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Singular Vectors and ψ -Dirichlet Numbers over Function Field

par SHREYASI DATTA et YEWEI XU

RÉSUMÉ. Nous montrons que contrairement aux nombres ψ -Dirichlet dans \mathbb{R} , les seuls nombres ψ -Dirichlet dans un corps de fonctions sur un corps fini sont les fonctions rationnelles. Nous prouvons également qu'il existe une quantité non dénombrable de vecteurs singuliers totalement irrationnels avec un grand exposant uniforme dans les surfaces quadratiques sur un corps de caractéristique positive.

ABSTRACT. We show that the only ψ -Dirichlet numbers in a function field over a finite field are rational functions, unlike ψ -Dirichlet numbers in \mathbb{R} . We also prove that there are uncountably many totally irrational singular vectors with large uniform exponent in quadratic surfaces over a positive characteristic field.

1. Introduction

1.1. ψ -Dirichlet numbers. Following [17], we define ψ -Dirichlet vectors in $\mathbb{F}_q((T^{-1}))^n$ and we denote the set of those vectors as $D(\psi)$. For the definitions of norms in $\mathbb{F}_q((T^{-1}))^n$, readers are referred to Section 1.4. Before defining $D(\psi)$, let us recall the Dirichlet's theorem in $\mathbb{F}_q((T^{-1}))^n$; see [7, Theorem 1.1]. For every $t \in \mathbb{N}$ there exists $\mathbf{0} \neq \mathbf{q} \in \mathbb{F}_q[T]^n$, $q_0 \in \mathbb{F}_q[T]$ satisfying

$$\begin{aligned} |\mathbf{q} \cdot \mathbf{x} + q_0| &< \frac{1}{e^{nt}}, \\ \|\mathbf{q}\| &\leq e^t. \end{aligned}$$

Definition 1.1. Let $\psi : [t_0, +\infty) \rightarrow \mathbb{R}_+$ be a function. A vector $\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{F}_q((T^{-1}))^n$ is said to be ψ -Dirichlet if for all sufficiently large $Q > 0$ there exists $\mathbf{0} \neq \mathbf{q} \in \mathbb{F}_q[T]^n$, $q_0 \in \mathbb{F}_q[T]$ satisfying the following system

$$\begin{aligned} (1.1) \quad |\mathbf{q} \cdot \mathbf{x} + q_0| &< \psi(Q), \\ \|\mathbf{q}\| &\leq Q. \end{aligned}$$

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Let $\psi_c(Q) = \frac{c}{Q^n}$. If $\mathbf{x} \in \mathbb{F}_q((T^{-1}))^n$ is ψ_c -Dirichlet for every $c > 0$, then \mathbf{x} is called singular vector. In recent years, ψ -Dirichlet vectors were studied in [16, 17, 18]. Even in the classical setting not much is known.

Diophantine approximation in function field has been a topic of interest since the work of Artin, [3], which developed the theory of continued fraction, and followed by Mahler's work in [21], which studied geometry of numbers in function field. For recent developments, we refer readers the survey [20], and to [1, 2, 4, 6, 9, 12, 13, 19] for a necessarily incomplete set of references. There are many interesting similarities and contrasts between the theory of Diophantine approximation over the real numbers and in function field over finite fields. The main theorems in this paper show both of these features.

In [17], for a non-increasing function $\psi(t) < \frac{1}{t}$ with $t \rightarrow t\psi(t)$ non-decreasing, it was shown that $D(\psi)$ in \mathbb{R} has zero-one law for Lebesgue measure depending on divergence or convergence of certain series involving ψ . Surprisingly, the same is not true over function field when $n = 1$ as we prove the following.

Theorem 1.2. *Let $\psi : [t_0, +\infty) \rightarrow \mathbb{R}_+$ be non-increasing. If $\psi(t) < \frac{1}{t}$ for sufficiently large t , then the ψ -Dirichlet points in $\mathbb{F}_q((T^{-1}))$ are only in $\mathbb{F}_q(T)$, i.e. $D(\psi) = \mathbb{F}_q(T)$.*

The above theorem shows that analogue of the main theorem in [17] over function field becomes drastically different than the real case. The main tool in proving Theorem 1.2 is the use of continued fraction expansion.

Remark 1.3. Theorem 2.4 in [8] shows that only *Dirichlet improvable* numbers in function field are rational functions. Our Theorem 1.2 generalizes the above mentioned result showing that even ψ -Dirichlet numbers are also only rational functions. We note that the technique of [8] is different than ours.

1.2. Plenty of singular vectors. The second part of this paper deals with singular vectors in submanifolds of function fields. Note that, if $(x_1, \dots, x_n) \in \mathbb{F}_q((T^{-1}))^n$ belongs to a rational affine hyperplane, then it must be singular. These are the most trivial singular vectors. In fact, the converse is also true when $n = 1$ ([8]). So, it is natural to introduce the following definition.

Definition 1.4. A vector is called totally irrational if it is not inside a rational affine hyperplane of $\mathbb{F}_q((T^{-1}))^n$.

For $n > 1$, in [11] Khintchine showed the existence of infinitely many totally irrational singular vectors in \mathbb{R}^n . Moreover, Kleinbock, Moshchevitin and Weiss in [14] showed that for real analytic submanifolds (of dimension greater than 2) which are not contained inside a rational affine subspace,

there are uncountably many totally irrational singular vectors. In this paper, we prove analogous result for certain submanifolds in $\mathbb{F}_q((T^{-1}))^n$.

As a special case of our Theorem 1.6, we prove the following theorem.

Theorem 1.5. *Suppose $\text{char}(\mathbb{F}_q((T^{-1}))) = p < \infty$. Let U be an open subset of $\mathbb{F}_q((T^{-1}))$. We consider S of the following two types:*

- $S = \{(x, y, p_3(x, y), \dots, p_n(x, y)) \mid x, y \in U\} \subset \mathbb{F}_q((T^{-1}))^n$, where each $p_i(x, y)$ is a degree 2 polynomial.
- $S = \{(x, y, p(x, y)) \mid x, y \in U\} \subset \mathbb{F}_q((T^{-1}))^3$, where $p(x, y) = \sum_{i=0}^m a_i^{p^i} x^{p^i} + \sum_{j=0}^n b_j^{p^j} y^{p^j}$.

Suppose that S is not contained inside any affine rational hyperplane, then there exist uncountably many totally irrational singular vectors in S .

The main challenge comes from the lack of understanding about intersections of a surface and an affine subspace in the function field setting. Another difficulty is due to total disconnectedness of function field. For real submanifolds, intersection of a connected analytic surface and an affine subspace is well understood due to [5, §2]. Both of these facts were used in [14] in a crucial manner. The proof in [14] relies on understanding how “semianalytic” sets can split into connected analytic sets. This becomes difficult in $\mathbb{F}_q((T^{-1}))^n$, as the notion of semianalyticity is not well defined due to the lack of order and the space $\mathbb{F}_q((T^{-1}))$ is totally disconnected. That is why we had to tackle case by case and we prove the theorem for a class of submanifolds which is smaller than the class of submanifolds that was taken in [14].

1.3. On uniform exponent. One can define $\widehat{\omega}(\cdot)$, as follows, which quantifies the singularity of a vector.

$$(1.2) \quad \widehat{\omega}(\mathbf{y}) := \sup \left\{ \omega \left| \begin{array}{l} \text{for all large enough } Q > 0, \\ \exists (q_0, \mathbf{q}) \in \mathbb{F}_q[T]^{n+1} \setminus \{0\} \\ \text{s.t. } \|\mathbf{q} \cdot \mathbf{y} + q_0\| \leq \frac{1}{Q^\omega}, \|\mathbf{q}\| \leq Q \end{array} \right. \right\}$$

Dirichlet’s Theorem ([7]) gives that $\widehat{\omega}(\mathbf{y}) \geq n$ for all $\mathbf{y} \in \mathbb{F}_q((T^{-1}))^n$. Our next theorem verifies that for certain analytic submanifolds in $\mathbb{F}_q((T^{-1}))^n$, there are plenty of totally irrational vectors whose exponents $\widehat{\omega}(\cdot)$ are infinity.

Theorem 1.6. *Let $\text{char}(\mathbb{F}_q((T^{-1}))) = p < \infty$. Let U be an open subset of $\mathbb{F}_q((T^{-1}))$. We consider S of the following two types:*

- $S = \{(x, y, p_3(x, y), \dots, p_n(x, y)) \mid x, y \in U\} \subset \mathbb{F}_q((T^{-1}))^n$, where each $p_i(x, y)$ is a degree 2 polynomial.
- $S = \{(x, y, p(x, y)) \mid x, y \in U\} \subset \mathbb{F}_q((T^{-1}))^3$ where $p(x, y) = \sum_{i=0}^m a_i^{p^i} x^{p^i} + \sum_{j=0}^n b_j^{p^j} y^{p^j}$.

Suppose that S is not contained inside any rational affine hyperplane, then there exist uncountably many totally irrational \mathbf{y} in S such that $\widehat{\omega}(\mathbf{y}) = \infty$.

Remark 1.7. In Lemma 3.12, we show that the above theorem is true for some higher dimensional submanifolds.

1.4. Norms and Definitions. In this section and in the following sections, we will use $|\cdot|$ (resp. $\|\cdot\|$) to denote the norm in $\mathbb{F}_q((T^{-1}))$ (resp. $\mathbb{F}_q((T^{-1}))^n$), unless otherwise mentioned. Let \mathbb{F}_q be a finite field of characteristic p with $q := p^r$ elements. Let $\mathbb{F}_q[T]$ be the ring of polynomials over \mathbb{F}_q and $\mathbb{F}_q(T)$ the field of rational functions. We define a norm $|\cdot|$ on $\mathbb{F}_q(T)$ as follow:

$$|0| := 0; \quad \left| \frac{P}{Q} \right| := e^{\deg P - \deg Q}$$

for all nonzero $P, Q \in \mathbb{F}_q[T]$, where e is Euler's number. Clearly $|\cdot|$ is a nontrivial, non-archimedean, and discrete absolute value in $\mathbb{F}_q(T)$. The completion field of $\mathbb{F}_q(T)$ with respect to this absolute value is $\mathbb{F}_q((T^{-1}))$, i.e. the field of Laurent series over \mathbb{F}_q . We will denote the absolute value of $\mathbb{F}_q((T^{-1}))$ by the same notation $|\cdot|$, is given as follows. Let $a \in \mathbb{F}_q((T^{-1}))$,

$$|a| := \begin{cases} 0 & \text{if } a = 0, \\ e^{k_0} & \text{if } a = \sum_{k \leq k_0} a_k T^k, k_0 \in \mathbb{Z}, a_k \in \mathbb{F}_q \text{ and } a_{k_0} \neq 0. \end{cases}$$

This clearly extends the absolute value $|\cdot|$ of $\mathbb{F}_q(T) \rightarrow \mathbb{F}_q((T^{-1}))$ and moreover, the extension remains non-archimedean and discrete. In the above, we call k_0 the degree of a , denoted $\deg a$. It is obvious that $\mathbb{F}_q[T]$ is discrete in $\mathbb{F}_q((T^{-1}))$. For any $n \in \mathbb{N}$, throughout $\mathbb{F}_q((T^{-1}))^n$ is assumed to be equipped with the supremum norm which is defined as $\|\mathbf{x}\| := \max_{1 \leq i \leq n} |x_i|$ for all $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{F}_q((T^{-1}))^n$, and with the topology induced by this norm. Clearly $\mathbb{F}_q[T]^n$ is discrete in $\mathbb{F}_q((T^{-1}))^n$. Since the topology on $\mathbb{F}_q((T^{-1}))^n$ considered here is the usual product topology on $\mathbb{F}_q((T^{-1}))^n$, it follows that $\mathbb{F}_q((T^{-1}))^n$ is locally compact as $\mathbb{F}_q((T^{-1}))$ is locally compact. Note this construction $\mathbb{F}_q[T] \subset \mathbb{F}_q(T) \subset \mathbb{F}_q((T^{-1}))$ is similar to $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. Let λ be the Haar measure on $\mathbb{F}_q((T^{-1}))^n$ which takes the value 1 on the closed unit ball $\|\mathbf{x}\| \leq 1$.

Following [15, Remark 3.4] and [23, §2.2], we recall the definition of orthogonality of vectors in $\mathbb{F}_q((T^{-1}))^n$.

Definition 1.8. We say $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ in $\mathbb{F}_q((T^{-1}))^n$ is orthogonal if

$$\|a_1 \mathbf{v}_1 + \dots + a_m \mathbf{v}_m\| = \max_{i=1}^m \{|a_i| \|\mathbf{v}_i\|\},$$

for every $a_i \in \mathbb{F}_q((T^{-1})), 1 \leq i \leq m$.

Moreover, if each $1 \leq i \leq m$, $\|\mathbf{v}_i\| = 1$, then we say the collection $\{\mathbf{v}_1, \dots, \mathbf{v}_m\}$ to be orthonormal.

Moreover, following [23, §2.2] we say that the subspaces V_1, \dots, V_m in $\mathbb{F}_q((T^{-1}))^n$ are orthogonal if

$$\|\mathbf{v}_1 + \dots + \mathbf{v}_m\| = \max_{i=1}^m \{\|\mathbf{v}_i\|\}, \text{ for every } \mathbf{v}_i \in V_i, 1 \leq i \leq m.$$

Let us define

$$\mathcal{X}_i := \{\mathbf{x} \in \mathbb{F}_q((T^{-1}))^n \mid x_i = 0\}.$$

Clearly, by the above definition of orthogonality of subspaces and by the definition of max norm, \mathcal{X}_i is orthogonal to x_i -axis := $\{\mathbf{x} \in \mathbb{F}_q((T^{-1}))^n \mid x_j = 0, j \neq i\}$. Given any affine subspace A in $\mathbb{F}_q((T^{-1}))^n$, we say A is orthogonal to x_i -axis if $A = \mathcal{X}_i + \mathbf{v}$, for some $\mathbf{v} \in \mathbb{F}_q((T^{-1}))^n$.

Next, let us recall the definition of $\mathbb{F}_q((T^{-1}))$ -analytic map. Let B be a ball in $\mathbb{F}_q((T^{-1}))$. We call $f : B \subset \mathbb{F}_q((T^{-1})) \rightarrow \mathbb{F}_q((T^{-1}))$ to be $\mathbb{F}_q((T^{-1}))$ -analytic on B if there exists a $b \in B$ such that $f(x) = \sum_{i=0}^{\infty} a_i(x-b)^i$, $a_i \in \mathbb{F}_q((T^{-1}))$. It is worth pointing out that unlike real analytic maps, if f is $\mathbb{F}_q((T^{-1}))$ -analytic on B , then for every $b \in B$ there exist $a_i \in \mathbb{F}_q((T^{-1}))$ such that $f(x) = \sum_{i=0}^{\infty} a_i(x-b)^i$. In particular, polynomials are $\mathbb{F}_q((T^{-1}))$ -analytic on any ball. Moreover, a map $\mathbf{f} = (f_1, \dots, f_n) : B \subset \mathbb{F}_q((T^{-1})) \rightarrow \mathbb{F}_q((T^{-1}))^n$ is analytic on B if each f_i is analytic on B .

2. ψ -Dirichlet numbers in function field

2.1. Continued fraction over function field. Suppose $a = \sum_{k \leq k_0} a_k T^k \in \mathbb{F}_q((T^{-1}))$ where $a_{k_0} \neq 0$, we call $[a] := \sum_{0 \leq k \leq k_0} a_k T^k$ the integer part of a and $\langle a \rangle = \sum_{k < 0} a_k T^k$ the fractional part of a . Note that $|[a]| = e^{k_0} \geq 1$ or we have $[a] = 0$. Also, note that $|\langle a \rangle| < 1$. This observation leads us to construct continued fraction expansion of a . An expression of the form $a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots}}$ where $a_0, a_1, a_2, \dots \in \mathbb{F}_q[T]$ is called a simple continued fraction; see [24, §1]. An expression of the form

$$\frac{p_n}{q_n} = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \dots + \frac{1}{a_n}}},$$

where $p_n, q_n, a_0, a_1, \dots, a_n \in \mathbb{F}_q[T]$ is called a finite continued fraction. An element of $\mathbb{F}_q(T)$, can be represented as a unique finite continued fraction. An $\alpha \in \mathbb{F}_q((T^{-1})) \setminus \mathbb{F}_q[T]$ can be represented as a simple continued fraction in the form of $[a_0, a_1, a_2, \dots]$ and we call the numbers $\frac{p_n}{q_n} = [a_0, a_1, \dots, a_n]$ the convergents of α . Note that $|q_n|$ is increasing as $n \rightarrow \infty$. The relation between two consecutive convergents is given by the following equation:

$$p_i q_{i+1} - p_{i+1} q_i = (-1)^{i+1} \text{ for } i \in \mathbb{Z}, i \geq -2.$$

Hence we have,

$$\left| \frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} \right| = \left| \frac{p_{n+1}q_n - p_nq_{n+1}}{q_nq_{n+1}} \right| = \left| \frac{\pm 1}{q_nq_{n+1}} \right| = \frac{1}{|q_n| \cdot |q_{n+1}|} \leq \frac{1}{|q_n|^2}.$$

In fact, by Equation 1.12 in [24] we have

$$(2.1) \quad \left| \alpha - \frac{p_n}{q_n} \right| = \left| \frac{1}{q_n q_{n+1}} \right|,$$

where $\frac{p_n}{q_n}$ is a convergent of $\alpha \in \mathbb{F}_q((T^{-1})) \setminus \mathbb{F}_q(T)$. We recall the following definition and theorem of best approximation [22, §1.2].

Definition 2.1. We say a rational $\frac{a}{b}$ is the best approximation to some $\alpha \in \mathbb{F}_q((T^{-1}))$ if for all $\frac{c}{d}$ such that $|d| \leq |b|$ we have $|b\alpha - a| \leq |d\alpha - c|$.

Theorem 2.2. Let $\alpha \in \mathbb{F}_q((T^{-1}))$ and let $(\frac{p_n}{q_n})_n$ be its convergents. Let $p, q \in \mathbb{F}_q[T]$ with $q \neq 0$ be two relatively prime polynomials. Then $\frac{p}{q}$ is the best approximation to α if and only if it is convergent to α .

We want to recall the following Lemma 2.1 from [17] which was stated for real numbers. The verbatim proof will give the following lemma for the function field. The proof uses the fact that convergents are best approximations, which we have by Theorem 2.2. In what follows $\frac{p_n}{q_n}$ are convergents of x .

Lemma 2.3. Let $\psi : [t_0, +\infty) \rightarrow \mathbb{R}_+$ be non-increasing. Then $x \in \mathbb{F}_q((T^{-1})) \setminus \mathbb{F}_q(T)$ is ψ -Dirichlet if and only if $|\langle q_{n-1}x \rangle| < \psi(|q_n|)$ for sufficiently large n .

2.2. Proof of Theorem 1.2. It is easy to see that $\mathbb{F}_q(T) \subset D(\psi)$. We want to show that $D(\psi) \subset \mathbb{F}_q(T)$. By Equation (2.1) for $x \in \mathbb{F}_q((T^{-1})) \setminus \mathbb{F}_q(T)$ we have $|\langle q_{n-1}x \rangle| = \frac{1}{|q_n|}$, $\forall n$. Since $\psi(t) < \frac{1}{t}$ for all large enough t , by Lemma 2.3 we conclude that there is no $x \in \mathbb{F}_q((T^{-1})) \setminus \mathbb{F}_q(T)$ such that x is ψ -Dirichlet.

3. Too many vectors with high uniform exponent

In this section, we study totally irrational singular vectors in submanifolds of $\mathbb{F}_q((T^{-1}))^n$. In dimension $n = 1$, Theorem 2.4 in [8] implies that the set of numbers y in $\mathbb{F}_q((T^{-1}))$ that are singular is $\mathbb{F}_q(T)$.

In order to state the main theorem of this section, we need to define the *irrationality measure* function as follows. We follow the definition in [14].

Definition 3.1. We define $\Phi : \mathbb{F}_q[T]^n \setminus \{0\} \rightarrow \mathbb{R}_+$ to be a proper function if the set $\{\mathbf{q} \in \mathbb{F}_q[T]^n \setminus \{0\} : \Phi(\mathbf{q}) \leq C\}$ is finite for any $C > 0$. For any arbitrary Φ and any $\mathbf{y} \in \mathbb{F}_q((T^{-1}))^n$, we define the irrationality measure function

$$\psi_{\Phi, \mathbf{y}}(t) := \min_{(q_0, \mathbf{q}) \in \mathbb{F}_q[T] \times \mathbb{F}_q[T]^n \setminus \{\mathbf{0}\}, \Phi(\mathbf{q}) \leq t} |\mathbf{q} \cdot \mathbf{y} + q_0|.$$

We can now state one of the main theorems of this section.

Theorem 3.2. *Let $\text{char}(\mathbb{F}_q((T^{-1}))) = p < \infty$. Let U be an open subset of $\mathbb{F}_q((T^{-1}))$. We consider S of the following two types:*

- $S = \{(x, y, p_3(x, y), \dots, p_n(x, y)) \mid x, y \in U\} \subset \mathbb{F}_q((T^{-1}))^n$, where each $p_i(x, y)$ is a degree 2 polynomial.
- $S = \{(x, y, p(x, y)) \mid x, y \in U\} \subset \mathbb{F}_q((T^{-1}))^3$ where $p(x, y) = \sum_{i=0}^n a_i^{p^i} x^{p^i} + \sum_{j=0}^n b_j^{p^j} y^{p^j}$.

Suppose that S is not contained inside any rational affine hyperplane. Then for any proper function $\Phi : \mathbb{F}_q[T]^n \setminus \{0\} \rightarrow \mathbb{R}_+$ and for any non-increasing function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, there exist uncountably many totally irrationals $\mathbf{y} \in S$ such that $\psi_{\Phi, \mathbf{y}}(t) \leq \phi(t)$ for all large enough t .

As an application of the previous theorem, we get Theorem 1.6.

Proof of Theorem 1.6. By taking $\Phi(\mathbf{q}) = \|\mathbf{q}\|$, Theorem 1.6 follows from Theorem 3.2. \square

It is also worth pointing out that in the above theorem, we can assume without loss of generality that U is a ball in $\mathbb{F}_q((T^{-1}))$. Hence for the rest of the paper we assume U is an open (closed) ball in $\mathbb{F}_q((T^{-1}))$.

Let us recall Theorem 1.1 from [14], which was proved for locally closed subsets of \mathbb{R}^n . The same proof verbatim will work for locally closed subsets in $\mathbb{F}_q((T^{-1}))^n$. It is noteworthy that the proof follows Khintchine's argument in [11]. We define $|A| := \max_{i=1}^{n+1} |a_i|$, where $A : a_1x_1 + \dots + a_nx_n = a_{n+1}$, and (a_1, \dots, a_{n+1}) is a primitive vector in $\mathbb{F}_q[T]^{n+1}$.

Theorem 3.3. *Let $S \subset \mathbb{F}_q((T^{-1}))^n$ be a nonempty locally closed subset. Let $\{L_1, L_2, \dots\}$ and $\{L'_1, L'_2, \dots\}$ be disjoint collections of distinct closed subsets of S , each of which is contained in a rational affine hyperplane in $\mathbb{F}_q((T^{-1}))^n$, and for each i let A_i be a rational affine hyperplane containing L_i , assume the following hold:*

- (a) $\bigcup_i L_i \cup \bigcup_j L'_j = \{x \in S : x \text{ is contained in a rational affine hyperplane}\};$
- (b) *For each i and each $\alpha > 0$, $L_i = \overline{\bigcup_{|A_j| > \alpha} L_i \cap L_j}$;*
- (c) *For each i , and for any finite subsets of indices F, F' with $i \notin F$, we have $L_i = \overline{L_i - (\bigcup_{k \in F} L_k \cup \bigcup_{k' \in F'} L'_{k'})}$;*
- (d) $\bigcup_i L_i$ is dense in S .

Then for arbitrary $\Phi : \mathbb{F}_q[T]^n \setminus \{0\} \rightarrow \mathbb{R}_+$ proper function and for any non-increasing function $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, there exist uncountably many totally irrationals $\mathbf{y} \in S$ such that $\psi_{\Phi, \mathbf{y}}(t) \leq \phi(t)$ for all large enough t .

We will refer to the property (a), (b), (c), and (d) defined above as “Property A”. Let us recall the following Theorem 2.1.1 in [10] which we are going to use throughout the rest of this section.

Theorem 3.4. *Let K be an arbitrary field and assume that for some m, n every $F_i(x, y)$ in $F(x, y) = (F_1(x, y), \dots, F_m(x, y))$ is in $K[[X, Y]] = K[[x_1, \dots, x_n, y_1, \dots, y_m]]$ satisfying $F_i(0, 0) = 0$ and further $\frac{\partial(F_1, \dots, F_m)}{\partial(y_1, \dots, y_m)} \Big|_{(0,0)} \neq 0$, in which $\frac{\partial(F_1, \dots, F_m)}{\partial(y_1, \dots, y_m)}$ is the Jacobian. Then there exists a unique $f(x) = (f_1(x), \dots, f_m(x))$ with every $f_i(x)$ in $K[[x]] = K[[x_1, \dots, x_m]]$ satisfying $f_i(0) = 0$ and further $F(x, f(x)) = 0$.*

3.1. When each $p_i(x, y)$ is a degree 2 polynomial and $\mathbb{F}_q((T^{-1}))$ is of any positive characteristic. Let us consider

$$S = \{(x, y, p_3(x, y), \dots, p_n(x, y)) \mid x, y \in U\},$$

where U is an open subset of $\mathbb{F}_q((T^{-1}))$, and

$$p_i(x, y) = b_{i,1}x^2 + b_{i,2}xy + b_{i,3}y^2 + b_{i,4}x + b_{i,5}y + b_{i,6}$$

with $b_{i,1}, b_{i,2}, \dots, b_{i,6} \in \mathbb{F}_q((T^{-1}))$ and $b_{i,1}, b_{i,2}, b_{i,3}$ not being zero simultaneously for $i = 3, \dots, n$. Let us take A to be a rational affine hyperplane in $\mathbb{F}_q((T^{-1}))^n$ and we assume that S is not contained inside A . We can define A by the linear equation $a_1x_1 + a_2x_2 + \dots + a_nx_n = a_{n+1}$, where $(a_1, a_2, \dots, a_{n+1}) \in \mathbb{F}_q[T]^{n+1}$ is primitive. Note that $S \cap A$ is given by the solutions to the equation;

$$(3.1) \quad f(x, y) := 0,$$

where

$$f(x, y) = \sum_{i=3}^n a_i p_i(x, y) + a_1x + a_2y - a_{n+1}.$$

We see that f is a polynomial of degree less than or equal to 2. Now note that

$$(3.2) \quad \frac{\partial f}{\partial x} = \sum_{i=3}^n (2a_i b_{i,1}x + a_i b_{i,2}y) + (a_1 + \sum_{i=3}^n a_i b_{i,4})$$

and

$$(3.3) \quad \frac{\partial f}{\partial y} = \sum_{i=3}^n (a_i b_{i,2}x + 2a_i b_{i,3}y) + (a_2 + \sum_{i=3}^n a_i b_{i,5}).$$

If $\frac{\partial f}{\partial x}(x_0, y_0) \neq 0$, then by Theorem 3.4 we get a neighborhood of (x_0, y_0) , where y is a $\mathbb{F}_q((T^{-1}))$ -analytic function of x . If $\frac{\partial f}{\partial y}(x_0, y_0) \neq 0$ then locally we can write x as a $\mathbb{F}_q((T^{-1}))$ -analytic function of y . Hence in order to find out all possible (x_0, y_0) such that there is no neighborhood of $(x_0, y_0, p_3(x_0, y_0), \dots, p_n(x_0, y_0)) \in S \cap A$ that is analytic curve in $S \cap A$, we consider the linear system

$$(3.4) \quad \begin{cases} \frac{\partial f}{\partial x} = 0; \\ \frac{\partial f}{\partial y} = 0. \end{cases}$$

The corresponding coefficient matrix $M \in \text{Mat}_{2 \times 2}(\mathbb{F}_q((T^{-1})))$ of the system is

$$\begin{bmatrix} \sum_{i=3}^n 2a_i b_{i,1} & \sum_{i=3}^n a_i b_{i,2} \\ \sum_{i=3}^n a_i b_{i,2} & \sum_{i=3}^n 2a_i b_{i,3} \end{bmatrix}.$$

First note that if a_3, \dots, a_n are zero, then $S \cap A$ is an analytic curve as the equation of A would be $a_1x + a_2y = a_{n+1}$. Therefore one of a_3, \dots, a_n must be nonzero, and without loss of generality we assume that $a_3 \neq 0$. Next let us denote,

$$b_k = \sum_{i=3}^n a_i b_{i,k}$$

for $k = 1, \dots, 6$. With this setting, we have the following lemma.

Lemma 3.5. *If $b_2^2 \neq 4b_1b_3$, then there are at most finitely many points of $A \cap S$ such that their neighbourhood is not an $\mathbb{F}_q((T^{-1}))$ -analytic curve in $\mathbb{F}_q((T^{-1}))^n$.*

Proof. Note that $\det(M) = 4b_1b_3 - b_2^2$. Hence by the hypothesis, we know that the system has only one solution. This completes the proof. \square

From the proof above we know that the key is to solve the following equations;

$$(3.5) \quad \begin{aligned} f(x, y) &= \sum_{i=3}^n a_i p_i(x, y) + a_1x + a_2y - a_{n+1} \\ &= b_1x^2 + b_2xy + b_3y^2 + (a_1 + b_4)x + (a_2 + b_5)y + (b_6 - a_{n+1}) = 0, \end{aligned}$$

$$(3.6) \quad \frac{\partial f}{\partial x}(x, y) = 2b_1x + b_2y + (a_1 + b_4) = 0,$$

$$(3.7) \quad \frac{\partial f}{\partial y}(x, y) = b_2x + 2b_3y + (a_2 + b_5) = 0.$$

Lemma 3.6. *If*

$$(3.8) \quad b_2^2 = 4b_1b_3$$

and $b_2 \neq 0$, then there are at most finitely many points of $A \cap S$ that its neighbourhood is not an $\mathbb{F}_q((T^{-1}))$ -analytic curve in $\mathbb{F}_q((T^{-1}))^n$.

Proof. Suppose that there exists no point that satisfies System (3.4), then the conclusion of the lemma holds trivially.

Now suppose that there exists a point (x_0, y_0) that satisfies System (3.4). Using Equation (3.8) we have,

$$\begin{aligned}
 (3.9) \quad & b_2(a_2 + b_5) \stackrel{(3.7)}{=} -b_2^2 x_0 - 2b_3(b_2 y_0) \\
 & \stackrel{(3.6)}{=} -b_2^2 x_0 + 2b_3(2b_1 x_0 + (a_1 + b_4)) \\
 & \stackrel{(3.8)}{=} 2b_3(a_1 + b_4).
 \end{aligned}$$

For any $x, y \in U$,

$$\begin{aligned}
 (3.10) \quad & b_2^2 f(x, y) \\
 & \stackrel{(3.5)}{=} b_2^2(b_1 x^2 + b_2 x y + b_3 y^2) + b_2^2((a_1 + b_4)x + (a_2 + b_5)y) \\
 & \quad + b_2^2(b_6 - a_{n+1}) \\
 & \stackrel{(3.8)}{=} b_3(2b_1 x + b_2 y)^2 + b_2^2((a_1 + b_4)x + (a_2 + b_5)y) + b_2^2(b_6 - a_{n+1}) \\
 & \stackrel{(3.9)}{=} b_3(2b_1 x + b_2 y)^2 + b_2(a_1 + b_4)(b_2 x + 2b_3 y) + b_2^2(b_6 - a_{n+1}).
 \end{aligned}$$

For any point that satisfies System (3.4),

$$\begin{aligned}
 (3.11) \quad & b_3(2b_1 x + b_2 y)^2 + b_2(a_1 + b_4)(b_2 x + 2b_3 y) + b_2^2(b_6 - a_{n+1}) \\
 & \stackrel{(3.6), (3.7)}{=} b_3(a_1 + b_4)^2 - b_2(a_1 + b_4)(a_2 + b_5) + b_2^2(b_6 - a_{n+1}) \\
 & \stackrel{(3.9)}{=} b_2^2(b_6 - a_{n+1}) - b_3(a_1 + b_4)^2.
 \end{aligned}$$

Suppose that $b_2^2(b_6 - a_{n+1}) \neq b_3(a_1 + b_4)^2$. We know that Equation (3.5) would never be satisfied for any points satisfying the System (3.4). Therefore there is no point in $S \cap A$ such that its neighborhood is not an $\mathbb{F}_q((T^{-1}))$ -analytic curve.

Now suppose that

$$(3.12) \quad b_2^2(b_6 - a_{n+1}) = b_3(a_1 + b_4)^2.$$

Let $g(x) := -\frac{2b_1 x + (a_1 + b_4)}{b_2}$ and γ be $\{(x, g(x), p_3(x, g(x)), \dots, p_n(x, g(x))) \mid x \in U\}$. Then clearly any point in γ satisfies (3.6), (3.7) and using (3.12) one can see that any point in γ also satisfies (3.5). Hence we have $\gamma \subseteq S \cap A$. Note here Equations (3.6) and (3.7) are essentially the same.

Now for any point in $S \cap A$, we have

$$f(x, y) = 0$$

$$\begin{aligned}
 & \stackrel{(3.10)}{\Rightarrow} b_3(2b_1 x + b_2 y)^2 + b_2(a_1 + b_4)(b_2 x + 2b_3 y) + b_2^2(b_6 - a_{n+1}) = 0 \\
 & \stackrel{(3.12), (3.8)}{\Rightarrow} b_3(2b_1 x + b_2 y + a_1 + b_4)^2 = 0 \\
 & \Rightarrow y = -\frac{2b_1 x + (a_1 + b_4)}{b_2}.
 \end{aligned}$$

The last equality holds because $b_3 \neq 0$ as $b_2 \neq 0$. Since γ is already an analytic curve, this completes the proof of the lemma. \square

Lemma 3.7. *If $b_2^2 = 4b_1b_3$ and $b_2 = 0$, then there are at most finitely many points of $A \cap S$ that its neighbourhood is not an $\mathbb{F}_q((T^{-1}))$ -analytic curve in $\mathbb{F}_q((T^{-1}))^n$.*

Proof. If $b_1 = 0, b_2 = 0, b_3 = 0$ and there exists one point whose neighborhood is not an $\mathbb{F}_q((T^{-1}))$ -analytic curve, then $S \cap A$ being nonempty implies $S \cap A = S$, because $f(x, y) = b_6 - a_{n+1} = 0$. This contradicts the standing assumption that S is not contained inside A .

If $\text{char}(\mathbb{F}_q((T^{-1}))) \neq 2$, then $b_1b_3 = 0$. Since b_1, b_2 and b_3 cannot be zero simultaneously, assume without loss of generality that $b_1 \neq 0$ and $b_3 = 0$.

Suppose that there exists no point that satisfies System (3.4), then the conclusion of the lemma holds. So let us assume that there exists a point (x_0, y_0) that satisfies System (3.4), then Equation (3.7) gives us $a_2 + b_5 = 0$. By Equation (3.5), we have

$$b_1x^2 + (a_1 + b_4)x + (b_6 - a_{n+1}) = 0.$$

At most two x can satisfy the above equation, say they are x_1 and x_2 respectively. Then

$$\gamma_1 = \{(x_1, y, p_3(x_1, y), \dots, p_n(x_1, y)) \mid y \in U\}$$

and

$$\gamma_2 = \{(x_2, y, p_3(x_2, y), \dots, p_n(x_2, y)) \mid y \in U\}$$

are both analytic curves and they are inside $S \cap A$.

In addition, since

$$f(x, y) = b_1x^2 + (a_1 + b_4)x + (b_6 - a_{n+1}),$$

we know that any point on $S \cap A$ must have an x -coordinate equal to x_1 or x_2 , which means that it is in γ_1 or γ_2 . In other words, $S \cap A = \gamma_1 \sqcup \gamma_2$. Thus the proof is complete for $\text{char}(\mathbb{F}_q((T^{-1}))) \neq 2$.

If $\text{char}(\mathbb{F}_q((T^{-1}))) = 2$, then $\frac{\partial f}{\partial x} = a_1 + b_4$ and $\frac{\partial f}{\partial y} = a_2 + b_5$. If either of the two is nonzero then the conclusion of the lemma holds. Otherwise, for every x, y we have

$$f(x, y) = b_1x^2 + b_3y^2 + (b_6 - a_{n+1}).$$

Since b_1, b_2 , and b_3 cannot be zero simultaneously, assume without loss of generality that $b_3 \neq 0$. Suppose $\frac{b_1}{b_3}$ is not square. Now if we have two points $(x_0, y_0, p_3(x_0, y_0), \dots, p_n(x_0, y_0))$ and $(x_1, y_1, p_3(x_1, y_1), \dots, p_n(x_1, y_1))$ in $S \cap A$, then

$$b_1x_0^2 + b_3y_0^2 = b_1x_1^2 + b_3y_1^2 \implies b_1(x_1 - x_0)^2 = b_3(y_1 - y_0)^2.$$

The above gives a contradiction to the assumption that $\frac{b_1}{b_3}$ is not a square. Hence in this case there could be at max one point in $S \cap A$.

Now let us assume $\frac{b_1}{b_3} = \alpha^2$ for some $\alpha \in \mathbb{F}_q((T^{-1}))$. Any point in $S \cap A$ satisfies the equation $b_1x^2 + b_3y^2 = (a_{n+1} - b_6)$. This is equivalent to $\alpha^2x^2 + y^2 = \beta$, where $\beta = \frac{a_{n+1} - b_6}{b_3}$. Suppose that $(x_0, y_0, p_3(x_0, y_0), \dots, p_n(x_0, y_0))$ is a point in $S \cap A$, which implies $\alpha^2x_0^2 + y_0^2 = \beta$. Suppose that

$$(x(t), y(t), p_3(x(t), y(t)), \dots, p_n(x(t), y(t)))$$

is in $S \cap A$. For any $t \in \mathbb{F}_q((T^{-1}))$, let $x(t) = x_0 + t$. Then the corresponding $y(t)$ can be given as $(y(t))^2 = \beta - \alpha^2(x(t))^2 = (\beta - \alpha^2x_0^2) - \alpha^2t^2 = y_0^2 + \alpha^2t^2$. Therefore $y(t) = y_0 + \alpha t$, and this shows that $S \cap A$ gives an analytic curve. \square

Combining Lemma 3.5, Lemma 3.6, and Lemma 3.7 we get the following proposition.

Proposition 3.8. *Let $S = \{(x, y, p_3(x, y), \dots, p_n(x, y)) \mid x, y \in U\}$, where U is an open set of $\mathbb{F}_q((T^{-1}))$, and each $p_i(x, y)$ is a degree 2 polynomial and A be an affine rational hyperplane in $\mathbb{F}_q((T^{-1}))^n$. Suppose that S is not contained inside A then there are at most finitely many points of $S \cap A$ such that its neighbourhood is not an $\mathbb{F}_q((T^{-1}))$ -analytic curve in $\mathbb{F}_q((T^{-1}))^n$.*

By the above proposition, we have that $S \cap A \setminus J = \bigcup_{\mathbf{z} \in S \cap A} \gamma(\mathbf{z})$, where J is a finite set of points which do not have an $\mathbb{F}_q((T^{-1}))$ -analytic curve as neighborhood, and $\gamma(\mathbf{z})$ is an $\mathbb{F}_q((T^{-1}))$ -analytic curve which is a neighborhood of \mathbf{z} in $S \cap A$. Also, note that $\gamma(\mathbf{z})$ is open and closed. Since $S \cap A \setminus J$ is a second countable space, we know that there exists a countable subcovering γ_j , i.e. $S \cap A \setminus J = \bigcup_i \gamma_i$.

Theorem 3.9. *Let S be as in the previous proposition. There exist $\{L_i\}$, $\{L'_j\}$, $\{A_j\}$ as mentioned in Theorem 3.3, that satisfy Property A.*

Proof. Let $\{A_i\}$ be the set of affine rational hyperplanes orthogonal to one of x -axis or y -axis in $\mathbb{F}_q((T^{-1}))^n$. Now let us define $L_i = S \cap A_i$, which are closed subsets and curves of S .

Next we define $\{L'_j\}$. For any affine rational hyperplane A that has a nonempty intersection with S , by Proposition 3.8, we have that $S \cap A$ is union of γ_j , excluding finitely many points. Let $\{L'_j\} = \{\gamma_j : \forall i, \gamma_j \not\subset L_i\}$. Now we want to verify that these collections satisfy four hypotheses of Theorem 3.3.

Property (a) of Theorem 3.3 follows directly from how we defined these sets.

First, let us consider $L_i = S \cap A_i$, such that $A_i : x = a$, $a \in \mathbb{F}_q(T)$. Now let us consider $A_{j_k} : y = \frac{b}{T^k}$, where $b \in \mathbb{F}_q[T]$. Since $L_i = \{(a, y, p_3(a, y), \dots, p_n(a, y)) \mid y \in U\}$, and $\{\frac{b}{T^k} \in U \mid k \geq m\}$ for some $m \in \mathbb{N}$, is dense in U , property (b) follows.

Let F and F' are as in hypothesis (c) of Theorem 3.3. Let

$$L_i = \{(a, y, p_3(a, y), \dots, p_n(a, y)) \mid y \in U\},$$

where $a \in \mathbb{F}_q(T)$. Note that each $L_i \cap L_k$ with $k \in F$ is either empty or consists of only a single point. For any $k' \in F'$, by the equation given above, $L_i \cap L'_{k'} = L_i \cap \gamma_{k'}$. We can write $\gamma_{k'}$ is a subset of $a_1x + a_2y + \sum_{i=3}^n a_i p_i(x, y) - a_{n+1} = 0$, and therefore, $L_i \cap \gamma_{k'}$ is a subset of $a_1a + a_2y + \sum_{i=3}^n a_i p_i(a, y) - a_{n+1} = 0$. Hence only finitely many solutions is possible and $L_i \cap L'_{k'}$ has no interior. The same proof will work when $L_i = A_i \cap S$ where A_i is orthogonal to y -axis.

To verify property (d) of Theorem 3.3 it is enough to observe that $S \cap A_i$ looks like

$$(x, a, p_3(x, a), \dots, p_n(x, a) \mid x \in U)$$

or

$$(b, y, p_3(b, y), \dots, p_n(b, y) \mid y \in U),$$

where $a, b \in \mathbb{F}_q(T)$. Clearly, they form a dense set in S . \square

3.2. A special case in higher degree $p(x, y)$.

Proposition 3.10. *Let $\text{char}(\mathbb{F}_q((T^{-1}))) = p < +\infty$. Let $S = \{(x, y, p(x, y)) \mid x, y \in U\}$, where $p(x, y) = \sum_{i=0}^m b_i^{p^i} x^{p^i} + \sum_{j=0}^n c_j^{p^j} y^{p^j}$ and U is an open subset in $\mathbb{F}_q((T^{-1}))$. There are at most finitely many points of $S \cap A$ such that its neighbourhood is not an $\mathbb{F}_q((T^{-1}))$ -analytic curve in $\mathbb{F}_q((T^{-1}))^3$.*

Proof. Let A be given by the equation $a_1x + a_2y + a_3z = a_4$. Without loss of generality let us assume that $S \cap A$ is nonempty and that there exists at least one point of $S \cap A$ whose neighbourhood is not an $\mathbb{F}_q((T^{-1}))$ -analytic curve in $\mathbb{F}_q((T^{-1}))^3$. This implies $a_3 \neq 0$, and we can also assume $a_3 = 1$ after normalization. This implies that $a_1 + b_0 = a_2 + c_0 = 0$. Suppose also without loss of generality that $m \geq n$, $b_m \neq 0$, and $c_n \neq 0$. The intersection $S \cap A$ is given by $f(x, y) = 0$, where

$$\begin{aligned} f(x, y) &= \sum_{i=1}^m b_i^{p^i} x^{p^i} + \sum_{j=1}^n c_j^{p^j} y^{p^j} - a_4 \\ &= \left(\sum_{i=1}^m b_i^{p^{i-1}} x^{p^{i-1}} + \sum_{j=1}^n c_j^{p^{j-1}} y^{p^{j-1}} \right)^p - a_4 \end{aligned}$$

Since $S \cap A$ is nonempty, a_4 must have a p -th root, say it is $a_{4,0}^p = a_4$. Therefore, we have that $S \cap A$ is given by

$$f_0(x, y) := \sum_{i=1}^m b_i^{p^{i-1}} x^{p^{i-1}} + \sum_{j=1}^n c_j^{p^{j-1}} y^{p^{j-1}} - a_{4,0}.$$

If $S \cap A$ has one point which has no neighbourhood that is an $\mathbb{F}_q((T^{-1}))$ -analytic curve, we have $b_1 = c_1 = 0$. Hence we can write $f_0(x, y) = 0$ as,

$$f_0(x, y) = \left(\sum_{i=2}^m b_i^{p^{i-2}} x^{p^{i-2}} + \sum_{j=2}^n c_j^{p^{j-2}} y^{p^{j-2}} \right)^p - a_{4,0}.$$

Since $S \cap A$ is nonempty $a_{4,0}$ must have a p -th root, say it is $a_{4,1}^p = a_{4,0}$. Since $f_0(x, y) = (f_1(x, y))^p$, where

$$f_1(x, y) := \sum_{i=2}^m b_i^{p^{i-2}} x^{p^{i-2}} + \sum_{j=2}^n c_j^{p^{j-2}} y^{p^{j-2}} - a_{4,1},$$

we have that $S \cap A$ is defined by $f_1(x, y) = 0$.

We use induction to derive the desired results. For any $k \in \mathbb{N}$ satisfying $1 \leq k \leq n-1$, $b_1 = \dots = b_{k-1} = 0$, and $c_1 = \dots = c_{k-1} = 0$, assume that $S \cap A$ is given by $f_k(x, y) = 0$, where

$$f_k(x, y) = \sum_{i=k+1}^m b_i^{p^{i-k-1}} x^{p^{i-k-1}} + \sum_{j=k+1}^n c_j^{p^{j-k-1}} y^{p^{j-k-1}} - a_{4,k}$$

for some $a_{4,k} \in \mathbb{F}_q((T^{-1}))$. We have $\frac{\partial f_k}{\partial x} = b_{k+1}$ and $\frac{\partial f_k}{\partial y} = c_{k+1}$. By System (3.4), we derive that $b_{k+1} = c_{k+1} = 0$. Now since

$$\begin{aligned} f_k(x, y) &= \sum_{i=k+2}^m b_i^{p^{i-k-1}} x^{p^{i-k-1}} + \sum_{j=k+2}^n c_j^{p^{j-k-1}} y^{p^{j-k-1}} - a_{4,k} \\ &= \left(\sum_{i=k+2}^m b_i^{p^{i-k-2}} x^{p^{i-k-2}} + \sum_{j=k+2}^n c_j^{p^{j-k-2}} y^{p^{j-k-2}} \right)^p - a_{4,k}, \end{aligned}$$

and by our assumption that $S \cap A$ is nonempty, we know that $a_{4,k}$ must have a p -th root, say it is $a_{4,(k+1)} \in \mathbb{F}_q((T^{-1}))$, i.e., $a_{4,(k+1)}^p = a_{4,k}$. Define

$$f_{k+1}(x, y) := \sum_{i=k+2}^m b_i^{p^{i-k-2}} x^{p^{i-k-2}} + \sum_{j=k+2}^n c_j^{p^{j-k-2}} y^{p^{j-k-2}} - a_{4,(k+1)}.$$

Then

$$f_k(x, y) = (f_{k+1}(x, y))^p.$$

This implies that the intersection $S \cap A$ is given by the equation $f_{k+1}(x, y) = 0$.

Repeating the steps above we see that the intersection $S \cap A$ is given by $f_{n-1}(x, y) = 0$. Also from the induction above, we see that

$$f_{n-1}(x, y) = c_n y - a_{4,n-1} + \sum_{i=n}^m b_i^{p^{i-n}} x^{p^{i-n}}.$$

This tells us that $S \cap A$ is completely given by the curve

$$\{(x, g(x), p(x, g(x))) \mid x \in U\},$$

where

$$g(x) := \frac{1}{c_n} \left(a_{4,n-1} - \sum_{i=n}^m b_i^{p^{i-n}} x^{p^{i-n}} \right), \text{ which is analytic.} \quad \square$$

The exact same proof as Theorem 3.9 with suitable changes gives the following theorem.

Theorem 3.11. *Let $S = \{(x, y, \sum_{i=0}^m b_i^{p^i} x^{p^i} + \sum_{j=0}^n c_j^{p^j} y^{p^j})\}$, where $x, y, b_i, c_j \in \mathbb{F}_q((T^{-1}))$, and not all b_i s or c_j s are being zero. Then there exists $\{L_i\}, \{L'_j\}, \{A_j\}$ that satisfies Property A.*

We now have everything we need to prove the main theorem of this section:

Proof of Theorem 3.2. Theorem 3.9 and Theorem 3.11 guarantee that the hypotheses of Theorem 3.3 are met by surfaces considered in Theorem 3.2. Therefore by Theorem 3.3, Theorem 3.2 follows. \square

3.3. Surface to higher dimensional manifold. Let λ_k be the Lebesgue measure in $B(0, 1)^k$. We say $\mathbf{f} : B(0, 1)^k \rightarrow \mathbb{F}_q((T^{-1}))^n$ is non-planar at \mathbf{x}_0 if, for every ball B such that $\mathbf{x}_0 \in B$, $\mathbf{f}(B)$ is not contained in any proper affine hyperplane. Then we say (\mathbf{f}, λ_k) is nonplanar if \mathbf{f} is nonplanar at λ_k almost every \mathbf{x}_0 . The following theorem is somewhat an analogue to Lemma 3.5 in [14].

Lemma 3.12. *Suppose that for $k \geq 3$ and $B(0, 1)$ is the open and closed ball in $\mathbb{F}_q((T^{-1}))$ of radius 1. Let $\mathbf{f} : B(0, 1)^k \rightarrow \mathbb{F}_q((T^{-1}))^n$ be a continuous map, and $M := \mathbf{f}(B(0, 1)^k)$. Suppose that (\mathbf{f}, λ_k) is nonplanar. Then there exists $\mathbf{y} \in B(0, 1)^{k-2}$ such that the surface $M_{\mathbf{y}} := \mathbf{f}_{\mathbf{y}}((B(0, 1)^2)$ is not contained inside an rational affine hyperplane, where $\mathbf{f}_{\mathbf{y}} : B(0, 1)^2 \rightarrow \mathbb{F}_q((T^{-1}))^n$, $\mathbf{f}_{\mathbf{y}}(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{f}(\mathbf{x}_1, \mathbf{x}_2, \mathbf{y})$.*

Proof. We will prove this by contradiction. Suppose for every $\mathbf{y} \in B(0, 1)^{k-2}$ there exists a rational affine subspace $A_{\mathbf{y}}$ such that $B(0, 1)^2 \times \{\mathbf{y}\} \subset \mathbf{f}^{-1}(A_{\mathbf{y}} \cap M)$ implies $B(0, 1)^k = \bigcup_A$ is a rational affine subspace $\mathbf{f}^{-1}(A \cap M)$. By Baire category theorem there is one A such that $\mathbf{f}^{-1}(A \cap M)$ contains an open ball inside $B(0, 1)^k$. This contradicts the fact that (\mathbf{f}, λ_k) is nonplanar. \square

Remark 3.13.

- (1) In the above lemma we don't need to consider \mathbf{f} to be analytic. We need a stronger assumption that (\mathbf{f}, λ) is nonplanar as compared to M being not inside an affine hyperplane.

(2) The above lemma shows it is enough to prove Theorem 1.6 for surfaces in certain cases.

3.4. Product of perfect sets. Let us recall that a subset of $\mathbb{F}_q((T^{-1}))$ is called perfect if it is compact and has no isolated points. The proof of the following proposition is the same as the proof of Theorem 1.6 in [14]. We still include the proof to make the paper self-contained.

Proposition 3.14. *Let $n \geq 2$ and let S_1, \dots, S_n be perfect subsets of $\mathbb{F}_q((T^{-1}))$ such that $\mathbb{F}_q(T) \cap S_1$ is dense in S_1 and $\mathbb{F}_q(T) \cap S_2$ is dense in S_2 . Let $S = \prod_{j=1}^n S_j$. Then there exist a collection of $\{L_i\}$, $\{L'_j\}$, $\{A_i\}$ of S that satisfy Property A.*

Proof. Let us take $\{A_i\}$ to be the collection

$$\{\mathcal{X}_1 + (a, 0, \dots, 0), a \in \mathbb{F}_q(T)\} \cup \{\mathcal{X}_2 + (0, b, 0, \dots, 0), b \in \mathbb{F}_q(T)\}.$$

Note by definition in Section 1.4, $\{A_i\}$ is the collection of affine rational hyperplanes that are orthogonal to either x_1 -axis or x_2 -axis. Note that $\mathcal{X}_1 + (a, 0, \dots, 0) = \{\mathbf{x} \in \mathbb{F}_q((T^{-1}))^n \mid x_1 = a\}$ and similarly $\mathcal{X}_2 + (0, b, 0, \dots, 0) = \{\mathbf{x} \in \mathbb{F}_q((T^{-1}))^n \mid x_2 = b\}$. Let us define $L_i = A_i \cap S$, and $\{L'_j\}$ be the collection of $S \cap A$, where A is affine rational hyperplane, and $L'_j \notin \{L_i\}$. For $L_i = \{\mathbf{x} \in \mathbb{F}_q((T^{-1}))^n \mid x_1 = a\} \cap S, a \in \mathbb{F}_q(T)$, suppose $(a, x_2, x_3, \dots, x_n) \in L_i$, then we have $x_2 \in S_2$. Since $\mathbb{F}_q(T) \cap S_2$ is dense in S_2 and S_2 is perfect, there exists $\frac{P_j}{Q_j} \in \mathbb{F}_q(T)$ such that $\frac{P_j}{Q_j} \rightarrow_{j \rightarrow \infty} x_2$ and $|Q_j| \rightarrow \infty$. Hence (b) of Property A is satisfied as $L_j = \{\mathbf{x} \in \mathbb{F}_q((T^{-1}))^n \mid x_2 = \frac{P_j}{Q_j}\} \cap S$ belong to the union in the right-hand side of (b) in Theorem 3.3. To the contrary to (c) in Theorem 3.3 suppose there exists

$$\mathbf{x} \in L_i \setminus \overline{L_i \setminus \bigcup_{k \in F, k' \in F'} L_k \cup L'_{k'}}.$$

This implies there exists an open set $\mathbf{x} \in V$ such that $V \cap L_i \setminus \bigcup_{k \in F, k' \in F'} L_k \cup L'_{k'} = \emptyset$. This implies $V \cap L_i \subset \bigcup_{k \in F, k' \in F'} L_k \cup L'_{k'}$. Since F, F' are finite, $i \notin F$, and S_j 's are perfect the last inclusion is not possible. Note that (a) follows from the construction and (d) follows from the hypothesis $\mathbb{F}_q(T) \cap S_i, i = 1, 2$ are dense in S_i . \square

Thus we have the following theorem combining Theorem 3.3 and Proposition 3.14.

Theorem 3.15. *Let $n \geq 2$ and let S_1, \dots, S_n be perfect subsets of $\mathbb{F}_q((T^{-1}))$ such that $(\mathbb{F}_q(T) \cap S_1)$ is dense in S_1 and $(\mathbb{F}_q(T) \cap S_2)$ is dense in S_2 . Let $S = \prod_{j=1}^n S_j$. Then there exist uncountably many totally irrational singular vectors in S .*

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