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Kähler differentials and $\mathbf{Z}_p$-extensions

par Laurent BERGER

Résumé. Soit K un corps p-adique, et soit K_∞/K une extension galoisienne qui est presque totalement ramifiée, et dont le groupe de Galois est un groupe de Lie p-adique de dimension 1. Nous montrons que K_∞ n'est pas dense dans $(\mathbf{B}_{\mathrm{dR}}^+/\operatorname{Fil}^2\mathbf{B}_{\mathrm{dR}}^+)^{\mathrm{Gal}(\overline{K}/K_\infty)}$. De plus, la restriction de θ à l'adhérence de K_∞ est injective, et l'image de celle-ci via θ est l'ensemble des vecteurs du complété p-adique de K_∞ qui sont C^1 de dérivée nulle pour l'action de $\mathrm{Gal}(K_\infty/K)$. L'ingrédient principal pour montrer ces résultats est la construction d'un réseau explicite de $\mathcal{O}_{K_\infty}$ qui est commensurable avec $\mathcal{O}_{K_\infty}^{d=0}$, où $d:\mathcal{O}_{K_\infty}\to\Omega_{\mathcal{O}_{K_\infty}/\mathcal{O}_K}$ est la différentielle canonique.

Abstract. Let K be a p-adic field, and let K_∞/K be a Galois extension that is almost totally ramified, and whose Galois group is a p-adic Lie group of dimension 1. We prove that K_∞ is not dense in $(\mathbf{B}_{\mathrm{dR}}^+/\operatorname{Fil}^2\mathbf{B}_{\mathrm{dR}}^+)^{\mathrm{Gal}(\overline{K}/K_\infty)}$. Moreover, the restriction of θ to the closure of K_∞ is injective, and the image of the closure via θ is the set of vectors of the p-adic completion of K_∞ that are C^1 with zero derivative for the action of $\mathrm{Gal}(K_\infty/K)$. The main ingredient for proving these results is the construction of an explicit lattice of $\mathcal{O}_{K_\infty}$ that is commensurable with $\mathcal{O}_{K_\infty}^{d=0}$, where $d:\mathcal{O}_{K_\infty}\to\Omega_{\mathcal{O}_{K_\infty}/\mathcal{O}_K}$ is the canonical differential.

Introduction

Let K be a p-adic field, namely a finite extension of $W(k)[1/p]$ where k is a perfect field of characteristic p. Let $\mathbf{C}$ be the p-adic completion of an algebraic closure $\overline{K}$ of K. Let K_∞/K be a Galois extension that is almost totally ramified, and whose Galois group is a p-adic Lie group of dimension 1. Let $\widehat{K}_\infty$ denote the p-adic completion of K_∞, let $\mathbf{B}_{\mathrm{dR}}(\widehat{K}_\infty)=\mathbf{B}_{\mathrm{dR}}(\mathbf{C})^{\mathrm{Gal}(\overline{K}/K_\infty)}$ be Fontaine's field of periods attached to K_∞/K, and for $n\geqslant 1$, let $\mathbf{B}_n(\widehat{K}_\infty)=\mathbf{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)/\operatorname{Fil}^n\mathbf{B}_{\mathrm{dR}}^+(\widehat{K}_\infty)$.

This note is motivated by Ponsinet's paper [7], in which he relates the study of universal norms for the extension K_∞/K to the question of whether K_∞ is dense in $\mathbf{B}_n(\widehat{K}_\infty)$ for $n\geqslant 1$. The density result holds for $n=1$ since $\mathbf{C}^{\mathrm{Gal}(\overline{K}/K_\infty)}=\widehat{K}_\infty$ by the Ax–Sen–Tate theorem.

Our main result is the following.

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Theorem A. *The field K_∞ is not dense in $\mathbf{B}_2(\widehat{K}_\infty)$.*

By the constructions of Fontaine and Colmez (see [2] and [4]), $\mathbf{B}_2(\mathbf{C}) = \mathbf{B}_{\mathrm{dR}}^+(\mathbf{C})/\operatorname{Fil}^2 \mathbf{B}_{\mathrm{dR}}^+(\mathbf{C})$ is the completion of $\overline{K}$ for a topology defined using the Kähler differentials $\Omega_{\mathcal{O}_{\bar{K}}/\mathcal{O}_K}$. Some partial results towards Theorem A have been proved by Iovita–Zaharescu in [6], by studying these Kähler differentials. Let $\Omega_{\mathcal{O}_{K_\infty}/\mathcal{O}_K}$ be the Kähler differentials of $\mathcal{O}_{K_\infty}/\mathcal{O}_K$ and let $d : \mathcal{O}_{K_\infty} \to \Omega_{\mathcal{O}_{K_\infty}/\mathcal{O}_K}$ be the differential. Our main technical result is the construction of a lattice of $\mathcal{O}_{K_\infty}$ that is commensurable with $\mathcal{O}_{K_\infty}^{d=0}$. Since the inertia subgroup of $\operatorname{Gal}(K_\infty/K)$ is a p-adic Lie group of dimension 1, there exists a finite subextension K_0/K of K_∞ such that K_∞/K_0 is a totally ramified $\mathbf{Z}_p$-extension. Let K_n be the n-th layer of this $\mathbf{Z}_p$-extension.

Theorem B. *The lattices $\sum_{n\geqslant 0} p^n \mathcal{O}_{K_n}$ and $\mathcal{O}_{K_\infty}^{d=0}$ are commensurable.*

In order to prove this, we use Tate's results on ramification in $\mathbf{Z}_p$-extensions. As a corollary of Theorem B, we can say more about the completion of K_∞ in $\mathbf{B}_2(\widehat{K}_\infty)$. The field $\widehat{K}_\infty$ is a Banach representation of the p-adic Lie group $\operatorname{Gal}(K_\infty/K)$. Let $c : \operatorname{Gal}(K_\infty/K_0) \to \mathbf{Z}_p$ be an isomorphism of p-adic Lie groups. If $x \in \widehat{K}_\infty$, we say that x is C^1 with zero derivative for the action of $\operatorname{Gal}(K_\infty/K)$ if $g(x) - x = \mathrm{o}(c(g))$ as $c(g) \to 0$.

Let $\theta : \mathbf{B}_2(\mathbf{C}) \to \mathbf{C}$ be the usual map from p-adic Hodge theory.

Theorem C. *The completion of K_∞ in $\mathbf{B}_2(\widehat{K}_\infty)$ is isomorphic via θ to the set of vectors of $\widehat{K}_\infty$ that are C^1 with zero derivative for the action of $\operatorname{Gal}(K_\infty/K)$.*

This is a field, and it is also the set of $y \in \widehat{K}_\infty$ that can be written as $y = \sum_{n\geqslant 0} p^n y_n$ with $y_n \in K_n$ and $y_n \to 0$.

We also prove that $d(\mathcal{O}_{K_\infty})$ contains no nontrivial p-divisible element (Corollary 3.5), and that $d : \mathcal{O}_{K_\infty} \to \Omega_{\mathcal{O}_{K_\infty}/\mathcal{O}_K}$ is not surjective (Corollary 3.6). These two statements are equivalent to Theorem A by the results of [6]; using our computations, we give a short independent proof.

1. Kähler differentials

Let K be a p-adic field. If L/K is a finite extension, let $\mathfrak{d}_{L/K} \subset \mathcal{O}_L$ denote its different.

Proposition 1.1. *Let K be a p-adic field, and let L/K be an algebraic extension.*

1. *If L/K is a finite extension, then $\Omega_{\mathcal{O}_L/\mathcal{O}_K} = \mathcal{O}_L/\mathfrak{d}_{L/K}$ as $\mathcal{O}_L$-modules.*
2. *If $M/L/K$ are finite extensions, then the map $\Omega_{\mathcal{O}_L/\mathcal{O}_K} \to \Omega_{\mathcal{O}_M/\mathcal{O}_K}$ is injective.*

(3) *If L/K is an algebraic extension, and $\omega_1, \omega_2 \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$, then there exists $x \in \mathcal{O}_L$ such that $\omega_2 = x\omega_1$ if and only if $\mathrm{Ann}(\omega_1) \subset \mathrm{Ann}(\omega_2)$.*

Proof. See for instance [3, §2]. □

Recall (see [1, §2]) that an algebraic extension L/K is deeply ramified if the set $\{\mathrm{val}_p(\mathfrak{d}_{F/K})\}_F$ is unbounded, as F runs through the set of finite extensions of K contained in L. Alternatively ([8, Remark 3.3]), L/K is deeply ramified if and only if $\widehat{L}$ is a perfectoid field. An extension K_∞/K as in the introduction is deeply ramified.

Corollary 1.2. *If L/K is deeply ramified, then $\Omega_{\mathcal{O}_L/\mathcal{O}_K} = L/\mathcal{O}_L$ as $\mathcal{O}_L$-modules.*

Proposition 1.3. *If L/K is deeply ramified, then $d : \mathcal{O}_L \to \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is surjective if and only if $d(\mathcal{O}_L)$ is p-divisible.*

Proof. Since L/K is deeply ramified, $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is isomorphic to $L/\mathcal{O}_L$ by Corollary 1.2. The claim now follows from the fact that a nonzero $\mathcal{O}_L$-submodule of $L/\mathcal{O}_L$ is equal to $L/\mathcal{O}_L$ if and only if it is p-divisible. □

Proposition 1.4. *Let L/K be a deeply ramified extension, and let $K' \subset L$ be a finite extension of K.*

(1) *$d : \mathcal{O}_L \to \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is surjective if and only if $d' : \mathcal{O}_L \to \Omega_{\mathcal{O}_L/\mathcal{O}_{K'}}$ is surjective.*
(2) *$\mathcal{O}_L^{d=0}$ and $\mathcal{O}_L^{d'=0}$ are commensurable.*

Proof. We have an exact sequence of $\mathcal{O}_L$-modules, compatible with d and d'

$$\mathcal{O}_L \otimes \Omega_{\mathcal{O}_{K'}/\mathcal{O}_K} \xrightarrow{f} \Omega_{\mathcal{O}_L/\mathcal{O}_K} \xrightarrow{g} \Omega_{\mathcal{O}_L/\mathcal{O}_{K'}} \to 0.$$

Let us prove (1). If $d : \mathcal{O}_L \to \Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is surjective, then clearly $d' : \mathcal{O}_L \to \Omega_{\mathcal{O}_L/\mathcal{O}_{K'}}$ is surjective. Conversely, there exists $r \geqslant 0$ such that $p^r \cdot \Omega_{\mathcal{O}_{K'}/\mathcal{O}_K} = \{0\}$. If $\omega \in \Omega_{\mathcal{O}_L/\mathcal{O}_K}$, write it as $\omega = p^r\omega_r$. By hypothesis, there exists $\alpha_r \in \mathcal{O}_L$ such that $\omega_r = d'\alpha_r$ in $\Omega_{\mathcal{O}_L/\mathcal{O}_{K'}}$. Hence $p^r(\omega_r - d\alpha_r) = 0$ in $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ so that $\omega = d(p^r\alpha_r)$. We now prove (2). The exact sequence above implies that $\mathcal{O}_L^{d=0} \subset \mathcal{O}_L^{d'=0}$. Conversely, if $x \in \mathcal{O}_L^{d'=0}$, then $dx \in \ker g = \mathrm{im}\, f$, so that $p^r \cdot dx = 0$. Hence $p^r \cdot \mathcal{O}_L^{d'=0} \subset \mathcal{O}_L^{d=0}$. □

Corollary 1.5. *In order to prove Theorem B, we can replace K by any finite subextension K' of K. In particular, we can assume that K_∞/K is a totally ramified $\mathbf{Z}_p$-extension.*

2. Ramification in $\mathbf{Z}_p$-extensions

Let K_∞/K be a totally ramified $\mathbf{Z}_p$-extension. We recall some of the results of [9, §3.1] concerning the ramification of K_∞/K and the action of $\mathrm{Gal}(K_\infty/K)$ on K_∞. Let K_n denote the n-th layer of K_∞/K, so that $[K_n : K] = p^n$.

Proposition 2.1. *There are constants a, b such that for all $n \geqslant 0$, we have $|\mathrm{val}_p(\mathfrak{d}_{K_n/K}) - n - b| \leqslant p^{-n}a$.*

Proof. See [9, §3.1]. □

The notation $\sum_{n\geqslant 0} p^n \mathcal{O}_{K_n}$ denotes the set of elements of K_∞ that are finite sums of elements of $p^n\mathcal{O}_{K_n}$.

Corollary 2.2. *There exists $n_0 \geqslant 0$ such that $\sum_{n\geqslant 0} p^{n+n_0}\mathcal{O}_{K_n} \subset \mathcal{O}_{K_\infty}^{d=0}$.*

Proposition 2.3. *There exists $c(K_\infty/K) > 0$ such that for all $n, k \geqslant 0$ and $x \in \mathcal{O}_{K_{n+k}}$, we have $\mathrm{val}_p(\mathrm{Nm}_{K_{n+k}/K_n}(x)/x^{[K_{n+k}:K_n]} - 1) \geqslant c(K_\infty/K)$.*

Proof. The result follows from the fact (see [10, 1.2.2]) that the extension K_∞/K is strictly APF. One can then apply 1.2.1, 1.2.2 and 1.2.3 of [10]. □

If $n \geqslant 0$ and $x \in K_\infty$, then $R_n(x) = p^{-k} \cdot \mathrm{Tr}_{K_{n+k}/K_n}(x)$ is independent of $k \gg 0$ such that $x \in K_{n+k}$, and is the normalized trace of x.

Proposition 2.4. *There exists $c_2 \in \mathbf{Z}_{\geqslant 0}$ such that $\mathrm{val}_p(R_n(x)) \geqslant \mathrm{val}_p(x) - c_2$ for all $n \geqslant 0$ and $x \in K_\infty$.*

Proof. See [9, §3.1] (including the remark at the bottom of p. 172). □

In particular, $R_n(\mathcal{O}_{K_\infty}) \subset p^{-c_2}\mathcal{O}_{K_n}$ for all $n \geqslant 0$. Let $K_0^\perp = K_0$ and for $n \geqslant 1$, let $K_n^\perp$ be the kernel of $R_{n-1} : K_n \to K_{n-1}$, let $R_n^\perp = R_n - R_{n-1}$, and $R_0^\perp = R_0$. Note that $K_n^\perp = \mathrm{im}(R_n^\perp : K_\infty \to K_n)$. If $x \in K_\infty$ and $i \geqslant 0$, then $R_n^\perp(x) = 0$ for $n \gg 0$, and $x = (\sum_{n\geqslant i+1} R_n^\perp(x)) + R_i(x)$. Proposition 2.4 implies that $R_n^\perp(\mathcal{O}_{K_\infty}) \subset p^{-c_2}\mathcal{O}_{K_n}$ for all $n \geqslant 0$. Let $\mathcal{O}_{K_n}^\perp = \mathcal{O}_{K_n} \cap K_n^\perp$.

Corollary 2.5. *If $i \geqslant 0$, we have $\mathcal{O}_{K_\infty} \subset (\bigoplus_{m\geqslant i+1} p^{-c_2}\mathcal{O}_{K_m}^\perp) \oplus p^{-c_2}\mathcal{O}_{K_i}$.*

Proof. If $x \in \mathcal{O}_{K_\infty}$, write $x = \sum_{m\geqslant i+1} R_m^\perp(x) + R_i(x)$. □

For $n \geqslant 0$, let g_n denote a topological generator of $\mathrm{Gal}(K_\infty/K_n)$.

Lemma 2.6. *There exists a constant c_3 such that for all $n \geqslant 1$ and all $x \in K_{n+1}^\perp$, we have $\mathrm{val}_p(x) \geqslant \mathrm{val}_p((1 - g_n)(x)) - c_3$.*

Proof. See [9, §3.1] (including the remark at the bottom of p. 172). □

3. The lattice $\mathcal{O}_{K_\infty}^{d=0}$

We now prove Theorem B. Thanks to Corollary 1.5, we assume that K_∞/K is a totally ramified $\mathbf{Z}_p$-extension. Let $\{\rho_n\}_{n\geqslant 0}$ be a norm compatible sequence of uniformizers of the K_n. Let $m_c \geqslant 0$ be the smallest integer such that $p^{m_c} \cdot c(K_\infty/K) \geqslant 1/(p-1)$ (where $c(K_\infty/K)$ was defined in Proposition 2.3).

Proposition 3.1. *We have* $\mathrm{val}_p(\rho_{n+1}^{pk} - \rho_n^k) \geqslant \mathrm{val}_p(k) - m_c$.

Proof. Note that if $x, y \in \mathbf{C}$ with $\mathrm{val}_p(x-y) \geqslant v$, then $\mathrm{val}_p(x^p - y^p) \geqslant \min(v+1, pv)$. Let $c = c(K_\infty/K)$ and $m = m_c$. We have $\mathrm{val}_p(\rho_{n+1}^p - \rho_n) \geqslant c$ by Proposition 2.3, so that $\mathrm{val}_p(\rho_{n+1}^{p^{j+1}} - \rho_n^{p^j}) \geqslant p^j c$ if $p^{j-1}c \leqslant 1/(p-1)$.

In particular, $\mathrm{val}_p(\rho_{n+1}^{p^{m+1}} - \rho_n^{p^m}) \geqslant p^m c \geqslant 1/(p-1)$, so that we have $\mathrm{val}_p(\rho_{n+1}^{p^{m+j+1}} - \rho_n^{p^{m+j}}) \geqslant j + 1/(p-1)$ if $j \geqslant 0$. This implies the result. □

Theorem 3.2. *There exists* $n_1 \geqslant 0$ *such that* $\mathcal{O}_{K_\infty}^{d=0} \subset \sum_{m \geqslant n_1} p^{m-n_1} \mathcal{O}_{K_m}$.

Proof. We prove the result with $n_1 = \lceil a - b + m_c + 2 \rceil$. Take $x \in \mathcal{O}_{K_n}^{d=0}$ and write $x = \sum_{i=0}^{p^n-1} x_i \rho_n^i$ with $x_i \in \mathcal{O}_K$, so that $dx = \sum_{i=0}^{p^n-1} i x_i \rho_n^{i-1} \cdot d\rho_n$. Since ρ_n is a uniformizer of $\mathcal{O}_{K_n}$, the $\mathcal{O}_{K_n}$-module $\Omega_{\mathcal{O}_{K_n}/\mathcal{O}_K} = \mathcal{O}_{K_n}/\mathfrak{d}_{K_n/K}$ (see Proposition 1.1) is generated by $d\rho_n$. If $dx = 0$, then $\sum_{i=0}^{p^n-1} i x_i \rho_n^{i-1}$ belongs to $\mathfrak{d}_{K_n/K}$ so that by Proposition 2.1 (and since $\mathrm{val}_p(\rho_n^{p^n}) \leqslant 1$), for all i we have

$$\mathrm{val}_p(x_i) \geqslant n - a + b - \mathrm{val}_p(i) - 1.$$

For $k \geqslant 1$, let

$$y_k = \sum_{p \nmid j} x_{p^{k-1}j} \rho_{n-(k-1)}^j + \sum_{\ell} x_{p^k \ell} (\rho_{n-(k-1)}^{p\ell} - \rho_{n-k}^{\ell}).$$

Note that $y_k \in \mathcal{O}_{K_{n-k+1}}$. Let us bound $\mathrm{val}_p(y_k)$. We have

$$\mathrm{val}_p(x_{p^{k-1}j} \rho_{n-(k-1)}^j) \geqslant n - a + b - k.$$

We also have $\mathrm{val}_p(x_{p^k \ell}) \geqslant n - a + b - k - \mathrm{val}_p(\ell) - 1$, and by Proposition 3.1

$$\mathrm{val}_p(\rho_{n-(k-1)}^{p\ell} - \rho_{n-k}^{\ell}) \geqslant \mathrm{val}_p(\ell) - m_c.$$

Hence $\mathrm{val}_p(y_k) \geqslant n - a + b - k - 1 - m_c$ and therefore $y_k \in p^{n-k+1-n_1} \mathcal{O}_{K_{n-k+1}}$. Finally, we have $x = y_1 + \cdots + y_{n-n_1} + \sum_\ell x_{p^{n-n_1}\ell} \rho_{n_1}^\ell$, and $\sum_\ell x_{p^{n-n_1}\ell} \rho_{n_1}^\ell$ belongs to $\mathcal{O}_{K_{n_1}}$, which implies the result. □

Remark 3.3. Compare with [5, Lemma 4.3.2].

Corollary 3.4. *We have* $\mathcal{O}_{K_\infty}^{d=0} \subset (\bigoplus_{m \geqslant n_1+1} p^{m-n_1-c_2} \mathcal{O}_{K_m}^{\perp}) \oplus p^{-c_2} \mathcal{O}_{K_{n_1}}$.

Proof. By Theorem 3.2, it is enough to prove that

$$p^n \mathcal{O}_{K_n} \subset \left(\bigoplus_{m \geqslant n_1+1} p^{m-c_2} \mathcal{O}_{K_m}^{\perp} \right) \oplus p^{n_1-c_2} \mathcal{O}_{K_{n_1}}$$

for all $n \geqslant n_1$. If $x \in p^n \mathcal{O}_{K_n}$, write $x = R_n^{\perp}(x) + R_{n-1}^{\perp}(x) + \cdots + R_{n_1+1}^{\perp}(x) + R_{n_1}(x)$. We have $R_{n-k}^{\perp}(x) \in p^{n-c_2} \mathcal{O}_{K_{n-k}}^{\perp} \subset p^{(n-k)-c_2} \mathcal{O}_{K_{n-k}}^{\perp}$ and likewise $R_{n_1}(x) \in p^{n-c_2} \mathcal{O}_{K_{n_1}} \subset p^{n_1-c_2} \mathcal{O}_{K_{n_1}}$. □

Corollary 3.5. *There are no nontrivial p-divisible elements in $d(\mathcal{O}_{K_\infty})$.*

Proof. By Propositions 1.3 and 1.4, we can assume that K_∞/K is a totally ramified $\mathbf{Z}_p$-extension. Let $\{\alpha_i\}_{i\geqslant 1}$ be a sequence of $\mathcal{O}_{K_\infty}$ such that $d\alpha_i = p\cdot d\alpha_{i+1}$ for all $i\geqslant 1$.

Using Corollary 2.5, write $\alpha_i = \sum \alpha_{i,m}$ with $\alpha_{i,m} = R_m^\perp(\alpha_i) \in p^{-c_2}\mathcal{O}_{K_m}^\perp$ for $m \geqslant n_1+1$ and $\alpha_{i,n_1} = R_{n_1}(\alpha_i) \in p^{-c_2}\mathcal{O}_{K_{n_1}}$. Since $p^k\alpha_{k+i} - \alpha_i \in \mathcal{O}_{K_\infty}^{d=0}$, Corollary 3.4 implies that $p^k\alpha_{k+i,m} - \alpha_{i,m} \in p^{m-n_1-c_2}\mathcal{O}_{K_m}$ for all $m\geqslant n_1$. Taking $k \gg 0$ now implies that $\alpha_{i,m} \in p^{m-n_1-c_2}\mathcal{O}_{K_m}$ for all $m \geqslant n_1$. Corollary 2.2 gives $p^{n_0+n_1+c_2}\alpha_i \in \mathcal{O}_{K_\infty}^{d=0}$. Taking $i = n_0+n_1+c_2+1$ gives $d\alpha_1 = 0$. ☐

Corollary 3.6. *The differential $d : \mathcal{O}_{K_\infty} \to \Omega_{\mathcal{O}_{K_\infty}/\mathcal{O}_K}$ is not surjective.*

Proof. This follows from Corollary 3.5 and Proposition 1.3. ☐

4. The completion of K_∞ in $\mathbf{B}_2(\mathbf{C})$

We now prove Theorems A and C. Since we are concerned with the completion of K_∞, we can once again replace K with a finite subextension of K_∞ and assume that K_∞/K is a totally ramified $\mathbf{Z}_p$-extension. Let $\widehat{K}_\infty^2$ denote the completion of K_∞ in $\mathbf{B}_2(\mathbf{C}) = \mathbf{B}_{\mathrm{dR}}^+(\mathbf{C})/\operatorname{Fil}^2\mathbf{B}_{\mathrm{dR}}^+(\mathbf{C})$, so that $R = \theta(\widehat{K}_\infty^2)$ is a subring of $\widehat{K}_\infty$. Let $\Gamma = \mathrm{Gal}(K_\infty/K)$, and let $c : \Gamma \to \mathbf{Z}_p$ be an isomorphism of p-adic Lie groups. Let w_2 be the valuation on K_∞ defined by $w_2(x) = \min\{n\in\mathbf{Z}$ such that $p^n x \in \mathcal{O}_{K_\infty}^{d=0}\}$. The restriction of the natural valuation of $\mathbf{B}_2(\mathbf{C})$ to K_∞ is w_2 (see [4, §1.4 and §1.5], or [2, Theorem 3.1]; the natural valuation on $\mathbf{B}_2(\mathbf{C})$ comes from its definition as the quotient of a certain Banach space, see ibid.).

The map $\theta : \mathbf{B}_2(\mathbf{C}) \to \mathbf{C}$ has the following property (see [4, §1.4]).

Lemma 4.1. *If $\{x_k\}_{k\geqslant 1}$ is a sequence of K_∞ that converges to $x\in\mathbf{B}_2(\mathbf{C})$ for w_2, then $\{x_k\}_{k\geqslant 1}$ is Cauchy for val_p, and $\theta(x) = \lim_{k\to+\infty} x_k$ for the p-adic topology.*

Let $M = \bigoplus_{n\geqslant 0} p^n\mathcal{O}_{K_n}^\perp$. Corollary 2.2 and Theorem 3.2 imply that M and $\mathcal{O}_{K_\infty}^{d=0}$ are commensurable. Hence $\widehat{K}_\infty^2$ is the M-adic completion of K_∞. Let w_2' be the M-adic valuation on K_∞, so that w_2' and w_2 are equivalent.

Lemma 4.2. *If $x\in K_\infty$, then $\mathrm{val}_p(R_n^\perp(x)) \geqslant w_2'(x) + n$.*

Proof. Write $x = \sum_{n\geqslant 0} R_n^\perp(x)$. If $x \in p^w M$, then $R_n^\perp(x) \in p^{n+w}\mathcal{O}_{K_n}$. ☐

Proposition 4.3. *Every element $x\in\widehat{K}_\infty^2$ can be written in one and only one way as $\sum_{n\geqslant 0} x_n^\perp$ where $x_n^\perp \in K_n^\perp$ and $p^{-n}x_n^\perp \to 0$ for val_p.*

Proof. Note that such a series converges for w_2. The map $R_n^\perp : K_\infty \to K_n^\perp$ sends $p^w M \subset K_\infty$ to $p^{w+n}\mathcal{O}_{K_n}^\perp$. It is uniformly continuous for the w_2-adic topology, so that it extends to a continuous map $R_n^\perp : \widehat{K}_\infty^2 \to K_n^\perp$.

Let $x \in \widehat{K}^2_\infty$ be the w_2-adic limit of $\{x_k\}_{k \geqslant 1}$ with $x_k \in K_\infty$. For a given k, the sequence $\{p^{-n} R_n^\perp(x_k)\}_{n \geqslant 0} \in \prod_{n \geqslant 0} K_n^\perp$ has finite support. As $k \to +\infty$, these sequences converge uniformly in $\prod_{n \geqslant 0} K_n^\perp$ to $\{p^{-n} R_n^\perp(x)\}_{n \geqslant 0}$, so that $p^{-n} R_n^\perp(x) \to 0$ as $n \to +\infty$. Hence $\sum_{n \geqslant 0} R_n^\perp(x)$ converges for w_2. Since $x_k = \sum_{n \geqslant 0} R_n^\perp(x_k)$ for all k, we have $x = \sum_{n \geqslant 0} R_n^\perp(x)$. Finally, if $x = \sum_{n \geqslant 0} x_n^\perp$ with $x_n^\perp \in K_n^\perp$ and $p^{-n} x_n^\perp \to 0$ for val_p, then $x_n^\perp = R_n^\perp(x)$ which proves unicity. □

Corollary 4.4. *The map $\theta : \widehat{K}^2_\infty \to \widehat{K}_\infty$ is injective.*

Proof. If $x_n^\perp \in K_n^\perp$ and $x_n^\perp \to 0$ and $\sum_{n \geqslant 0} x_n^\perp = 0$ in $\widehat{K}_\infty$, then $x_n^\perp = 0$ for all n. □

Corollary 4.5. *The ring R is the set of $y \in \widehat{K}_\infty$ that can be written as $y = \sum_{n \geqslant 0} p^n y_n$ with $y_n \in K_n$ and $y_n \to 0$.*

Proposition 4.6. *The ring R is a field, and $R = \{x \in \widehat{K}_\infty$ such that $g(x) - x = \mathrm{o}(c(g))$ as $g \to 1$ in $\Gamma\}$.*

Proof. The fact that R is a field results from the second statement, since $g(1/x) - 1/x = (x - g(x))/(xg(x))$. Take $y = \sum_{n \geqslant 0} p^n y_n$ with $y_n \in K_n$ and $y_n \to 0$. If $m \geqslant 1$, then for all $k \gg 0$, we have $y_n \in p^{m+n} \mathcal{O}_{K_n}$. We can write $y = x_k + \sum_{n \geqslant k} p^n y_n$ and then $(g-1)(y) \in p^{k+m} \mathcal{O}_{K_\infty}$ if $g \in \mathrm{Gal}(K_\infty/K_k)$. This proves one implication.

Conversely, take $x \in \widehat{K}_\infty$ such that $g(x) - x = \mathrm{o}(c(g))$. Write $x = \sum_{k \geqslant 0} x_k$ with $x_0 = R_0(x) \in K_0$ and $x_k = R_k^\perp(x) \in K_k^\perp$ for all $k \geqslant 1$. For $n \geqslant 0$, let g_n denote a topological generator of $\mathrm{Gal}(K_\infty/K_n)$. Take $m \geqslant 0$, and $n \gg 0$ such that we have $\mathrm{val}_p((g_n - 1)(x)) \geqslant m + n$. We have $(1 - g_n)(x) = \sum_{k \geqslant n+1} (1 - g_n) x_k$, so that by Lemma 2.6 and Proposition 2.4:

$$\begin{aligned} \mathrm{val}_p(x_{n+1}) &\geqslant \mathrm{val}_p((1 - g_n)(x_{n+1})) - c_3 \\ &\geqslant \mathrm{val}_p((1 - g_n)(x)) - c_2 - c_3 \\ &\geqslant n + m - c_2 - c_3. \end{aligned}$$

This implies the result. □

Remark 4.7. Proposition 4.6 says that R is the set of vectors of $\widehat{K}_\infty$ that are C^1 with zero derivative (flat to order 1) for the action of Γ.

Theorem A follows from Corollary 4.4 since $\theta : \mathbf{B}_2(\widehat{K}_\infty) \to \widehat{K}_\infty$ is not injective. Finally, Corollary 4.4, Corollary 4.5, and Proposition 4.6 imply Theorem C.

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