JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

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Tome 33, nº 3.1 (2021), p. 733-778.

<http://jtnb.centre-mersenne.org/item?id=JTNB_2021__33_3.1_733_0>

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Controlling $\boldsymbol{\lambda}$-invariants for the double and triple product $\boldsymbol{p}$-adic $\boldsymbol{L}$-functions

par Daniel DELBOURGO et Hamish GILMORE

Résumé. À la fin des années 1990, Vatsal a montré qu'une congruence modulo p^ν entre deux formes modulaires implique une congruence entre leurs fonctions L p-adiques. Nous prouvons des énoncés analogues pour les fonctions L p-adiques $\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g})$ et $\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g} \otimes \mathbf{h})$ associées aux produits double et triple de formes modulaires : la première est de nature cyclotomique, tandis que l'autre est définie sur l'espace des poids.

Comme corollaire, nous obtenons des formules de transition reliant les invariants λ analytiques des représentations de Galois congruentes pour $V_\mathbf{f} \otimes V_\mathbf{g}$ et $V_\mathbf{f} \otimes V_\mathbf{g} \otimes V_\mathbf{h}$ respectivement.

Abstract. In the late 1990s, Vatsal showed that a congruence modulo p^ν between two modular forms implied a congruence between their respective p-adic L-functions. We prove an analogous statement for both the double product and triple product p-adic L-functions, $\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g})$ and $\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g} \otimes \mathbf{h})$: the former is cyclotomic in its nature, while the latter is over the weight-space. As a corollary, we derive transition formulae relating analytic λ-invariants of congruent Galois representations for $V_\mathbf{f} \otimes V_\mathbf{g}$, and for $V_\mathbf{f} \otimes V_\mathbf{g} \otimes V_\mathbf{h}$, respectively.

1. Introduction

A major theme at the Iwasawa 2019 conference was recent progress on the Iwasawa theory of motives arising from tensor products of newforms. Fix a prime $p > 2$. The principal objects at play here are:

(i) the analytic p-adic L-function which interpolates the normalised critical values, and
(ii) the algebraic p-adic L-function which is traditionally the characteristic power series of some large Selmer group.

The so-called "Main Conjecture" predicts that they are equal, up to a unit of course.

Question. *How do the analytic and algebraic λ-invariants appearing in the Main Conjecture vary as we switch between two p^ν-congruent $G_\mathbb{Q}$-representations?*

Manuscrit reçu le 26 février 2020, révisé le 16 juillet 2020, accepté le 18 septembre 2020.
2010 *Mathematics Subject Classification.* 11F33, 11F67, 11G40, 11R23.
Mots-clefs. Iwasawa theory, p-adic L-functions, automorphic forms.

We shall provide an answer for the analytic p-adic L-functions attached to double and triple product Galois representations, in certain common situations at least. The algebraic version of our transition formulae will be addressed in future work.

For a pure motive M defined over $\mathbb{Q}$ that has good ordinary reduction at p, there is a precise recipe of Coates and Perrin-Riou [3] describing the (conjectural) behaviour of its analytic p-adic L-function, $\mathbf{L}_p(M, \cdot\,, s)$, at a critical point $s = 1$. Throughout we shall tacitly fix embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$ where $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}}}_p$ denotes the Tate field, both of which are needed for p-adic interpolation. At each Dirichlet character χ of conductor p^{n_χ}, the p-adic L-function should satisfy

$$\mathbf{L}_p(M, \chi, 1) = \iota_p \circ \iota_\infty^{-1} \left(\mathcal{E}_p(M, \chi^{-1}, 1) \cdot \frac{L(M, \chi, 1)}{\Omega_\infty^{\mathrm{sign}(\chi)}(M)} \right)$$

for a suitably chosen pair of archimedean periods $\Omega_\infty^\pm(M) \in \mathbb{C}^\times$, and where the multiplier term $\mathcal{E}_p(M, \chi^{-1}, s)$ is introduced fully in (4.14) of [3] and consists of a Gauss sum, an Euler factor at p, and a power of the unit root of Frobenius.

The Main Goal. *Let* $(\mathbf{f}^{(\mathrm{I})}, \mathbf{g}^{(\mathrm{I})}, \mathbf{h}^{(\mathrm{I})})$ *and* $(\mathbf{f}^{(\mathrm{II})}, \mathbf{g}^{(\mathrm{II})}, \mathbf{h}^{(\mathrm{II})})$ *denote triples of newforms of suitable weight, character and level. We want to prove an implication*

$$\text{“}T_p(M^{(\mathrm{I})}) \equiv T_p(M^{(\mathrm{II})}) \bmod p^\nu \Longrightarrow \mathbf{L}_p(M^{(\mathrm{I})}, \cdot\,, 1) \equiv \mathbf{L}_p(M^{(\mathrm{II})}, \cdot\,, 1) \bmod p^\nu\text{”}$$

for the double product motives $M^{(\star)} = M(\mathbf{f}^{(\star)} \otimes \mathbf{g}^{(\star)})$ *and for the triple product motives* $M^{(\star)} = M(\mathbf{f}^{(\star)} \otimes \mathbf{g}^{(\star)} \otimes \mathbf{h}^{(\star)})$*, with* $T_p(\,\cdot\,)$ *denoting their p-adic realisations.*

Note for $M^{(\star)} = M(\mathbf{f}^{(\star)})$ with $\star \in \{\mathrm{I}, \mathrm{II}\}$ the above is a theorem of Vatsal [24], who established the existence of canonical periods $\Omega_\infty^\pm(M^{(\star)}) \in \mathbb{C}^\times$ such that if one normalises each $\mathbf{L}_p(M(\mathbf{f}^{(\star)}), \cdot\,)$ using his periods, the congruences hold modulo p^ν. It would therefore be worthwhile to recall Vatsal's congruences in a bit more detail, but we must outline some standard definitions and terminology first.

Let $\mathbb{Q}_{\mathrm{cyc}}$ denote the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. If one writes μ_{p^n} for the group of p^n-th roots of unity, there is a decomposition

$$G_\infty := \mathrm{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q}) \cong \mathbb{Z}_p^\times \cong \mathbb{F}_p^\times \times (1 + p\mathbb{Z}_p) \cong \Delta \times \Gamma_{\mathrm{cyc}}$$

where $\Delta := \mathrm{Gal}(\mathbb{Q}(\mu_p)/\mathbb{Q})$, and the group $\Gamma_{\mathrm{cyc}} := \mathrm{Gal}(\mathbb{Q}_{\mathrm{cyc}}/\mathbb{Q}) \cong \mathbb{Z}_p$.

For a discrete valuation ring R of residue characteristic p, let us define the (cyclotomic) Iwasawa algebras

$$\Lambda_{\mathrm{cyc}} := R[\![\Gamma]\!] = \varprojlim_{n \geqslant 1} R\big[\Gamma/\Gamma^{p^n}\big] \quad \text{and} \quad R[\![G_\infty]\!] := \Lambda_{\mathrm{cyc}}[\Delta] \cong \bigoplus_{j=0}^{p-2} R[\![\Gamma]\!]_{(\omega^j)}$$

with $\omega : \Delta \cong \mathbb{F}_p^\times \overset{\sim}{\to} \mu_{p-1}$ obtained from the Teichmüller character mod p. Now fix a topological generator γ_0 of Γ. By linearity and continuity, the mapping $\gamma_0 \mapsto X+1$ induces isomorphisms $\Lambda_{\mathrm{cyc}} \overset{\sim}{\to} R[\![X]\!]$ and $R[\![G_\infty]\!] \overset{\sim}{\to} \bigoplus_{j=0}^{p-2} R[\![X]\!]_{(\omega^j)}$.

Definition 1.1. Let ϖ be a uniformiser of R, and choose $\beta(X) \in R[\![X]\!][1/\varpi]$.

(i) If the power series $\beta(X) = \sum_{n=0}^\infty c_n(\beta) \cdot X^n$, then the integer invariant $\mu(\beta) = \mu_\varpi(\beta)$ is the largest power of ϖ such that $c_n(\beta) \in \varpi^{\mu(\beta)} \cdot R$ for all $n \geqslant 0$.

(ii) The non-negative integer $\lambda(\beta)$ equals the number of zeroes (counted with multiplicity) of $\beta(X)$, viewed as a function on the open p-adic unit disk inside $\mathbb{C}_p$. One can also take
$$\lambda(\beta) := \mathrm{rank}_{R/\varpi[\![X]\!]} \left(\frac{R[\![X]\!]}{\langle \varpi, \varpi^{-\mu(\beta)} \cdot \beta(X) \rangle} \right),$$
and both are equivalent.

Suppose we are given two newforms $\mathbf{f}^{(\mathrm{I})}$ and $\mathbf{f}^{(\mathrm{II})}$ of weight $k > 1$, character ψ, and of levels $N_{\mathbf{f}}^{(\mathrm{I})}$ and $N_{\mathbf{f}}^{(\mathrm{II})}$ respectively, such that their Fourier coefficients satisfy
$$a_n(\mathbf{f}^{(\mathrm{I})}) \equiv a_n(\mathbf{f}^{(\mathrm{II})}) \pmod{p^\nu} \text{ at each } n \in \mathbb{N} \text{ with } \gcd(n, N_{\mathbf{f}}^{(\mathrm{I})} N_{\mathbf{f}}^{(\mathrm{II})}) = 1.$$
By enlarging R if necessary, one may assume that R contains $a_n(\mathbf{f}^{(\star)})$ for all n. The following result concerns congruences between the Mazur–Tate–Teitelbaum [20] p-adic L-functions $\mathbf{L}_p(\mathbf{f}^{(\star)}, \omega^j) \in \Lambda_{\mathrm{cyc}}$, and was instrumental in Greenberg and Vatsal's subsequent work on the Iwasawa Main Conjecture for elliptic curves [12].

Vatsal's Theorem ([24, Proposition 1.7])**.** *At each ω^j-branch with $j \in \{0, \dots, p-2\}$:*

(i) $\mathbf{L}_{p,S_{\mathbf{f}}}(\mathbf{f}^{(\mathrm{I})}, \omega^j) \equiv \mathbf{L}_{p,S_{\mathbf{f}}}(\mathbf{f}^{(\mathrm{II})}, \omega^j) \mod p^\nu \cdot \Lambda_{\mathrm{cyc}}$, *and*

(ii) $\lambda(\mathbf{L}_p(\mathbf{f}^{(\mathrm{I})}, \omega^j)) = \lambda(\mathbf{L}_p(\mathbf{f}^{(\mathrm{II})}, \omega^j)) + \sum_{l \in S_{\mathbf{f}}} \mathbf{v}_l^{(\mathrm{II})}(\omega^j) - \mathbf{v}_l^{(\mathrm{I})}(\omega^j)$

where $S_{\mathbf{f}}$ consists of the primes dividing $N_{\mathbf{f}}^{(\mathrm{I})} \cdot N_{\mathbf{f}}^{(\mathrm{II})}$, and $\mathbf{v}_l^{(\star)}(\omega^j)$ denotes the λ-invariant of the power series that interpolates the Euler factor $L_l(\mathbf{f}^{(\star)} \otimes \omega^j, s)$ at a prime l.

Strictly speaking, this is not quite the statement that Vatsal proves in [24] but it is an easy exercise, involving the $S_{\mathbf{f}}$-depletions of the newforms $\mathbf{f}^{(\mathrm{I})}$ and $\mathbf{f}^{(\mathrm{II})}$, to show that it follows from his congruences (e.g. see [7, Sections 4.1–4.2] for a discussion). He also assumes irreducibility of the residual Galois representations $\bar{\rho}_{\mathbf{f}^{(\star)}}$ and the torsion-freeness of some H^1-groups, the details of which we ignore for brevity.

Emerton, Pollack and Weston [9] later generalised this construction to allow $\mathbf{f}$ to vary within a Hida family, and showed that the λ-invariant was stable along the branches of a certain Hecke algebra, $\mathbb{T}_\Sigma(\bar{\rho})$, parameterising the deformation. Recently the theory has been extended to cover anticyclotomic λ-invariants in the work of Castella, Kim and Longo [1], and also to treat non-commutative p-adic Lie extensions (with a meta-abelian structure) by the first-named author in [5, 6]. Further generalisations of Vatsal's original ideas can be found in [2, 7, 8, 19, 22].

1.1. Statement of the main results. There are three basic approaches one can take in constructing p-adic L-functions for tensor products of modular forms:

- the *Betti realisation* approach adopted by Mazur–Tate–Teitelbaum, Vatsal, and others [20, 24, 25], which utilises modular symbols;
- the *étale realisation* approach of Perrin-Riou [4, 18], which converts Euler systems directly into p-adic L-functions; or
- the *de Rham realisation* approach of Hida and Panchishkin [14, 21], which involves both the Rankin convolution and Petersson inner product.

In the Betti approach, the two main ingredients are a "mod p multiplicity-one" theorem and Ihara's Lemma. The multiplicity-one result is used to show that the μ-invariant is stable amongst families of p-congruent modular symbols, whilst Ihara's Lemma allows one to change between different level structures.

This paper follows the de Rham approach, which has the advantage of being completely explicit in nature. It also carries the disadvantage that the associated periods may not be canonical with respect to the Iwasawa Main Conjecture, hence the μ-invariants of our automorphic p-adic L-functions can sometimes be negative. Here the rôle of mod p multiplicity-one is played by holomorphic projection [13], while Ihara's Lemma is replaced with an explicit calculation involving depletions of χ-twisted modular forms (see Theorem 2.5 and Proposition 2.11, respectively).

1.1.1. *The double product.* Let $(\mathbf{f}, \mathbf{g}^{(\mathrm{I})})$ and $(\mathbf{f}, \mathbf{g}^{(\mathrm{II})})$ denote pairs of newforms of weight $(k_1, k_2) \geqslant \underline{1}$ with $k_1 > k_2$, levels $(N_{\mathbf{f}}, N_{\mathbf{g}}^{(\mathrm{I})}), (N_{\mathbf{f}}, N_{\mathbf{g}}^{(\mathrm{II})})$ respectively, and nebentypes (ψ_1, ψ_2). We also assume they are p-ordinary, i.e. $a_p(\mathbf{f}), a_p(\mathbf{g}^{(\star)}) \in \mathcal{O}_{\mathbb{C}_p}^\times$. Using the results of Hida and Panchishkin [14, 21], for each choice of $\star \in \{\mathrm{I}, \mathrm{II}\}$ there exists a p-adic L-function $\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}^{(\star)}) \in \Lambda_{\mathrm{cyc}}[\Delta][1/p]$ interpolating

$$\iota_p \circ \iota_\infty^{-1} \left(\mathcal{E}_p(\mathbf{f} \otimes \mathbf{g}^{(\star)}, \chi^{-1}, n + k_2) \cdot \frac{L(\mathbf{f} \otimes \mathbf{g}^{(\star)}, n + k_2)}{(2\pi i)^{1-k_2} \cdot \Omega_\infty(\mathbf{f})} \right)$$
$$\text{with } \Omega_\infty(\mathbf{f}) = \langle \mathbf{f}, \mathbf{f} \rangle_{\mathrm{Pet}},$$

at all integers $n \in \{0, \dots, k_1 - k_2 - 1\}$ and special characters of the form $\chi\kappa_{\mathrm{cyc}}^n$ where χ is of finite order, and $\kappa_{\mathrm{cyc}} : G_\infty \overset{\sim}{\to} \mathbb{Z}_p^\times$ is the p-th cyclotomic character.

Remark. If $\mathbf{f}_E$ is the weight two newform arising from an elliptic curve $E_{/\mathbb{Q}}$, then it is an easy exercise to show that

$$\Omega_\infty(\mathbf{f}_E) = \frac{\deg(X_0(N_{\mathbf{f}_E}) \twoheadrightarrow E)}{4\pi^2\sqrt{-1} \cdot r_E^2} \times \int_{E(\mathbb{C})^+} \omega_E \cdot \int_{E(\mathbb{C})^-} \omega_E$$

where ω_E is the differential associated to a minimal Weierstrass equation for $E_{/\mathbb{Z}}$, and $r_E \in \mathbb{Q}^\times$ denotes the Manin constant for the modular parameterisation.

Let $\rho_{\mathbf{g}^{(\star)}} : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ be the p-adic Galois representation attached to $\mathbf{g}^{(\star)}$ by the work of Deligne if $k_2 \geqslant 2$, and by Deligne–Serre if $k_2 = 1$. We assume that

$$\rho_{\mathbf{g}^{(\mathrm{I})}}\Big|_{G_{\mathbb{Q}_l}} \cong \rho_{\mathbf{g}^{(\mathrm{II})}}\Big|_{G_{\mathbb{Q}_l}} \bmod p^{\nu_2} \quad \text{at all primes } l \nmid N_{\mathbf{g}}^{(\mathrm{I})} \cdot N_{\mathbf{g}}^{(\mathrm{II})},$$

which is equivalent to saying

$$a_n(\mathbf{g}^{(\mathrm{I})}) \equiv a_n(\mathbf{g}^{(\mathrm{II})}) \bmod p^{\nu_2}$$

if $\gcd(n, N_{\mathbf{g}}^{(\mathrm{I})} N_{\mathbf{g}}^{(\mathrm{II})}) = 1$. For stupid reasons, we must also suppose that ψ_1 is trivial or a quadratic character.

Theorem 1.2. *At each branch $j \in \{0, \dots, p-2\}$, let $\mu_{\mathrm{cyc}}^{(j)}$ denote the minimum of the μ-invariants for $\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{I})}, \omega^j)$ and $\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{II})}, \omega^j)$. If $p > k_1 - 2$, then*

(i) $\mathbf{L}_{p,S_\mathbf{g}}(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{I})}, \omega^j) \equiv \mathbf{L}_{p,S_\mathbf{g}}(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{II})}, \omega^j) \mod p^{\mu_{\mathrm{cyc}}^{(j)} + \nu_2} \cdot \Lambda_{\mathrm{cyc}}$, *and*

(ii) $\lambda(\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{I})}, \omega^j)) = \lambda(\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{II})}, \omega^j)) + \sum_{l \in S_\mathbf{g}} \mathbf{e}_l^{(\mathrm{II})}(\omega^j) - \mathbf{e}_l^{(\mathrm{I})}(\omega^j)$

where $S_\mathbf{g}$ consists of the primes dividing $N_{\mathbf{g}}^{(\mathrm{I})} \cdot N_{\mathbf{g}}^{(\mathrm{II})}$, and $\mathbf{e}_l^{(\star)}(\omega^j)$ is the λ-invariant of the power series interpolating the Euler factor $L_l(\mathbf{f} \otimes \mathbf{g}^{(\star)} \otimes \omega^j, s)$ at a prime l.

There is a nice application of this result towards the Iwasawa Main Conjecture. By the work of Kings, Loeffler and Zerbes [18, Definition 3.3.2], there exist one-cocycles

$$\mathrm{Eis}_{\text{ét},b,N}^{[\mathbf{f},\mathbf{g}^{(\star)},r]} \in H^1_{\text{ét}}\left(\mathbb{Z}[1/Np], T_p(\mathbf{f} \otimes \mathbf{g}^{(\star)})^* \otimes \kappa_{\mathrm{cyc}}^{-r}\right) \qquad \text{for } 0 \leqslant r \leqslant k_2 - 2,\ b \in \mathbb{Z}/N\mathbb{Z}$$

called Rankin–Eisenstein classes, that map to a component $\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}^{(\star)}, \omega^j)$. Applying Theorem 11.6.4 of [18] which relies on the existence of these classes then outside of the critical range, one obtains a divisibility of power series

$$\mathrm{char}_{\Lambda_{\mathrm{cyc}}}\left(\widetilde{H}^2\left(\mathbb{Z}[1/S], T_p(\mathbf{f}\otimes\mathbf{g}^{(\star)})^*\otimes\Lambda_\Gamma(-j);\Delta^{(\mathbf{f})}\right)_{(\omega^j)}\right) \Big| \ \mathrm{Tw}_{1+j}(\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j))$$

where the left-hand side is described fully in Proposition 11.2.9 of [18] and arises naturally from Nekovář's theory of Selmer complexes (in fact, it is helpful to think of the $\widetilde{H}^2(\cdots)$-cohomology intuitively as being a cyclotomic Selmer group).

If we now write $\lambda^{\mathrm{alg}}(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j)$ for the λ-invariant of $\mathrm{char}_{\Lambda_{\mathrm{cyc}}}(\widetilde{H}^2(\cdots))_{(\omega^j)}$ and likewise $\lambda^{\mathrm{an}}(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j)$ for the λ-invariant of $\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j)$, then their divisibility theorem implies that $\lambda^{\mathrm{alg}}(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j)\leqslant\lambda^{\mathrm{an}}(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j)$; moreover

$$\left\{\text{zeroes of }\mathrm{char}_{\Lambda_{\mathrm{cyc}}}(\widetilde{H}^2(\cdots;\Delta^{(\mathbf{f})})_{(\omega^j)})\right\}\subset\left\{\text{zeroes of }\mathrm{Tw}_{1+j}(\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j))\right\}$$

for all $j\in\{0,\dots,p-2\}$, and at either choice of $\star\in\{\mathrm{I},\mathrm{II}\}$.

Conjecture 1.3. *At branches $j\in\{0,\dots,p-2\}$, there is a transition formula*

$$\lambda^{\mathrm{alg}}(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{I})},\omega^j)=\lambda^{\mathrm{alg}}(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{II})},\omega^j)+\sum_{l\in S_{\mathbf{g}}}\mathbf{e}_l^{(\mathrm{II})}(\omega^j)-\mathbf{e}_l^{(\mathrm{I})}(\omega^j).$$

This algebraic prediction is currently work in progress of the first-named author. Assuming its validity, one can show if the Iwasawa Main Conjecture is true for one motive, $M(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{I})})$ say, it must be true for the p^{ν_2}-congruent motive $M(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{II})})$. Unfortunately we have not yet found a method to switch between two dominant weight newforms $\mathbf{f}^{(\mathrm{I})}$ and $\mathbf{f}^{(\mathrm{II})}$, if they are congruent to each other modulo p^{ν_1}.

1.1.2. ***The triple product.*** We shall now add an extra pair of forms into the discussion: let $(\mathbf{f},\mathbf{g}^{(\mathrm{I})},\mathbf{h}^{(\mathrm{I})})$ and $(\mathbf{f},\mathbf{g}^{(\mathrm{II})},\mathbf{h}^{(\mathrm{II})})$ denote triples of newforms of weight $\underline{k}=(k_1,k_2,k_3)$, levels $(N_{\mathbf{f}},N_{\mathbf{g}}^{(\star)},N_{\mathbf{h}}^{(\star)})$ and nebentypes (ψ_1,ψ_2,ψ_3). We further suppose that these triples are p-ordinary, so that $a_p(\mathbf{f}),a_p(\mathbf{g}^{(\star)}),a_p(\mathbf{h}^{(\star)})\in\mathcal{O}_{\mathbb{C}_p}^{\times}$. There exist primitive Λ-adic families $(\mathbf{F},\mathbf{G}^{(\star)},\mathbf{H}^{(\star)})$ passing through $(\mathbf{f},\mathbf{g}^{(\star)},\mathbf{h}^{(\star)})$ at each choice of $\star\in\{\mathrm{I},\mathrm{II}\}$. For technical reasons only, we impose the conditions:

(T1) The primitive characters satisfy $\psi_1\psi_2\psi_3=\mathbf{1}$.
(T2) $\bar{\rho}_{\mathbf{F}_1}:G_{\mathbb{Q}}\to\mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is absolutely irreducible and p-distinguished;
(T3) $\gcd(N_{\mathbf{f}},N_{\mathbf{g}}^{(\star)},N_{\mathbf{h}}^{(\star)})$ is a square-free integer for both choices $\star\in\{\mathrm{I},\mathrm{II}\}$;

(T4) $\epsilon(1/2, \Pi^{(\star)}_{\underline{k},l}) = 1$ at primes $l | N_{\mathbf{f}} N^{(\star)}_{\mathbf{g}} N^{(\star)}_{\mathbf{h}}$ and unbalanced $\underline{k} = (k_1, k_2, k_3)$, where $\Pi^{(\star)}_{\underline{k}}$ is the representation attached to $\mathbf{F} \otimes \mathbf{G}^{(\star)} \otimes \mathbf{H}^{(\star)}$ at each $\underline{k}$.

These hypotheses are required in [10, 15] to guarantee the existence of a triple product p-adic L-function, $\mathbf{L}_p(\mathbf{F} \otimes \mathbf{G}^{(\star)} \otimes \mathbf{H}^{(\star)})$, interpolating the special values

$$\iota_p \circ \iota_\infty^{-1} \left(\mathcal{E}_p(\mathbf{F}_{k_1} \otimes \mathbf{G}^{(\star)}_{k_2} \otimes \mathbf{H}^{(\star)}_{k_3} \otimes \chi^{-1}_{\underline{k}}) \cdot \frac{L(\mathbf{F}_{k_1} \otimes \mathbf{G}^{(\star)}_{k_2} \otimes \mathbf{H}^{(\star)}_{k_3} \otimes \chi^{-1}_{\underline{k}}, \frac{k_1+k_2+k_3-2}{2})}{(-1)^{k_1} \cdot \Omega_\infty(\mathbf{F}_{k_1})^2} \right)$$

at $\underline{k} = (k_1, k_2, k_3)$ with $k_1 > k_2 + k_3 - 1$, where $\chi_{\underline{k}}$ is the unitarization of $\det(\Pi^{(\star)}_{\underline{k}})^{1/2}$.

Remark. To consider congruences here we will treat the following situation. Assume there exists a p-adic line $\mathcal{V}$ in the ambient weight-space for $(\mathbf{F}, \mathbf{G}^{(\star)}, \mathbf{H}^{(\star)})$, such that for all primes $l \nmid N^{(\mathrm{I})}_{\mathbf{g}} \cdot N^{(\mathrm{II})}_{\mathbf{g}} \cdot N^{(\mathrm{I})}_{\mathbf{h}} \cdot N^{(\mathrm{II})}_{\mathbf{h}}$ and unbalanced $\underline{k} = (k_1, k_2, k_3) \in \mathcal{V}$:

(i) $\rho_{\mathbf{G}^{(\mathrm{I})}_{k_2}}\Big|_{G_{\mathbb{Q}_l}} \cong \rho_{\mathbf{G}^{(\mathrm{II})}_{k_2}}\Big|_{G_{\mathbb{Q}_l}} \mod p^{\nu_2}$, and

(ii) $\rho_{\mathbf{H}^{(\mathrm{I})}_{k_3}}\Big|_{G_{\mathbb{Q}_l}} \cong \rho_{\mathbf{H}^{(\mathrm{II})}_{k_3}}\Big|_{G_{\mathbb{Q}_l}} \mod p^{\nu_3}$.

Whenever this line is parameterised by a finite flat extension $\mathbb{I}^{\mathcal{V}}$ of $\mathcal{O}_{K,p}[\![1+p\mathbb{Z}_p]\!]$, then we call $\mathcal{V}$ a *congruence line of type* (p^{ν_2}, p^{ν_3}) for the triples $(\mathbf{F}, \mathbf{G}^{(\star)}, \mathbf{H}^{(\star)})$. Let $\mathbf{L}^{\mathcal{V}}_p(\mathbf{F} \otimes \mathbf{G}^{(\star)} \otimes \mathbf{H}^{(\star)}) \in \mathbb{I}^{\mathcal{V}}$ denote the restriction of the p-adic L-function to $\mathcal{V}$.

Example. Consider two modular elliptic curves $E^{(\mathrm{I})}$ and $E^{(\mathrm{II})}$ over $\mathbb{Q}$, whose p-adic Galois representations $\rho_{E^{(\star)},p} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Z}_p)$ satisfy the congruences $\rho_{E^{(\mathrm{I})},p}\big|_{G_{\mathbb{Q}_l}} \cong \rho_{E^{(\mathrm{II})},p}\big|_{G_{\mathbb{Q}_l}} \pmod{p^{\nu_2}}$ at all prime numbers $l \nmid \mathrm{cond}_{\mathbb{Q}}(E^{(\mathrm{I})}) \cdot \mathrm{cond}_{\mathbb{Q}}(E^{(\mathrm{II})})$. Let $\mathbf{G}^{(\mathrm{I})} \in \mathbb{I}_2[\![q]\!]$ and $\mathbf{G}^{(\mathrm{II})} \in \mathbb{I}_2[\![q]\!]$ be Hida families passing through $E^{(\mathrm{I})}$ and $E^{(\mathrm{II})}$ respectively, and assume that $\mathbf{F} \in \mathbb{I}_1[\![q]\!]$ and $\mathbf{H}^{(\mathrm{I})} = \mathbf{H}^{(\mathrm{II})} \in \mathbb{I}_3[\![q]\!]$ denote arbitrary primitive $\mathbb{I}_i$-adic forms. Then we can choose our p-adic line in weight-space to be the set

$$\mathcal{V} = \{(k, 2, k-2) \,|\, k \in \mathbb{D}_{\mathbf{F}} \cap \mathbb{D}_{\mathbf{H}^{(\star)}}\}$$

where $\mathbb{D}_{\mathbf{F}} \subset \mathbb{Z}_p$ (resp. $\mathbb{D}_{\mathbf{H}^{(\star)}}$) is the disk of convergence for $\mathbf{F}$ (resp. $\mathbf{H}^{(\mathrm{I})} = \mathbf{H}^{(\mathrm{II})}$), and the specialisation map

$$\phi_{\mathcal{V}} : \mathbb{I}_1 \widehat{\otimes}_{\mathcal{O}_{K,p}} \mathbb{I}_2 \widehat{\otimes}_{\mathcal{O}_{K,p}} \mathbb{I}_3 \twoheadrightarrow \mathbb{I}^{\mathcal{V}}$$

is induced by sending $(X_1, X_2, X_3) \mapsto (X_{\mathcal{V}}, 0, \frac{X_{\mathcal{V}}+1}{(1+p)^2} - 1)$.

As it is non-standard, we should define the (weight) λ-invariant in this context. Since $\mathbb{I}^{\mathcal{V}}$ is a finite extension of $\Lambda_{\rm wt} := \mathcal{O}_{K,p}[\![1+p\mathbb{Z}_p]\!] \cong \mathcal{O}_{K,p}[\![X]\!]$, one can consider its normal closure $\mathbb{I}^{\mathcal{V},\rm cl}$ and the field of fractions $\mathcal{K}^{\mathcal{V}} = \mathrm{Frac}(\mathbb{I}^{\mathcal{V},\rm cl})$. We then define

$$\lambda^{\rm wt}(\beta) := \left[\mathcal{K}^{\mathcal{V}} : \mathcal{F}_{\rm wt}\right]^{-1} \times \left(\text{the number of zeroes of } \prod_{\sigma\in \mathrm{Gal}(\mathcal{K}^{\mathcal{V}}/\mathcal{F}_{\rm wt})} \beta^{\sigma}\right)$$

for each $\beta \in \mathbb{I}^{\mathcal{V}}$, where $\mathcal{F}_{\rm wt}$ is the field of fractions of $\Lambda_{\rm wt}$ (note $\prod_\sigma \beta^\sigma \in \mathcal{O}_{K,p}[\![X]\!]$). Let us denote by $\mu_{\rm wt}^{(\mathcal{V})}$ the minimum value of the weight μ-invariant amongst the two p-adic L-functions, namely $\mathbf{L}_p^{\mathcal{V}}(\mathbf{F}\otimes\mathbf{G}^{(\mathrm{I})}\otimes\mathbf{H}^{(\mathrm{I})})$ and $\mathbf{L}_p^{\mathcal{V}}(\mathbf{F}\otimes\mathbf{G}^{(\mathrm{II})}\otimes\mathbf{H}^{(\mathrm{II})})$.

Theorem 1.4. *If the weights $\underline{k} = (k_1,k_2,k_3)$ satisfying $k_1 > k_2+k_3-1$ and $p \nmid \frac{(k_1-2)!}{(\frac{k_1+k_2+k_3}{2}-2)!}$ are dense in $\mathrm{Spec}(\mathbb{I}^{\mathcal{V}})$, and if ψ_1 is trivial or quadratic, then*

(i) $\mathbf{L}_{p,S_{\mathbf{g},\mathbf{h}}}^{\mathcal{V}}(\mathbf{F}\otimes\mathbf{G}^{(\mathrm{I})}\otimes\mathbf{H}^{(\mathrm{I})}) \equiv \mathbf{L}_{p,S_{\mathbf{g},\mathbf{h}}}^{\mathcal{V}}(\mathbf{F}\otimes\mathbf{G}^{(\mathrm{II})}\otimes\mathbf{H}^{(\mathrm{II})}) \bmod p^{\mu_{\rm wt}^{(\mathcal{V})}+\min\{\nu_2,\nu_3\}}$, *and*

(ii) $\lambda^{\rm wt}(\mathbf{L}_p^{\mathcal{V}}(\mathbf{F}\otimes\mathbf{G}^{(\mathrm{I})}\otimes\mathbf{H}^{(\mathrm{I})})) = \lambda^{\rm wt}(\mathbf{L}_p^{\mathcal{V}}(\mathbf{F}\otimes\mathbf{G}^{(\mathrm{II})}\otimes\mathbf{H}^{(\mathrm{II})})) + \sum_{l\in S_{\mathbf{g},\mathbf{h}}} \mathbf{w}_{l,\mathcal{V}}^{(\mathrm{II})} - \mathbf{w}_{l,\mathcal{V}}^{(\mathrm{I})}$

where $S_{\mathbf{g},\mathbf{h}}$ consists of primes dividing $N_{\mathbf{g}}^{(\mathrm{I})}\cdot N_{\mathbf{g}}^{(\mathrm{II})}\cdot N_{\mathbf{h}}^{(\mathrm{I})}\cdot N_{\mathbf{h}}^{(\mathrm{II})}$, and $\mathbf{w}_{l,\mathcal{V}}^{(\star)}$ is the $\lambda^{\rm wt}$-invariant for the $\mathbb{I}^{\mathcal{V}}$-adic factor $L_l(\mathbf{F}_{k_1}\otimes\mathbf{G}_{k_2}^{(\star)}\otimes\mathbf{H}_{k_3}^{(\star)}\otimes\chi_{\underline{k}}^{-1}, \frac{k_1+k_2+k_3-2}{2})|_{\underline{k}\in\mathcal{V}}$.

As discussed in the above example, a good source of these congruence lines $\mathcal{V}$ is given by specialising $\mathbf{G}^{(\star)}$ at a fixed weight k_2 at which there exists a mod p^{ν_2} congruence between $\mathbf{G}_{k_2}^{(\mathrm{I})}$ and $\mathbf{G}_{k_2}^{(\mathrm{II})}$, and taking the weights (k_1,k_2,k_1-k_2) with k_1 denoting the free variable. One can therefore obtain congruences between the p-adic L-functions $\mathbf{L}_{p,S_{\mathbf{g},\mathbf{h}}}(\mathbf{F}_{k_1}\otimes\mathbf{G}_{k_2}^{(\mathrm{I})}\otimes\mathbf{H}_{k_1-k_2})$ and $\mathbf{L}_{p,S_{\mathbf{g},\mathbf{h}}}(\mathbf{F}_{k_1}\otimes\mathbf{G}_{k_2}^{(\mathrm{II})}\otimes\mathbf{H}_{k_1-k_2})$. By symmetry, the same thing works when the rôles of $\mathbf{G}^{(\star)}$ and $\mathbf{H}^{(\star)}$ are reversed.

The reader will notice that there is no cyclotomic variable appearing here, although by recent work of Hsieh and Yamana on exceptional p-adic zeroes [16], this extra variable can certainly be introduced. The techniques presented in our paper should carry over to the four-variable (quaternionic) setting, thereby enabling us to prove transition formulae for the cyclotomic λ-invariant at balanced $(k_1,k_2,k_3)\in\mathcal{V}$.

We should also mention the results of Darmon, Rotger and others, which relate specialisations of $\mathbf{L}_p(\mathbf{F}\otimes\mathbf{G}^{(\star)}\otimes\mathbf{H}^{(\star)})$ to generalised Kato classes [4] in global Galois cohomology with coefficients in $T_p(\mathbf{F}_{k_1}\otimes\mathbf{G}_{k_2}^{(\star)}\otimes\mathbf{H}_{k_3}^{(\star)})$. In particular, at weight $(k_1, k_2, k_3) = (2, 1, 1)$ they obtain the key information on the Birch and Swinnerton-Dyer Conjecture for elliptic curves E. Therefore given the existence of a congruence line $\mathcal{V}$ of type (p^{ν_2}, p^{ν_3}) containing $(2, 1, 1)$ as a point, one could use a balanced version of Theorem 1.4 to produce non-trivial congruences between the values of $L(E, \rho_2^{(\mathrm{I})} \otimes \rho_3^{(\mathrm{I})}, s)$ and $L(E, \rho_2^{(\mathrm{II})} \otimes \rho_3^{(\mathrm{II})}, s)$ at $s = 1$, for twists by degree four Artin representations $\rho_2^{(\star)} \otimes \rho_3^{(\star)}$ which are self-dual and congruent.

1.1.3. ***Brief plan of the paper.*** In Section 2 we study projections of $\mathcal{C}^\infty$-modular forms of the type $\mathbf{g} \cdot \delta_w^{(r)}(\mathbf{h})$, where the differential operator $\delta_w = \frac{1}{2\pi i}\left(\frac{w}{2iy} + \frac{\partial}{\partial z}\right)$. If $\mathbf{h}$ is an Eisenstein series then these projections are related to double products, while if $\mathbf{h}$ is a cuspidal eigenform then they are essentially triple product L-values. In Section 3, by writing these critical values in terms of a linear functional $\mathcal{L}_{\mathbf{f}}^{(r,\varepsilon)}(\,\cdot\,)$ acting on the space of nearly holomorphic forms, one can then read off congruences amongst the L-values in terms of congruences between the original modular forms. This is an ad hoc approach and we apologise in advance for the very ugly formulae!

Conventions. We employ the following terminology throughout this article:

- If $\chi : \mathbb{Z} \to \mathbb{C}$ is any Dirichlet character, then we write $\chi_{(p)}$ for its p-part and similarly we use $\chi^{(p)}$ to denote its non-p-part, so that $\chi = \chi_{(p)} \cdot \chi^{(p)}$;
- If F is a number field or local field then $\mathcal{O}_F$ will be its ring of integers, and we say that two expansions $H, H^\dagger \in \mathcal{O}_F[\![q]\!]$ are congruent modulo p^ν if their q^n-coefficients satisfy $a_n(H) \equiv a_n(H^\dagger) \bmod p^\nu$ for every $n \geqslant 0$;
- If $\mathbb{I}$ denotes the normal closure of $\Lambda_{\mathrm{wt}} := \mathcal{O}_K[\![1 + p\mathbb{Z}_p]\!]$ inside of $\mathrm{Frac}(\Lambda_{\mathrm{wt}})$, then we assume $K/\mathbb{Q}_p$ is chosen large enough to ensure $\mathbb{I} \cap \overline{\mathbb{Q}}_p = \mathcal{O}_K$, and that the algebraic points $\mathrm{Spec}\,\mathbb{I}(\mathcal{O}_K)^{\mathrm{alg}}$ are Zariski dense in $\mathrm{Spec}\,\mathbb{I}(\overline{\mathbb{Q}}_p)$;
- For an integer $N \geqslant 1$ coprime to p and a Dirichlet character χ modulo N, we use $\mathbb{T}^{\mathrm{ord}}(N, \chi; \mathbb{I})$ to indicate the Hecke algebra acting on $\mathcal{S}^{\mathrm{ord}}(N, \chi; \mathbb{I})$, the space of ordinary $\mathbb{I}$-adic cusp forms of tame level N and character χ.

Acknowledgements. The first-named author was largely inspired by a series of talks given by Victor Rotger and Shunsuke Yamana at Iwasawa 2019 in Bordeaux. He also thanks the local conference organisers, Denis Benois and Pierre Parent, for their hospitality. The second author is supported by a University of Waikato PhD Scholarship and this paper forms a part of his PhD dissertation [11].

2. A lowbrow study of Petersson inner products

Let F_1, G_2, G_3 be modular forms of levels N_1, N_2, N_3, weights $k_1, k_2, k_3 > 0$ and nebentypes ψ_1, ψ_2, ψ_3 respectively. We shall assume that F_1 and G_2 are cusp forms, that the primitive characters satisfy $\psi_2 \cdot \psi_3 = \psi_1^{-1}$, and thirdly that $k_1 > k_2 + k_3 - 1$. Our main goal here is to derive an explicit expression for quotients of the type

$$\frac{\langle F_1^{\sharp}, \mathrm{Tr}_{\widetilde{N}/N_0}(\mathrm{Hol}_\infty(G_2 \cdot \delta_w^{(r)}(G_3))\big|_{k_1} W_{\widetilde{N}}^{\varepsilon})\rangle_{N_0}}{\langle F_1, F_1\rangle_{N_1}}, \quad \varepsilon \in \{0,1\} \tag{2.1}$$

where the various operators, levels and inner products above will be defined shortly (the precise formulae for these ratios will be given in Propositions 2.12 and 2.13). We need to study these projections in some detail, as the critical values of both the double and triple product L-functions can be represented via integrals of this type.

2.1. Preliminaries on modular forms. We begin with some terminology. Choose an integer $N \geqslant 1$, a Dirichlet character ψ modulo N, and a weight $k > 0$. One writes $\mathcal{M}_k(N, \psi)$ for the vector space of modular forms of weight k, level N and character ψ, while the notation $\mathcal{S}_k(N, \psi)$ refers to the subspace of cusp forms. If $F, G \in \mathcal{M}_k(N, \psi)$ one of which is a cusp form, then we normalise the Petersson inner product[1] by taking

$$\langle F, G\rangle_N := \int_{\Gamma_0(N)\backslash\mathfrak{H}} \overline{F(z)} G(z) y^k \cdot \frac{\mathrm{d}x\mathrm{d}y}{y^2}.$$

Here $\mathfrak{H}$ denotes the standard upper half-plane $\{z = x+iy \in \mathbb{C} \mid y = \mathrm{Im}(z) > 0\}$. An advantage of making this choice is that if $M|N$ and F exists at level M, then

$$\langle F, G\rangle_N = \langle F, \mathrm{Tr}_M^N(G)\rangle_M$$

where the trace mapping $\mathrm{Tr}_M^N : \mathcal{M}_k(N, \psi) \to \mathcal{M}_k(M, \psi)$ is given by the summation

$$\mathrm{Tr}_M^N(G) := \sum_{\gamma \in \Gamma_0(N)\backslash\Gamma_0(M)} \bar{\psi}(\gamma) \cdot G\Big|_k \gamma.$$

Recall the three modular forms F_1, G_2, G_3 of levels N_1, N_2, N_3 mentioned above.

Notations.

(a) For each $i \in \{1,2,3\}$, we factorise the level into $N_i = p^{e_i} \cdot N_i^{(p)}$ with $e_i = \mathrm{ord}_p(N_i)$, and where $N_i^{(p)}$ is the corresponding tame (prime-to-p) level.

[1] This normalisation differs from [23, Section 2] in that we do not divide by the volume of a fundamental domain for $\Gamma_0(N)\backslash\mathfrak{H}$, which means that our inner product will be level-dependent.

(b) We set $\widetilde{N} := \mathrm{lcm}(N_1, N_2, N_3)$, which one decomposes into $\widetilde{N} = p^{\tilde{e}} \cdot \widetilde{N}^{(p)}$.

(c) Lastly let us choose $N_0 := p \cdot \mathrm{lcm}(N_1^{(p)}, N_2^{(p)}, N_3^{(p)}) = p^{1-\tilde{e}} \cdot \widetilde{N} \in p \cdot \mathbb{Z}_p^{\times}$.

Note that F_1 belongs to $\mathcal{S}_{k_1}(N_1, \psi_1)$ with q-expansion $F_1(q) = \sum_{n=1}^{\infty} a_n(F_1) q^n$, so there exists a conjugate form $F_1^{\sharp} \in \mathcal{S}_{k_1}(N_1, \psi_1^{-1})$ with $F_1^{\sharp}(q) = \sum_{n=1}^{\infty} \overline{a_n(F_1)} q^n$. We shall further suppose that F_1 is a newform of conductor N_1, so that

$$F_1\Big|_{k_1} W_{N_1} = \epsilon_1 \cdot F_1^{\sharp} \quad \text{where } W_{N_1} = \begin{pmatrix} 0 & -1 \\ N_1 & 0 \end{pmatrix} \text{ and } \epsilon_1 \in \mathbb{C},\ \big|\epsilon_1\big|_{\infty} = 1.$$

For simplicity, throughout this paper we assume that $F_1^{\sharp} = F_1$ *and* $\psi_1^2 = \mathbf{1}$.

Let us write $V_d : \sum a_n q^n \mapsto \sum a_n q^{nd}$ for the d-th degeneracy mapping, and as usual $U_p : \sum a_n q^n \mapsto \sum a_{pn} q^n$ means the p-th Hecke operator if p divides the level.

Lemma 2.1. *If* $p \nmid N_1$ *so that* $e_1 = 0$, *then for an arbitrary* $G \in \mathcal{M}_{k_1}(\widetilde{N}, \psi_1)$,

$$\Big\langle F_1^{\sharp}, \mathrm{Tr}_{N_0}^{\widetilde{N}}(G) \Big\rangle_{N_0} = \epsilon_1 p^{1 - \frac{(k_1-2)(\tilde{e}-2)}{2}} \left(\frac{\widetilde{N}^{(p)}}{N_1} \right)^{\frac{k_1}{2}} \cdot \sum_{d \left| \frac{N_0}{N_1} \right.} \mathfrak{c}_{d, \widetilde{N}, \tilde{e}}(G) \cdot \Big\langle F_1\big|_{k_1} V_{\frac{N_0}{N_1}}, F_1\big|_{k_1} V_d \Big\rangle_{N_0}$$

where each form $G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1} \in \mathcal{M}_{k_1}(N_0, \psi_1)$ *has been decomposed into a sum*

$$G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1} = \sum_{d \left| \frac{N_0}{N_1} \right.} \mathfrak{c}_{d, \widetilde{N}, \tilde{e}}(G) \cdot F_1\big|_{k_1} V_d + G_{\widetilde{N}, \tilde{e}}^{(\perp)} \quad \text{for scalars } \mathfrak{c}_{d, \widetilde{N}, \tilde{e}}(G) \in \mathbb{C},$$

and here the modular form $G_{\widetilde{N}, \tilde{e}}^{(\perp)}$ *is obtained by projecting* $G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1}$ *onto the orthogonal complement of the* F_1*-isotypic subspace inside* $\mathcal{M}_{k_1}(N_0, \psi_1)$.

Proof. As the ratio $\widetilde{N}/N_0 = p^{\tilde{e}-1}$ is a power of p and $p|N_0$, one deduces that

$$\mathrm{Tr}_{N_0}^{\widetilde{N}}(G) = p^{(1-k_1/2)(\tilde{e}-1)} \times G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1} \circ W_{N_0}.$$

Applying this standard identity to our inner product:

$$\left\langle F_1^\sharp, \mathrm{Tr}^{\widetilde{N}}_{N_0}(G)\right\rangle_{N_0} = p^{(1-k_1/2)(\tilde{e}-1)} \times \left\langle F_1^\sharp, G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1} \circ W_{N_0}\right\rangle_{N_0}$$
$$= (-1)^{k_1} p^{(1-k_1/2)(\tilde{e}-1)} \times \left\langle F_1^\sharp\big|_{k_1} W_{N_0}, G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1}\right\rangle_{N_0}$$
$$= (-1)^{k_1} p^{1-\frac{(k_1-2)(\tilde{e}-2)}{2}} \left(\frac{\widetilde{N}^{(p)}}{N_1}\right)^{\frac{k_1}{2}} \times \left\langle F_1^\sharp\big|_{k_1} W_{N_1} \circ V_{p\cdot\frac{\widetilde{N}^{(p)}}{N_1}}, G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1}\right\rangle_{N_0}$$

and the last line follows because $(\,\cdot\,)\Big|_{k_1} W_{N_0} = (p\cdot\frac{\widetilde{N}^{(p)}}{N_1})^{k_1/2}\cdot(\,\cdot\,)\Big|_{k_1} W_{N_1} \circ V_{p\cdot\frac{\widetilde{N}^{(p)}}{N_1}}$. However $F_1^\sharp\big|_{k_1} W_{N_1} = \overline{\epsilon_1}\cdot(-1)^{k_1} \times F_1$ and also $p\cdot\frac{\widetilde{N}^{(p)}}{N_1} = \frac{N_0}{N_1}$, in which case

$$\left\langle F_1^\sharp, \mathrm{Tr}^{\widetilde{N}}_{N_0}(G)\right\rangle_{N_0} = \epsilon_1 p^{1-\frac{(k_1-2)(\tilde{e}-2)}{2}} \left(\frac{\widetilde{N}^{(p)}}{N_1}\right)^{\frac{k_1}{2}} \times \left\langle F_1\big|_{k_1} V_{\frac{N_0}{N_1}}, G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1}\right\rangle_{N_0}.$$

Finally our assumption that $F_1^\sharp = F_1$ implies that the F_1-isotypic subspace inside $\mathcal{M}_{k_1}(N_0,\psi_1)$ is spanned by the normalised eigenforms $F_1\big|_{k_1} V_d$ as d runs through the divisors of N_0/N_1; we may therefore write

$$G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1} = \sum_{d\left|\frac{N_0}{N_1}\right.} \mathfrak{c}_{d,\widetilde{N},\tilde{e}}(G)\cdot F_1\big|_{k_1} V_d + G^{(\perp)}_{\widetilde{N},\tilde{e}}$$

for the particular choice of scalars, $\mathfrak{c}_{d,\widetilde{N},\tilde{e}}(G)$, obtained by projecting $G\big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1}$ onto each basis element $F_1\big|_{k_1} V_d$. Since the modular form $G^{(\perp)}_{\widetilde{N},\tilde{e}}$ is orthogonal to $F_1\big|_{k_1} V_{\frac{N_0}{N_1}}$ under the Petersson inner product at level N_0, the result now follows. □

2.2. Expansions of nearly holomorphic functions. The strategy over the next two sections is to show for $G_2 \in \mathcal{S}_{k_2}(N_2,\psi_2)$ and $G_3 \in \mathcal{M}_{k_3}(N_3,\psi_3)$ as before, that the modular forms

$$\mathrm{Hol}_\infty(G_2\cdot\delta^{(r)}_{k_1-k_2-2r}(G_3)) \quad \text{with } r=(k_1-k_2-k_3)/2 \in \mathbb{Z}_{\geqslant 0}$$

behave well under mod p^ν congruences, in the sense that if we replace G_2 and G_3 by p^ν-congruent forms then $\mathrm{Hol}_\infty((\,\cdot\,)\cdot\delta^{(r)}_{k_1-k_2-2r}(\,\cdot\,))$ preserves these congruences. We first recall properties of the Maass–Shimura differential operator "$\delta^{(r)}_w$" from [23], and then in Section 2.3 we give some background on the projection mapping "Hol_∞".

Let $w, r \geqslant 0$ be integers, and consider the operator $\delta_w := \frac{1}{2\pi i}(\frac{w}{2iy} + \frac{\partial}{\partial z})$ where as usual $\frac{\partial}{\partial z} = \frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})$ for all $z = x + iy$. One can take an r-fold composition

$$\delta_w^{(r)} := \delta_{w+2r-2} \circ \cdots \circ \delta_{w+2} \circ \delta_w$$

with the convention that if $r = 0$, then $\delta_w^{(0)}$ just refers to the identity operator.

If G is a holomorphic modular form of weight w, level N and character ψ, then $\delta_w^{(r)}(G)$ has weight $w + 2r$, level N and character ψ although it may no longer be holomorphic; in fact $\delta_w^{(r)}(G)$ is an element of $\{\sum_{j=0}^{r} y^{-j} \cdot h_j \,|\, h_j \text{ is holomorphic}\}$. It follows that $\delta_w^{(r)}(G) \in \mathcal{C}^\infty(\mathfrak{H})$ belongs to the larger space of $\mathcal{C}^\infty$-modular forms, denoted by $\mathcal{M}_{w+2r}^\infty(\Gamma(N))$, and exhibits "moderate growth" in the sense of [13, 21]. Specifically, a form $H \in \mathcal{M}_w^\infty(\Gamma(N))$ is said to have *moderate growth at* $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ if for all $z \in \mathfrak{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$, the complex integrals

$$(2.2) \quad \int_{\tau = u+iv \in \mathfrak{H}} (H\big|_w \gamma)(\tau) \cdot (\overline{\tau} - z)^{-w-2r} \big|\overline{\tau} - z\big|_\infty^{-2s} (\mathrm{Im}(\tau))^{w+2r+s} \cdot \frac{\mathrm{d}u\mathrm{d}v}{v^2}$$

are absolutely convergent, and admit an analytic continuation to the point $s = 0$.

Definition 2.2. Let $R \subset \mathbb{C}$ be a commutative ring, and $\mathfrak{p} \lhd R$ a prime ideal.

(i) For each $t \geqslant 0$, denote by $\mathcal{N}_{w,\mathrm{pol}}^{\infty,t}(\Gamma(N); R)$ the R-submodule of $\mathcal{M}_w^\infty(\Gamma(N))$ consisting of $\mathcal{C}^\infty$-modular forms, $H(z)$, with Fourier expansions of the type

$$H(z) = \sum_{m \in N^{-1}\mathbb{Z}} e^{-2\pi m y} \cdot \mathcal{P}_H\left(\frac{1}{4\pi y}, m\right) \cdot e^{2\pi i m x}$$

where $z = x + iy \in \mathfrak{H}$ and for all $m \in N^{-1}\mathbb{Z}$, the coefficient terms $\mathcal{P}_H(X, m) \in R[X]$ satisfy $\deg(\mathcal{P}_H) \leqslant t$.

(ii) We similarly define $\mathcal{N}_{w,\mathrm{pol}}^{\infty,t}(N, \psi; R) := \mathcal{N}_{w,\mathrm{pol}}^{\infty,t}(\Gamma(N); R) \cap \mathcal{M}_w^\infty(N, \psi)$.

(iii) If $H(z), H^\dagger(z) \in \mathcal{N}_{w,\mathrm{pol}}^{\infty,t}(\Gamma(N); R)$ and there exists $\nu \geqslant 1$ such that

$$\mathcal{P}_H(X, m) - \mathcal{P}_{H^\dagger}(X, m) \in \mathfrak{p}^\nu \cdot R[X] \quad \text{for every } m \in N^{-1}\mathbb{Z},$$

then we say that H is congruent to $H^\dagger$ modulo $\mathfrak{p}^\nu$, and write $H \equiv H^\dagger \pmod{\mathfrak{p}^\nu \cdot R}$.

For example, if $R = \mathcal{O}_K$ is the ring of integers of some number field K and if one considers a classical form $G = \sum_{n=0}^\infty a_n(G) q^n \in \mathcal{M}_w(N, \psi) \cap \mathcal{O}_K[\![q]\!]$, then clearly $\mathcal{P}_G(X, m) = a_m(G)$ if $m \in \mathbb{Z}_{\geqslant 0}$, while $\mathcal{P}_G(X, m) = 0$ if $m \notin \mathbb{Z}_{\geqslant 0}$. We therefore have a natural containment $\mathcal{M}_w(N, \psi) \cap \mathcal{O}_K[\![q]\!] \subset \mathcal{N}_{w,\mathrm{pol}}^{\infty,0}(N, \psi; \mathcal{O}_K)$. Furthermore, the definition of mod $\mathfrak{p}^\nu$-congruent forms introduced above generalises the standard notion of modulo $\mathfrak{p}^\nu$ congruences used for series expansions in $\mathcal{O}_K[\![q]\!]$.

Lemma 2.3.

(a) *For a commutative ring R as above, the differential operator $\delta_w^{(r)}$ sends the nearly holomorphic forms $\mathcal{N}_{w,\mathrm{pol}}^{\infty,t}(\Gamma(N);R)$ into $\mathcal{N}_{w+2r,\mathrm{pol}}^{\infty,t+r}(\Gamma(N);R)$, and by restriction sends $\mathcal{N}_{w,\mathrm{pol}}^{\infty,t}(N,\psi;R)$ into $\mathcal{N}_{w+2r,\mathrm{pol}}^{\infty,t+r}(N,\psi;R)$.*

(b) *If $H(z), H^\dagger(z) \in \mathcal{M}_w(N,\psi)$ are $\mathfrak{p}^\nu$-congruent forms with R-coefficients, then one also obtains congruences*

$$\delta_w^{(r)}(H) \equiv \delta_w^{(r)}(H^\dagger) \pmod{\mathfrak{p}^\nu \cdot R}$$

at all integers $r \geqslant 0$, in the spirit of Definition 2.2(iii).

Proof. Let us deal with part (a) first. Recall from [18] that a $\mathcal{C}^\infty$-modular form $G(z) \in \mathcal{M}_w^\infty(\Gamma(N))$ can be always expanded as a Fourier series of the type

$$G(z) = \sum_{m\in N^{-1}\mathbb{Z}} A_G(y,m)\cdot e^{2\pi imx} \quad \text{with } z = x+iy,$$

and each term $A_G(y,m) \in \mathcal{C}^\infty(\mathbb{R}^+)$. Applying the operator $\frac{\partial}{\partial z}$ to $G(z)$ then yields

$$\frac{\partial G(z)}{\partial z} = \sum_{m\in N^{-1}\mathbb{Z}} \left(m\pi i\cdot A_G(y,m) - \frac{i}{2}A_G'(y,m)\right)\cdot e^{2\pi imx}$$

with $A_G'(y,m) = \frac{\mathrm{d}A_G(y,m)}{\mathrm{d}y}$, so that as an element of $\mathcal{M}_{w+2}^\infty(\Gamma(N))$ we find that

$$\delta_w(G(z)) = \sum_{m\in N^{-1}\mathbb{Z}} \left(\left(\frac{m}{2} - \frac{w}{4\pi y}\right)\cdot A_G(y,m) - \frac{1}{4\pi}A_G'(y,m)\right)\cdot e^{2\pi imx}.$$

In the specific situation with $G \in \mathcal{N}_{w,\mathrm{pol}}^{\infty,t}(\Gamma(N);R)$, one can further write

$$A_G(y,m) = e^{-2\pi my}\cdot \mathcal{P}_G\left(\frac{1}{4\pi y}, m\right) \quad \text{where } \mathcal{P}_G(X,m) = \sum_{j=0}^{t} \beta_j(m)\cdot X^j \in R[X].$$

A straightforward calculation reveals that

$$A_G'(y,m) = -2\pi e^{-2\pi my}\cdot\left(\sum_{j=0}^{t} m\beta_j(m)\cdot(4\pi y)^{-j} + 2\cdot\sum_{j=1}^{t} j\beta_j(m)\cdot(4\pi y)^{-j-1}\right),$$

in which case

$$
\begin{aligned}
&\delta_w(G(z)) \\
&= \sum_{m\in N^{-1}\mathbb{Z}} \left(\left(\frac{m}{2}-\frac{w}{4\pi y}\right)\cdot e^{-2\pi m y}\mathcal{P}_G\left(\frac{1}{4\pi y},m\right)-\frac{1}{4\pi}A'_G(y,m)\right)\cdot e^{2\pi i m x} \\
&= \sum_{m\in N^{-1}\mathbb{Z}} e^{-2\pi m y}\cdot\left(m\beta_0(m)+\sum_{j=1}^{t}(m\beta_j(m)+(j-1-w)\beta_{j-1}(m))\cdot(4\pi y)^{-j}\right. \\
&\qquad\qquad\left. + (t-w)\beta_t(m)\cdot(4\pi y)^{-t-1}\right)\cdot e^{2\pi i m x}.
\end{aligned}
$$

Consequently for every $m\in N^{-1}\mathbb{Z}$, we set $\mathcal{P}_{\delta_w(G)}(X,m)$ equal to the polynomial

$$
m\beta_0(m)+\sum_{j=1}^{t}(m\beta_j(m)+(j-1-w)\beta_{j-1}(m))\cdot X^j+(t-w)\beta_t(m)\cdot X^{t+1}
$$

so in particular, $\mathcal{P}_{\delta_w(G)}(X,m)\in R[X]$ with $\deg(\mathcal{P}_{\delta_w(G)})\leqslant t+1$, hence

$$
\delta_w(G(z))=\sum_{m\in N^{-1}\mathbb{Z}} e^{-2\pi m y}\cdot\mathcal{P}_{\delta_w(G)}\left(\frac{1}{4\pi y},m\right)\cdot e^{2\pi i m x}\in\mathcal{N}^{\infty,t+1}_{w+2,\mathrm{pol}}(\Gamma(N);R).
$$

It follows that $\delta_w:\mathcal{N}^{\infty,t}_{w,\mathrm{pol}}((\Gamma(N);R)\to\mathcal{N}^{\infty,t+1}_{w+2,\mathrm{pol}}(\Gamma(N);R)$, and then applying an inductive argument to $\delta_w^{(r)}=\delta_{w+2r-2}\circ\cdots\circ\delta_{w+2}\circ\delta_w$ for increasing values of $r>0$, we conclude that $\delta_w^{(r)}:\mathcal{N}^{\infty,t}_{w,\mathrm{pol}}(\Gamma(N);R)\to\mathcal{N}^{\infty,t+r}_{w+2r,\mathrm{pol}}(\Gamma(N);R)$ as asserted in (a).

To show that statement (b) is true, let us in greater generality suppose that:

$$
H(z)=\sum_{m\in\mathbb{Z}} e^{-2\pi m y}\cdot\mathcal{P}_H\left(\frac{1}{4\pi y},m\right)\cdot e^{2\pi i m x},\qquad \mathcal{P}_H(X,m)=\sum_{j=0}^{t}\beta_j(m)\cdot X^j;
$$

$$
H^\dagger(z)=\sum_{m\in\mathbb{Z}} e^{-2\pi m y}\cdot\mathcal{P}_{H^\dagger}\left(\frac{1}{4\pi y},m\right)\cdot e^{2\pi i m x},\quad \mathcal{P}_{H^\dagger}(X,m)=\sum_{j=0}^{t}\beta_j^\dagger(m)\cdot X^j.
$$

The condition $H\equiv H^\dagger\ (\mathrm{mod}\,\mathfrak{p}^\nu\cdot R)$ is by definition equivalent to the family of congruences $\beta_j(m)\equiv\beta_j^\dagger(m)\ (\mathrm{mod}\,\mathfrak{p}^\nu\cdot R)$ for every $m\in\mathbb{Z}$ and $j\in\{0,\ldots,t\}$. Adopting the same argument as in part (a), it directly follows that

$$
\delta_w(H(z))=\sum_{m\in\mathbb{Z}} e^{-2\pi m y}\cdot\mathcal{P}_H^\delta\left(\frac{1}{4\pi y},m\right)\cdot e^{2\pi i m x},\mathcal{P}_H^\delta(X,m)=\sum_{j=0}^{t+1}\beta_j^\delta(m)\cdot X^j
$$

where

$$\beta_j^\delta(m) = \begin{cases} (t-w)\beta_t(m) & \text{if } j = t+1 \\ m\beta_j(m) + (j-1-w)\beta_{j-1}(m) & \text{if } 0 < j < t+1 \\ m\beta_0(m) & \text{if } j = 0. \end{cases}$$

Likewise for the second Fourier expansion,

$$\delta_w(H^\dagger(z)) = \sum_{m\in\mathbb{Z}} e^{-2\pi my} \cdot \mathcal{P}^\delta_{H^\dagger}\left(\frac{1}{4\pi y}, m\right) \cdot e^{2\pi imx},$$

$$\mathcal{P}^\delta_{H^\dagger}(X, m) = \sum_{j=0}^{t+1} \beta_j^{\dagger,\delta}(m) \cdot X^j$$

where

$$\beta_j^{\dagger,\delta}(m) = \begin{cases} (t-w)\beta_t^\dagger(m) & \text{if } j = t+1 \\ m\beta_j^\dagger(m) + (j-1-w)\beta_{j-1}^\dagger(m) & \text{if } 0 < j < t+1 \\ m\beta_0^\dagger(m) & \text{if } j = 0. \end{cases}$$

The implication "$\beta_j(m) \equiv \beta_j^\dagger(m) \pmod{\mathfrak{p}^\nu} \implies \beta_j^\delta(m) \equiv \beta_j^{\dagger,\delta}(m) \pmod{\mathfrak{p}^\nu}$" is now obvious since the indices $m, j, w, t \in \mathbb{Z}$, whence $\delta_w(H) \equiv \delta_w(H^\dagger)$ $(\mathrm{mod}\, \mathfrak{p}^\nu \cdot R)$. Finally, recalling that $\delta_w^{(r)} = \delta_{w+2r-2} \circ \cdots \circ \delta_{w+2} \circ \delta_w$ and iterating this process above $(r-1)$-times more, one establishes that $\delta_w^{(r)}(H) \equiv \delta_w^{(r)}(H^\dagger)$ $(\mathrm{mod}\, \mathfrak{p}^\nu \cdot R)$. $\square$

2.3. Projecting Eisenstein series and cusp forms. Proceeding further with our calculation of the inner product in (2.1), we shall require some background on the operator "$\mathrm{Hol}_\infty(\,\cdot\,)$" which appears in the automorphic theory.

Throughout G_2 is a cusp form of weight k_2, level N_2 and character ψ_2.

Definition 2.4. If $H(z) = \sum_{m\in\mathbb{Z}} A_H(y, m) \cdot e^{2\pi imx} \in \mathcal{M}_w^\infty(N, \psi)$ denotes an arbitrary $\mathcal{C}^\infty$-modular form with $w \geqslant 2$ and $A_H(y, m) \in \mathcal{C}^\infty(\mathbb{R}^+)$, then we define

$$\mathrm{Hol}_\infty(H) := \sum_{n=0}^{\infty} a(n, H) \cdot q^n \in \mathbb{C}[\![q]\!]$$

where at each integer $n > 0$, the n-th Fourier coefficient is given by

$$a(n, H) = \lim_{s\to 0^+}\left(\frac{(4\pi n)^{w-1}}{\Gamma(w-1)} \cdot \int_0^\infty A_H(y, n) e^{-2\pi ny} y^{w+s-2} \cdot \mathrm{d}y\right).$$

Theorem 2.5 (Gross–Zagier and Panchishkin [13, 21])**.** *Let us suppose that $H(z) \in \mathcal{M}_w^\infty(N,\psi)$ is a $\mathcal{C}^\infty$-modular form which exhibits the two extra properties:*

(i) *the coefficients $A_H(y,m) = 0$ for all $m \leqslant 0$, and*
(ii) *$H\big|_w \gamma \in \mathcal{M}_w^\infty(\Gamma(N)), \gamma \in \mathrm{SL}_2(\mathbb{Z})$ has moderate growth, cf.* (2.2).

Then $a(0,H) = 0$, moreover $\mathrm{Hol}_\infty(H)$ belongs to $\mathcal{M}_w(N,\psi)$ i.e. it is a classical holomorphic modular form, and lastly it satisfies the inner product identity

$$\big\langle F, \mathrm{Hol}_\infty(H)\big\rangle_N = \big\langle F, H\big\rangle_N \quad \text{at every } F \in \mathcal{S}_w(N,\psi).$$

2.3.1. ***The double product case.*** The first case we treat relates to the double product L-function $L(F_1 \otimes G_2, s)$. Consider the Eisenstein series in [23, (2.3)] of weight $w \geqslant 0$, character η^{-1} and level N, given by the infinite series

$$(2.3)\quad E^*_{w,N}(z,s,\eta) = \sum_{\Gamma_\infty\backslash\Gamma_0(N)} \eta(\gamma)\cdot(cz+d)^{-w}\big|cz+d\big|_\infty^{-2s}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

For technical reasons, our formulae become tidier if we renormalise these series via

$$(2.4)\qquad \mathbf{E}^*_{w,N}(z,\eta) := \frac{N^{w/2}}{2}\cdot\frac{\Gamma(w)}{(2\pi i)^w}\cdot\zeta_N(w,\eta)\times E^*_{w,N}(z,0,\eta).$$

Henceforth let us assume that $r, w \in \mathbb{Z}$ satisfy both $w = k_1 - k_2 - 2r \geqslant 0$ and $r \geqslant 0$.

Proposition 2.6. *Setting $N = \widetilde{N}$, $\eta = \psi_3$ and $\breve{G}_3 = \mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z,\psi_3)$, then*

$$\mathcal{H} = \mathrm{Hol}_\infty(G_2\cdot\delta^{(r)}_{k_1-k_2-2r}(\breve{G}_3)\big|_{k_1-k_2} W_{\widetilde{N}}) \in \mathcal{M}_{k_1}(\widetilde{N},\psi_2\psi_3)$$

has the q-expansion $\mathcal{H}(z) = \sum_{n=1}^\infty a(n,\mathcal{H})\cdot q^n$, where

$$a(n,\mathcal{H}) = \sum_{n=\xi_2+\xi_3>0} a_{\xi_2}(G_2)\cdot\sum_{\xi_3=b\cdot c} b^{k_1-k_2-2r-1}\cdot\psi_3(c)\cdot P_{-r}(\xi_3,n)$$

and for $s \in \mathbb{Z}_{\leqslant 0}$, the rational polynomial "$P_s(-,-)$" is given by

$$P_s(X,Y) = \sum_{j=0}^{-s}(-1)^j\binom{-s}{j}\frac{\Gamma(k_1-k_2+s)}{\Gamma(k_1-k_2+s-j)}\frac{\Gamma(k_1-1-j)}{\Gamma(k_1-1)}\cdot X^{-s-j}Y^j.$$

Proof. Firstly applying [23, (2.9)], one has the identity

$$E^*_{w+2r,\widetilde{N}}(z,-r,\eta) = \frac{\Gamma(w)}{\Gamma(w+r)}(-4\pi y)^r\cdot\delta_w^{(r)}\left(E^*_{w,\widetilde{N}}(z,0,\eta)\right).$$

If one has $r = 0$ then $E^*_{w+2r,\widetilde{N}}(z, 0, \eta)$ is of holomorphic type, while if $r > 0$ then it is nearly holomorphic and has moderate growth, so that Theorem 2.5 is applicable. After rearranging the above equation, it follows directly that

$$\delta_w^{(r)}\left(E^*_{w,\widetilde{N}}(z,0,\eta)\right)\Big|W_{\widetilde{N}} = (-4\pi)^{-r}\cdot\frac{\Gamma(w+r)}{\Gamma(w)}\times\left(y^{-r}\cdot E^*_{w+2r,\widetilde{N}}(z,-r,\eta)\right)\Big|W_{\widetilde{N}}$$

and then combining it with Panchishkin's definitions [21, (4.3), (4.6) and (4.13)],

$$\left(y^{-r}\cdot E^*_{w+2r,\widetilde{N}}(z,-r,\eta)\right)\Big|_{w+2r}W_{\widetilde{N}} = \frac{2\cdot\zeta_N(w,\eta)^{-1}}{\widetilde{N}^{w/2}\cdot\Gamma(w+r)}\cdot\frac{(2\pi i)^w}{(-4\pi)^{-r}}\cdot\mathcal{E}_{w+2r}(-r,\eta).$$

Here $\mathcal{E}_{w+2r}(s,\eta)$ denotes the Eisenstein series introduced in [21, (4.13)]: in particular at $s = -r$, the $\mathcal{C}^\infty$-function $\mathcal{E}_{w+2r}(-r,\eta)$ has the Fourier expansion

$$(4\pi y)^{-r}\cdot\sum_{\xi_3=1}^{\infty}\left(\sum_{\xi_3=b\cdot c} b^{w-1}\eta(c)\sum_{j=0}^{r}(-1)^j\binom{r}{j}\frac{\Gamma(w+r)}{\Gamma(w+r-j)}\cdot(4\pi\xi_3 y)^{r-j}\right)e^{2\pi i\xi_3 z}.$$

Writing out everything in terms of our renormalised Eisenstein series $\mathbf{E}^*_{w,N}(z,\cdot)$, one finds that $\delta_w^{(r)}(\mathbf{E}^*_{w,\widetilde{N}}(z,\eta))\Big|_{w+2r}W_{\widetilde{N}}$ coincides with $\mathcal{E}_{w+2r}(-r,\eta)$, in which case

$$\mathrm{Hol}_\infty(G_2\cdot\delta_w^{(r)}(\mathbf{E}^*_{w,\widetilde{N}}(z,\eta))\Big|_{w+2r}W_{\widetilde{N}}) = \mathrm{Hol}_\infty(G_2\cdot\mathcal{E}_{w+2r}(-r,\eta)).$$

We next apply the integral operator $\frac{(4\pi n)^{k_1-1}}{\Gamma(k_1-1)}\cdot\int_0^\infty A_H(y,n)e^{-2\pi ny}y^{k_1-2}\cdot\mathrm{d}y$ to the n-th Fourier coefficient of the form

$$H(z) = G_2\cdot\mathcal{E}_{w+2r}(-r,\eta) = \sum_{m=1}^{\infty}A_H(y,m)\cdot e^{2\pi imx}$$

and then exploit the well known identity

$$\text{(2.5)}\quad \frac{(4\pi n)^{k_1-1}}{\Gamma(k_1-1)}\cdot\int_0^\infty((4\pi y)^{-j}e^{-2\pi ny})\cdot e^{-2\pi ny}y^{k_1-2}\cdot\mathrm{d}y = n^j\cdot\frac{\Gamma(k_1-j-1)}{\Gamma(k_1-1)}.$$

A tedious calculation, but essentially identical to the one given in [21, Section 5], allows us to conclude that

$$\begin{aligned}&\mathrm{Hol}_\infty(G_2\cdot\mathcal{E}_{w+2r}(-r,\eta))\\&\qquad=\sum_{n=1}^{\infty}\left(\sum_{n=\xi_2+\xi_3>0}a_{\xi_2}(G_2)\cdot\sum_{\xi_3=b\cdot c}b^{w-1}\eta(c)\cdot P_{-r}(\xi_3,n)\right)q^n.\end{aligned}$$

The automorphy properties follow directly from Theorem 2.5 since each translate $G_2\cdot\mathcal{E}_{w+2r}(-r,\eta)\Big|_{k_1}\gamma$ has moderate growth for $\gamma\in\mathrm{SL}_2(\mathbb{Z})$, and secondly the Fourier coefficients $A_H(y,n)$ of the form $H=G_2\cdot\mathcal{E}_{w+2r}(-r,\eta)$ vanish at every $n\leqslant 0$. □

Corollary 2.7. *Suppose $G_2^{(\mathrm{I})},G_2^{(\mathrm{II})}\in\mathcal{S}_{k_2}(N_2,\psi_2)$ have expansions in $\mathcal{O}_K[\![q]\!]$ for a given number field K, that they satisfy the p-adic congruence*

$$G_2^{(\mathrm{I})}\equiv G_2^{(\mathrm{II})}\pmod{p^{\nu_2}}$$

at some integer $\nu_2\geqslant 1$, and that $\breve{G}_3=\mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z,\psi_3)$. If $p>k_1-2$, then

$$\mathrm{Hol}_\infty(G_2^{(\mathrm{I})}\cdot\delta^{(r)}_{k_1-k_2-2r}(\breve{G}_3)\big|_{k_1-k_2}W_{\widetilde{N}})\equiv\mathrm{Hol}_\infty(G_2^{(\mathrm{II})}\cdot\delta^{(r)}_{k_1-k_2-2r}(\breve{G}_3)\big|_{k_1-k_2}W_{\widetilde{N}})$$

modulo $p^{\nu_2}\cdot\mathcal{O}_K[\![q]\!]$, provided the integer r lies in the range $0\leqslant r\leqslant\frac{1}{2}(k_1-k_2)$.

Proof. We use the Fourier expansions given in the preceding result for both $G_2=G_2^{(\mathrm{I})}$ and $G_2=G_2^{(\mathrm{II})}$, and observe that $P_{-r}(X,Y)\in\mathbb{Z}_p[X,Y]$ as $p>k_1-2$. □

2.3.2. ***The triple product case.*** The next case relates to $L(F_1\otimes G_2\otimes G_3,s)$. Here there are no Eisenstein series to contend with, and their rôle is replaced by the holomorphic form G_3 of weight $w=k_3$, level N_3 and nebentypus $\psi_3=(\psi_1\psi_2)^{-1}$.

Proposition 2.8. *If $G_3\in\mathcal{M}_w(N_3,\overline{\psi_1\psi_2};R)$ for a given subring $R\subset\mathbb{C}$, then*

$$\mathcal{G}=\mathrm{Hol}_\infty(G_2\cdot\delta_w^{(r)}(G_3))\quad\text{at each } r=(k_1-k_2-w)/2\in\mathbb{Z}_{\geqslant 0}$$

is a cusp form of weight k_1, level $\widetilde{N}$ and character $\overline{\psi_1}$; furthermore, it has the q-expansion $\mathcal{G}(z)=\sum_{n=1}^{\infty}a(n,\mathcal{G})\cdot q^n$, where

$$a(n,\mathcal{G})=\sum_{n=\xi_2+\xi_3>0}a_{\xi_2}(G_2)\cdot\sum_{j=0}^{r}\frac{\Gamma(k_1-1-j)}{\Gamma(k_1-1)}\cdot\beta_j^{(r)}(\xi_3)\cdot n^j$$

and $\mathcal{P}_{\delta_w^{(r)}(G_3)}(X,m)=\sum_{j=0}^{r}\beta_j^{(r)}(m)\cdot X^j\in R[X]$ in the sense of Definition 2.2(i).

Proof. One simply points out that $G_2 \cdot \delta_w^{(r)}(G_3)$ has the Fourier expansion

$$(G_2 \cdot \delta_w^{(r)}(G_3))(z) = \sum_{n=0}^{\infty} \left(\sum_{n=\xi_2+\xi_3>0} a_{\xi_2}(G_2) \cdot \sum_{j=0}^{r} \beta_j^{(r)}(\xi_3) \cdot (4\pi y)^{-j} \right) \cdot e^{2\pi i n z}$$

which we hit it with the operator $\mathrm{Hol}_\infty(\,\cdot\,)$, and then repeatedly use (2.5). The property that G_2 is a cusp form directly implies $\mathcal{G}$ vanishes at cusps too. $\square$

Corollary 2.9. *If $G_2^{(\mathrm{I})}, G_2^{(\mathrm{II})} \in \mathcal{S}_{k_2}(N_2, \psi_2)$ and $G_3^{(\mathrm{I})}, G_3^{(\mathrm{II})} \in \mathcal{M}_{k_3}(N_3, \psi_3)$ have expansions in $\mathcal{O}_K[\![q]\!]$ for a given number field K, if they satisfy respectively*

$$G_2^{(\mathrm{I})} \equiv G_2^{(\mathrm{II})} \pmod{p^{\nu_2}} \text{ and } G_3^{(\mathrm{I})} \equiv G_3^{(\mathrm{II})} \pmod{p^{\nu_3}} \quad \text{for some } \nu_2, \nu_3 \geqslant 1,$$

and lastly if the prime $p \nmid \frac{(k_1-2)!}{(k_1-2-r)!}$, then

$$\mathrm{Hol}_\infty(G_2^{(\mathrm{I})} \cdot \delta_{k_1-k_2-2r}^{(r)}(G_3^{(\mathrm{I})})) \equiv \mathrm{Hol}_\infty(G_2^{(\mathrm{II})} \cdot \delta_{k_1-k_2-2r}^{(r)}(G_3^{(\mathrm{II})})) \mod p^{\min\{\nu_2,\nu_3\}}$$

provided again that the integer r lies inside the range $0 \leqslant r \leqslant \frac{1}{2}(k_1 - k_2)$.

Proof. From Lemma 2.3 (b), $\delta_{k_1-k_2-2r}^{(r)}(G_3^{(\mathrm{I})}) \equiv \delta_{k_1-k_2-2r}^{(r)}(G_3^{(\mathrm{II})}) \pmod{p^{\nu_3}}$ and using the Fourier expansions which are calculated in the preceding proposition, the result follows immediately. $\square$

2.4. The effect of Σ-depletion and χ-twisting. In the following discussion $\mathbf{g}^{(\mathrm{I})}$ and $\mathbf{g}^{(\mathrm{II})}$ denote primitive Hecke eigenforms of weight k, character ψ, and levels $N_{\mathbf{g}}^{(\mathrm{I})}$ and $N_{\mathbf{g}}^{(\mathrm{II})}$ respectively (note that we treat both $p \nmid N_{\mathbf{g}}^{(\mathrm{I})} \cdot N_{\mathbf{g}}^{(\mathrm{II})}$ and $p \mid N_{\mathbf{g}}^{(\mathrm{I})} \cdot N_{\mathbf{g}}^{(\mathrm{II})}$). We shall further suppose the coefficients in their q-expansions satisfy:

$$(2.6) \quad a_n(\mathbf{g}^{(\mathrm{I})}) \equiv a_n(\mathbf{g}^{(\mathrm{II})}) \pmod{p^\nu} \qquad \text{for all } n \in \mathbb{N} \text{ with } \gcd(n, N_{\mathbf{g}}^{(\mathrm{I})} N_{\mathbf{g}}^{(\mathrm{II})}) = 1.$$

Let $\Sigma \subset \mathrm{Spec}(\mathbb{Z})$ be a finite set containing the primes dividing $N_{\mathbf{g}}^{(\mathrm{I})} N_{\mathbf{g}}^{(\mathrm{II})}$, but not p.

Definition 2.10.

(a) If $\star \in \{\mathrm{I}, \mathrm{II}\}$, then $\mathbf{g}_\Sigma^{(\star)}$ indicates the depleted cusp form

$$\mathbf{g}_\Sigma^{(\star)}(z) = \sum_{n=1}^{\infty} a_n(\mathbf{g}_\Sigma^{(\star)}) \cdot q^n \in \mathcal{S}_k(N_\Sigma^{(\star)}, \psi), N_\Sigma^{(\star)} = \mathrm{lcm}(N_{\mathbf{g}}^{(\star)}, \prod_{l \in \Sigma} l^2)$$

where $a_n(\mathbf{g}_\Sigma^{(\star)}) = a_n(\mathbf{g}^{(\star)})$ if $\mathrm{supp}(n) \cap \Sigma = \emptyset$, and $a_n(\mathbf{g}_\Sigma^{(\star)}) = 0$ if $\mathrm{supp}(n) \cap \Sigma \neq \emptyset$.

(b) For a Dirichlet character χ of conductor $p^{n_\chi} \geqslant 1$, and choosing $\star \in \{\mathrm{I}, \mathrm{II}\}$, we define χ-twisted cusp forms by $\mathbf{g}_\chi^{(\star)} := \mathbf{g}^{(\star)} \otimes \chi$ and $\mathbf{g}_{\Sigma,\chi}^{(\star)} := (\mathbf{g}^{(\star)} \otimes \chi)_\Sigma = \mathbf{g}_\Sigma^{(\star)} \otimes \chi$.

If we set $\widetilde{N}_{\Sigma,\chi} := \mathrm{lcm}(p^{2n_\chi}, N_\Sigma^{(\mathrm{I})}, N_\Sigma^{(\mathrm{II})})$ then both $\mathbf{g}_{\Sigma,\chi}^{(\mathrm{I})}$ and $\mathbf{g}_{\Sigma,\chi}^{(\mathrm{II})}$ are cuspidal Hecke eigenforms of weight k and character $\psi\chi^2$, each of whose levels divides $\widetilde{N}_{\Sigma,\chi}$. Furthermore, their q-expansions automatically satisfy

$$\mathbf{g}_\chi^{(\star)} = \sum_{n=1}^\infty \chi(n) \cdot a_n(\mathbf{g}^{(\star)}) \cdot q^n \quad \text{and} \quad \mathbf{g}_{\Sigma,\chi}^{(\star)} = \sum_{n=1}^\infty \chi(n) \cdot a_n(\mathbf{g}_\Sigma^{(\star)}) \cdot q^n$$

provided that the conductor $p^{n_\chi} \geqslant \max\left\{ \left|N_{\mathbf{g}}^{(\mathrm{I})}\right|_p^{-\frac{1}{2}}, \left|N_{\mathbf{g}}^{(\mathrm{II})}\right|_p^{-\frac{1}{2}} \right\}$.

Proposition 2.11. *If $\mathbf{g}^{(\mathrm{I})}$ and $\mathbf{g}^{(\mathrm{II})}$ satisfy* (2.6)*, then at all characters χ of p-power conductor and for each finite set $\Sigma \supset \mathrm{supp}(N_{\mathbf{g}}^{(\mathrm{I})} \cdot N_{\mathbf{g}}^{(\mathrm{II})}) - \{p\}$,*

$$\mathbf{g}_{\Sigma,\chi}^{(\mathrm{I})}\Big|_k W_{\widetilde{N}} \equiv \mathbf{g}_{\Sigma,\chi}^{(\mathrm{II})}\Big|_k W_{\widetilde{N}} \pmod{p^\nu} \quad \text{if } \widetilde{N}_{\Sigma,\chi} \big| \widetilde{N} \text{ and } \mathrm{ord}_p(\widetilde{N}_{\Sigma,\chi}) = \mathrm{ord}_p(\widetilde{N}),$$

*as a congruence between (on both sides) a p-*integral *linear sum of eigenforms*[2]*.*

Proof. For a rational prime l, if l does not divide the level we write T_l for the l-th Hecke operator, whilst if l does divide the level we shall use the notation U_l. For $m \in \mathbb{N}$ coprime to the level, the m-th diamond operator is denoted by $\langle m \rangle$ and for an integer $d \geqslant 1$, one writes V_d for the degeneracy map (as we did in Section 2.1). Let us begin by remarking that for each $\star \in \{\mathrm{I}, \mathrm{II}\}$,

$$\mathbf{g}_{\Sigma,\chi}^{(\star)} = \mathbf{g}_\chi^{(\star)}\Big|_k \prod_{\substack{l\in\Sigma,\\ l \nmid N_{\mathbf{g}}^{(\star)}}} (1 - T_l \cdot V_l + l^{k-1} \cdot \langle l \rangle \cdot V_{l^2}) \cdot \prod_{\substack{l\in\Sigma,\\ l \| N_{\mathbf{g}}^{(\star)}}} (1 - U_l \cdot V_l) \tag{2.7}$$

which gives an alternative construction of these Σ-depleted, χ-twisted cusp forms. To prove our result, it is necessary to establish that the composition of operators

$$(\,\cdot\,)\Big|_k \prod_{\substack{l\in\Sigma,\\ l \nmid N_{\mathbf{g}}^{(\star)}}} (1 - T_l \cdot V_l + l^{k-1} \cdot \langle l \rangle \cdot V_{l^2}) \cdot \prod_{\substack{l\in\Sigma,\\ l \| N_{\mathbf{g}}^{(\star)}}} (1 - U_l \cdot V_l)\Big|_k W_{\widetilde{N}}$$

acting on newforms of weight k and character $\psi\chi^2$ preserves the integral structure.

---

[2] By work of Vatsal [25, Proposition 4.5], the canonical motivic periods associated to $\mathbf{g}_{\Sigma,\chi}^{(\star)}$ and $\mathbf{g}_\chi^{(\star)}$ are known to differ from each other by a p-adic unit, at least in the case where $a_p(\mathbf{g}^{(\star)}) \in \mathcal{O}_{\mathbb{C}_p}^\times$.

Fix a choice of $\star \in \{\mathrm{I}, \mathrm{II}\}$. Let us assume that l is a rational prime number, and M denotes a multiple of $N_{\mathbf{g}}^{(\star)}$ such that l^2 divides M. Then for a "weight k" action,

$$\begin{aligned}(1 - U_l \cdot V_l) \cdot W_M &= W_M - U_l \cdot V_l \cdot W_M \\ &= l^{k/2} \cdot W_{M/l} \cdot V_l - l^{-k/2} \cdot U_l \cdot W_{M/l}\end{aligned}$$

because at such a weight, we have $W_M = l^{k/2} \cdot W_{M/l} \cdot V_l$ and $V_l \cdot W_M = l^{-k/2} \cdot W_{M/l}$. One therefore deduces

$$\begin{aligned}(1 - U_l \cdot V_l) \cdot W_M &= l^{k/2} \cdot W_{M/l} \cdot V_l - l^{-k/2} \cdot W_{M/l} \cdot U_l^* \\ &= W_{M/l} \cdot (l^{k/2} \cdot V_l - l^{-k/2} \cdot U_l^*)\end{aligned} \tag{2.8}$$

where $(\,\cdot\,)^*$ indicates the adjoint Hecke operator. Analogously, one calculates that

$$\begin{aligned}&(1 - T_l \cdot V_l + l^{k-1}\langle l\rangle \cdot V_{l^2}) \cdot W_M \\ &\qquad = W_M - T_l \cdot V_l \cdot W_M + l^{k-1}\langle l\rangle \cdot V_{l^2} \cdot W_M \\ &\qquad = l^k \cdot W_{M/l^2} \cdot V_{l^2} - l^{-k/2} \cdot T_l \cdot W_{M/l} + l^{k-1}(l^2)^{-k/2}\langle l\rangle \cdot W_{M/l^2}\end{aligned}$$

as $W_M = l^k \cdot W_{M/l^2} \cdot V_{l^2}$, $V_l \cdot W_M = l^{-k/2} \cdot W_{M/l}$ and $V_{l^2} \cdot W_M = (l^2)^{-k/2} W_{M/l^2}$. We then obtain a string of equalities

$$\begin{aligned}&(1 - T_l \cdot V_l + l^{k-1} \cdot \langle l\rangle \cdot V_{l^2}) \cdot W_M \\ &\qquad = l^k \cdot W_{M/l^2} \cdot V_{l^2} - T_l \cdot W_{M/l^2} \cdot V_l + l^{-1} \cdot \langle l\rangle \cdot W_{M/l^2} \\ &\qquad = l^k \cdot W_{M/l^2} \cdot V_{l^2} - W_{M/l^2} \cdot T_l^* \cdot V_l + l^{-1} \cdot W_{M/l^2} \cdot \langle l\rangle^* \\ &\qquad = W_{M/l^2} \cdot \Big(l^k \cdot V_{l^2} - T_l^* \cdot V_l + l^{-1} \cdot \langle l^{-1}\rangle\Big)\end{aligned} \tag{2.9}$$

and these three lines follow from the respective identities: $l^{-k/2} \cdot W_{M/l} = W_{M/l^2} \cdot V_l$, $T_l \cdot W_{M/l^2} = W_{M/l^2} \cdot T_l^*$ and $\langle l\rangle^* = \langle l^{-1}\rangle$, applied in a consecutive order.

Returning to the description in (2.7), our calculations in Equations (2.8–2.9) imply via an inductive argument that

$$\begin{aligned}&\mathbf{g}_\chi^{(\star)}\Big|_k \prod_{\substack{l\in\Sigma,\\ l\nmid N_{\mathbf{g}}^{(\star)}}} (1 - T_l \cdot V_l + l^{k-1}\langle l\rangle \cdot V_{l^2}) \cdot \prod_{\substack{l\in\Sigma,\\ l\| N_{\mathbf{g}}^{(\star)}}} (1 - U_l \cdot V_l) \cdot \Big|_k W_{\widetilde{N}_{\Sigma,\chi}} \\ &= \mathbf{g}_\chi^{(\star)}\Big|_k W_{\widetilde{M}_{\Sigma,\chi}} \cdot \prod_{\substack{l\in\Sigma,\\ l\nmid N_{\mathbf{g}}^{(\star)}}} \Big(l^k \cdot V_{l^2} - T_l^* \cdot V_l + l^{-1} \cdot \langle l^{-1}\rangle\Big) \cdot \prod_{\substack{l\in\Sigma,\\ l\| N_{\mathbf{g}}^{(\star)}}} (l^{k/2} \cdot V_l - l^{-k/2} \cdot U_l^*)\end{aligned} \tag{2.10}$$

with the level of the W-operator being decreased to

$$\widetilde{M}_{\Sigma,\chi} := \widetilde{N}_{\Sigma,\chi} \cdot \prod_{\substack{l\in\Sigma,\\ l\nmid N_{\mathbf{g}}^{(\star)}}} l^{-2} \cdot \prod_{\substack{l\in\Sigma,\\ l\| N_{\mathbf{g}}^{(\star)}}} l^{-1} = N_{\mathbf{g}^{(\star)}\otimes\chi} \times M_{\Sigma,\mathbf{g}}^{(\star)} \text{ for some } M_{\Sigma,\mathbf{g}}^{(\star)} \in \mathbb{N}\cap\mathbb{Z}_p^\times.$$

Under this weight k action, we may factorise

$$W_{\widetilde{M}_{\Sigma,\chi}} = \left(M_{\Sigma,\mathbf{g}}^{(\star)}\right)^{k/2} \cdot W_{N_{\mathbf{g}^{(\star)}\otimes\chi}} \cdot V_{M_{\Sigma,\mathbf{g}}^{(\star)}}$$

and one readily deduces that

$$\begin{aligned}
(2.11)\quad \mathbf{g}_\chi^{(\star)}\Big|_k W_{\widetilde{M}_{\Sigma,\chi}} &= \left(M_{\Sigma,\mathbf{g}}^{(\star)}\right)^{k/2} \cdot \left(\mathbf{g}^{(\star)}\otimes\chi\Big|_k W_{N_{\mathbf{g}^{(\star)}\otimes\chi}}\right)\Big|_k V_{M_{\Sigma,\mathbf{g}}^{(\star)}} \\
&= \left(M_{\Sigma,\mathbf{g}}^{(\star)}\right)^{k/2} \cdot \left(\psi(p^{2n_\chi})\chi(N_{\mathbf{g}}^{(\star)})\frac{\tau(\chi)^2}{p^{n_\chi}}\epsilon_{\mathbf{g}}^{(\star)} \cdot (\mathbf{g}^{(\star),\sharp}\otimes\chi^{-1})\right)\Bigg|_k V_{M_{\Sigma,\mathbf{g}}^{(\star)}}
\end{aligned}$$

where $\epsilon_{\mathbf{g}}^{(\star)} \in \mathbb{C}$, $\left|\epsilon_{\mathbf{g}}^{(\star)}\right|_\infty = 1$ satisfies $\mathbf{g}^{(\star)}\big|_k W_{N_{\mathbf{g}}^{(\star)}} = \epsilon_{\mathbf{g}}^{(\star)} \cdot \mathbf{g}^{(\star),\sharp}$ (see [21, (1.24)]). If we define the algebraic number

$$\mathcal{Z}_{\Sigma,\chi}^{(\star)} := \left(M_{\Sigma,\mathbf{g}}^{(\star)}\right)^{k/2} \cdot \psi(p^{2n_\chi})\chi\left(N_{\mathbf{g}}^{(\star)}\right)\frac{\tau(\chi)^2}{p^{n_\chi}}\epsilon_{\mathbf{g}}^{(\star)}$$

which is a p-adic unit as $\frac{\tau(\chi)^2}{p^{n_\chi}}, \epsilon_{\mathbf{g}}^{(\star)} \in \mathcal{O}_{\mathbb{C}_p}^\times$, Equations (2.8) and (2.10–2.11) imply

$$\begin{aligned}
\mathbf{g}_{\Sigma,\chi}^{(\star)}\Big|_k W_{\widetilde{N}_{\Sigma,\chi}} &= \mathcal{Z}_{\Sigma,\chi}^{(\star)} \cdot (\mathbf{g}^{(\star),\sharp}\otimes\chi^{-1})\Big|_k V_{M_{\Sigma,\mathbf{g}}^{(\star)}}\Big|_k \\
&\prod_{l\in\Sigma,\, l\nmid N_{\mathbf{g}}^{(\star)}} \left(l^k\cdot V_{l^2} - T_l^*\cdot V_l + l^{-1}\cdot\langle l^{-1}\rangle\right)\Big|_k \prod_{l\in\Sigma,\, l\| N_{\mathbf{g}}^{(\star)}} (l^{k/2}\cdot V_l - l^{-k/2}\cdot U_l^*).
\end{aligned}$$

The right-hand side of the above equation is clearly a p-integral combination of eigenforms with algebraic integer q-expansions, therefore the left-hand side is too. To pass from $\mathbf{g}_{\Sigma,\chi}^{(\star)}\big|_k W_{\widetilde{N}_{\Sigma,\chi}}$ to the cusp form $\mathbf{g}_{\Sigma,\chi}^{(\star)}\big|_k W_{\widetilde{N}}$, one employs the identity

$$\mathbf{g}_{\Sigma,\chi}^{(\star)}\Big|_k W_{\widetilde{N}} = \left(\widetilde{N}/\widetilde{N}_{\Sigma,\chi}\right)^{k/2} \cdot \left(\mathbf{g}_{\Sigma,\chi}^{(\star)}\Big|_k W_{\widetilde{N}_{\Sigma,\chi}}\right)\Big|_k V_{\widetilde{N}/\widetilde{N}_{\Sigma,\chi}}$$

and observes that the quotient $\widetilde{N}/\widetilde{N}_{\Sigma,\chi} \in \mathbb{N}\cap\mathbb{Z}_p^\times$ since $\mathrm{ord}_p(\widetilde{N}_{\Sigma,\chi}) = \mathrm{ord}_p(\widetilde{N})$.

Finally, those congruences asserted in the statement of the proposition now follow from the system of congruences

$$\chi^{-1}(n)\cdot\overline{a_n(\mathbf{g}_{\mathbf{\Sigma}}^{(\mathrm{I})})} \equiv \chi^{-1}(n)\cdot\overline{a_n(\mathbf{g}_{\mathbf{\Sigma}}^{(\mathrm{II})})} \pmod{p^\nu}$$

which hold at integers $n \geqslant 1$ by (2.6), and the proof is complete. □

2.5. Finishing off the inner product calculation. Let us return to our earlier computation of the numerator from (2.1), namely we must evaluate

$$\langle F_1^\sharp, \mathrm{Tr}_{\widetilde{N}/N_0}(\mathrm{Hol}_\infty(G_2 \cdot \delta_w^{(r)}(G_3))\big|_{k_1} W_{\widetilde{N}}^{\varepsilon})\rangle_{N_0}, \quad \varepsilon \in \{0,1\}$$

for forms F_1, G_2, G_3 of level N_1, N_2, N_3, weight k_1, k_2, k_3 and nebentypus ψ_1, ψ_2, ψ_3 with $\psi_2 \cdot \psi_3 = \psi_1^{-1}$. Throughout we will again suppose that $F_1^\sharp = F_1$ and $\psi_1^2 = \mathbf{1}$.

In particular, after dividing through by the period $\langle F_1, F_1\rangle_{N_1}$, one wants to see how this quantity varies when we replace G_2 and G_3 with p^ν-congruent forms. We shall treat the same two cases as in Section 2.3, corresponding to the double product $L(F_1 \otimes G_2, s)$ and the triple product $L(F_1 \otimes G_2 \otimes G_3, s)$, respectively.

2.5.1. ***The double product case.*** Assume we are given newforms $\mathbf{g}^{(\mathrm{I})}$ and $\mathbf{g}^{(\mathrm{II})}$ of common weight $k = k_2 > 0$, common character ψ, and conductors $N_{\mathbf{g}}^{(\mathrm{I})}$ and $N_{\mathbf{g}}^{(\mathrm{II})}$. Let us further suppose (2.6) holds for their q-expansions with $\nu = \nu_2$, i.e.

$$a_n(\mathbf{g}^{(\mathrm{I})}) \equiv a_n(\mathbf{g}^{(\mathrm{II})}) \ (\mathrm{mod}\, p^{\nu_2}) \quad \text{for all } n \in \mathbb{N} \text{ with } \gcd(n, N_{\mathbf{g}}^{(\mathrm{I})} N_{\mathbf{g}}^{(\mathrm{II})}) = 1.$$

We shall carefully select the subset $\Sigma \subset \mathrm{Spec}(\mathbb{Z})$ of primes in order to satisfy the three conditions:

(i) $\mathrm{supp}(N_{\mathbf{g}}^{(\mathrm{I})} N_{\mathbf{g}}^{(\mathrm{II})}) - \{p\} \subset \Sigma$,
(ii) $\#\Sigma < \infty$ and
(iii) $p \notin \Sigma$.

Let χ denote a character of conductor $p^{n_\chi} \geqslant 1$. If we set $\widetilde{N} = \mathrm{lcm}(N_1, \widetilde{N}_{\Sigma,\chi})$ and $\psi_2 = \bar{\psi}\chi^{-2}$, one may consider $\mathbf{g}_{\Sigma,\chi}^{(\mathrm{I})}\big|_{k_2} W_{\widetilde{N}}$ and $\mathbf{g}_{\Sigma,\chi}^{(\mathrm{II})}\big|_{k_2} W_{\widetilde{N}}$ as belonging to the vector space $\mathcal{S}_{k_2}(\widetilde{N}, \psi_2)$; they have p-integral q-expansions by Proposition 2.11, and their Fourier coefficients lie in some finite algebraic extension of $\mathbb{Q}$.

Now for any integer r in the range $0 \leqslant 2r \leqslant k_1 - k_2$, just as in (2.4) one can define

$$\breve{G}_3(z) := \mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z, \psi_3)$$

where $\psi_3 = (\psi_1\psi_2)^{-1} = \psi_1 \cdot \psi \cdot \chi^2$, and the level of the Eisenstein series equals $\widetilde{N}$. It follows for each choice of $\star \in \{\mathrm{I}, \mathrm{II}\}$, the product of the two modular forms

$$G^{(\star)} = \mathbf{g}_{\Sigma,\chi}^{(\star)} \cdot \delta_{k_1-k_2-2r}^{(r)}(\breve{G}_3) \in \mathcal{M}_{k_1}^{\infty}(\widetilde{N}, (\psi_2\psi_3)^{-1})$$

is such that $G^{(\star)}\big|_{k_1}\gamma$ has moderate growth at every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, in which case

$$\mathcal{H}^{(\star)} := \mathrm{Hol}_\infty(G^{(\star)})\Big|_{k_1} W_{\widetilde{N}} = \mathrm{Hol}_\infty\left(\mathbf{g}_{\Sigma,\chi}^{(\star)}\Big|_{k_2} W_{\widetilde{N}} \cdot \delta_{k_1-k_2-2r}^{(r)}(\breve{G}_3)\Big|_{k_1-k_2} W_{\widetilde{N}}\right)$$

is an element of $\mathcal{M}_{k_1}(\widetilde{N}, \psi_2\psi_3)$.

Let $\mathcal{O}_{K,\chi}$ denote the integral extension of $\mathbb{Z}$ generated by the Fourier coefficients $a_n(\mathbf{g}_\Sigma^{(\star)})$ and the character values $\chi(n)$, for all positive integers n and $\star \in \{\mathrm{I}, \mathrm{II}\}$. Note that in the context of Lemma 2.1, each of the holomorphic modular forms

$$\mathcal{H}^{(\star)}\Big|_{k_1} U_p^{\tilde{e}-1} = \mathrm{Hol}_\infty(G^{(\star)})\Big|_{k_1} W_{\widetilde{N}} \circ U_p^{\tilde{e}-1} \in \mathcal{M}_{k_1}(N_0, \psi_2\psi_3) \cap \mathcal{O}_{K,\chi}[\![q]\!]$$

can be decomposed into its F_1-isotypic and non-F_1-isotypic components via

$$\mathcal{H}^{(\star)}\Big|_{k_1} U_p^{\tilde{e}-1} = \sum_{d \left| \frac{N_0}{N_1}\right.} \mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \cdot F_1\big|_{k_1} V_d + \mathcal{H}^{(\star),(\perp)}_{\widetilde{N},\tilde{e}}$$

for scalars $\mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \in \mathcal{O}_{K,\chi}$. If we define $\widetilde{M} := \widetilde{N}/\widetilde{N}_{\Sigma,\chi} \in \mathbb{N} \cap \mathbb{Z}_p^\times$, using Proposition 2.11 one finds that

$$\mathbf{g}^{(\mathrm{I})}_{\Sigma,\chi}\Big|_{k_2} W_{\widetilde{N}} \equiv \mathbf{g}^{(\mathrm{II})}_{\Sigma,\chi}\Big|_{k_2} W_{\widetilde{N}} \pmod{p^{\nu_2}}$$

and moreover, if the prime $p > k_2 - 1$, then Corollary 2.7 implies

$$\mathcal{H}^{(\mathrm{I})} \equiv \mathcal{H}^{(\mathrm{II})} \pmod{p^{\nu_2}}. \tag{2.12}$$

We next apply the results in Section 2.1 to this pair of congruent modular forms.

Proposition 2.12. *If* $\varepsilon = 0$ *and* $G^{(\star)} = \mathbf{g}^{(\star)}_{\Sigma,\chi} \cdot \delta^{(r)}_{k_1-k_2-2r}(\mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z, \psi_3))$ *as above for either* $\star \in \{\mathrm{I}, \mathrm{II}\}$ *with the prime* $p \notin \Sigma$, $p > k_2 - 1$ *and* $p \nmid N_1$, *then*

$$\frac{\langle F_1^\sharp, \mathrm{Tr}^{\widetilde{N}}_{N_0}(\mathrm{Hol}_\infty(G^{(\star)})|_{k_1} W^{\varepsilon}_{\widetilde{N}})\rangle_{N_0}}{\langle F_1, F_1\rangle_{N_1}} \tag{2.13}$$
$$= \epsilon_1 \cdot p^{1-\frac{(k_1-2)(\tilde{e}-2)}{2}} \cdot \left(\frac{\widetilde{N}^{(p)}}{N_1}\right)^{\frac{k_1}{2}} \times \sum_{d \left| \frac{N_0}{N_1}\right.} \mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \cdot \frac{\langle F_1|_{k_1} V_{\frac{N_0}{N_1}}, F_1|_{k_1} V_d\rangle_{N_0}}{\langle F_1, F_1\rangle_{N_1}}$$

where $\widetilde{N} = \mathrm{lcm}(N_1, p^{2n_\chi}, N_\Sigma^{(\mathrm{I})}, N_\Sigma^{(\mathrm{II})})$, $\widetilde{N}^{(p)} = \big|\widetilde{N}\big|_p \cdot \widetilde{N}$ *and lastly* $N_0 = p \cdot \widetilde{N}^{(p)}$. *Moreover the congruences* $\mathfrak{c}^{(\mathrm{I})}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \equiv \mathfrak{c}^{(\mathrm{II})}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \pmod{p^{\nu_2}}$ *hold at integers* $d \big| \frac{N_0}{N_1}$.

Proof. Most of these assertions follow upon applying Lemma 2.1 directly to the forms

$$G = \mathrm{Hol}_\infty\Big(\mathbf{g}^{(\mathrm{I})}_{\Sigma,\chi} \cdot \delta^{(r)}_{k_1-k_2-2r}(\breve{G}_3)\Big) \quad \text{and} \quad G = \mathrm{Hol}_\infty\Big(\mathbf{g}^{(\mathrm{II})}_{\Sigma,\chi} \cdot \delta^{(r)}_{k_1-k_2-2r}(\breve{G}_3)\Big).$$

The levels $\widetilde{N}$, $\widetilde{N}^{(p)}$ and N_0 are easily determined from their descriptions in Section 2.1. We should point out that the q-expansions of $\mathcal{H}^{(\mathrm{I})}$ and $\mathcal{H}^{(\mathrm{II})}$ take values in $\mathcal{O}_{K,\chi}$ by Propositions 2.6 and 2.11, hence so do the q-expansions of the N_0-level modular forms $\mathcal{H}^{(\mathrm{I})}\big|_{k_1}U_p^{\tilde{e}-1}$ and $\mathcal{H}^{(\mathrm{II})}\big|_{k_1}U_p^{\tilde{e}-1}$. Finally, one may combine (2.12) together with the implication

$$\mathcal{H}^{(\mathrm{I})} \equiv \mathcal{H}^{(\mathrm{II})} \pmod{p^{\nu_2}} \implies \mathcal{H}^{(\mathrm{I})}\big|_{k_1}U_p^{\tilde{e}-1} \equiv \mathcal{H}^{(\mathrm{II})}\big|_{k_1}U_p^{\tilde{e}-1} \pmod{p^{\nu_2}}$$

to conclude that the F_1-isotypic parts of $\mathcal{H}^{(\mathrm{I})}\big|_{k_1}U_p^{\tilde{e}-1}$ and $\mathcal{H}^{(\mathrm{II})}\big|_{k_1}U_p^{\tilde{e}-1}$ are similarly congruent modulo $p^{\nu_2}\cdot\mathcal{O}_{K,\chi}[\![q]\!]$, whence $\mathfrak{c}^{(\mathrm{I})}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \equiv \mathfrak{c}^{(\mathrm{II})}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \pmod{p^{\nu_2}}$. □

2.5.2. ***The triple product case.*** Alternatively, suppose one is given cusp forms $\mathbf{g}^{(\mathrm{I})}, \mathbf{g}^{(\mathrm{II})}$ of weight k_2, character ψ_2, and that their respective levels are $N_{\mathbf{g}}^{(\mathrm{I})}, N_{\mathbf{g}}^{(\mathrm{II})}$. In addition, we suppose that $\mathbf{h}^{(\mathrm{I})}, \mathbf{h}^{(\mathrm{II})}$ are modular forms of weight $k_3 = k_1 - k_2 - 2r$, character $\psi_3 = \overline{\psi_1\psi_2}$, with levels $N_{\mathbf{h}}^{(\mathrm{I})}$ and $N_{\mathbf{h}}^{(\mathrm{II})}$ respectively. One further assumes:

$$a_n(\mathbf{g}^{(\mathrm{I})}) \equiv a_n(\mathbf{g}^{(\mathrm{II})}) \pmod{p^{\nu_2}} \quad \text{if } \gcd(n, N_{\mathbf{g}}^{(\mathrm{I})}N_{\mathbf{g}}^{(\mathrm{II})}) = 1, \text{ and} \tag{2.14}$$

$$a_n(\mathbf{h}^{(\mathrm{I})}) \equiv a_n(\mathbf{h}^{(\mathrm{II})}) \pmod{p^{\nu_3}} \quad \text{if } \gcd(n, N_{\mathbf{h}}^{(\mathrm{I})}N_{\mathbf{h}}^{(\mathrm{II})}) = 1. \tag{2.15}$$

We shall now choose the set of rational primes Σ to satisfy the three modified conditions:

(i) $\mathrm{supp}(N_{\mathbf{g}}^{(\mathrm{I})}N_{\mathbf{g}}^{(\mathrm{II})}N_{\mathbf{h}}^{(\mathrm{I})}N_{\mathbf{h}}^{(\mathrm{II})}) - \{p\} \subset \Sigma$,
(ii) $\#\Sigma < \infty$ and
(iii) $p \notin \Sigma$.

Notations.

(a) If we construct a "suitably large enough" level by taking

$$\widetilde{N} := \mathrm{lcm}\left(N_1, N_{\mathbf{g}}^{(\mathrm{I})}, N_{\mathbf{g}}^{(\mathrm{II})}, N_{\mathbf{h}}^{(\mathrm{I})}, N_{\mathbf{h}}^{(\mathrm{II})}, \prod_{l\in\Sigma} l^2\right)$$

then the Σ-depleted forms $\mathbf{g}_\Sigma^{(\mathrm{I})}, \mathbf{g}_\Sigma^{(\mathrm{II})}, \mathbf{h}_\Sigma^{(\mathrm{I})}, \mathbf{h}_\Sigma^{(\mathrm{II})}$ will each exist at this top level $\widetilde{N}$.

(b) Let $K = K(\mathbf{g}_\Sigma, \mathbf{h}_\Sigma)$ denote the number field generated by the q-coefficients of the depleted modular forms $\mathbf{g}_\Sigma^{(\mathrm{I})}, \mathbf{g}_\Sigma^{(\mathrm{II})}, \mathbf{h}_\Sigma^{(\mathrm{I})}$ and $\mathbf{h}_\Sigma^{(\mathrm{II})}$.

(c) We shall write $\mathcal{O}_K = \mathcal{O}_K(\mathbf{g}_\Sigma, \mathbf{h}_\Sigma)$ for the ring of integers of $K(\mathbf{g}_\Sigma, \mathbf{h}_\Sigma)$.

Proposition 2.13. *If $\varepsilon = 1$ and $G^{(\star)} = \mathbf{g}_\Sigma^{(\star)} \cdot \delta^{(r)}_{k_1-k_2-2r}(\mathbf{h}_\Sigma^{(\star)})$ for $\star \in \{\mathrm{I}, \mathrm{II}\}$ with $p \notin \Sigma$, $p \nmid \frac{(k_1-2)!}{(k_1-2-r)!}$ and $p \nmid N_1$, then $G^{(\mathrm{I})}$ and $G^{(\mathrm{II})}$ belong to $\mathcal{N}^{\infty,r}_{k_1,\mathrm{pol}}(\widetilde{N}, \psi_1^{-1}; \mathcal{O}_K)$ and they both satisfy* (2.13)*, where $\mathcal{H}^{(\star)} =$*

$\mathrm{Hol}_\infty(G^{(\star)})\Big|_{k_1} W^2_{\widetilde{N}}$ *and*

$$\mathcal{H}^{(\star)}\Big|_{k_1} U_p^{\tilde{e}-1} = \sum_{d\left|\frac{N_0}{N_1}\right.} \mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}(\mathcal{H})\cdot F_1\big|_{k_1} V_d + \mathcal{H}^{(\star),(\perp)}_{\widetilde{N},\tilde{e}}, \quad \mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \in \mathcal{O}_K(\mathbf{g}_\Sigma, \mathbf{h}_\Sigma).$$

Moreover the congruences $\mathfrak{c}^{(\mathrm{I})}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \equiv \mathfrak{c}^{(\mathrm{II})}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}) \pmod{p^{\min\{\nu_2,\nu_3\}}}$ *hold for* $d\big|\frac{N_0}{N_1}$.

Proof. The forms above satisfy $\mathbf{h}^{(\star)}_\Sigma \in \mathcal{M}_{k_3}(\widetilde{N}, \psi_3; \mathcal{O}_K) \subset \mathcal{N}^{\infty,0}_{k_3,\mathrm{pol}}(\widetilde{N}, \psi_3; \mathcal{O}_K)$ so that $\delta^{(r)}_{k_1-k_2-2r}(\mathbf{h}^{(\star)}_\Sigma) \in \mathcal{N}^{\infty,r}_{k_1-k_2,\mathrm{pol}}(\widetilde{N}, \psi_3; \mathcal{O}_K)$ by Lemma 2.3(a); consequently

$$G^{(\star)} = \mathbf{g}^{(\star)}_\Sigma \cdot \delta^{(r)}_{k_1-k_2-2r}(\mathbf{h}^{(\star)}_\Sigma) \in \mathcal{N}^{\infty,r}_{k_1,\mathrm{pol}}(\widetilde{N}, \psi_2\psi_3; \mathcal{O}_K),$$

and combining (2.14–2.15) with Lemma 2.3 (b) implies $G^{(\mathrm{I})} \equiv G^{(\mathrm{II})} \bmod p^{\min\{\nu_2,\nu_3\}}$. From Corollary 2.9 with $G_2^{(\star)} = \mathbf{g}^{(\star)}_\Sigma$ and $G_3^{(\star)} = \mathbf{h}^{(\star)}_\Sigma$, it follows directly that

$$\mathrm{Hol}_\infty(G^{(\mathrm{I})}) \equiv \mathrm{Hol}_\infty(G^{(\mathrm{II})}) \mod p^{\min\{\nu_2,\nu_3\}} \cdot \mathcal{O}_K[\![q]\!].$$

One next applies Lemma 2.1 to the pair of cusp forms $G = \mathrm{Hol}_\infty(G^{(\mathrm{I})})\big|_{k_1} W_{\widetilde{N}}$ and $G = \mathrm{Hol}_\infty(G^{(\mathrm{II})})\big|_{k_1} W_{\widetilde{N}}$. By copying the same argument as in the previous proof, the required congruences are a consequence of the implication

$$\mathcal{H}^{(\mathrm{I})} \equiv \mathcal{H}^{(\mathrm{II})} \mod p^{\min\{\nu_2,\nu_3\}} \implies \mathcal{H}^{(\mathrm{I})}\Big|_{k_1} U_p^{\tilde{e}-1} \equiv \mathcal{H}^{(\mathrm{II})}\Big|_{k_1} U_p^{\tilde{e}-1} \mod p^{\min\{\nu_2,\nu_3\}}$$

and the property that taking the F_1-isotypic projection will respect congruences (because the module $\mathcal{M}_{k_1}(N_0, \psi_1^{-1}) \cap \mathcal{O}_K[\![q]\!]$ contains a basis consisting of Hecke eigenforms whose q-expansion coefficients also lie in the ring of integers $\mathcal{O}_K$). □

2.5.3. *Determining the* $\frac{\langle F_1|V_-, F_1|V_d\rangle}{\langle F_1, F_1\rangle}$***'s explicitly.*** For both the double product and triple product cases, our special value formulae each involve (2.13). It therefore remains to evaluate the ratio of $\langle F_1\big|_{k_1} V_{\frac{N_0}{N_1}}, F_1\big|_{k_1} V_d\rangle_{N_0}$ to $\langle F_1, F_1\rangle_{N_1}$ as the integer "d" runs through the divisors of $\frac{N_0}{N_1}$. Firstly applying [23, Lemma 1],

$$\frac{\langle F_1\big|_{k_1} V_{\frac{N_0}{N_1}}, F_1\big|_{k_1} V_d\rangle_{N_0}}{\langle F_1, F_1\rangle_{N_0}} = \mathrm{Res}_{s=k_1}\left(\frac{D(s, F_1^\sharp\big|_{k_1} V_{\frac{N_0}{N_1}}, F_1\big|_{k_1} V_d)}{D(s, F_1^\sharp, F_1)}\right)$$

where the convolution L-series $D(s,\mathcal{F},\mathcal{G}) := \sum_{n=1}^{\infty} a_n(\mathcal{F})a_n(\mathcal{G})\cdot n^{-s}$ for $\mathrm{Re}(s) \gg 0$. By assumption $F_1^\sharp = F_1$, so we may factorise the ratio of L-functions above into

$$\frac{D(s, F_1^\sharp|_{k_1} V_{\frac{N_0}{N_1}}, F_1|_{k_1} V_d)}{D(s, F_1^\sharp, F_1)} = \left(\frac{N_0}{N_1}\right)^{-s} \cdot \prod_{l \mid \frac{N_0}{dN_1}} \frac{\sum_{j=0}^{\infty} a_{l^j}(F_1) a_{l^{j+t_{l,d}}}(F_1)\cdot l^{-js}}{\sum_{j=0}^{\infty} a_{l^j}(F_1)^2 \cdot l^{-js}}$$

with the integer exponent $t_{l,d} := \mathrm{ord}_l(N_0) - \mathrm{ord}_l(dN_1) \geqslant 0$.

Lemma 2.14. *If the prime l divides into N_0/dN_1, then*

$$\frac{\sum_{j=0}^{\infty} a_{l^j}(F_1) a_{l^{j+t_{l,d}}}(F_1)\cdot l^{-jk_1}}{\sum_{j=0}^{\infty} a_{l^j}(F_1)^2 \cdot l^{-jk_1}} = \begin{cases} \frac{a_{l^{t_{l,d}}}(F_1) - l^{k_1-2} a_{l^{t_{l,d}-2}}(F_1)}{1+\psi_1(l)\cdot l^{-1}} & \text{if } t_{l,d} \geqslant 2 \\ \frac{a_l(F_1)}{1+\psi_1(l)\cdot l^{-1}} & \text{if } t_{l,d} = 1 \\ 1 & \text{if } t_{l,d} = 0. \end{cases}$$

Proof. At each prime l, let us factorise the Hecke polynomial for F_1 into $X^2 - a_l(F_1)X + \psi_1(l)\cdot l^{k_1-1} = (X-\alpha_l)(X-\alpha_l')$ where we choose $\alpha_l' = 0$ if $l|N_1$. Then quoting verbatim from (3.1) of [23], for any integer $t \geqslant 0$:

$$\begin{aligned} & Y_l(s) \times \sum_{j=0}^{\infty} a_{l^j}(F_1) a_{l^{j+t}}(F_1)\cdot l^{-js} \\ &= \begin{cases} a_{l^t}(F_1) - a_{l^{t-1}}(F_1)a_l(F_1)\alpha_l\alpha_l' \cdot l^{-s} + a_{l^{t-2}}(F_1)(\alpha_l\alpha_l')^3\cdot l^{-2s} & \text{if } t \geqslant 2 \\ a_l(F_1) - a_l(F_1)\alpha_l\alpha_l'\cdot l^{-s} & \text{if } t = 1 \\ 1 - (\alpha_l\alpha_l')^2 \cdot l^{-2s} & \text{if } t = 0, \end{cases} \end{aligned}$$

and the Euler factor[3] here is defined by

$$Y_l(s) := (1-\alpha_l^2 \cdot l^{-s})(1-\alpha_l'^2\cdot l^{-s})(1-\alpha_l\alpha_l'\cdot l^{-s})^2.$$

Putting $s = k_1$ and utilising the identities $\alpha_l + \alpha_l' = a_l(F_1)$ and $\alpha_l\alpha_l' = \psi_1(l)\cdot l^{k_1-1}$, the required quotient can be readily computed from this expression, firstly at $t = t_{l,d}$ and secondly at $t = 0$. We will leave these details as an exercise for the reader. $\square$

[3] In general, given two distinct cusp forms $\mathcal{F} = \sum_{n=1}^{\infty} a_n(\mathcal{F})\cdot q^n$ and $\mathcal{G} = \sum_{n=1}^{\infty} a_n(\mathcal{G})\cdot q^n$, the Euler factor $Y_l(s) = (1-\alpha_l\beta_l\cdot l^{-s})(1-\alpha_l\beta_l'\cdot l^{-s})(1-\alpha_l'\beta_l\cdot l^{-s})(1-\alpha_l'\beta_l'\cdot l^{-s})$ where α_l, α_l' (resp. β_l, β_l') denote the Weil numbers of $\mathcal{F}$ (resp. $\mathcal{G}$); moreover the actual formula for $\sum_{j=0}^{\infty} a_{l^j}(\mathcal{F})a_{l^{j+t}}(\mathcal{G})\cdot l^{-js}$ involves $\alpha_l, \alpha_l', \beta_l, \beta_l'$, and only simplifies to the above when $\mathcal{F} = \mathcal{G}$.

Corollary 2.15. *For each positive divisor d of N_0/N_1, one has the identity*

$$\frac{\left\langle F_1\big|_{k_1} V_{\frac{N_0}{N_1}}, F_1\big|_{k_1} V_d \right\rangle_{N_0}}{\left\langle F_1, F_1 \right\rangle_{N_1}} = \prod_{l|N_1} l^{\mathrm{ord}_l(N_0)-\mathrm{ord}_l(N_1)} \times \prod_{l|N_0, l\nmid N_1} (l+1)\cdot l^{\mathrm{ord}_l(N_0)-1}$$
$$\times \left(\frac{N_0}{N_1}\right)^{-k_1} \times \prod_{l\| \frac{N_0}{dN_1}} \frac{a_l(F_1)}{1+\psi_1(l)\cdot l^{-1}} \times \prod_{l^2 \left| \frac{N_0}{dN_1}\right.} \frac{a_{l^{t_{l,d}}}(F_1) - l^{k_1-2} a_{l^{t_{l,d}-2}}(F_1)}{1+\psi_1(l)\cdot l^{-1}}.$$

Proof. The result follows upon splitting up the quotient into a product

$$\frac{\left\langle F_1\big|_{k_1} V_{\frac{N_0}{N_1}}, F_1\big|_{k_1} V_d \right\rangle_{N_0}}{\left\langle F_1, F_1 \right\rangle_{N_1}} = \frac{\left\langle F_1\big|_{k_1} V_{\frac{N_0}{N_1}}, F_1\big|_{k_1} V_d \right\rangle_{N_0}}{\left\langle F_1, F_1 \right\rangle_{N_0}} \times \frac{\left\langle F_1, F_1 \right\rangle_{N_0}}{\left\langle F_1, F_1 \right\rangle_{N_1}}$$

and using the above lemma to compute the first ratio, whilst it is well known that

$$\frac{\left\langle F_1, F_1 \right\rangle_{N_0}}{\left\langle F_1, F_1 \right\rangle_{N_1}} = [\Gamma_0(N_1) : \Gamma_0(N_0)] = \frac{\prod_{l|N_0} l^{\mathrm{ord}_l(N_0)} + l^{\mathrm{ord}_l(N_0)-1}}{\prod_{l|N_1} l^{\mathrm{ord}_l(N_1)} + l^{\mathrm{ord}_l(N_1)-1}}. \qquad \square$$

3. Variation between the analytic λ-invariants

The technical portion of the paper is complete, and we now use these formulae to study the λ-invariant for both the double and triple product p-adic L-functions. A nice feature of our inner product expression is that the special values of both types of p-adic L-function can be treated on an equal footing, using the same ideas. However let us begin by streamlining the existing notation to avoid clutter later.

Definition 3.1.

(a) For $\varepsilon \in \{0,1\}$ and an integer $r \in \{0,\dots,\lfloor k_1/2 \rfloor\}$, one defines a linear functional $\mathcal{L}_{F_1}^{(r,\varepsilon)} = \mathcal{L}_{F_1}^{(r,\varepsilon)}(p, N_0, N_1, \widetilde{N}) : \mathcal{N}_{k_1,\mathrm{pol}}^{\infty,r}(\widetilde{N}, \psi_1^{-1}) \to \mathbb{C}$ by

$$\mathcal{L}_{F_1}^{(r,\varepsilon)}(H) := \epsilon_1^{-1} p^{\frac{(k_1-2)(\tilde{e}-2)}{2}-1} \left(\frac{\widetilde{N}^{(p)}}{N_1}\right)^{-\frac{k_1}{2}} \left(\frac{N_0}{N_1}\right)^{k_1}$$
$$\cdot \frac{\left\langle F_1^{\sharp}, \mathrm{Tr}_{N_0}^{\widetilde{N}}\left(\mathrm{Hol}_\infty(H)\big|_{k_1} W_{\widetilde{N}}^{\varepsilon}\right)\right\rangle_{N_0}}{\left\langle F_1, F_1 \right\rangle_{N_1}}$$

where $F_1\big|_{k_1} W_{N_1} = \epsilon_1 \cdot F_1^{\sharp}$, and the levels $\widetilde{N} = p^{\tilde{e}} \cdot \widetilde{N}^{(p)}$, $N_0 = p \cdot \widetilde{N}^{(p)}$ are as before.

(b) At each positive divisor d of N_0/N_1, we will introduce the algebraic number

$$\mathbf{X}_d(N_0,N_1) := \prod_{l|N_1} l^{\mathrm{ord}_l(N_0)-\mathrm{ord}_l(N_1)} \times \prod_{l|N_0, l\nmid N_1} (l+1)\cdot l^{\mathrm{ord}_l(N_0)-1}$$
$$\times \prod_{l\| \frac{N_0}{dN_1}} \frac{a_l(F_1)}{1+\psi_1(l)\cdot l^{-1}} \times \prod_{l^2\left| \frac{N_0}{dN_1}\right.} \frac{a_{l^{t_{l,d}}}(F_1) - l^{k_1-2}a_{l^{t_{l,d}-2}}(F_1)}{1+\psi_1(l)\cdot l^{-1}}$$

with the identical choice of exponent $t_{l,d} = \mathrm{ord}_l(N_0) - \mathrm{ord}_l(dN_1)$ from Section 2.5.

For instance, using these definitions above, one may repackage (2.13) into the more succinct form

$$\mathcal{L}_{F_1}^{(r,\varepsilon)}(G^{(\star)}) = \sum_{d\left|\frac{N_0}{N_1}\right.} \mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}(\mathcal{H})\cdot \mathbf{X}_d(N_0,N_1) \tag{3.1}$$

where $\mathcal{H}^{(\star)} = \mathrm{Hol}_\infty(G^{(\star)})\big|_{k_1} W_{\widetilde{N}}^{1+\varepsilon}$ at either choice of $\star\in\{\mathrm{I},\mathrm{II}\}$.

The $\mathbf{X}_d(N_0,N_1)$'s each have bounded denominators, and are independent of the $\mathcal{C}^\infty$-modular form $G^{(\star)}$. Furthermore, if $G^{(\star)} = \mathbf{g}^{(\star)}_{\Sigma,\chi}\cdot \delta^{(r)}_{k_1-k_2-2r}(\mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z,\psi_3))$ or if $G^{(\star)} = (\mathbf{g}^{(\star)}_{\Sigma}\cdot\delta^{(r)}_{k_1-k_2-2r}(\mathbf{h}^{(\star)}_{\Sigma}))\big|_{k_1} W_{\widetilde{N}}$, corresponding to the double product and triple product cases respectively, then the scalars $\mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}(\mathcal{H})$ are algebraic integers which are congruent to each other as one switches between $\star=\mathrm{I}$ and $\star=\mathrm{II}$.

Although we shall treat the double and triple product separately, the underlying methods are basically the same. *In both situations $F_1=\mathbf{f}$ will be a weight k_1 newform of level N_1, $p\nmid N_1$ and nebentypus ψ_1, where $\mathbf{f}^\sharp=\mathbf{f}$ and $\psi_1^2=\mathbf{1}$.* In addition, it is now necessary to assume that the cusp form $\mathbf{f}$ is ordinary at p.

3.1. The double product p-adic L-function. For two eigenforms F and G of weights $k_1>k_2$ and characters η_1,η_2, the L-function attached to $F\otimes G$ equals

$$\Psi(s,F,G) := \frac{\Gamma(s)\Gamma(s+1-k_2)}{(2\pi)^{2s}} \times \zeta(2s+2-k_1-k_2,\eta_1\eta_2)\cdot D(s,F,G) \tag{3.2}$$

with $\mathrm{Re}(s)\gg 0$, and this admits an analytic continuation to the complex plane. We write $\Psi_\Sigma(s,F,G)$ for the L-function stripped of Euler factors at primes $l\in\Sigma$.

Throughout assume we are given newforms $\mathbf{g}^{(\mathrm{I})},\mathbf{g}^{(\mathrm{II})}$ of weight k_2, character ψ, with conductors $N_{\mathbf{g}}^{(\mathrm{I})},N_{\mathbf{g}}^{(\mathrm{II})}$ respectively, and which satisfy:

$$a_n(\mathbf{g}^{(\mathrm{I})}) \equiv a_n(\mathbf{g}^{(\mathrm{II})}) \pmod{p^{\nu_2}} \quad \textit{for all } n\in\mathbb{N} \textit{ with } \gcd(n, N_{\mathbf{g}}^{(\mathrm{I})}N_{\mathbf{g}}^{(\mathrm{II})})=1.$$

We again choose the set Σ so that $\mathrm{supp}(N_{\mathbf{g}}^{(\mathrm{I})}N_{\mathbf{g}}^{(\mathrm{II})}) - \{p\} \subset \Sigma$, $\#\Sigma < \infty$ and $p \notin \Sigma$.

Proposition 3.2. *If χ has conductor $p^{n_\chi} \geqslant \max\left\{\left|N_{\mathbf{g}}^{(\mathrm{I})}\right|_p^{-\frac{1}{2}}, \left|N_{\mathbf{g}}^{(\mathrm{II})}\right|_p^{-\frac{1}{2}}\right\}$, then*

$$\mathcal{L}_{\mathbf{f}}^{(r,0)}\left(\mathbf{g}_{\Sigma,\chi}^{(\star)} \cdot \delta_{k_1-k_2-2r}^{(r)}(\mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z,\overline{\psi}_1\psi\chi^2))\right)$$
$$= \frac{(\widetilde{N}^{(p)})^{k_1-k_2/2-r}N_1^{-k_1/2}}{\epsilon_1 \cdot 2 \cdot (2i)^{k_1-1}} \times p^{n_\chi(2k_1-k_2-2r-2)+1} \cdot \frac{\Psi_\Sigma(k_1-1-r,\mathbf{f},\mathbf{g}^{(\star)} \otimes \chi)}{(2\pi i)^{1-k_2} \cdot \langle \mathbf{f},\mathbf{f}\rangle_{N_1}}$$

at each integer r in the range $0 \leqslant 2r < k_1 - k_2$, and for either choice of $\star \in \{\mathrm{I},\mathrm{II}\}$.

Proof. Recall that $\psi_3 = \overline{\psi}_1 \cdot \psi \cdot \chi^2$ and $\widetilde{N} = \mathrm{lcm}(N_1, \widetilde{N}_{\Sigma,\chi}) = p^{\tilde{e}} \cdot \widetilde{N}^{(p)}$. An essential starting point is the following formula[4] of Shimura [23, Theorem 2],

$$D(k_1-1-r,\mathbf{f},\mathbf{g}_{\Sigma,\chi}^{(\star)}) = \frac{(-1)^r(4\pi)^{k_1-1} \cdot \Gamma(k_1-k_2-2r)}{\Gamma(k_1-1-r)\cdot\Gamma(k_1-k_2-r)}$$
$$\times \left\langle \mathbf{f}^\sharp, \mathbf{g}_{\Sigma,\chi}^{(\star)} \cdot \delta_{k_1-k_2-2r}^{(r)}(E^*_{k_1-k_2-2r,\widetilde{N}}(z,\psi_3))\right\rangle_{\widetilde{N}}$$

where $E^*_{k_1-k_2-2r,\widetilde{N}}(z,\eta)$ denotes the $\mathcal{C}^\infty$-modular form defined in (2.3), and $D(s,\mathbf{f},\mathbf{g}_{\Sigma,\chi}^{(\star)})$ coincides with the Σ-depleted convolution L-function

$$D_\Sigma(s,\mathbf{f},\mathbf{g}_\chi^{(\star)}) = \sum_{\substack{n=1,\\ \mathrm{supp}(n)\cap\Sigma=\emptyset}}^{\infty} a_n(\mathbf{f})a_n(\mathbf{g}^{(\star)})\chi(n)\cdot n^{-s}, \quad \mathrm{Re}(s) \gg 0.$$

Reconciling the different normalisation of Eisenstein series in (2.3–2.4), one may rephrase Shimura's identity above into an equivalent form

$$\left\langle \mathbf{f}^\sharp, \mathbf{g}_{\Sigma,\chi}^{(\star)} \cdot \delta_{k_1-k_2-2r}^{(r)}(\mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z,\psi_3))\right\rangle_{\widetilde{N}}$$
$$= \frac{(-1)^r}{(4\pi)^{k_1-1}} \cdot \frac{\widetilde{N}^{\frac{k_1-k_2-2r}{2}}}{2(2\pi i)^{k_1-k_2-2r}} \times \Gamma(k_1-1-r)$$
$$\cdot\,\Gamma(k_1-k_2-r)\cdot\zeta_{\widetilde{N}}(k_1-k_2-2r,\psi_3)\cdot D_\Sigma(k_1-1-r,\mathbf{f},\mathbf{g}_\chi^{(\star)})$$
$$= (4\pi^2)^{k_1-1-r}\cdot\frac{(-1)^r}{(4\pi)^{k_1-1}}\cdot\frac{\widetilde{N}^{\frac{k_1-k_2-2r}{2}}}{2(2\pi i)^{k_1-k_2-2r}} \times \Psi_\Sigma(k_1-1-r,\mathbf{f},\mathbf{g}_\chi^{(\star)}).$$

In fact, the terms directly before $\Psi_\Sigma(\cdots)$ can be simplified to $(2i)^{k_2-k_1}\cdot\frac{\widetilde{N}^{\frac{k_1-k_2-2r}{2}}}{2\pi^{1-k_2}}$, which means that if $G^{(\star)} = \mathbf{g}_{\Sigma,\chi}^{(\star)}\cdot\delta_{k_1-k_2-2r}^{(r)}(\mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z,\psi_3))$

[4] His normalisation of the Petersson inner product differs from ours by $\mathrm{vol}(\Gamma_1(\widetilde{N})\backslash\mathfrak{H})^{-1}$.

then

$$\frac{\langle \mathbf{f}^\sharp, G^{(\star)}\rangle_{\widetilde{N}}}{\langle \mathbf{f},\mathbf{f}\rangle_{N_1}} = \frac{\widetilde{N}^{\frac{k_1-k_2-2r}{2}}}{2(2i)^{k_1-k_2}} \times \frac{\Psi_\Sigma(k_1-1-r,\mathbf{f},\mathbf{g}_\chi^{(\star)})}{\pi^{1-k_2}\cdot\langle \mathbf{f},\mathbf{f}\rangle_{N_1}}.$$

Focussing on the left-hand side, as $G^{(\star)}\big|_{k_1}\gamma$ has moderate growth for all $\gamma\in \mathrm{SL}_2(\mathbb{Z})$ it follows from Theorem 2.5 that

$$\frac{\langle \mathbf{f}^\sharp, G^{(\star)}\rangle_{\widetilde{N}}}{\langle \mathbf{f},\mathbf{f}\rangle_{N_1}} = \frac{\langle \mathbf{f}^\sharp, \mathrm{Hol}_\infty(G^{(\star)})\rangle_{\widetilde{N}}}{\langle \mathbf{f},\mathbf{f}\rangle_{N_1}} = \frac{\langle \mathbf{f}^\sharp, \mathrm{Tr}_{\widetilde{N}/N_0}(\mathrm{Hol}_\infty(G^{(\star)}))\rangle_{N_0}}{\langle \mathbf{f},\mathbf{f}\rangle_{N_1}}$$

and so by Definition 3.1 (a),

$$\begin{aligned}\mathcal{L}_{\mathbf{f}}^{(r,0)}(G^{(\star)}) &= \epsilon_1^{-1}\cdot p^{\frac{(k_1-2)(\tilde{e}-2)}{2}-1}\cdot\left(\frac{\widetilde{N}^{(p)}}{N_1}\right)^{-\frac{k_1}{2}}\cdot\left(\frac{N_0}{N_1}\right)^{k_1}\times\frac{\langle \mathbf{f}^\sharp, G^{(\star)}\rangle_{\widetilde{N}}}{\langle \mathbf{f},\mathbf{f}\rangle_{N_1}}\\ &= \epsilon_1^{-1}\cdot p^{\frac{(k_1-2)(\tilde{e}-2)}{2}-1}\cdot\left(\frac{\widetilde{N}^{(p)}}{N_1}\right)^{-\frac{k_1}{2}}\cdot\left(\frac{N_0}{N_1}\right)^{k_1}\cdot\frac{\widetilde{N}^{\frac{k_1-k_2-2r}{2}}}{2(2i)^{k_1-k_2}}\\ &\qquad\times\frac{\Psi_\Sigma(k_1-1-r,\mathbf{f},\mathbf{g}_\chi^{(\star)})}{\pi^{1-k_2}\cdot\langle \mathbf{f},\mathbf{f}\rangle_{N_1}}.\end{aligned}$$

Provided that $p^{2n_\chi}\geqslant \max\left\{\big|N_{\mathbf{g}}^{(\mathrm{I})}\big|_p^{-1},\big|N_{\mathbf{g}}^{(\mathrm{II})}\big|_p^{-1}\right\}$, the p-part of the level of both cusp forms $\mathbf{g}_{\Sigma,\chi}^{(\mathrm{I})}$ and $\mathbf{g}_{\Sigma,\chi}^{(\mathrm{II})}$ equals p^{2n_χ}: thus $\widetilde{e}=2n_\chi$, $\widetilde{N}=p^{2n_\chi}\cdot\widetilde{N}^{(p)}$ and $N_0=p\cdot\widetilde{N}^{(p)}$. Substituting these values into our formula, the result follows after a clean-up. ☐

Let K be the number field generated by the Fourier coefficients of $\mathbf{f},\mathbf{g}^{(\mathrm{I})},\mathbf{g}^{(\mathrm{II})}$. Since the newform $\mathbf{f}$ is p-ordinary, we can factorise its Hecke polynomial at p into

$$X^2-a_p(\mathbf{f})X+\psi_1(p)\cdot p^{k_1-1}=(X-\alpha_p)(X-\alpha_p')$$

where $\big|\alpha_p\big|_p=1$ and $\big|\alpha_p'\big|_p=p^{1-k_1}<1$. Now applying the results of Hida and Panchishkin [14, 21], for each choice of $\star\in\{\mathrm{I},\mathrm{II}\}$ there exists a p-adic L-function $\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\star)})\in\mathcal{O}_{K,p}[\![\mathbb{Z}_p^\times]\!][1/p]$ interpolating

$$\begin{aligned}&\chi x_p^s\left(\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\star)})\right)\\ &\qquad=\psi(p)^{n_\chi}\cdot\frac{\tau(\overline{\chi})^2\cdot p^{n_\chi(k_2+2s-1)}}{(-1)^s\cdot\alpha_p^{2n_\chi}}\cdot\mathcal{A}(s,\overline{\chi})\times\frac{\Psi(k_2+s,\mathbf{f},\mathbf{g}_{\Sigma,\chi}^{(\star)})}{(2\pi i)^{1-k_2}\cdot\langle \mathbf{f},\mathbf{f}\rangle_{N_1}}\end{aligned}$$

at all integers $s\in\{0,\dots,k_1-k_2-1\}$. Here $\tau(\chi)=\sum_{j=1}^{p^{n_\chi}}\chi(n)e^{2\pi ij/p^{n_\chi}}$ denotes a Gauss sum for χ, and the p-Euler factor term $\mathcal{A}(s,\overline{\chi})$ is equal to 1 whenever $\chi\neq\mathbf{1}$.

Remarks.

(i) If one changes variable by instead setting $s = k_1 - k_2 - r - 1$, then for $\chi \neq \mathbf{1}$ the above becomes

$$\chi x_p^s \left(\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\star)})\right) = \psi(p)^{n_\chi} \cdot \frac{\tau(\bar{\chi})^2 \cdot p^{n_\chi(2k_1-k_2-2r-3)}}{(-1)^{k_1-k_2-r-1} \cdot \alpha_p^{2n_\chi}} \times \frac{\Psi(k_1-1-r, \mathbf{f}, \mathbf{g}_{\Sigma,\chi}^{(\star)})}{(2\pi i)^{1-k_2} \cdot \langle \mathbf{f}, \mathbf{f} \rangle_{N_1}}.$$

(ii) The formula in Proposition 3.2 can similarly be expressed in the form

$$\mathcal{L}_{\mathbf{f}}^{(r,0)}(G^{(\star)}) = \frac{(\widetilde{N}^{(p)})^{k_1-k_2/2-r} N_1^{-k_1/2}}{\epsilon_1 \cdot 2 \cdot (2i)^{k_1-1}} \cdot p^{n_\chi(2k_1-k_2-2r-2)+1} \times \frac{\Psi(k_1-1-r, \mathbf{f}, \mathbf{g}_{\Sigma,\chi}^{(\star)})}{(2\pi i)^{1-k_2} \cdot \langle \mathbf{f}, \mathbf{f} \rangle_{N_1}}.$$

(iii) Consequently, $(-1)^s \cdot \chi x_p^s(\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\star)})) = p^{-1} \cdot \Xi_{r,\chi} \times \mathcal{L}_{\mathbf{f}}^{(r,0)}\left(G^{(\star)}\right)$ where

$$\Xi_{r,\chi} := \left(\frac{\psi(p)}{\alpha_p^2}\right)^{n_\chi} \cdot \frac{\tau(\bar{\chi})^2}{p^{n_\chi}} \times \frac{\epsilon_1 \cdot 2 \cdot (2i)^{k_1-1}}{(\widetilde{N}^{(p)})^{k_1-k_2/2-r} N_1^{-k_1/2}}$$

is actually a p-adic unit.

There is a natural decomposition $\mathbb{Z}_p^\times \cong \mathbb{F}_p^\times \times (1 + p\mathbb{Z}_p)$, and let $\omega : \mathbb{Z}_p^\times \twoheadrightarrow \mu_{p-1}$ be the Teichmüller character, so that $\omega(a) \equiv a \pmod p$ and $\omega(1+p\mathbb{Z}_p) = \{1\}$. One can split the Iwasawa algebra up into $\mathbb{F}_p^\times$-eigenfactors

$$\mathcal{O}_{K,p}[\![\mathbb{Z}_p^\times]\!] \cong \bigoplus_{j=0}^{p-2} \mathcal{O}_{K,p}[\![1+p\mathbb{Z}_p]\!]_{(\omega^j)} \xrightarrow{\sim} \bigoplus_{j=0}^{p-2} \mathcal{O}_{K,p}[\![X]\!]_{(\omega^j)}$$

where the last isomorphism arises by sending $1 + p \in \mathbb{Z}_p^\times$ to the polynomial $X + 1$. For each $j \in \mathbb{Z}$ and $\star \in \{\mathrm{I}, \mathrm{II}\}$, we will write $\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\star)}, \omega^j)$ for the image of the Hida–Panchishkin p-adic L-function inside the ω^j-eigenspace $\mathcal{O}_{K,p}[\![X]\!][1/p]_{(\omega^j)}$. Let us also choose a local parameter, ϖ, for the discrete valuation ring $\mathcal{O}_{K,p}$.

Theorem 3.3. *At each $j \in \{0, \ldots, p-2\}$, let us define $\mu_{\mathrm{I,II}}^{(j)}$ to be the minimum of $\mu_\varpi(\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\mathrm{I})}, \omega^j))$ and $\mu_\varpi(\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\mathrm{II})}, \omega^j))$. If the prime $p > k_1 - 2$, then one obtains a congruence of Σ-imprimitive p-adic L-functions*

$$\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\mathrm{I})}, \omega^j) \equiv \mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\mathrm{II})}, \omega^j) \mod \varpi^{\mathfrak{e}_p \nu_2 + \mu_{\mathrm{I,II}}^{(j)}} \cdot \mathcal{O}_{K,p}[\![X]\!]_{(\omega^j)}$$

where the ramification index $\mathfrak{e}_p \in \mathbb{N}$ satisfies $\langle \varpi \rangle^{\mathfrak{e}_p} = p \cdot \mathcal{O}_{K,p}$.

Proof. We first pick an integer $s = k_1 - k_2 - r - 1 \geqslant 0$ to Tate twist by. Consider the $\mathcal{O}_{\mathbb{C}_p}$-module, $\mathbb{L}^{(j,r)}$, generated by the special values $\mathcal{L}_{\mathbf{f}}^{(r,0)}(G_\chi^{(\star)})$ where

$$G_\chi^{(\star)} := \mathbf{g}_{\Sigma,\chi}^{(\star)} \cdot \delta_{k_1-k_2-2r}^{(r)}(\mathbf{E}^*_{k_1-k_2-2r,\widetilde{N}}(z, \bar{\psi}_1 \psi \chi^2)) \in \mathcal{M}_{k_1}^{\infty}(\widetilde{N}, \psi_1), \quad \text{and}$$

χ ranges over non-trivial characters of conductor $p^{n_\chi} \geqslant \max\Big\{ \big|N_{\mathbf{g}}^{(\mathrm{I})}\big|_p^{-\frac{1}{2}}, \big|N_{\mathbf{g}}^{(\mathrm{II})}\big|_p^{-\frac{1}{2}} \Big\}$ such that $\chi\big|_{\mathbb{F}_p^\times} = \omega^j$.

Using the identity $\chi x_p^s(\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\star)})) = \pm p^{-1} \Xi_{r,\chi} \cdot \mathcal{L}_{\mathbf{f}}^{(r,0)}(G^{(\star)})$ in Remark (iii), and also because $\big|\Xi_{r,\chi}\big|_p^{-1} = 1$, it follows that $\mathbb{L}^{(j,r)} = \varpi^{\mathfrak{c}_p + \mu_{\mathrm{I,II}}^{(j)}} \cdot \mathcal{O}_{\mathbb{C}_p}$ where $\mu_{\mathrm{I,II}}^{(j)} = \min_{\star \in \{\mathrm{I,II}\}} \Big\{ \mu_\varpi(\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\star)}, \omega^j)) \Big\} \in \mathbb{Z} \cup \{\pm\infty\}$. From a naive perspective only three possibilities can ever happen:

(a) $\mathbb{L}^{(j,r)} = \{0\}$,
(b) $\mathbb{L}^{(j,r)} = \varpi^{\mathfrak{c}_p + \mu_{\mathrm{I,II}}^{(j)}} \cdot \mathcal{O}_{\mathbb{C}_p}$ with $\mu_{\mathrm{I,II}}^{(j)} \neq \pm\infty$, or alternatively
(c) $\mathbb{L}^{(j,r)} = \mathbb{C}_p$.

In case (a) one has $\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\mathrm{I})}, \omega^j) = \mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\mathrm{II})}, \omega^j) = 0$ and therefore $\mu_\varpi(\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\star)}, \omega^j)) = +\infty$, so the congruence is vacuously true and moreover content-free. On the other hand, if we are in case (c) then $\mu_\varpi(\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\star)}, \omega^j)) = -\infty$, which would then imply that the ω^j-branches of $\mathbf{L}_p(\mathbf{f} \otimes \mathbf{g}_\Sigma^{(\star)})$ arise from an unbounded p-adic measure. This directly contradicts the work in [14, 21] and so never occurs!

This leaves us to deal with the interesting case (b). Recall from (3.1) that the linear functional degenerates into a finite sum

$$\mathcal{L}_{F_1}^{(r,0)}(G_\chi^{(\star)}) = \sum_{d \left| \frac{N_0}{N_1} \right.} \mathfrak{c}_{d,\widetilde{N},\tilde{e}}^{(\star)}(\mathcal{H}_\chi) \cdot \mathbf{X}_d(N_0, N_1)$$

where $\mathcal{H}_\chi^{(\star)} = \mathrm{Hol}_\infty(G_\chi^{(\star)})\big|_{k_1} W_{\widetilde{N}}$, and the $\mathbf{X}_d(N_0, N_1)$'s are independent of $G_\chi^{(\star)}$.

Applying Proposition 2.12, one has congruences $\mathfrak{c}_{d,\widetilde{N},\tilde{e}}^{(\mathrm{I})}(\mathcal{H}_\chi) \equiv \mathfrak{c}_{d,\widetilde{N},\tilde{e}}^{(\mathrm{II})}(\mathcal{H}_\chi)$ $(\bmod\, p^{\nu_2})$ at every $d \big| \frac{N_0}{N_1}$ and finite order character χ on $\mathbb{Z}_p^\times$. As an immediate consequence

$$\mathcal{L}_{F_1}^{(r,0)}(G_\chi^{(\mathrm{I})}) - \mathcal{L}_{F_1}^{(r,0)}(G_\chi^{(\mathrm{II})}) \in \varpi^{\mathfrak{c}_p + \mu_{\mathrm{I,II}}^{(j)}} \cdot p^{\nu_2} \cdot \mathcal{O}_{\mathbb{C}_p},$$

i.e. $\chi x_p^s(\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{I})})-\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{II})}))\in\varpi^{\mathfrak{e}_p\nu_2+\mu_{\mathrm{I,II}}^{(j)}}\cdot\mathcal{O}_{\mathbb{C}_p}$ at almost all characters[5] $\chi:\mathbb{Z}_p^\times\to\overline{\mathbb{Q}}_p^\times$ such that $\chi\big|_{\mathbb{F}_p^\times}=\omega^j$. The rest now follows by p-adic continuity. $\square$

Let us instead consider primitive versions of these double product L-functions, namely $\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{I})},\omega^j)$ and $\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{II})},\omega^j)$ which both belong to $\mathcal{O}_{K,p}[\![X]\!][1/p]_{(\omega^j)}$. For either choice of $\star\in\{\mathrm{I},\mathrm{II}\}$, they are related to their Σ-imprimitive cousins via

$$\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\star)},\omega^j)=\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j)\times\prod_{l\in\Sigma}E_l(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j)\tag{3.3}$$

where each term $E_l(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j)\in\mathcal{O}_{K,p}[\![X]\!]$ p-adically interpolates the Euler factor $L_l(\mathbf{f}\otimes\mathbf{g}^{(\star)}\otimes\chi\omega^j,s)$ as χ ranges over finite order characters on $1+p\mathbb{Z}_p\subset\mathbb{Z}_p^\times$.

Definition 3.4. At each prime l and branch $j\in\{0,\dots,p-2\}$, let us define the non-negative integer $\mathbf{e}_l^{(\star)}(\omega^j):=$ the λ-invariant of $E_l(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j)$.

Theorem 3.5. *If the prime $p>k_1-2$, then*

$$\lambda(\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{I})},\omega^j))=\lambda(\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{II})},\omega^j))+\sum_{l\mid N_{\mathbf{g}}^{(\mathrm{I})}N_{\mathbf{g}}^{(\mathrm{II})}}\mathbf{e}_l^{(\mathrm{II})}(\omega^j)-\mathbf{e}_l^{(\mathrm{I})}(\omega^j).$$

Proof. Firstly, we note that the Euler factors $E_l(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j)$ in (3.3) for primes $l\in\Sigma$ each have unit content, and therefore possess a trivial μ-invariant. If $\mu_{\mathrm{I,II}}^{(j)}\in\mathbb{Z}\cup\{+\infty\}$ denotes the minimum of the μ-invariants for $\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{I})},\omega^j)$ and $\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}^{(\mathrm{II})},\omega^j)$, then by Theorem 3.3 one has

$$\varpi^{-\mu_{\mathrm{I,II}}^{(j)}}\cdot\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{I})},\omega^j)\equiv\varpi^{-\mu_{\mathrm{I,II}}^{(j)}}\cdot\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{II})},\omega^j)\mod\varpi^{\mathfrak{e}_p\cdot\nu_2}\cdot\mathcal{O}_{K,p}[\![X]\!].$$

Moreover as $\mathfrak{e}_p\cdot\nu_2\geqslant 1$, we can then deduce that

$$\begin{aligned}\lambda(\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{I})},\omega^j))&=\mathrm{rank}_{\mathbb{F}[\![X]\!]}\left(\mathcal{O}_{K,p}[\![X]\!]\Big/\langle\varpi,\varpi^{-\mu_{\mathrm{I,II}}^{(j)}}\cdot\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{I})},\omega^j)\rangle\right)\\&=\mathrm{rank}_{\mathbb{F}[\![X]\!]}\left(\mathcal{O}_{K,p}[\![X]\!]\Big/\langle\varpi,\varpi^{-\mu_{\mathrm{I,II}}^{(j)}}\cdot\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{II})},\omega^j)\rangle\right)\\&=\lambda(\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\mathrm{II})},\omega^j))\end{aligned}$$

where $\mathbb{F}=\mathcal{O}_{K,p}/\langle\varpi\rangle$ indicates the residue field. Finally, using (3.3) in tandem with the additivity of the λ-invariant, clearly one has a relation

$$\lambda(\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}_\Sigma^{(\star)},\omega^j))=\lambda(\mathbf{L}_p(\mathbf{f}\otimes\mathbf{g}^{(\star)},\omega^j))+\mathbf{e}_l^{(\star)}(\omega^j).$$

[5] This containment is also true for the missing characters, which can be seen by exploiting the p-adic density of finite order characters χ with $\chi\big|_{\mathbb{F}_p^\times}=\omega^j$ inside the parameter space $1+p\mathbb{Z}_p$.

The result follows upon observing that $\mathbf{e}_l^{(\mathrm{I})}(\omega^j) = \mathbf{e}_l^{(\mathrm{II})}(\omega^j)$ at any prime $l \in \Sigma$ such that $l \nmid N_{\mathbf{g}}^{(\mathrm{I})} N_{\mathbf{g}}^{(\mathrm{II})}$, because here $E_l(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{I})}, \omega^j) \equiv E_l(\mathbf{f} \otimes \mathbf{g}^{(\mathrm{II})}, \omega^j) \bmod \varpi^{\mathfrak{e}_p \cdot \nu_2}$. $\square$

3.2. The triple product *p*-adic *L*-function. We shall closely follow the notation employed by Fukunaga and Hsieh in [10, 15]. In particular, $\mathbb{I}_i$ denotes a normal finite flat extension of the algebra $\Lambda_{\mathrm{wt}} = \mathcal{O}_K[\![\Gamma^{\mathrm{wt}}]\!]$ at each $i \in \{1, 2, 3\}$, with $\Gamma^{\mathrm{wt}} = 1 + p\mathbb{Z}_p$ and $[K : \mathbb{Q}_p] < \infty$. Let us fix a triple of $\mathbb{I}_i$-adic forms $(\mathbf{F}_1, \mathbf{G}^{(2)}, \mathbf{G}^{(3)})$ such that $\mathbf{F}_1 := \mathbf{G}^{(1)} \in \mathcal{S}^{\mathrm{ord}}(C_1, \psi_1; \mathbb{I}_1)$ and also $\mathbf{G}^{(i)} \in \mathcal{S}^{\mathrm{ord}}(C_i, \psi_i; \mathbb{I}_i)$ for $i = 2, 3$ are each primitive families in the sense of Hida [14], and have expansions in $\mathbb{I}_i[\![q]\!]$.

For a choice of index $i \in \{1, 2, 3\}$, we consider the set of non-zero continuous $\mathcal{O}_K$-algebraic homorphisms $\mathfrak{X}_i := \{\mathcal{Q}_m^{(i)} : \mathbb{I}_i \to \overline{\mathbb{Q}}_p\}_{m \in \mathbb{N}}$. Now given such a formal series $\mathbf{G}^{(i)} \in \mathbb{I}_i[\![q]\!]$ as described above, at every $m \geqslant 1$ one can take its specialisation

$$\mathbf{G}^{(i)}(m) := \sum_{n=0}^{\infty} \mathcal{Q}_m^{(i)}(a_n(\mathbf{G}^{(i)})) \cdot q^n \in \overline{\mathbb{Q}}_p[\![q]\!]$$

which yields a normalised p-stabilised newform of weight $k^{(i)}(m)$, level $p^{e^{(i)}(m)} C_i$ and character $\psi_i \omega^{-k^{(i)}(m)} \epsilon_m^{(i)}$, where $\epsilon_m^{(i)}$ is the restriction of $\mathcal{Q}_m^{(i)}$ to $\Gamma^{\mathrm{wt}} \subset \Lambda_{\mathrm{wt}}$.

Definition 3.6. If $\mathcal{R} = \mathbb{I}_1 \widehat{\otimes}_{\mathcal{O}_K} \mathbb{I}_2 \widehat{\otimes}_{\mathcal{O}_K} \mathbb{I}_3$ is the three-parameter weight algebra, then the unbalanced domain $\mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$ of interpolation points for $\mathcal{R}$ is given by

$$\mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1} := \left\{ \underline{\mathcal{Q}} = (\mathcal{Q}_{m_1}^{(1)}, \mathcal{Q}_{m_2}^{(2)}, \mathcal{Q}_{m_3}^{(3)}) \in \mathfrak{X}_1 \times \mathfrak{X}_2 \times \mathfrak{X}_3 \;\middle|\; \begin{array}{l} k_1 + k_2 + k_3 \equiv 0 \pmod 2 \\ k_1 > k_2 + k_2 - 1, \;\; k_1 \geqslant 2 \end{array} \right\}$$

where we abbreviate $(k^{(1)}(m_1), k^{(2)}(m_2), k^{(3)}(m_3))$ by instead using (k_1, k_2, k_3).

Let $\Pi'_{\underline{\mathcal{Q}}}$ be the product of the automorphic representations $\pi_{\mathbf{G}^{(i)}(m)}$ on $\mathrm{GL}_2(\mathbb{A})$ associated to the triple $(\mathbf{F}_1, \mathbf{G}^{(2)}, \mathbf{G}^{(3)})(\underline{\mathcal{Q}})$, and define $\Pi_{\underline{\mathcal{Q}}} := \Pi'_{\underline{\mathcal{Q}}} \otimes (\chi_{\underline{\mathcal{Q}}})_{\mathbb{A}}$ with

$$\chi_{\underline{\mathcal{Q}}} = \omega^{-\frac{k^{(1)}(m_1) + k^{(2)}(m_2) + k^{(3)}(m_3)}{2}} \cdot (\epsilon_m^{(1)} \epsilon_m^{(2)} \epsilon_m^{(3)})^{\frac{1}{2}} \quad \text{at every point } \underline{\mathcal{Q}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}.$$

Passing from the automorphic viewpoint to the setting of Galois representations, one has an identification of complex L-series

$$L(\Pi_{\underline{\mathcal{Q}}}, s) = \Gamma(\Pi_{\underline{\mathcal{Q}}, \infty}, s) \cdot \prod_{l \in \mathrm{Spec}\, \mathbb{Z}} L_l\Big(\mathbf{F}_1(m) \otimes \mathbf{G}^{(2)}(m) \otimes \mathbf{G}^{(3)}(m) \otimes \chi_{\underline{\mathcal{Q}}}, s + \frac{w-1}{2}\Big)$$

where $\Gamma(\Pi_{\underline{\mathcal{Q}},\infty}, s) = \Gamma_{\mathbb{C}}(s+w/2)\cdot\prod_{i=1}^{3}\Gamma_{\mathbb{C}}(s+1-k_i^*)$ is the factor at infinity, $w = k^{(1)}(m_1) + k^{(2)}(m_2) + k^{(3)}(m_3) - 2$, and each $k_i^* = w/2 + 1 - k^{(i)}(m_i)$.

The following conditions (which are copied directly from those given in [10]) will guarantee us the existence of a p-adic L-function attached to $\mathbf{F}_1 \otimes \mathbf{G}^{(2)} \otimes \mathbf{G}^{(3)}$.

Hypothesis (T1). The primitive characters satisfy $\psi_1\psi_2\psi_3 = \mathbf{1}$.

Hypothesis (T2). The residual Galois representation $\bar{\rho}_{\mathbf{F}_1} : G_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is absolutely irreducible, and the semi-simplification of $\bar{\rho}_{\mathbf{F}_1}\big|_{G_{\mathbb{Q}_p}} \cong \theta_1 \oplus \theta_2$ with $\theta_1 \neq \theta_2$.

Hypothesis (T3). The value of $\gcd(C_1, C_2, C_3)$ is a square-free integer.

Hypothesis (T4). At each $\underline{\mathcal{Q}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$ and $l\big|C_1C_2C_3$, one has $\epsilon(1/2, \Pi_{\underline{\mathcal{Q}},l}) = +1$ where $\epsilon(s, \Pi_{\underline{\mathcal{Q}},l})$ denotes the local ϵ-factor at a prime l, as defined by Ikeda in [17].

Theorem 3.7 (Hsieh–Fukunaga [10, 15])**.** *Under the Hypotheses* (T1)–(T4)*, there exists a unique element $\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}^{(2)},\mathbf{G}^{(3)}} \in \mathcal{R}$ satisfying the interpolation property*

$$(\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}^{(2)},\mathbf{G}^{(3)}}(\underline{\mathcal{Q}}))^2 = \mathcal{E}_{\mathbf{F}_1(m)}(\Pi_{\underline{\mathcal{Q}},p}) \cdot \frac{L(\Pi_{\underline{\mathcal{Q}}}, 1/2)}{\sqrt{-1}^{2k^{(1)}(m_1)} \cdot \Omega^2_{\mathbf{F}_1(m)}}$$

at all unbalanced points $\underline{\mathcal{Q}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$, where the p-Euler factor $\mathcal{E}_{\mathbf{F}_1(m)}(\Pi_{\underline{\mathcal{Q}},p})$ and the canonical period $\Omega_{\mathbf{F}_1(m)}$ are given in [10, (3.3.1) and Definition 3.3.4], *respectively.*

To avoid possible confusion later on, the element $\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}^{(2)},\mathbf{G}^{(3)}}$ is the square-root of the p-adic L-function, $\mathbf{L}_p(\mathbf{F}_1, \mathbf{G}^{(2)}, \mathbf{G}^{(3)})$, originally mentioned in the Introduction. Therefore any congruence modulo p^ν one can prove for the former automatically implies the same mod p^ν congruence holds for the latter. The construction of $\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}^{(2)},\mathbf{G}^{(3)}}$ from [10] involves gluing "$\mathbf{G}^{(2)} \cdot \delta_\bullet^{(r)}(\mathbf{G}^{(3)})(\underline{\mathcal{Q}})$" along the unbalanced points $\mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$ to produce an interpolating family $\mathbf{H}^{\mathrm{aux}} \in \mathcal{S}^{\mathrm{ord}}(N, \psi_{1,(p)}\overline{\psi}_1^{(p)}; \mathbb{I}_1) \otimes_{\mathbb{I}_1} \mathcal{R}$. One then sets

$$L^{\mathbf{F}_1}_{\mathbf{G}^{(2)},\mathbf{G}^{(3)}} := \text{the first Fourier coefficient of } \eta_{\mathbf{F}_1} \cdot 1_{\mathbf{F}_1} \cdot \mathrm{Tr}_{N/C_1}(\mathbf{H}^{\mathrm{aux}})$$

with $N := C_1C_2C_3$, and where the operators $\eta_{\mathbf{F}_1}, 1_{\mathbf{F}_1}$ will be introduced shortly (in fact $\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}^{(2)},\mathbf{G}^{(3)}}$ and $L^{\mathbf{F}_1}_{\mathbf{G}^{(2)},\mathbf{G}^{(3)}}$ differ from each other by a very simple $\mathcal{R}$-unit).

3.2.1. ***The basic congruences set-up.*** At the risk of bombarding the reader with too many superscripts, suppose that we are given two primitive $\mathbb{I}_i$-adic triples

$$(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{I})}, \mathbf{G}^{(3),(\mathrm{I})}) \quad \text{and} \quad (\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{II})}, \mathbf{G}^{(3),(\mathrm{II})})$$

where $\mathbf{F}_1$ has level $N_1 = C_1$, and the families $\mathbf{G}^{(i),(\star)}$ have level equal to $C_i^{(\star)}$. *Assume there exists a one-dimensional subset (i.e. line) $\mathcal{V} \subset \mathfrak{X}_1 \times \mathfrak{X}_2 \times \mathfrak{X}_3$ in the parameter space, such that for all unbalanced points $\underline{\mathcal{Q}} \in \mathcal{V} \cap \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$:*

$$(3.4)\ \underline{\mathcal{Q}}\Big(a_n(\mathbf{G}^{(2),(\mathrm{I})})\Big) \equiv \underline{\mathcal{Q}}\Big(a_n(\mathbf{G}^{(2),(\mathrm{II})})\Big) \pmod{p^{\nu_2}} \text{ if } \gcd(n, C_2^{(\mathrm{I})} C_2^{(\mathrm{II})}) = 1,$$

$$(3.5)\ \underline{\mathcal{Q}}\Big(a_n(\mathbf{G}^{(3),(\mathrm{I})})\Big) \equiv \underline{\mathcal{Q}}\Big(a_n(\mathbf{G}^{(3),(\mathrm{II})})\Big) \pmod{p^{\nu_3}} \text{ if } \gcd(n, C_3^{(\mathrm{I})} C_3^{(\mathrm{II})}) = 1.$$

We also suppose the image of the specialisations $\phi_{\mathcal{V}} : \mathcal{R} \to \bigoplus_{\underline{\mathcal{Q}} \in \mathcal{V} \cap \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}} \underline{\mathcal{Q}}(\mathcal{R})$ glues into a one-parameter algebra, $\mathbb{I}^{\mathcal{V}} \cong \overline{\phi_{\mathcal{V}}(\mathcal{R})}$, of finite-type over Λ_{wt}.

Let us write $\mu_{\mathrm{wt}}^{(\mathcal{V})} \in \mathbb{Z} \cup \{-\infty, +\infty\}$ for the minimum of the (weight) μ-invariants associated to $\phi_{\mathcal{V}}\left(\mathbf{L}_p(\mathbf{F}_1, \mathbf{G}^{(2),(\star)}, \mathbf{G}^{(3),(\star)})\right) \in \mathbb{I}^{\mathcal{V}}$ over both choices of $\star \in \{\mathrm{I}, \mathrm{II}\}$. The theorem immediately below is the primary technical result in this section.

Theorem 3.8. *If both triples $(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{I})}, \mathbf{G}^{(3),(\mathrm{I})})$ and $(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{II})}, \mathbf{G}^{(3),(\mathrm{II})})$ satisfy Hypotheses* (T1)–(T4)*, if the congruences* (3.4)–(3.5) *hold for $\nu_2, \nu_3 \geqslant 1$, if the points $\underline{\mathcal{Q}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$ with $p \nmid \frac{(k_1-2)!}{(k_1-2-r)!}$ are dense in* $\mathrm{Spec}(\mathbb{I}^{\mathcal{V}})$*, and if $\psi_1^2 = \mathbf{1}$, then*

$$\phi_{\mathcal{V}}\left(\mathbf{L}_{p,\Sigma}(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{I})}, \mathbf{G}^{(3),(\mathrm{I})})\right) \equiv \phi_{\mathcal{V}}\left(\mathbf{L}_{p,\Sigma}(\mathbf{F}_1, \mathbf{G}^{(2),(\mathrm{II})}, \mathbf{G}^{(3),(\mathrm{II})})\right)$$

modulo $p^{\mu_{\mathrm{wt}}^{(\mathcal{V})} + \min\{\nu_2, \nu_3\}}.\mathbb{I}^{\mathcal{V}}$, where the finite set $\Sigma := \mathrm{supp}(C_2^{(\mathrm{I})} C_2^{(\mathrm{II})} C_3^{(\mathrm{I})} C_3^{(\mathrm{II})})$.

In particular, this is equivalent to Theorem 1.4 (i) stated in the Introduction. Moreover let us recall that the Σ-imprimitive p-adic L-function factorises into

$$\mathbf{L}_{p,\Sigma}(\mathbf{F}_1, \mathbf{G}^{(2),(\star)}, \mathbf{G}^{(3),(\star)}) = \mathbf{L}_p(\mathbf{F}_1, \mathbf{G}^{(2),(\star)}, \mathbf{G}^{(3),(\star)}) \times \prod_{l \in \Sigma} E_l^{(\star)}(\mathbf{F}_1, \mathbf{G}^{(2)}, \mathbf{G}^{(3)})$$

where $E_l^{(\star)}(\cdot)$ interpolates $L_l(\mathbf{F}_1(m)\otimes\mathbf{G}^{(2),(\star)}(m)\otimes\mathbf{G}^{(3),(\star)}(m)\otimes\chi_{\underline{\mathcal{Q}}},\frac{w}{2})$ on $\mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$. Applying an identical argument to that used in the proof of Theorem 3.5,

$$\begin{aligned}
&\lambda^{\mathrm{wt}}\circ\phi_{\mathcal{V}}\left(\mathbf{L}_p(\mathbf{F}_1,\mathbf{G}^{(2),(\mathrm{I})},\mathbf{G}^{(3),(\mathrm{I})})\right)+\sum_{l\in\Sigma}\lambda^{\mathrm{wt}}\circ\phi_{\mathcal{V}}\left(E_l^{(\mathrm{I})}(\mathbf{F}_1,\mathbf{G}^{(2)},\mathbf{G}^{(3)})\right)\\
&=\lambda^{\mathrm{wt}}\circ\phi_{\mathcal{V}}\left(\mathbf{L}_{p,\Sigma}(\mathbf{F}_1,\mathbf{G}^{(2),(\mathrm{I})},\mathbf{G}^{(3),(\mathrm{I})})\right)\\
&\overset{\text{by } 3.5}{=}\lambda^{\mathrm{wt}}\circ\phi_{\mathcal{V}}\left(\mathbf{L}_{p,\Sigma}(\mathbf{F}_1,\mathbf{G}^{(2),(\mathrm{II})},\mathbf{G}^{(3),(\mathrm{II})})\right)\\
&=\lambda^{\mathrm{wt}}\circ\phi_{\mathcal{V}}\Big(\mathbf{L}_p(\mathbf{F}_1,\mathbf{G}^{(2),(\mathrm{II})},\mathbf{G}^{(3),(\mathrm{II})})\Big)+\sum_{l\in\Sigma}\lambda^{\mathrm{wt}}\circ\phi_{\mathcal{V}}\Big(E_l^{(\mathrm{II})}(\mathbf{F}_1,\mathbf{G}^{(2)},\mathbf{G}^{(3)})\Big)
\end{aligned}$$

and Theorem 1.4 (ii) now follows as an immediate corollary.

Remarks. The strategy we adopt to establish Theorem 3.8 has *three steps*:

(1) At each point $\underline{\mathcal{Q}}\in\mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$ and $\star\in\{\mathrm{I},\mathrm{II}\}$, we will express the special value $\underline{\mathcal{Q}}(L^{\mathbf{F}_1}_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}})$ in terms of $\underline{\mathcal{Q}}(a_1(\eta_{\mathbf{F}_1}\cdot 1_{\mathbf{F}_1}\cdot\mathrm{Tr}_{\widetilde{N}/C_1}(\mathbf{H}_\Sigma^{\mathrm{aux},(\star)})))$. Note that by construction, both the Σ-depleted families $\mathbf{H}_\Sigma^{\mathrm{aux},(\star)}\in\mathcal{S}^{\mathrm{ord}}(\widetilde{N},\psi_{1,(p)}\overline{\psi}_1^{(p)};\mathbb{I}_1)\otimes_{\mathbb{I}_1}\mathcal{R}$ exist at the top-most level $\widetilde{N}:=\mathrm{lcm}(C_1C_2^{(\mathrm{I})}C_3^{(\mathrm{I})},C_1C_2^{(\mathrm{II})}C_3^{(\mathrm{II})},\prod_{l\in\Sigma}l^2)$.

(2) By replacing the original triple $(\mathbf{F}_1,\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)})$ with the twisted triple
$$(\mathbf{F}_1\otimes(\omega^{-k^{(1)}(m)}\epsilon_m^{(1)})^{-1/2},\mathbf{G}_\Sigma^{(2),(\star)}\otimes(\omega^{-k^{(1)}(m)}\epsilon_m^{(1)})^{1/2},\mathbf{G}_\Sigma^{(3),(\star)}),$$
we relate $\underline{\mathcal{Q}}\big(a_1(\eta_{\mathbf{F}_1}\cdot 1_{\mathbf{F}_1}\cdot\mathrm{Tr}_{\widetilde{N}/C_1}(\mathbf{H}_\Sigma^{\mathrm{aux},(\star)}))\big)$ to the special value of our functional $\mathcal{L}_{F_1}^{(r,1)}(\underline{\mathcal{Q}}(\mathbf{G}_\Sigma^{(2),(\star)})\cdot\delta_{k_3}^{(r)}(\underline{\mathcal{Q}}(\mathbf{G}_\Sigma^{(3),(\star)}))|\mathbf{??})$ with $F_1^\alpha=\underline{\mathcal{Q}}(\mathbf{F}_1)\otimes(\omega^{-k^{(1)}(m)}\epsilon_m^{(1)})^{-1/2}$, $\underline{k}=(k_1,k_2,k_3)$, $r=(k_1-k_2-k_3)/2$, and "**??**" a combination of Hecke operators.

(3) Finally, upon exploiting the congruence preserving properties of the linear functionals $\mathcal{L}_{F_1}^{(r,1)}(-|\mathbf{??})$ and the Zariski density of $\mathcal{V}\cap\mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$ inside of $\mathrm{Spec}(\mathbb{I}^{\mathcal{V}})$, the mod $p^{\min\{\nu_2,\nu_3\}}$-congruences between $\underline{\mathcal{Q}}(\mathbf{H}_\Sigma^{\mathrm{aux},(\mathrm{I})})$ and $\underline{\mathcal{Q}}(\mathbf{H}_\Sigma^{\mathrm{aux},(\mathrm{II})})$ will produce mod $p^{\mu_{\mathrm{wt}}^{(\mathcal{V})}+\min\{\nu_2,\nu_3\}}$-congruences between the respective triple product L-values.

3.2.2. ***Step (1).*** Let us begin by reviewing the important properties of $\mathbf{H}^{\mathrm{aux},(\star)}$. In fact this family is obtained from a secondary $\mathcal{R}$-adic family, $\mathbf{H}^{\mathrm{ord},(\star)}$, through

$$\mathbf{H}^{\mathrm{aux},(\star)}=\sum_{I\subset\Sigma_{1,0}^{\mathrm{IIb}}}(-1)^{\#I}\cdot\frac{\psi_{1,(p)(n_I/d_1)}\langle n_I/d_1\rangle_{\mathbb{I}_1}d_1}{\beta_I(\mathbf{F}_1)\cdot n_I}\circ\mathbf{H}^{\mathrm{ord},(\star)}\Big|U_{d_1/n_I}$$

where the sets $I, \Sigma_{1,0}^{\mathrm{IIb}}$ and the positive integers n_I, d_1 can be found in [10, Section 4]. Each $\beta_I(\mathbf{F}_1) \in \mathbb{I}_1^{\times}$ is a distinguished root of $X^2 - a_l(\mathbf{F}_1)X + \psi_1(l) \cdot l^{-1}\langle l\rangle_{\mathbb{I}_1}$ at the primes $l \big| C_1 C_2^{(\star)} C_3^{(\star)}$, in which case the denominator $\beta_I(\mathbf{F}_1) \cdot n_I$ must be a unit.

Definition 3.9. The operator $\Upsilon_{N,\mathbf{F}_1}^{\mathrm{aux}} \in \mathrm{End}_{\mathbb{I}_1}(\mathcal{S}^{\mathrm{ord}}(N, \psi_{1,(p)}\overline{\psi}_1^{(p)}; \mathbb{I}_1) \otimes_{\mathbb{I}_1} \mathcal{R})$ is obtained via the formula

$$\mathbf{H}\Big|\Upsilon_{N,\mathbf{F}_1}^{\mathrm{aux}} := \sum_{I \subset \Sigma_{1,0}^{\mathrm{IIb}}} (-1)^{\#I} \cdot \frac{\psi_{1,(p)}(n_I/d_1)\langle n_I/d_1\rangle_{\mathbb{I}_1} d_1}{\beta_I(\mathbf{F}_1) \cdot n_I} \circ \mathbf{H}\Big|U_{d_1/n_I}.$$

If we instead deplete our families by omitting the q^n-coefficients involving those integers n such that $\mathrm{supp}(n) \cap \Sigma \neq \emptyset$, then analogously $\mathbf{H}_{\Sigma}^{\mathrm{aux},(\star)} = \mathbf{H}_{\Sigma}^{\mathrm{ord},(\star)} | \Upsilon_{\widetilde{N},\mathbf{F}_1}^{\mathrm{aux}}$. Now by its very definition,

$$L_{\mathbf{G}_{\Sigma}^{(2),(\star)},\mathbf{G}_{\Sigma}^{(3),(\star)}}^{\mathbf{F}_1} := a_1\Big(\eta_{\mathbf{F}_1} \cdot 1_{\mathbf{F}_1} \cdot \mathrm{Tr}_{\widetilde{N}/C_1}(\mathbf{H}_{\Sigma}^{\mathrm{aux},(\star)})\Big) \text{ (e.g. see [10, Section 4.2.5])}$$

where $\eta_{\mathbf{F}_1} \in \mathbb{I}_1$ generates the annihilator of the congruence module attached to $\mathbf{F}_1$, while $1_{\mathbf{F}_1} \in \mathbb{T}^{\mathrm{ord}}(C_1, \psi_1; \mathbb{I}_1)_{\mathbf{m}_{\mathbf{F}_1}} \otimes_{\mathbb{I}_1} \mathrm{Frac}(\mathbb{I}_1)$ is the idempotent element[6] which cuts the $\mathbf{F}_1$-isotypic part out from $\mathcal{S}^{\mathrm{ord}}(C_1, \psi_{1,(p)}\overline{\psi}_1^{(p)}; \mathbb{I}_1)$. Therefore at every $\underline{\mathcal{Q}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$,

$$\underline{\mathcal{Q}}\Big(L_{\mathbf{G}_{\Sigma}^{(2),(\star)},\mathbf{G}_{\Sigma}^{(3),(\star)}}^{\mathbf{F}_1}\Big) = \mathcal{Q}_{m_1}^{(1)}(\eta_{\mathbf{f}_1}) \times \underline{\mathcal{Q}}\Big(a_1\Big(1_{\mathbf{F}_1} \cdot \mathrm{Tr}_{\widetilde{N}/C_1}(\mathbf{H}_{\Sigma}^{\mathrm{ord},(\star)} | \Upsilon_{\widetilde{N},\mathbf{F}_1}^{\mathrm{aux}})\Big)\Big) \tag{3.6}$$

and the next stage is to relate the right-hand side of this to the functional $\mathcal{L}_{F_1}^{(r,1)}$.

3.2.3. ***Step (2).*** Before we can proceed further, a word of caution: for a fixed unbalanced point $\underline{\mathcal{Q}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$, the specialisation $\underline{\mathcal{Q}}(\mathbf{F}_1) = \mathcal{Q}_{m_1}^{(1)}(\mathbf{F}_1)$ has the character $\psi_1 \omega^{-k^{(1)}(m)} \epsilon_m^{(1)}$, which in general is *not* quadratic. Consequently the theory we developed in Section 2 cannot be directly applied to the classical eigenform $\underline{\mathcal{Q}}(\mathbf{F}_1)$.

To salvage the argument, we replace the triple $(\mathbf{F}_1, \mathbf{G}_{\Sigma}^{(2),(\star)}, \mathbf{G}_{\Sigma}^{(3),(\star)})$ with its modified version $(\mathbf{F}_1 \otimes (\omega^{-k^{(1)}(m)}\epsilon_m^{(1)})^{-1/2}, \mathbf{G}_{\Sigma}^{(2),(\star)} \otimes (\omega^{-k^{(1)}(m)}\epsilon_m^{(1)})^{1/2}, \mathbf{G}_{\Sigma}^{(3),(\star)})$, which works fine for *even* $k^{(1)}(m)$. If the original triple satisfies (T1)–(T4), it is easy to check the modified version does too. Furthermore, it follows readily that

$$F_1^{\alpha} := \underline{\mathcal{Q}}(\mathbf{F}_1 \otimes (\omega^{-k^{(1)}(m)}\epsilon_m^{(1)})^{-1/2}) \in \mathcal{S}_{k^{(1)}(m)}(pC_1, \psi_1; \mathcal{O}_{K,\epsilon_m^{(1)}})$$

---

[6] Hsieh and Fukunaga consider $\eta_{\check{\mathbf{F}}_1}$ and $1_{\check{\mathbf{F}}_1}$ where $\check{\mathbf{F}}_1 := \mathbf{F}_1 \Big| \big[\overline{\psi_1^{(p)}}\big]$; however our condition $\psi_1^2 = \mathbf{1}$ implies $\mathbf{F}_1$ and $\check{\mathbf{F}}_1$ share the same character, so we supress notation and ignore this switch.

must be an ordinary p-stabilised newform. If $k^{(1)}(m) > 2$ then we can assume it is principal series at p, in which case

$$F_1^{\alpha}(z) = F_1(z) - \psi_1(p)p^{k^{(1)}(m)-1}\alpha^{-1} \cdot F_1(pz)$$

where the underlying newform $F_1 \in \mathcal{S}_{k^{(1)}(m)}(C_1, \psi_1)$ is exactly as in Section 2.

Remarks.

(a) If $k^{(1)}(m) = 2$ and F_1^{α} is Steinberg at p, then $F_1^{\alpha} = F_1$ is already a newform of level pC_1, and we cannot apply the calculations in Section 2 to it.

(b) Replacing $(\mathbf{F}_1, \mathbf{G}_{\Sigma}^{(2),(\star)}, \mathbf{G}_{\Sigma}^{(3),(\star)})$ by the modified (twisted) triple above has no effect on the triple product L-function as the Galois representation is unchanged, however $L^{\mathbf{F}_1}_{\mathbf{G}_{\Sigma}^{(2),(\star)}, \mathbf{G}_{\Sigma}^{(3),(\star)}}$ is essentially a square-root so it might flip its sign around.

By the previous discussion, after first modifying $(\mathbf{F}_1, \mathbf{G}_{\Sigma}^{(2),(\star)}, \mathbf{G}_{\Sigma}^{(3),(\star)})$ one may then assume $F_1^{\alpha} = \underline{\mathcal{Q}}(\mathbf{F}_1)$ has exact level pC_1 and character ψ_1, such that $\psi_1^2 = \mathbf{1}$. To simplify the notation suppose that we have fixed a point $\underline{\mathcal{Q}} \in \mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$, and define $(k_1, k_2, k_3) = (k^{(1)}(m), k^{(2)}(m), k^{(3)}(m))$, $N_1 = C_1$, and $N_i = p^{e^{(i)}(m)}C_i$ for $i = 2, 3$. We shall also require the depleted Hecke eigenforms

$$\mathbf{g}_{\Sigma}^{(\star)} := \underline{\mathcal{Q}}_{m_2}^{(2)}(\mathbf{G}_{\Sigma}^{(2),(\star)}) \quad \text{and} \quad \mathbf{h}_{\Sigma}^{(\star)} := \underline{\mathcal{Q}}_{m_3}^{(3)}(\mathbf{G}_{\Sigma}^{(3),(\star)})\Big|\Theta_{\underline{\mathcal{Q}}}$$

in the context of Section 2.5, where $\Theta_{\underline{\mathcal{Q}}} = \psi_{1,(p)} \cdot \omega^{-(k_1-k_2-k_3)/2} \cdot (\overline{\epsilon_m^{(1)}\epsilon_m^{(2)}\epsilon_m^{(3)}})^{1/2}$ and the twisting operation "$\cdot\,|\Theta_{\underline{\mathcal{Q}}}$" sends $\sum_{n=1}^{\infty} c_n \cdot q^n \mapsto \sum_{n=1}^{\infty} c_n \Theta_{\underline{\mathcal{Q}}}(n) \cdot q^n$.

Lemma 3.10. *If $\underline{\mathcal{Q}}$ is unbalanced of weight (k_1, k_2, k_3) and $k_1 \in 2 \cdot \mathbb{Z}_{\geqslant 2}$, then*

$$\underline{\mathcal{Q}}(a_1(1_{\mathbf{F}_1} \cdot \mathrm{Tr}_{\widetilde{N}/N_1}(\mathbf{H}_{\Sigma}^{\mathrm{ord},(\star)}))) = \mathfrak{u}_{\underline{\mathcal{Q}}} \cdot \mathcal{L}_{F_1}^{(r,1)}\left(\mathbf{g}_{\Sigma}^{(\star)} \cdot \delta_{k_3}^{(r)}(\mathbf{h}_{\Sigma}^{(\star)})\Big|_{k_1}\left(\frac{1}{p} \cdot \mathrm{id} - \frac{\psi_1(p)}{p^2\alpha} \cdot U_p^*\right)\right)$$

with $\mathfrak{u}_{\underline{\mathcal{Q}}} \in \mathcal{O}_{\mathbb{C}_p}^{\times}$ independent of $\star \in \{\mathrm{I}, \mathrm{II}\}$, and U_p^* is the adjoint of U_p at level $\widetilde{N}$.

Proof. We start by using a convenient formula of Hida in [14, Lemma 9.1], which implies that the specialised coefficient

$$\underline{\mathcal{Q}}(a_1(1_{\mathbf{F}_1} \cdot \mathrm{Tr}_{\widetilde{N}/N_1}(\mathbf{H}_{\Sigma}^{\mathrm{ord},(\star)}))) = \frac{\left\langle \underline{\mathcal{Q}}(\mathbf{F}_1)^{\sharp}, e^{\mathrm{ord}} \cdot \underline{\mathcal{Q}}(\mathbf{H}_{\Sigma}^{\mathrm{ord},(\star)})\big|_{k_1} W_{\widetilde{N}} \right\rangle_{\widetilde{N}}}{\left\langle \underline{\mathcal{Q}}(\mathbf{F}_1)^{\sharp}, \underline{\mathcal{Q}}(\mathbf{F}_1)\big|_{k_1} W_{\widetilde{N}} \right\rangle_{\widetilde{N}}}.$$

Here the idempotent $e^{\mathrm{ord}} = \lim_{n\to\infty} U_p^{n!}$ and $\underline{\mathcal{Q}}(\mathbf{F}_1) = F_1^\alpha$ as before, whilst from [10, Lemma 4.2.3] we know that $\underline{\mathcal{Q}}(\mathbf{H}_\Sigma^{\mathrm{ord},(\star)})$ coincides with

$$e^{\mathrm{ord}} \cdot \mathrm{Hol}_\infty \left(\underline{\mathcal{Q}}_{m_2}^{(2)}(\mathbf{G}_\Sigma^{(2),(\star)}) \cdot \delta_{k_3}^{(r_{\underline{\mathcal{Q}}})} \underline{\mathcal{Q}}_{m_3}^{(3)}(\mathbf{G}_\Sigma^{(3),(\star)}) \Big| \Theta_{\underline{\mathcal{Q}}} \right) = e^{\mathrm{ord}} \cdot \mathrm{Hol}_\infty \left(\mathbf{g}_\Sigma^{(\star)} \cdot \delta_{k_3}^{(r)}(\mathbf{h}_\Sigma^{(\star)}) \right)$$

with $r = r_{\underline{\mathcal{Q}}} = (k_1 - k_2 - k_3)/2$.

As an immediate consequence, one deduces that

$$\underline{\mathcal{Q}}(a_1(1_{\mathbf{F}_1} \cdot \mathrm{Tr}_{\widetilde{N}/N_1}(\mathbf{H}_\Sigma^{\mathrm{ord},(\star)}))) = \frac{\left\langle (F_1^\alpha)^\sharp, e^{\mathrm{ord}} \cdot \mathrm{Hol}_\infty \left(\mathbf{g}_\Sigma^{(\star)} \cdot \delta_{k_3}^{(r)}(\mathbf{h}_\Sigma^{(\star)}) \right) \Big|_{k_1} W_{\widetilde{N}} \right\rangle_{\widetilde{N}}}{\left\langle (F_1^\alpha)^\sharp, F_1^\alpha \big|_{k_1} W_{\widetilde{N}} \right\rangle_{\widetilde{N}}}.$$

To deal with the denominator first, applying [14, Lemma 5.3 (vi)] it can be shown

$$\begin{aligned}\left\langle (F_1^\alpha)^\sharp, F_1^\alpha \big|_{k_1} W_{\widetilde{N}} \right\rangle_{\widetilde{N}} &= (-1)^{k_1} \left\langle (F_1^\alpha)^\sharp \big|_{k_1} W_{\widetilde{N}}, F_1^\alpha \right\rangle_{\widetilde{N}} \\ &= (-1)^{k_1} p^{\left(\frac{2-k_1}{2}\right)\tilde{e}} \mathfrak{u}_\dagger \cdot \left\langle F_1, F_1 \right\rangle_{N_1}\end{aligned}$$

where the term $\mathfrak{u}_\dagger$ is composed of Euler factors/Gauss sums[7], and is a p-adic unit.

To study the numerator term, if we write "$\mathbf{gh}$" as shorthand for $\mathbf{g}_\Sigma^{(\star)} \cdot \delta_{k_3}^{(r)}(\mathbf{h}_\Sigma^{(\star)})$ then because the p-stabilised newform F_1^α is p-ordinary,

$$\begin{aligned}\left\langle (F_1^\alpha)^\sharp, e^{\mathrm{ord}} \cdot \mathrm{Hol}_\infty (\mathbf{gh}) \big|_{k_1} W_{\widetilde{N}} \right\rangle_{\widetilde{N}} &= \left\langle (F_1^\alpha)^\sharp, \mathrm{Hol}_\infty (\mathbf{gh}) \big|_{k_1} W_{\widetilde{N}} \right\rangle_{\widetilde{N}} \\ &\overset{\text{by } 2.5}{=} \left\langle (F_1^\alpha)^\sharp, \mathbf{gh} \big|_{k_1} W_{\widetilde{N}} \right\rangle_{\widetilde{N}} \\ &= \left\langle F_1, \mathbf{gh} \big|_{k_1} W_{\widetilde{N}} \right\rangle_{\widetilde{N}} - \frac{\psi_1(p) p^{k_1 - 1}}{\alpha} \left\langle F_1 |_{k_1} V_p, \mathbf{gh} \big|_{k_1} W_{\widetilde{N}} \right\rangle_{\widetilde{N}}\end{aligned}$$

and the last equality follows since $(F_1^\alpha)^\sharp(q) = F_1(q) - \overline{\frac{\psi_1(p)p^{k_1-1}}{\alpha}} \cdot F_1(q^p)$ if $k_1 > 2$. Now $\left\langle F_1|_{k_1} V_p, \mathbf{gh}\big|_{k_1} W_{\widetilde{N}} \right\rangle_{\widetilde{N}} = p^{-k_1} \left\langle F_1, \mathbf{gh}\big|_{k_1} W_{\widetilde{N}} \circ U_p \right\rangle_{\widetilde{N}}$ while $W_{\widetilde{N}} \circ U_p = U_p^* \circ W_{\widetilde{N}}$, in which case

$$\left\langle (F_1^\alpha)^\sharp, e^{\mathrm{ord}} \cdot \mathrm{Hol}_\infty (\mathbf{gh}) \big|_{k_1} W_{\widetilde{N}} \right\rangle_{\widetilde{N}} = \left\langle F_1, \mathbf{gh} \Bigg|_{k_1} \left(\mathrm{id} - \frac{\psi_1(p)}{p\alpha} \cdot U_p^* \right) \circ W_{\widetilde{N}} \right\rangle_{\widetilde{N}}.$$

[7] In fact, the term $\mathfrak{u}_\dagger = \eta(p)^{\tilde{e}} \cdot \psi_\infty(-1) \cdot W'(F_1^\alpha) \cdot S(P) \cdot \prod_{\mathfrak{q}\in\Sigma_1} \tau(\eta'^{-1}\psi'^{-1}) \cdot \prod_{\mathfrak{v}\in\Sigma} \frac{\eta\eta'(d_\mathfrak{v})}{|\eta\eta'(d_\mathfrak{v})|}$ in the notation of [14, Section 5]; one then carefully checks each individual term is a unit of $\mathcal{O}_{\mathbb{C}_p}$.

Therefore, combining together the numerator and denominator calculations:

$$\underline{\mathcal{Q}}(a_1(1_{\mathbf{F}_1}\cdot \mathrm{Tr}_{\widetilde{N}/N_1}(\mathbf{H}_\Sigma^{\mathrm{ord},(\star)})))$$
$$= \frac{p^{\left(\frac{k_1-2}{2}\right)\tilde{e}}}{(-1)^{k_1}\mathfrak{u}_\dagger}\cdot\frac{\left\langle F_1, \mathbf{gh}\Big|_{k_1}\left(\mathrm{id}-\frac{\psi_1(p)}{p\alpha}\cdot U_p^*\right)\circ W_{\widetilde{N}}\right\rangle_{\widetilde{N}}}{\left\langle F_1,F_1\right\rangle_{N_1}}.$$

On the other hand, carefully rearranging the factors in Definition 3.1 (a) one finds

$$\mathcal{L}_{F_1}^{(r,1)}\left(\mathbf{gh}\Big|_{k_1}(\mathrm{id}-\frac{\psi_1(p)}{p\alpha}\cdot U_p^*)\right)$$
$$=\epsilon_1^{-1}\cdot p^{1+\left(\frac{k_1-2}{2}\right)\tilde{e}}\cdot\left(\frac{N_0^{(p)}}{N_1}\right)^{\frac{k_1}{2}}\times\frac{\left\langle F_1,\mathbf{gh}\Big|_{k_1}(\mathrm{id}-\frac{\psi_1(p)}{p\alpha}\cdot U_p^*)\circ W_{\widetilde{N}}\right\rangle_{\widetilde{N}}}{\left\langle F_1,F_1\right\rangle_{N_1}}$$

and then setting $\mathfrak{u}_{\underline{\mathcal{Q}}}:=\epsilon_1\cdot(\frac{N_0^{(p)}}{N_1})^{-\frac{k_1}{2}}\cdot(-1)^{k_1}\cdot\mathfrak{u}_\dagger^{-1}\in\mathcal{O}_{\mathbb{C}_p}^\times$, the result is proven. $\square$

Of course, we want the value of $a_1(\eta_{\mathbf{F}_1}\cdot 1_{\mathbf{F}_1}\cdot\mathrm{Tr}_{\widetilde{N}/N_1}(\mathbf{H}_\Sigma^{\mathrm{aux},(\star)}))$ at a point $\underline{\mathcal{Q}}$ not the value of $a_1(1_{\mathbf{F}_1}\cdot\mathrm{Tr}_{\widetilde{N}/N_1}(\mathbf{H}_\Sigma^{\mathrm{ord},(\star)}))$ at $\underline{\mathcal{Q}}$, but they are closely connected. Comparing the preceding lemma with Definition 3.9, then at even weight $k_1>2$

$$\underline{\mathcal{Q}}(a_1(\eta_{\mathbf{F}_1}\cdot 1_{\mathbf{F}_1}\cdot\mathrm{Tr}_{\widetilde{N}/N_1}(\mathbf{H}_\Sigma^{\mathrm{aux},(\star)})))$$
$$=\mathfrak{u}_{\underline{\mathcal{Q}}}\cdot\mathcal{Q}_{m_1}^{(1)}(\eta_{\mathbf{F}_1})\times\mathcal{L}_{F_1}^{(r,1)}\left(\mathbf{g}_\Sigma^{(\star)}\cdot\delta_{k_3}^{(r)}(\mathbf{h}_\Sigma^{(\star)})\Big|_{k_1}\underline{\mathcal{Q}}(\Upsilon_{\widetilde{N},\mathbf{F}_1}^{\mathrm{aux}})\circ\left(\frac{1}{p}\cdot\mathrm{id}-\frac{\psi_1(p)}{p^2\alpha}\cdot U_p^*\right)\right).$$

Moreover by its construction $L^{\mathbf{F}_1}_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}}=a_1(\eta_{\mathbf{F}_1}\cdot 1_{\mathbf{F}_1}\cdot\mathrm{Tr}_{\widetilde{N}/N_1}(\mathbf{H}_\Sigma^{\mathrm{aux},(\star)}))$, and so we may summarise the various calculations of *Step (2)* in the following way.

Corollary 3.11. *If $\underline{\mathcal{Q}}\in\mathfrak{X}_{\mathcal{R}}^{\mathbf{F}_1}$ has weight $\underline{k}=(k_1,k_2,k_3)$ and $k_1\in 2\cdot\mathbb{Z}_{\geqslant 2}$, then the special value of $L^{\mathbf{F}_1}_{\mathbf{G}_\Sigma^{(2),(\star)},\mathbf{G}_\Sigma^{(3),(\star)}}$ at the unbalanced point $\underline{\mathcal{Q}}$ is equal to*

$$p^{-2}\cdot\mathfrak{u}_{\underline{\mathcal{Q}}}\cdot\mathcal{Q}_{m_1}^{(1)}(\eta_{\mathbf{F}_1})\times\mathcal{L}_{F_1}^{(r,1)}\left(\mathbf{g}_\Sigma^{(\star)}\cdot\delta_{k_3}^{(r)}(\mathbf{h}_\Sigma^{(\star)})\Big|_{k_1}\underline{\mathcal{Q}}(\Upsilon_{\widetilde{N},\mathbf{F}_1}^{\mathrm{aux}})\circ\left(p\cdot\mathrm{id}-\frac{\psi_1(p)}{\alpha}\cdot U_p^*\right)\right).$$

The operator $\underline{\mathcal{Q}}(\Upsilon_{\widetilde{N},\mathbf{F}_1}^{\mathrm{aux}})\circ(p\cdot\mathrm{id}-\frac{\psi_1(p)}{\alpha}\cdot U_p^*)$ is the mysterious "**??**" mentioned in the remarks after Theorem 3.8.

3.2.4. ***Step (3).*** The final task is to prove the congruences for $\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}^{(2),(\star)}_\Sigma,\mathbf{G}^{(3),(\star)}_\Sigma}$ by reading them off at enough unbalanced specialisations $\underline{\mathcal{Q}}$ which are Zariski dense. An important initial observation is that

$$\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}^{(2),(\star)}_\Sigma,\mathbf{G}^{(3),(\star)}_\Sigma} = (-\psi_{1,(p)}(-1))^{-1/2} \cdot L^{\mathbf{F}_1}_{\mathbf{G}^{(2),(\star)}_\Sigma,\mathbf{G}^{(3),(\star)}_\Sigma} \times \prod_{l|N} \mathfrak{f}_l^{-1/2}$$

where the factors $\mathfrak{f}_l \in \mathcal{R}^\times$ are given in [10, Proposition 5.1.4], but are not required here. Thus to prove a congruence for the $\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}^{(2),(\star)}_\Sigma,\mathbf{G}^{(3),(\star)}_\Sigma}$'s over the one-dimensional set $\mathcal{V}$, it is necessary and sufficient to show the same congruence for the $L^{\mathbf{F}_1}_{\mathbf{G}^{(2),(\star)}_\Sigma,\mathbf{G}^{(3),(\star)}_\Sigma}$'s. Because each $\mathcal{L}^{\mathbf{F}_1}_{\mathbf{G}^{(2),(\star)}_\Sigma,\mathbf{G}^{(3),(\star)}_\Sigma}$ is a square-root, one has an equality of μ-invariants

$$\mu \circ \phi_{\mathcal{V}} \left(\mathbf{L}_p(\mathbf{F}_1, \mathbf{G}^{(2),(\star)}_\Sigma, \mathbf{G}^{(3),(\star)}_\Sigma)\right) = 2 \cdot \mu \circ \phi_{\mathcal{V}}\left(L^{\mathbf{F}_1}_{\mathbf{G}^{(2),(\star)}_\Sigma,\mathbf{G}^{(3),(\star)}_\Sigma}\right) \quad \text{at either } \star \in \{\mathrm{I}, \mathrm{II}\},$$

which means $\underline{\mathcal{Q}}(L^{\mathbf{F}_1}_{\mathbf{G}^{(2),(\star)}_\Sigma,\mathbf{G}^{(3),(\star)}_\Sigma})$ takes values in $p^{\mu^{(\mathcal{V})}_{\mathrm{wt}}/2} \cdot \mathcal{O}_{\mathbb{C}_p}$ for all $\underline{\mathcal{Q}} \in \mathcal{V} \cap \mathfrak{X}^{\mathbf{F}_1}_{\mathcal{R}}$. It follows directly from Corollary 3.11 that for each $\star \in \{\mathrm{I}, \mathrm{II}\}$,

$$\mathcal{L}^{(r,1)}_{F_1}\left(\mathbf{g}^{(\star)}_\Sigma \cdot \delta^{(r)}_{k_3}(\mathbf{h}^{(\star)}_\Sigma)\Big|_{k_1} \underline{\mathcal{Q}}(\Upsilon^{\mathrm{aux}}_{\widetilde{N},\mathbf{F}_1}) \circ \left(p \cdot \mathrm{id} - \frac{\psi_1(p)}{\alpha} \cdot U_p^*\right)\right)$$

lies inside $\mathcal{Q}^{(1)}_{m_1}(\eta_{\mathbf{F}_1})^{-1} p^{2+\mu^{(\mathcal{V})}_{\mathrm{wt}}/2} \cdot \mathcal{O}_{\mathbb{C}_p}$, provided that $\underline{\mathcal{Q}} \in \mathcal{V} \cap \mathfrak{X}^{\mathbf{F}_1}_{\mathcal{R}}$ with $k_1 \in 2 \cdot \mathbb{Z}_{\geqslant 2}$.

Remarks.

(i) By (3.1), the functional values below degenerate into

$$\mathcal{L}^{(r,1)}_{F_1}\left(\mathbf{g}^{(\star)}_\Sigma \cdot \delta^{(r)}_{k_3}(\mathbf{h}^{(\star)}_\Sigma)\right) = \sum_{d \left| \frac{N_0}{N_1}\right.} \mathfrak{c}^{(\star)}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}_\Sigma) \cdot \mathbf{X}_d(N_0, N_1)$$

where $\mathcal{H}^{(\star)}_\Sigma = \mathrm{Hol}_\infty(\mathbf{g}^{(\star)}_\Sigma \cdot \delta^{(r)}_{k_3}(\mathbf{h}^{(\star)}_\Sigma))\big|_{k_1} W^2_{\widetilde{N}} = (-1)^{k_1} \cdot \mathrm{Hol}_\infty(\mathbf{g}^{(\star)}_\Sigma \cdot \delta^{(r)}_{k_3}(\mathbf{h}^{(\star)}_\Sigma))$.

(ii) Applying Proposition 2.13 at divisors $d \left| \frac{N_0}{N_1}\right.$ and if $p \nmid \frac{(k_1-2)!}{(k_1-2-r)!}$, one has

$$\mathfrak{c}^{(\mathrm{I})}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}_\Sigma) \equiv \mathfrak{c}^{(\mathrm{II})}_{d,\widetilde{N},\tilde{e}}(\mathcal{H}_\Sigma) \pmod{p^{\min\{\nu_2,\nu_3\}}}.$$

Since the composition of operators $\mathfrak{R}_{\underline{\mathcal{Q}}} := \underline{\mathcal{Q}}(\Upsilon^{\mathrm{aux}}_{\widetilde{N},\mathbf{F}_1}) \circ (p \cdot \mathrm{id} - \frac{\psi_1(p)}{\alpha} \cdot U_p^*)$ does not introduce any new denominators involving p, it follows from these remarks that

$$\mathcal{L}^{(r,1)}_{F_1}(\mathbf{g}^{(\mathrm{I})}_\Sigma \cdot \delta^{(r)}_{k_3}(\mathbf{h}^{(\star)}_\Sigma)\Big|_{k_1} \mathfrak{R}_{\underline{\mathcal{Q}}}) - \mathcal{L}^{(r,1)}_{F_1}(\mathbf{g}^{(\mathrm{II})}_\Sigma \cdot \delta^{(r)}_{k_3}(\mathbf{h}^{(\star)}_\Sigma)\Big|_{k_1} \mathfrak{R}_{\underline{\mathcal{Q}}})$$

belongs to $\mathcal{Q}^{(1)}_{m_1}(\eta_{\mathbf{F}_1})^{-1}p^{2+\min\{\nu_2,\nu_3\}+\mu^{(\mathcal{V})}_{\rm wt}/2}\cdot\mathcal{O}_{\mathbb{C}_p}$ at all the points $\underline{\mathcal{Q}}\in\mathcal{V}\cap\mathfrak{X}^{\mathbf{F}_1}_{\mathcal{R}}$ satisfying $k_1\in 2\cdot\mathbb{Z}_{\geqslant 2}$ and $p\nmid\frac{(k_1-2)!}{(k_1-2-r_{\underline{\mathcal{Q}}})!}$. Reversing the previous chain of reasoning,

$$\underline{\mathcal{Q}}\Big(L^{\mathbf{F}_1}_{\mathbf{G}^{(2),(\mathrm{I})}_{\Sigma},\mathbf{G}^{(3),(\star)}_{\Sigma}}\Big)-\underline{\mathcal{Q}}\Big(L^{\mathbf{F}_1}_{\mathbf{G}^{(2),(\mathrm{II})}_{\Sigma},\mathbf{G}^{(3),(\star)}_{\Sigma}}\Big)\in p^{\min\{\nu_2,\nu_3\}+\mu^{(\mathcal{V})}_{\rm wt}/2}\cdot\mathcal{O}_{\mathbb{C}_p}$$

hence both $\underline{\mathcal{Q}}(\mathbf{L}_p(\mathbf{F}_1\otimes\mathbf{G}^{(2),(\mathrm{I})}_{\Sigma}\otimes\mathbf{G}^{(3),(\mathrm{I})}_{\Sigma}))$ and $\underline{\mathcal{Q}}(\mathbf{L}_p(\mathbf{F}_1\otimes\mathbf{G}^{(2),(\mathrm{II})}_{\Sigma}\otimes\mathbf{G}^{(3),(\mathrm{II})}_{\Sigma}))$ are congruent to each other modulo $p^{\mu^{(\mathcal{V})}_{\rm wt}+\min\{\nu_2,\nu_3\}}$.

Lastly as $p\neq 2$, we use the density of those $\underline{\mathcal{Q}}\in\mathcal{V}\cap\mathfrak{X}^{\mathbf{F}_1}_{\mathcal{R}}$ with $p\nmid\frac{(k_1-2)!}{(k_1-2-r_{\underline{\mathcal{Q}}})!}$ and $2|k_1$ inside $\mathrm{Spec}(\mathbb{I}^{\mathcal{V}})$ to obtain the full congruence, and Theorem 3.8 is proved.

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