be a normalized cusp form which is an eigenform for all  $T_p$ ,  $p \nmid N$  with the eigenvalue  $a_p(f)$ . Let  $\alpha_p, \beta_p$  be the Satake parameters at  $p$  so that  $a_p(f) = \alpha_p + \beta_p$  and  $\alpha_p\beta_p = \chi(p)p^{k-1}$ . Let  $\pi_f$  be the automorphic representation associated to  $f$  and  $\pi_{f,p}$  the local component at  $p$ . Since  $\pi_{f,p}$  is a principal series representation, there exist characters  $\chi_i : \mathbb{Q}_p^{\times} \rightarrow \mathbb{C}^{\times}$ ,  $i = 1, 2$  such that  $\pi_{f,p} \simeq \pi(\chi_1, \chi_2)$ . We may put  $\chi_1(p^{-1})p^{\frac{k-1}{2}} = \alpha_p$ ,  $\chi_2(p^{-1})p^{\frac{k-1}{2}} = \beta_p$ . We define  $p$ -stabilized forms as



follows:

$$f_{\alpha_p}(z) := f(z) - \beta_p f(pz), \quad f_{\beta_p}(z) := f(z) - \alpha_p f(pz).$$

Then  $f_{\alpha_p}$  and  $f_{\beta_p}$  belong to  $M_k(\Gamma_0(pN), \chi)$ , and these are Hecke eigenforms for  $T_\ell$ ,  $\ell \nmid pN$  and  $U_p$ -eigenforms with the eigenvalues  $\alpha_p, \beta_p$  respectively.

This can be checked via local computation as follows. By using the strong approximation theorem, it is easy to see that  $F_{f(pz)}(g) = F_f(g \cdot \text{diag}(1, p))$ . Hence the local component of  $F_{f_{\alpha_p}}$  (respectively  $F_{f_{\beta_p}}$ ) at  $p$  corresponds to  $\chi_1(p)f_1$  (respectively  $\chi_2(p)f_2$ ) of Section 5. By the computation of Section 5 again, we have

$$U_p f_{\alpha_p}(z) = p^{\frac{k}{2}-1} p^{\frac{1}{2}} \chi_2^{-1}(p) f_{\alpha_p}(z) = \alpha_p f_{\alpha_p}(z)$$

and

$$U_p f_{\beta_p}(z) = p^{\frac{k}{2}-1} p^{\frac{1}{2}} \chi_1^{-1}(p) f_{\beta_p}(z) = \beta_p f_{\beta_p}(z).$$

We say the above  $f$  is  $p$ -ordinary if  $\text{ord}_p(a_p(f)) = 0$ . If so is  $f$ , then we may assume that  $\text{ord}_p(\alpha_p) = 0$  and  $\text{ord}_p(\beta_p) > 0$ . The above computations show us that  $f_{\alpha_p}$  can be embedded into a Hida family if  $f$  is  $p$ -ordinary. Note that  $f_{\beta_p}$  can be embedded into a Coleman family (cf. [7]).

**9.2.  $GS_{p_4}$ -case.** Let  $\nu : G = GS_{p_4} \rightarrow GL_1$  be the similitude character.

Put  $Sp_4 := \text{Ker}\nu$ . In this case, we have  $G^{\text{der}} = Sp_4$ ,  $K = \left\{ \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in$

$Sp_4(\mathbb{R}) \right\} \simeq U(2)(\mathbb{R})$ ,  $K_{\mathbb{C}} \simeq GL_2(\mathbb{C})$  and the corresponding Hermitian sym-

metric space is the Siegel upper half-plane  $\mathcal{H}_2 = \{Z \in M_2(\mathbb{C}) \mid {}^t Z = Z, \text{Im}(Z) > 0\}$ . For a pair of positive integers  $\underline{k} = (k_1, k_2)$  such that  $k_1 \geq k_2$ , we define the algebraic representation  $\lambda_{\underline{k}}$  of  $GL_2(\mathbb{C})$  by

$$\lambda_{\underline{k}} = \text{Sym}^{k_1-k_2} \text{St}_2 \otimes \det^{k_2} \text{St}_2$$

where  $\text{St}_2$  is the standard representation of dimension 2 over  $\mathbb{C}$ . Then the corresponding automorphic factor is defined by

$$J_{\underline{k}}(\gamma, Z) = \lambda_{\underline{k}}(CZ + D)$$

for  $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_4(\mathbb{R})$  and  $Z \in \mathcal{H}_2$ . For an integer  $N \geq 1$ , we define

a principal congruence subgroup  $\Gamma(N)$  to be the group consisting of the elements  $g \in Sp_4(\mathbb{Z})$  such that  $g \equiv 1 \pmod N$ . We also define the level of  $\Gamma$  to be the minimal  $N$  satisfying (i)  $g\Gamma g^{-1}$  contains  $\Gamma(N)$  for some  $g \in GSp_4(\mathbb{Q})$  and (ii) all divisors of the denominator or the numerator of the entries of  $g$  divide  $N$ . If  $\Gamma$  is of level  $N$ , then the closure  $\bar{\Gamma}$  in  $Sp_4(\mathbb{A}_f)$  satisfies that the  $p$ -component of  $\bar{\Gamma}$  is  $Sp_4(\mathbb{Z}_p)$  for all  $p \nmid N$ . For a parabolic subgroup  $P$ , let

$I_P = \Gamma_P(p)$  be the group consisting of the elements  $g \in \mathrm{Sp}_4(\mathbb{Z})$  such that  $(g \bmod p) \in P(\mathbb{F}_p)$ . For a discrete subgroup  $\Gamma$  of level  $N$ , put  $\Gamma_{I_P} := \Gamma \cap I_P$ .

For a discrete subgroup  $\Gamma$  of level  $N$  and a Dirichlet character  $\chi : \Gamma \rightarrow \mathbb{C}^\times$ , we define  $M_{\underline{k}}(\Gamma, \chi) := M_{\lambda_{\underline{k}}}(\Gamma, \chi)$ . A function of this space is called a Siegel modular form of weight  $\underline{k}$  and level  $\Gamma$  with a character  $\chi$ . If  $k = k_1 = k_2$ , put  $M_k(\Gamma, \chi) := M_{(k,k)}(\Gamma, \chi)$  for short. For a prime  $p \nmid N$  and  $t'_1 = \mathrm{diag}(1, 1, p, p)$ ,  $t'_2 = \mathrm{diag}(1, p, p, p^2)$ , we define the action of two Hecke operators  $T_{p,i} = [\Gamma t'_i \Gamma]$ ,  $i = 1, 2$  on  $M_k := \bigcup_{(\Gamma, \chi)} M_k(\Gamma, \chi)$  by

$$\begin{aligned} T_{p,i}f(Z) &:= p^{i(\frac{k_1+k_2}{2}-3)} \sum_{\alpha \in \Gamma \backslash \Gamma t'_i \Gamma} ((\nu(t'_i)^{-\frac{1}{2}}\alpha)^{-1}f)(Z) \\ &= p^{i(\frac{k_1+k_2}{2}-3)} \sum_{\alpha \in \Gamma \backslash \Gamma t'_i \Gamma} J_{\underline{k}}(\nu(t'_i)^{-\frac{1}{2}}\alpha, Z)^{-1}f(\nu(t'_i)^{-\frac{1}{2}}\alpha Z). \end{aligned}$$

Then  $T_{p,i}f \in M_k$ , but in general, these operators do not preserve  $M_{\underline{k}}(\Gamma, \chi)$  (see Lemma 3.1 of [17]). Note that  $\nu(t'_i)^{-\frac{1}{2}}\alpha \in \mathrm{Sp}_4(\mathbb{R})$ . If  $p \nmid N$ , then we define the action of  $U_{p,i}^{P,\mathrm{global}} = [\Gamma_{I_P} t'_i \Gamma_{I_P}]$ ,  $i = 1, 2$  on  $M_k(\Gamma_{I_P}, \chi)$  by

$$(9.1) \quad U_{p,i}^{P,\mathrm{global}}f(Z) := p^{i(\frac{k_1+k_2}{2}-3)} \sum_{\alpha \in \Gamma_{I_P} \backslash \Gamma_{I_P} t'_i \Gamma_{I_P}} ((\nu(t'_i)^{-\frac{1}{2}}\alpha)^{-1}f)(Z).$$

In what follows, we will discuss about a  $p$ -stabilized form of Saito-Kurokawa lift. Let  $\Gamma_0(N)$  be the subgroup of  $\mathrm{Sp}_4(\mathbb{Z})$  consisting of the elements  $g = \begin{pmatrix} A_g & B_g \\ C_g & D_g \end{pmatrix}$  such that  $C_g \equiv 0_2 \pmod N$ . Fix a finite character  $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ . Then we define (and denote it by  $\chi$  again) the character on  $\Gamma_0(N)$  associated to  $\chi$  by  $\chi(g) := \chi(\det(D_g))$  for  $g \in \Gamma_0(N)$ . For a normalized elliptic newform  $f$  of weight  $2k - 2 \geq 2$  and level  $N$  with the character  $\chi^2$ , there exists a cusp form  $F = SK(f)$  in  $S_{k-1}(\Gamma_0(N), \chi)$  by [11] so that  $F$  is an eigenform for all  $T_{p,i}$ ,  $p \nmid N$  with the eigenvalue  $a_{p,i}(F)$  and these eigenvalues are written as

$$a_{p,1}(F) = \chi(p)(p^{k-1} + p^{k-2}) + a_p(f),$$

$$a_{p,2}(F) = a_p(f)\chi(p)(p^{k-2} + p^{k-3}) + 2\chi^2(p)p^{2k-4} - (p^2 + 1)\chi^2(p)p^{2k-6}$$

where  $a_p(f)$  is the eigenvalue of  $f$  for  $T_p$  in Section 9.1. Let  $\pi_f$  (respectively  $\Pi_F$ ) be the automorphic representation associated to  $f$  (respectively  $F = SK(f)$ ) and  $\pi_{f,p}$  (respectively  $\Pi_{F,p}$ ) the local component at  $p$ . Let  $\chi' : \mathbb{A}^\times \rightarrow \mathbb{C}^\times$  be the character corresponding to  $\chi$  and denote by  $\chi'_p$  its local component at  $p$ . Since  $\pi_{f,p}$  is a principal series representation, there exist characters  $\chi_i : \mathbb{Q}_p^\times \rightarrow \mathbb{C}^\times$ ,  $i = 1, 2$  such that  $\pi_{f,p} \simeq \pi(\chi_1, \chi_2)$  with the central character  $\chi_1\chi_2 = \chi_p'^{-2}$ . Hence we have  $\chi_1\chi_2(p) = \chi_p'^2(p^{-1})$ . Then

by [16], we have  $\Pi_{F,p} \simeq \chi_1 \chi'_1 1_{\text{GL}_2} \rtimes \chi_1^{-1} \chi'_p{}^{-2}$  which is the case IIb in Section 7.1.2. Then the eigenvalues of  $U_{p,i}^B, i = 1, 2$  in the local setting are

$$(p^{\frac{3}{2}}\delta, p^2\delta\beta), (p^{\frac{3}{2}}\alpha, p^2\alpha\beta), (p^{\frac{3}{2}}\beta, p^2\beta\delta), (p^{\frac{3}{2}}\beta, p^2\beta\alpha)$$

where  $\alpha = \chi_1(p^{-1}), \beta = p^{\frac{1}{2}}\chi'(p), \gamma = p^{-\frac{1}{2}}\chi'(p)$  and  $\delta = \chi_2(p^{-1})$ . We may put  $\chi_1(p^{-1})p^{\frac{2k-3}{2}} = \alpha_p(f), \chi_2(p^{-1})p^{\frac{2k-3}{2}} = \beta_p(f)$  where  $\alpha_p(f), \beta_p(f)$  are eigenvalues of  $T_p$  for  $f$ . From (9.1) and the observation in Section 7 (7.1-(iii) and 7.1.2 Case IIb),

$$(9.2) \quad (\beta_p - U_{p,1}^{B,\text{global}})(\chi'(p)p^{k-1} - U_{p,1}^{B,\text{global}})F$$

is a  $p$ -stabilized form with the eigenvalues  $(\alpha_p, p^{k-2}\alpha_p\chi'(p))$  for  $U_{p,i}^{B,\text{global}}, i = 1, 2$ . We take the Fourier expansion

$$F = \sum_{T \in \text{Sym}^2(\mathbb{Z})_{>0}} a(T)e^{2\pi\sqrt{-1}\text{tr}(TZ)}$$

where  $\text{Sym}^2(\mathbb{Z})_{>0}$  is the subset of  $M_2(\mathbb{Q})$  consisting of all symmetric matrices which are positive and semi-integral. Then we have

$$(9.3) \quad (\beta_p - U_{p,1}^{B,\text{global}})(\chi'(p)p^{k-1} - U_{p,1}^{B,\text{global}})F = \sum_{T>0} (p^{k-1}\chi'(p)\beta_p a(T) - (\chi'(p)p^{k-1} + \beta_p)a(pT) + a(p^2T))e^{2\pi\sqrt{-1}\text{tr}(TZ)},$$

since we can choose a complete system of representatives for  $\Gamma_{I,B} \backslash \Gamma_{I_B} t'_1 \Gamma_{I_B}$  to be the same as in the case Siegel parabolic and in that case we can compute the action easily (cf. [2]).

Similarly,  $(\beta_p - U_{p,1}^{P,\text{global}})(\chi'(p)p^{k-1} - U_{p,1}^{P,\text{global}})F$  is also a  $p$ -stabilized form with respect to  $\widehat{P} = Q$  with the eigenvalues  $\alpha_p$  for  $U_{p,1}^{P,\text{global}}$  (see 7.2-(iii) and 7.2.2 Case IIb). As explained above, we have the same expansion. Therefore we have

$$\begin{aligned} & (\beta_p - U_{p,1}^{B,\text{global}})(\chi'(p)p^{k-1} - U_{p,1}^{B,\text{global}})F \\ &= (\beta_p - U_{p,1}^{P,\text{global}})(\chi'(p)p^{k-1} - U_{p,1}^{P,\text{global}})F \end{aligned}$$

for  $t'_1$ . If  $k \geq 2, F$  has  $p$ -integral (algebraic) coefficients, then so does  $(\beta_p - U_{p,1}^{B,\text{global}})(\chi'(p)p^{k-1} - U_{p,1}^{B,\text{global}})F$ . We further assume that  $\text{ord}_p(\alpha_p) = 0$  (this implies  $\text{ord}_p(\beta_p) > 0$ ). Then by using the relation of Hecke operators (cf. p.228, Example 4.2.10 of [1]), we see that  $\text{ord}_p(a(p^2T_0)) = 0$  if  $\text{ord}_p(a(T_0)) = 0$  for some  $T_0 \in \text{Sym}^2(\mathbb{Z})_{>0}$ . Hence under this assumption (the existence of such  $T_0$ ),  $(\beta_p - U_{p,1}^{B,\text{global}})(\chi'(p)p^{k-1} - U_{p,1}^{B,\text{global}})F$  is not only preserving  $p$ -integrality, but also non-vanishing modulo  $p$ .

## References

- [1] A. N. ANDRIANOV, *Quadratic forms and Hecke operators*, Springer Berlin (1987).
- [2] S. BÖCHERER, *Siegfried On the Hecke operator  $U(p)$* . With an appendix by Ralf Schmidt. *J. Math. Kyoto Univ.* **45**, 4 (2005), 807–829.
- [3] A. BOREL, *Automorphic  $L$ -functions. Automorphic forms, representations and  $L$ -functions* (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., (1979) 27–61.
- [4] A. BOREL AND H. JACQUET, *Automorphic forms and automorphic representations*. With a supplement "On the notion of an automorphic representation" by R. P. Langlands. Proc. Sympos. Pure Math., XXXIII, Automorphic forms, representations and  $L$ -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, 189–207.
- [5] C. J. BUSHNELL AND P. C. KUTZKO, *Smooth representations of reductive  $p$ -adic groups: structure theory via types*. Proc. London Math. Soc. (3) **77** (1998), no. 3, 582–634.
- [6] W. CASSELMAN, *Introduction to admissible representations of  $p$ -adic groups*, available at his homepage.
- [7] ROBERT F. COLEMAN, *Classical and overconvergent modular forms*. *Invent. Math.* **124**, 1-3 (1996), 215–241.
- [8] P. GARRETT, *Representations with Iwahori-fixed vectors*. Note available at his homepage <http://www.math.umn.edu/~garrett/m/v/>.
- [9] M. GORESKEY, *Compactifications and cohomology of modular varieties. Harmonic analysis, the trace formula, and Shimura varieties*, , Clay Math. Proc., **4**, Amer. Math. Soc., Providence, RI, (2005), 551–582.
- [10] R. HOWE *Harish-Chandra homomorphisms for  $p$ -adic groups*, **59** of CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC, (1985). With the collaboration of A. Moy.
- [11] T. IBUKIYAMA, *Saito-Kurokawa liftings of level  $N$  and practical construction of Jacobi forms*, *Kyoto J. Math.* **52**, 1 (2012), 141–178.
- [12] H. JACQUET, *Sur les représentations des groupes réductifs  $p$ -adiques*, *C. R. Acad. Sci. Paris Ser. A-B* **280** (1975), Aii, A1271–A1272.
- [13] D. KEYS, *Principal series representations of special unitary groups over local fields*, *Compositio Math.* **51**, 1 (1984), 115–130.
- [14] B. MAZUR, *An "infinite fern" in the universal deformation space of Galois representations*, *Collect. Math.*, **48**, 1-2 (1997), 155–193. Journées Arithmétiques (Barcelona, 1995).
- [15] J. S. MILNE, *Introduction to Shimura varieties*. In *Harmonic analysis, the trace formula, and Shimura varieties*, **4** of Clay Math. Proc., (2005) 265–378. Amer. Math. Soc., Providence, RI.
- [16] B. ROBERTS AND R. SCHMIDT, *Local newforms for  $GSp(4)$ . Lecture Notes in Mathematics, 1918*, Springer, Berlin, (2007), viii+307 pp.
- [17] R. SALVATI MANNI AND J. TOP, *Cusp forms of weight 2 for the group  $\Gamma(4, 8)$* , *Amer. J. Math.* **115**, (1993), 455–486.
- [18] I. SATAKE, *Algebraic structures of symmetric domains* Kano Memorial Lectures, 4. Iwanami Shoten, Tokyo; Princeton University Press, Princeton, N.J., (1980). xvi+321 pp.
- [19] G. SHIMURA, *Introduction to the arithmetic theory of automorphic functions*, Reprint of the 1971 original. Publications of the Mathematical Society of Japan, **11**, Kano Memorial Lectures, 1. Princeton University Press, Princeton, NJ, (1994).
- [20] C. SKINNER AND E. URBAN, *Sur les déformations  $p$ -adiques de certaines représentations automorphes*. *J. Inst. Math. Jussieu* **5**, 4 (2006), 629–698.
- [21] J. TILOUINE, *Nearly ordinary rank four Galois representations and  $p$ -adic Siegel modular forms*. With an appendix by Don Blasius. *Compos. Math.* **142**, 5 (2006), 1122–1156.

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