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A geometric view on Iwasawa theory

par Adel BETINA et Mladen DIMITROV

Résumé. Cet article prolonge notre étude de la géométrie de la courbe p-adique de Hecke en un point défini par une forme modulaire cuspidale f de poids 1 à multiplication complexe et irrégulière en p, et des implications en théories d'Iwasawa et de Hida. Les nouveaux résultats incluent la détermination des coefficients de Fourier de certaines formes modulaires p-adiques non-classiques appartenant à l'espace propre généralisé de f, en termes de logarithmes p-adiques de nombres algébriques. Nous calculons aussi le « mystérieux » bi-rapport des filtrations p-ordinaires des familles de Hida contenant f.

Abstract. This article extends our study of the geometry of the p-adic eigencurve at a point defined by a weight 1 cuspform f irregular at p and having complex multiplication, and the implications in Iwasawa and in Hida theories. The novel results include the determination of the Fourier coefficients of certain non-classical p-adic modular forms belonging to the generalized eigenspace of f, in terms of p-adic logarithms of algebraic numbers. We also compute the "mysterious" cross-ratios of the p-ordinary filtrations of the Hida families containing f.

Introduction

Historical background. In the 1960s Kubota and Leopoldt used Kummer's congruences involving Bernoulli numbers to define the p-adic zeta function $\zeta_p \in \mathbb{Z}_p[\![\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]\!]$ and Iwasawa formulated his Main Conjecture postulating that ζ_p generates the characteristic ideal of a certain $\mathbb{Z}_p[\![\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]\!]$-module controlling the class group growth of the number fields contained in the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$. Around the same time J.-P. Serre [45] observed that ordinary Eisenstein series vary p-adic analytically in the weight, hence can be interpolated over the p-adic analytic weight space $\mathcal{W}(\mathbb{C}_p) = \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{C}_p^\times)$, thus providing a modular

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interpretation of the Kubota–Leopoldt p-adic zeta as a constant term of a $\mathbb{Z}_p[\![\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]\!]$-adic Eisenstein series.

In the 1980s H. Hida gave a new impulse to the subject by p-adically interpolating in [38] ordinary cuspforms of weight at least 2. He introduced a space of $\mathbb{Z}_p[\![\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]\!]$-adic cuspform which is in a perfect duality of finite free $\mathbb{Z}_p[\![\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]\!]$-modules with the corresponding Hecke algebra and proved a Control Theorem (see [39] when p is odd and [31] when $p = 2$). Furthermore, Hida showed that every eigenform as above is the specialization of a unique, up to Galois conjugacy, $\mathbb{Z}_p[\![\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]\!]$-adic eigenform, also called a Hida family. The Galois theoretic properties of these families, namely the restriction at the decomposition group at p, were studied by B. Mazur and A. Wiles who also made a crucial, for us, extension of Hida Theory to allow classical weight one forms.

Coming back to Iwasawa theory, Eisenstein families play a prominent role in the proof of the Main Conjecture over $\mathbb{Q}$ by Mazur and Wiles. They proved one divisibility using Ribet's Eisenstein ideal method and then deduced the equality via the class number formula. In their own words, while being "explicit" from a certain modular perspective, the approach does not allow to determine whether or not ζ_p admits multiple zeros (one merely knows that if that were the case then the characteristic series of the Iwasawa module would have a zero of the same multiplicity). Instead, they relied on the Ferrero–Greenberg Theorem [29] showing that the "trivial" zeros of ζ_p are simple.

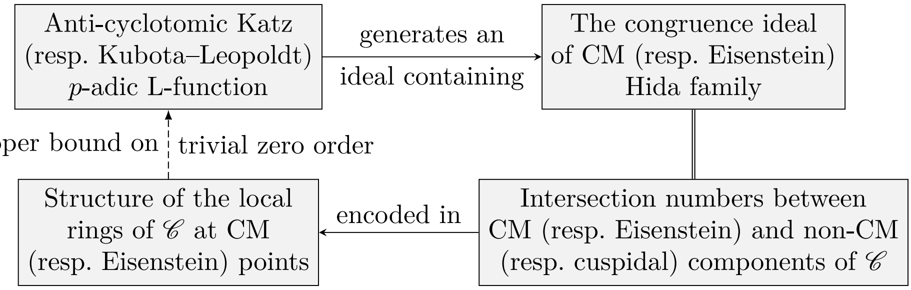
Another remarkable class of Hida families have complex multiplication (CM) by an imaginary quadratic field K, and are obtained by p-adic interpolation of classical theta series. Their adjoint p-adic L-functions is essentially equal to a Katz anti-cyclotomic p-adic L-function, and the corresponding anti-cyclotomic Main Conjecture over K has been proven by K. Rubin and independently by H. Hida and J. Tilouine. Similarly to the Eisenstein case, both proofs proceed by proving one divisibility and by invoking the class number formula, but without determining the possible zeros and their multiplicities. The question, analogous to the Ferrero–Greenberg Theorem, of whether the trivial zeros of Katz' anti-cyclotomic p-adic L-functions are simple, has remained open while being reformulated in terms of certain Iwasawa modules, via the mysterious bridge envisioned by Iwasawa. In [9] we use Hida Theory together with Mazur's Galois Deformations Theory, to show that these trivial zeros are indeed simple, provided that a certain anti-cyclotomic $\mathscr{L}$-invariant does not vanish, as predicted by the Four Exponentials Conjecture in Transcendence Theory.

Explicit results in Hida theory have been notoriously difficult to obtain since the Hecke–Hida algebras are finite and flat over $\mathbb{Z}_p[\![\mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})]\!]$, a complete local algebra of Krull dimension 2 isomorphic to $\mathbb{Z}_p[\![X]\!]$ by a theorem of J.-P. Serre. What has made our approach successful is the passage to the discrete valuation ring $\mathbb{Q}_p[\![X]\!]$, and even to its strict henselization $\Lambda = \overline{\mathbb{Q}}_p[\![X]\!]$, allowing the use of the rigid geometry tools brought to the modular world by R. Coleman and B. Mazur in the 1990s. A central object in this theory is the Coleman–Mazur eigencurve $\mathscr{C} \to \mathcal{W}$ first introduced in [20] (under some technical assumptions which were later removed by K. Buzzard [14]) as the rigid analytic curve parametrizing systems of Hecke eigenvalues of overconvergent p-adic modular forms of finite slope (see Section 1 for a detailed presentation).

Intersection numbers and p-adic L-functions. The local geometry at classical points has significant impact on Iwasawa theory since p-adic L-functions have acquired a new variable "the weight" in addition to the cyclotomic variable considered in classical Iwasawa theory. R. Greenberg and G. Stevens [33] have brilliantly illustrated how the weight variable can be used to shed light on problems involving a priori only the cyclotomic variable, such as the Mazur–Tate–Teitelbaum Conjecture [43] on the central trivial zeros of the p-adic L-function of modular forms. More recently J. Bellaïche [2] pushed the idea of using the weight variable even further to palliate the shortage of critical values needed for the construction of p-adic L-functions for modular forms of critical slope, provided that $\mathscr{C}$ is smooth at such points.

At points where the weight map $\mathsf{w} : \mathscr{C} \to \mathcal{W}$ is étale, one can use the weight space $\mathcal{W}$ to parametrize a Coleman family, whereas at singular points it is a challenge in itself to attach a p-adic L-function to a family. Let us recall that Hida's Control Theorem [39] for ordinary forms, extended by Coleman [18] to all forms of non-critical slope, implies that $\mathsf{w} : \mathscr{C} \to \mathcal{W}$ is étale at classical non-critical slope p-regular points of weight $\geqslant 2$. J. Bellaïche and one of us showed in [6] that $\mathscr{C}$ is smooth at classical points of weight 1 which are p-regular, using Galois deformations and Transcendence Theory, namely the Baker–Brumer Theorem on the linear independence of logarithms of algebraic numbers, to elucidate the geometry of the eigencurve. The geometry of $\mathscr{C}$ at points which are either irregular at p or have critical slope is expected to be more complicated, and Section 2 presents the state of the art on such matters.

Another link between the geometry of $\mathscr{C}$ and Iwasawa theory, is the expectation that local intersection numbers of $\mathscr{C}$ should be directly related to the adjoint p-adic L-functions, as shown in Hida's trilogy [35, 36, 37] and vast subsequent research by numerous authors. In particular, a vanishing of the adjoint p-adic L-functions, including for trivial reasons, should detect interesting geometric phenomena. This has been the leitmotiv in [10], resp. [9], where the geometry of $\mathscr{C}$ at certain Eisenstein, resp. CM, weight 1 points irregular at p is related to trivial zeros of the Kubota–Leopoldt, resp. Katz anti-cyclotomic, p-adic L-function.



In the CM case summarized in Section 2.6, we determine the completed local ring of $\mathscr{C}$ at the weight 1 point and use the congruence ideal between CM and non-CM families passing through that point, to provide an upper bound for the order of the trivial zero of the corresponding branch $\zeta_{\varphi}^{-} \in \overline{\mathbb{Z}}_p[\![X]\!]$ of the anti-cyclotomic Katz p-adic L-function. The exactness of this upper bound is predicted by widely accepted conjectures in Transcendence Theory. Furthermore, thanks to the Six Exponentials Theorem we know that at least one amongst ζ_{φ}^{-} or $\zeta_{\overline{\varphi}}^{-}$ has a simple trivial zero.

In the Eisenstein case, treated in [10] and summarized in Section 2.4, one can determine the local geometry of $\mathscr{C}$ at a weight 1 Eisenstein cuspidal overconvergent point unconditionally, and deduce from there Gross' formula for the derivative of the Kubota–Leopoldt p-adic zeta at a trivial zero. The non-trivial zeros all occur at cuspidal overconvergent Eisenstein points having non-classical weight. By assuming Greenberg's pseudo-null conjecture, C. Wang-Erickson and P. Wake showed in [46] that the cuspidal eigencurve is smooth at these points if, and only if, the zeros are simple. The question about the étaleness of the weight map at these points is still open.

Regarding generalizations to groups of higher rank, in a collaboration with S.-C. Shih [11], we recently extended the main results of [10] to weight one Eisenstein points on Hilbert eigenvarieties. The investigation of Eisenstein Hida families in this setting goes at least back to Wiles' work [47] on the Iwasawa Main Conjecture over totally real number fields, and also plays a prominent role in the proof [23, 24] of the Gross–Stark Conjecture on the derivative of the Deligne–Ribet p-adic L-function at a trivial zero.

Overconvergent generalized eigenforms. Fourier coefficients of classical eigenforms are related to arithmetic functions such as the partition function or the Dedekind eta function, and are motivic in nature as the corresponding two-dimensional Galois representations occur in the étale cohomology of proper smooth varieties. Amongst non-classical overconvergent forms, the closest in nature to a classical eigenform f, are those belonging to its generalized eigenspace $S^{\dagger}_{\mathsf{w}(f)}[\![f]\!]$. The very exclusive club of genuine overconvergent generalized eigenforms $S^{\dagger}_{\mathsf{w}(f)}[\![f]\!]_0$ is a natural supplement of the classical subspace in $S^{\dagger}_{\mathsf{w}(f)}[\![f]\!]$. In his quest [1, 2] to attach p-adic L-functions to classical eigenforms of critical slope, J. Bellaïche classified the possible such examples in weight $k \geqslant 2$, and concluded that conjecturally the only genuine overconvergent generalized eigenforms are critical CM forms, whose Fourier coefficients were recently computed by Hsu [40]. The first to take up the task in weight 1 were H. Darmon, A. Lauder and V. Rotger [21] who expressed the Fourier coefficients of a certain p-adic overconvergent weight 1 generalized eigenform in terms of p-adic logarithms of algebraic numbers in ring class fields of real quadratic fields. The underlying classical weight 1 form has real multiplication (RM) and, according to [6], it is the only case in which a p-regular weight 1 form admits a genuine overconvergent generalized eigenspace. Such generalized eigenforms could provide an approach to Hilbert's twelfth problem asking for an "explicit Class Field Theory" for real quadratic fields (analogous to the theory of complex multiplication for imaginary quadratic fields).

In order to state our main result concerning the p-irregular CM case, we need to fix some notations which will be used throughout the paper. We denote by $\mathrm{G}_L = \mathrm{Gal}(\overline{L}/L)$ the absolute Galois group of a field L. The choice of an embedding $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ allows one to see $\mathrm{G}_{\mathbb{Q}_p}$ as a decomposition subgroup of $\mathrm{G}_{\mathbb{Q}}$. Let K be an imaginary quadratic field in which p splits. Given a finite order character $\psi : \mathrm{G}_K \to \overline{\mathbb{Q}}^\times \xrightarrow{\iota_p} \overline{\mathbb{Q}}_p^\times$, we consider the anti-cyclotomic character $\varphi = \psi \cdot \bar{\psi}^{-1}$, where $\bar{\psi}$ denotes the *internal* Galois conjugate of ψ by the complex conjugation $\tau \in \mathrm{G}_{\mathbb{Q}} \backslash \mathrm{G}_K$. We assume that $\varphi_{|\mathrm{G}_{\mathbb{Q}_p}}$ is trivial, but φ is not. Under these assumptions, the newform on $\mathrm{GL}_2/\mathbb{Q}$ obtained by automorphic induction from K to $\mathbb{Q}$ of ψ is a weight 1 theta series θ_ψ of level N and central character $\varepsilon = \varepsilon_K \cdot \psi \circ \mathrm{Ver}$. Here ε_K denotes the quadratic Dirichlet character attached to $K/\mathbb{Q}$, and Ver denotes the transfer homomorphism. Let $f = \sum_{n \geqslant 1} a_n q^n$ denote the unique p-stabilization of θ_ψ (see (2.5)).

Throughout the paper, we make the following assumption on the anti-cyclotomic $\mathscr{L}$-invariants introduced in [9, Section 1]

$$\mathscr{L}_{\text{-}}(\varphi) \cdot \mathscr{L}_{\text{-}}(\bar{\varphi}) \cdot (\mathscr{L}_{\text{-}}(\varphi) + \mathscr{L}_{\text{-}}(\bar{\varphi})) \neq 0, \tag{0.1}$$

and we choose a square root ξ of $\mathscr{L}_{\text{-}}(\bar{\varphi})\mathscr{L}_{\text{-}}(\varphi)^{-1}\mathscr{S}_\varphi^{-1}$, where the slope $\mathscr{S}_\varphi$ is defined in (3.1). One should observe that (0.1) is true for φ quadratic, whereas in all other cases the Schanuel Conjecture predicts that $\mathscr{L}_{\text{-}}(\varphi)$ and $\mathscr{L}_{\text{-}}(\bar{\varphi})$ are algebraically independent. Finally, we consider the p-adic logarithms $\mathscr{L}_{\mathfrak{l}}$ (for $\ell \neq p$ splitting in K as $\mathfrak{l}\bar{\mathfrak{l}}$) and $\mathscr{L}_{\psi,\ell}$ (for ℓ ramified or inert in K) of some explicit ℓ-units defined in (3.2) and (3.6), respectively.

Theorem A. *There exists a basis* $\{f^{\dagger}_{\mathscr{F}}, f^{\dagger}_{\Theta}\}$ *of* $S^{\dagger}_{\mathsf{w}(f)}[\![f]\!]_0$ *whose* q*-expansion is as follows:*

(i) *For any prime* $\ell \neq p$ *splitting in* K *as* $\mathfrak{l}\cdot\bar{\mathfrak{l}}$, *one has*

$$a_\ell(f^{\dagger}_{\mathscr{F}}) = 0,\ a_\ell(f^{\dagger}_{\Theta}) = (\mathscr{L}_{\mathfrak{l}} - \mathscr{L}_{\bar{\mathfrak{l}}})\cdot(\psi(\mathfrak{l}) - \psi(\bar{\mathfrak{l}})).$$

(ii) *For any prime* $\ell | N$ *not splitting in* K, *one has*

$$a_\ell(f^{\dagger}_{\Theta}) = 0,\ a_\ell(f^{\dagger}_{\mathscr{F}}) = 2\psi(\mathfrak{l})\xi\mathscr{L}_{\psi,\ell}\cdot\frac{\mathscr{L}_{\text{-}}(\varphi)}{\mathscr{L}_{\text{-}}(\varphi)+\mathscr{L}_{\text{-}}(\bar{\varphi})}.$$

(iii) *For any prime* $\ell \nmid N$ *inert in* K, *one has*

$$a_\ell(f^{\dagger}_{\Theta}) = 0,\ \textit{and}\ a_\ell(f^{\dagger}_{\mathscr{F}}) = 2\xi\mathscr{L}_{\psi,\ell}\cdot\frac{\mathscr{L}_{\text{-}}(\varphi)}{\mathscr{L}_{\text{-}}(\varphi)+\mathscr{L}_{\text{-}}(\bar{\varphi})}.$$

(iv) *Any form* $\sum_{n\geqslant 1} a^{\dagger}_n\cdot q^n \in S^{\dagger}_{\mathsf{w}(f)}[\![f]\!]_0$ *satisfies* $a^{\dagger}_1 = a^{\dagger}_p = 0$ *and the following recursive relations:*

$$a^{\dagger}_{mn} = a_m a^{\dagger}_n + a_n a^{\dagger}_m,\ \textit{for all}\ (n,m)=1,\ \textit{and}$$

$$a^{\dagger}_{\ell^r} = \begin{cases} a_\ell a^{\dagger}_{\ell^{r-1}} + a_{\ell^{r-1}} a^{\dagger}_\ell - \varepsilon(\ell) a^{\dagger}_{\ell^{r-2}}, & \textit{for all primes}\ \ell\nmid Np\ \textit{and all}\ r\geqslant 2,\\ r a_\ell^{r-1} a^{\dagger}_\ell, & \textit{othewise}.\end{cases}$$

Most of the paper is organized around the proof of the above Theorem. In Section 1 we summarize the basic properties of $\mathscr{C}$ while making a detour to define a geometric q-expansion of a Coleman family at a cusp of the ordinary locus using the overconvergent modular sheaf constructed by V. Pilloni [44]. Using the resulting q-expansion Principle we prove a perfect Hida duality between the space of Coleman families and the corresponding Hecke algebra. Exploiting this duality and the results of [9] on the local geometry of $\mathscr{C}$ at f, allows us in Section 3 to compute infinitesimally the Fourier coefficients of all families passing through f and to obtain Theorem A. The resulting formulas involve p-adic logarithms of algebraic numbers in the field cut out by the adjoint Galois representation attached to f.

A mysterious cross-ratio. Let us first formulate the precise problem. We let f be a p-ordinary stabilization of a newform of weight $k\geqslant 1$ and level $\Gamma_1(N)$, and denote by $\alpha\neq 0$ its U_p-eigenvalue. If f is p-regular then one knows that (see Section 2.1 for more details)

- f belongs to a unique, up to Galois conjugacy, Hida family $\mathscr{F}$ and the $\Lambda = \mathcal{O}^{\wedge}_{\mathcal{W},\mathsf{w}(f)} \simeq \overline{\mathbb{Q}}_p[\![X]\!]$-algebra $\mathcal{O}^{\wedge}_{\mathscr{C},f}$ is isomorphic to $\overline{\mathbb{Q}}_p[\![Y]\!]$, where $Y^e = X$ for some $e\geqslant 1$,
- the corresponding Galois representation $\rho_{\mathscr{F}} : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}_p[\![Y]\!])$ is p-ordinary and its $\mathrm{G}_{\mathbb{Q}_p}$-stable line reduces modulo (Y) to the unique $\mathrm{G}_{\mathbb{Q}_p}$-stable line in ρ_f such that the arithmetic Frobenius Frob_p acts by α on the (unramified) quotient.

If f is irregular at p and $k=1$, then the restriction of ρ_f to $\mathrm{G}_{\mathbb{Q}_p}$ is scalar, given by an unramified character sending Frob_p to α. Galois conjugacy classes of Hida families $\mathscr{F}$ containing f are in bijection with the irreducible components of $\mathscr{C}$ containing f, and

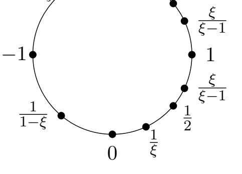
- either $\mathcal{O}^{\wedge}_{\mathscr{F},f}$ is not a regular ring (e.g. it is not even normal) and $\rho_{\mathscr{F}}$ might not even admit an ordinary filtration,
- or $\mathcal{O}^{\wedge}_{\mathscr{F},f} \simeq \overline{\mathbb{Q}}_p[\![Y]\!]$ is a discrete valuation ring and thus $\rho_{\mathscr{F}}$ does admit a (unique) ordinary filtration yielding, when reduced modulo (Y), a well defined line in ρ_f, i.e. an element of $\mathbb{P}(\rho_f)$.

The choice of basis for ρ_f allows one to identify $\mathbb{P}(\rho_f)$ with $\mathbb{P}^1(\overline{\mathbb{Q}}_p)$ and each of the finitely many "regular" Hida families containing f picks a well-defined element in it. We will now illustrate this phenomenon with two familiar examples.

Suppose first that f is a weight 1 Eisenstein series which is irregular at p. It is natural to chose a basis in which ρ_f is reducible and semi-simple. There are two Eisenstein Hida families containing f having residual slope 0 and ∞, respectively. The main result in [10] shows that there is a unique cuspidal Hida family $\mathscr{F}$ containing f, whose residual slope belongs to $\overline{\mathbb{Q}}_p^\times$. Since one can rescale the vectors of the basis, all values in $\overline{\mathbb{Q}}_p^\times = \mathbb{P}^1(\overline{\mathbb{Q}}_p) \setminus \{0,\infty\}$ are allowed, the forbidden values 0 and ∞ corresponding to the two $\mathrm{G}_{\mathbb{Q}}$-stable lines. One can then recover $\rho_{\mathscr{F}}$ as the universal ordinary deformation of ρ_f endowed with *any* non-$\mathrm{G}_{\mathbb{Q}}$-stable line. A tame analogue of such deformation problems was used by F. Calegari and M. Emerton in [15] to establish an $R=T$ theorem for the weight 2 Hecke algebra at Eisenstein primes.

Assume now that f is a weight 1 cuspform irregular at p and having complex multiplication. The situation is then more rigid as ρ_f is irreducible. As ρ_f is odd, a canonical pair of elements of $\mathbb{P}(\rho_f)$ is given by the eigenspaces for the complex conjugation $\tau \in \mathrm{G}_{\mathbb{Q}}$. Recall the notations introduced immediately before Theorem A, in particular the number $\xi \in \overline{\mathbb{Q}}_p^\times$.

Theorem B. *Under the assumption* (0.1)*, the ordinary lines of the four families containing f are pairwise distinct and their cross-ratio belongs to $\{-1, 2, \frac{1}{2}\}$. Moreover, the cross-ratio of the line fixed by the complex conjugation and the three lines in ρ_f obtained by reducing the ordinary lines of $\mathscr{F}$ and the two CM families containing f, belongs to* $\{\xi, \frac{1}{\xi}, 1-\xi, \frac{1}{1-\xi}, \frac{\xi}{\xi-1}, \frac{\xi-1}{\xi}\}$.



1. Background on the p-adic eigencurve $\mathscr{C}$

1.1. Overconvergent modular forms. Let p be any prime number. For an integer $N \geqslant 4$ relatively prime to p, we let $\mathscr{X}$ be the proper smooth modular curve of level $\Gamma_1(N)$ over $\mathbb{Z}_p$ and $\mathscr{E} \to \mathscr{X}$ be the universal generalized elliptic curve. The fiber of $\mathscr{E}$ above any cusp is given by a certain Néron polygon endowed with $\Gamma_1(N)$-level structure.

The invertible sheaf ω on $\mathscr{X}$ is defined as the pull-back of the relative differentials $\Omega_{\mathscr{E}/\mathscr{X}}$ along the zero section of $\mathscr{E} \to \mathscr{X}$. The space of classical modular forms of weight $k \in \mathbb{Z}_{\geqslant 1}$, level $\Gamma_1(N)$ and coefficients in a $\mathbb{Z}_p$-algebra A is defined as $M_k(N; A) = \mathrm{H}^0(\mathscr{X}_A, \omega_A^{\otimes k})$. The A-module $M_k(N; A)$ is functorial in A and commutes with flat base change.

Let X^{an} be the rigid analytification of the generic fibre $X = \mathscr{X}_{\mathbb{Q}_p}$ of $\mathscr{X}$. Note that the properness of $\mathscr{X}$ over $\mathbb{Z}_p$ implies that X^{an} is also the rigid space in the sense of Raynaud. The analytification of the line bundle ω is a line bundle on X^{an} and will still be denoted by ω.

The ordinary locus X^{ord} is the complement of the supersingular residue classes in X^{an} and can be characterized as the locus where the truncated valuation of the Hasse invariant is 0. More generally, for $v \in \mathbb{Q}_{\geqslant 0}$, let $X(v)$ denote the strict overconvergent neighborhood in X^{an} of the ordinary locus $X^{\mathrm{ord}} = X(0)$ where the (truncated) valuation of the Hasse invariant is $\leqslant v$. The space Katz p-adic modular forms of weight $k \in \mathbb{Z}$ is the infinite dimensional $\mathbb{C}_p$-vector space $\mathrm{H}^0(X^{\mathrm{ord}}_{\mathbb{C}_p}, \omega_{\mathbb{C}_p}^{\otimes k})$, whereas the space of p-adic overconvergent modular forms was defined by Coleman as the Fréchet $\mathbb{C}_p$-vector space:

$$M_k^{\dagger} = \varinjlim_{v>0} \mathrm{H}^0(X(v)_{\mathbb{C}_p}, \omega_{\mathbb{C}_p}^{\otimes k}).$$

The weight space $\mathcal{W}_p$ is the rigid space over $\mathbb{Q}_p$ representing homomorphisms $\mathbb{Z}_p^\times \to \mathbb{G}_m$. We consider $\mathbb{Z}$ as a subset of $\mathcal{W}_p$ by sending $k \in \mathbb{Z}$ to the algebraic character $x \mapsto x^k$. The space $\mathcal{W}_p$ is a disjoint union, indexed by the characters of $(\mathbb{Z}/2p\mathbb{Z})^\times$, of copies of the rigid open unit disk $\{|z-1|_p < 1\}$ representing homomorphisms $1 + 2p\mathbb{Z}_p \to \mathbb{G}_m$ (via the image of the topological generator $1+2p$). The latter is admissibly covered by the closed disks $\mathcal{B}_m = \{|z-1|_p \leqslant p^{-1/n_m}\}$, where the increasing sequence of positive integers $(n_m)_{m\geqslant 1}$ is chosen so that the following holds. For any $m \in \mathbb{Z}_{\geqslant 1}$ there exists a (unique) character

$$\widetilde{\kappa}_m : \mathbb{Z}_p^\times \cdot (1 + p^m \mathcal{O}_{\mathbb{C}_p}) \longrightarrow \mathcal{O}(\mathcal{B}_m)^\times$$

extending the universal character $\kappa_m : \mathbb{Z}_p^\times \to \mathcal{O}(\mathcal{B}_m)^\times$ and whose restriction to $(1 + p^m \mathcal{O}_{\mathbb{C}_p})$ is analytic.

The invertible sheaf ω is representable by $\mathrm{Hom}_{\mathscr{X}}(\mathcal{O}_{\mathscr{X}}, \omega)$ and we denote by $\pi : \mathcal{I} = \mathrm{Isom}_{\mathscr{X}}(\mathcal{O}_{\mathscr{X}}, \omega) \to \mathscr{X}$ the corresponding $\mathbb{G}_m$-torsor. The fibers of the rigid analytification $\pi^{\mathrm{an}} : \mathcal{I}^{\mathrm{an}} \to X^{\mathrm{an}}$ are naturally isomorphic to

$\mathbb{C}_p^\times$. For $k \in \mathbb{Z}$ one can recover ω^k as $\pi_*^{\mathrm{an}}(\mathcal{O}_{\mathcal{I}^{\mathrm{an}}})[k]$, where $[k]$ means the k-equivariant sections for the action of $\mathbb{G}_m$.

In [44, Section 3], Pilloni showed that there exists $v_m > 0$ and an invertible sheaf ω_m on $X(v_m) \times \mathcal{B}_m$ specializing, for any $k \in \mathbb{Z}_{\geqslant 1} \cap \mathcal{B}_m$, to the automorphic line bundle $\omega^{\otimes k}$ on $X(v_m)$. Namely, he constructed an open $\mathcal{J}_m$ of $\mathcal{I}^{\mathrm{an}}_{|X(v_m)}$ endowed with $\mathbb{Z}_p^\times$-action such that

$$\omega_m = \left(\pi_*^{\mathrm{an}}\mathcal{O}_{\mathcal{J}_m} \mathbin{\widehat{\otimes}} \mathcal{O}_{\mathcal{B}_m}\right)[\kappa_m],$$

where $[\kappa_m]$ means the κ_m-equivariant sections for the action of $\mathbb{Z}_p^\times$. More precisely, by *loc. cit.* $\mathcal{J}_m$ is locally isomorphic for the étale topology on X^{ord} to the union of disks

$$\mathbb{Z}_p^\times \cdot (1 + p^m \mathcal{O}_{\mathbb{C}_p}) = \bigcup_{y \in (\mathbb{Z}/p^m\mathbb{Z})^\times} \mathrm{B}(\widetilde{y}, p^{-m}) \subset \mathbb{C}_p^\times,$$

where $\widetilde{y} \in \mathbb{Z}_p^\times$ denotes a lift of y and $\mathrm{B}(\widetilde{y}, p^{-m}) = \{z \in \mathbb{C}_p, |z - \widetilde{y}|_p \leqslant p^{-m}\}$. By definition ω_m is locally isomorphic for the étale topology on X^{ord} to the eigenspace of $\mathcal{O}(\mathcal{B}_m)$-valued locally analytic functions on $\mathbb{Z}_p^\times \cdot (1 + p^m\mathcal{O}_{\mathbb{C}_p})$ which are κ_m-equivariant for the action of $\mathbb{Z}_p^\times$. This space is clearly generated by the section corresponding to $\widetilde{\kappa}_m$.

Let $\mathcal{U}$ be an open affinoid of $\mathcal{W}_p$ which is contained in some $\mathcal{B}_m$. For $v > 0$ sufficiently small, we let $\omega_{\mathcal{U}}$ denote the corresponding invertible sheaf on $X(v) \times \mathcal{U}$. The correspondence U_p is defined on the locus $X(\frac{p}{p+1})$ where the canonical subgroup of $\mathscr{E}$ exists. It sends $X(v)$ on $X(\frac{v}{p})$ and the resulting endomorphism of the $\mathcal{O}(\mathcal{U})$-Banach module $\mathrm{H}^0(X(v) \times \mathcal{U}, \omega_{\mathcal{U}})$ is compact. As $\mathcal{U}$ is reduced, after possibly shrinking it around any given point, the $\mathcal{O}(\mathcal{U})$-Banach module $\mathrm{H}^0(X(v) \times \mathcal{U}, \omega_{\mathcal{U}})$ admits slope decomposition (see [34, Section 2.3]). It follows that the subset $\mathrm{H}^0(X(v) \times \mathcal{U}, \omega_{\mathcal{U}})^{\leqslant s}$ of elements of slope at most $s \in \mathbb{Q}_{\geqslant 0}$ is a $\mathcal{O}(\mathcal{U})$-submodule which is locally free and of finite type. The $\mathcal{O}(\mathcal{U})$-module of Coleman families is defined as

$$M_{\mathcal{U}}^{\dagger,\leqslant s} = \varinjlim_{v>0} \mathrm{H}^0(X(v) \times \mathcal{U}, \omega_{\mathcal{U}})^{\leqslant s}.$$

The weight space $\mathcal{W}_p$ can be admissibly covered by countably many open affinoids $\mathcal{U}$ as above for each of which $M_{\mathcal{U}}^{\dagger,\leqslant s}$ is free of finite type over $\mathcal{O}(\mathcal{U})$.

Remark 1.1. One can extend the definition to tame levels $N \leqslant 3$ by considering the analogous objects on the modular curve of level $\Gamma_1(4N)$ and then taking $\Gamma_1(N)$-invariants. Note however that this will not be needed in Section 3 where Theorems A and B are proven, as $N \geqslant 23$ in that case.

1.2. *q*-expansions of Coleman families. In this subsection we describe how one can attach to a Coleman family a geometric q-expansion which interpolates the q-expansions of its classical specializations and satisfies the q-expansion Principle.

The generalized elliptic curve Tate(q) over $\mathbb{Z}_p[\![q]\!]$, with 1-gon special fiber and natural $\Gamma_1(N)$-level structure defines a morphism $\mathrm{Spec}(\mathbb{Z}_p[\![q]\!]) \to \mathscr{X}$ (the section $q=0$ corresponds to ∞). Let $\mathrm{d}^\times t$ be the canonical differential of Tate(q), i.e., a canonical $\mathcal{O}^\wedge_{\mathscr{X},\infty}$-basis of the completed stalk $\omega^\wedge_\infty$ of ω along the section $\infty : \mathrm{Spec}(\mathbb{Z}_p) \to \mathscr{X}$. For $k \in \mathbb{Z}$, one can identify $(\omega^\wedge_\infty)^{\otimes k}$ with $\mathcal{O}^\wedge_{\mathscr{X},\infty} = \mathbb{Z}_p[\![q]\!]$ using the canonical basis $(\mathrm{d}^\times t)^{\otimes k}$, allowing one to geometrically define the q-expansion of an overconvergent modular form.

Proposition 1.2. *There exists a neighborhood $\mathcal{V}$ for the étale topology of the cusp $\infty \in X^{\mathrm{ord}}$ and a generator of $\omega_{\mathcal{U}}(\mathcal{V}\times\mathcal{U})$ specializing at any $k \in \mathbb{Z}_{\geqslant 1} \cap \mathcal{U}$ to the canonical differential $(\mathrm{d}^\times t)^{\otimes k}$. In particular, one can attach to any Coleman family $\mathscr{F} \in M^{\dagger,\leqslant s}_{\mathcal{U}}$ a q-expansion $\sum_{n\geqslant 0} a_n(\mathscr{F})q^n \in \mathcal{O}(\mathcal{U})[\![q]\!]$ interpolating the q-expansions of its classical specializations. Moreover, the q-expansion Principle holds, i.e., the q-expansion map is injective.*

Proof. As already observed, $\omega_{\mathcal{U}}$ is locally isomorphic for the étale topology on $X^{\mathrm{ord}} \ni \infty$ to the eigenspace of $\mathcal{O}(\mathcal{U})$-valued locally analytic functions on $\mathbb{Z}_p^\times \cdot (1+p^m\mathcal{O}_{\mathbb{C}_p})$ which are $\mathbb{Z}_p^\times$-equivariant with respect to the action of the universal character $\kappa_{\mathcal{U}} : \mathbb{Z}_p^\times \to \mathcal{O}(\mathcal{U})^\times$, a basis being given by the locally analytic character $\widetilde{\kappa}_{\mathcal{U}}$. Here m is chosen so that $\mathcal{U} \subset \mathcal{B}_m$.

Since Tate(q) is ordinary at p, one can choose a neighborhood $\mathcal{V}$ for the étale topology of the cusp ∞ with a local trivialization of ω given by $\mathrm{d}^\times t$ and such that the section $\widetilde{\kappa}_{\mathcal{U}}$ generates $\omega_{\mathcal{U}}(\mathcal{V}\times\mathcal{U})$. The specialization of $\widetilde{\kappa}_{\mathcal{U}}$ at any $k \in \mathbb{Z}_{\geqslant 1} \cap \mathcal{U}$ corresponds under this construction to the canonical differential $(\mathrm{d}^\times t)^{\otimes k}$ generating $\omega^{\otimes k}(\mathcal{V})$, thus providing the sought-for p-adic analytic interpolation. This yields the desired trivialization $\omega_{\mathcal{U}}(\mathcal{V}\times\mathcal{U}) \simeq \mathcal{O}(\mathcal{V}) \mathbin{\widehat{\otimes}} \mathcal{O}(\mathcal{U})$ together with the natural injection $\mathcal{O}(\mathcal{V}) \mathbin{\widehat{\otimes}} \mathcal{O}(\mathcal{U}) \hookrightarrow \mathcal{O}(\mathcal{U})[\![q]\!]$ given by the localization $\mathcal{O}(\mathcal{V}) \to \mathcal{O}^\wedge_{X^{\mathrm{ord}},\infty} = \overline{\mathbb{Q}}_p[\![q]\!]$ at ∞. In this manner we have associated to any Coleman family $\mathscr{F} \in M^{\dagger,\leqslant s}_{\mathcal{U}}$ a q-expansion $\sum_{n\geqslant 0} a_n(\mathscr{F})q^n \in \mathcal{O}(\mathcal{U})[\![q]\!]$ interpolating the q-expansions of its classical specializations, and thus satisfying a q-expansion Principle. $\square$

Remark 1.3. Using similar techniques one can prove a statement analogous to Proposition 1.2 at an arbitrary cusp of X^{ord}.

1.3. Basic global properties of $\mathscr{C}$. The eigencurve $\mathscr{C}$ of tame level N is admissibly covered by the affinoids attached to the $\mathcal{O}(\mathcal{U})$-algebras $\mathcal{T}^{\leqslant s}_{\mathcal{U}}$ generated by the Hecke operators T_ℓ, $\langle \ell \rangle$, $\ell \nmid Np$ and U_p acting on $M^{\dagger,\leqslant s}_{\mathcal{U}}$, where $s \in \mathbb{Q}_{\geqslant 0}$ is arbitrary and the open affinoids $\mathcal{U}$ form an admissible cover of $\mathcal{W}_p$ as in Section 1.1.

Henceforth we will use the weight space $\mathcal{W}$ representing the continuous homomorphisms:

$$\mathbb{Z}_p^\times \times (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{G}_m,$$

which is endowed with *shifted* forgetful map to $\mathcal{W}_p$ and is locally generated over the latter by the diamond operators $\langle a \rangle$, $a \in (\mathbb{Z}/N\mathbb{Z})^\times$. The shift is made so that $k \in \mathbb{Z}$ henceforth corresponds to the character $x \mapsto x^{k-1}$ of $\mathcal{W}$, and on the level of Iwasawa algebras is given by the automorphism of $\mathbb{Z}_p[\![1+2p\mathbb{Z}_p]\!]$ sending $[1+2p]$ to $(1+2p)[1+2p]$.

The eigencurve $\mathscr{C}$ is reduced, and it follows from its construction that there exists a flat and locally finite morphism $\mathsf{w} : \mathscr{C} \to \mathcal{W}$, called the weight map. Moreover w is proper by [26]. Thanks to the above shift, the classical weight 1 forms, which are the focal point of our study, map under w to finite order characters equal to the determinant of the corresponding Galois representation (pre-composed with the Artin reciprocity map).

By construction of $\mathscr{C}$, there exist bounded global sections $\{T_\ell, U_p\}_{\ell \nmid Np} \subset \mathcal{O}^+_{\mathscr{C}}(\mathscr{C})$ such that the usual application "system of eigenvalues"

$$x \in \mathscr{C}(\overline{\mathbb{Q}}_p) \longmapsto \{T_\ell(x), U_p(x)\}_{\ell \nmid Np}$$

is injective, and produces all systems of eigenvalues for $\{T_\ell, U_p\}_{\ell \nmid Np}$ acting on the space of overconvergent forms with coefficients in $\overline{\mathbb{Q}}_p$, of tame level N, having weight in $\mathcal{W}(\overline{\mathbb{Q}}_p)$ and a non-zero U_p-eigenvalue.

A fundamental arithmetic tool in the study of the geometry of $\mathscr{C}$ is the universal 2-dimensional pseudo-character

$$\tau_{\mathscr{C}} : \mathrm{G}_{\mathbb{Q}} \longrightarrow \mathcal{O}_{\mathscr{C}}(\mathscr{C}), \tag{1.1}$$

which is unramified at all $\ell \nmid Np$ and such that $\tau_{\mathscr{C}}$ maps an arithmetic Frobenius Frob_ℓ to T_ℓ. This pseudo-character interpolates p-adically the traces of semi-simple p-adic Galois representations attached to the classical points of $\mathscr{C}$. While these Galois representations are De Rham at p, the semi-simple p-adic Galois representation attached to an arbitrary specialization of $\tau_{\mathscr{C}}$ is only trianguline at p.

1.4. Classical points of $\mathscr{C}$. A point of $\mathcal{W}$ is said to be classical if its restriction to some open subgroup of $\mathbb{Z}_p^\times$ is given by the homomorphism $(x \mapsto x^{k-1})$, for some $k \in \mathbb{Z}_{\geqslant 1}$ (such characters are locally algebraic). The subset $\mathcal{W}^{\mathrm{cl}} \subset \mathcal{W}(\overline{\mathbb{Q}}_p)$ of classical weights is very Zariski dense in the sense that for any affinoid $\mathcal{U}$ of $\mathcal{W}$, the intersection $\mathcal{W}^{\mathrm{cl}} \cap \mathcal{U}$ is either empty or is Zariski dense in $\mathcal{U}$. A classical point of $\mathscr{C}$ always maps to a point of $\mathcal{W}^{\mathrm{cl}}$, but the converse in not necessarily true.

In order to describe the classical points of $\mathscr{C}$ let us first recall the notion of a p-stabilization. Let $f(z) = \sum_{n \geqslant 0} a_n e^{2i\pi nz}$ be a primitive normalized eigenform of weight $k \in \mathbb{Z}_{\geqslant 1}$, central character ε, and level $\Gamma_1(Mp^t)$, with $t \in \mathbb{Z}_{\geqslant 0}$ and M dividing N. We distinguish the following two cases.

(i) If $t = 0$, then the Hecke polynomial $X^2 - a_p X + \varepsilon(p)p^{k-1}$ of f at p has two (necessarily non-zero, but not necessarily distinct) roots, denoted α and β. The corresponding p-stabilizations $f_\alpha(z) =$

$f(z)-\beta f(pz)$ and $f_\beta = f(z)-\alpha f(pz)$ both have level $\Gamma_1(M)\cap\Gamma_0(p)$ and define points f_α and f_β in $\mathscr{C}^{\mathrm{cl}}$. If those points are distinct, we call them p-regular, if not, p-irregular. By an abuse of language we sometime say that f itself is regular or irregular at p.

(ii) If $t > 0$, then f is already a U_p-eigenvector with eigenvalue $\alpha = a_p$. If f has finite slope (i.e., $\alpha \neq 0$), then it defines a point $f_\alpha = f \in \mathscr{C}^{\mathrm{cl}}$. As the Hecke polynomial equals $X(X-\alpha)$, the point f is regular at p.

The set of classical points $\mathscr{C}^{\mathrm{cl}} \subset \mathscr{C}(\overline{\mathbb{Q}}_p)$ consists of all points f_α as above (considered with coefficients in $\overline{\mathbb{Q}}_p$ via ι_p) as $k \in \mathbb{Z}_{\geqslant 1}$, $t \in \mathbb{Z}_{\geqslant 0}$ and M dividing N vary. By Coleman's Control Theorem $\mathscr{C}^{\mathrm{cl}}$ is very Zariski dense in $\mathscr{C}$.

The p-adic valuation of the U_p-eigenvalue of a point in $\mathscr{C}(\overline{\mathbb{Q}}_p)$ is called its *slope*. The slope of a classical weight k point cannot exceed $k-1$ and is called *critical* when the equality is reached. The points having slope 0 are called *ordinary*. The locus $\mathscr{C}^{\mathrm{ord}}$ of $\mathscr{C}$ where $|U_p|_p = 1$ is open and closed in $\mathscr{C}$, and is called the ordinary locus. A formal model of $\mathscr{C}^{\mathrm{ord}}$ is given by the universal p-ordinary reduced Hida Hecke algebra of tame level N generated by the Hecke operators T_ℓ for all primes $\ell \nmid Np$ and U_p.

1.5. Hecke operators at primes dividing the level. For each $\ell | N$ the module $M_{\mathcal{U}}^{\dagger,\leqslant s}$ is endowed with a $\mathcal{O}(\mathcal{U})$-linear operator U_ℓ, commuting with $\mathcal{T}_{\mathcal{U}}^{\leqslant s}$. It can be either defined geometrically as a correspondence between modular curves of levels prime to p, or by the usual formulas on q-expansions, i.e., using p-adic interpolation of classical forms. By adding those operators to $\mathcal{T}_{\mathcal{U}}^{\leqslant s}$ one can define the full eigencurve $\mathscr{C}^{\mathrm{full}}$, whose ordinary part is directly related to the Hida Hecke algebras in their most classical definition. The advantage of working with the full Hecke algebra would become transparent in Section 1.6. The disadvantage is that this bigger algebra is not necessarily reduced. Luckily one does not have to choose when working in a neighborhood of a classical point corresponding to newform, as the next proposition shows that the two are locally isomorphic

Proposition 1.4. *Any $f \in \mathscr{C}^{\mathrm{cl}}$ corresponding to a newform of tame level N has an affinoid neighborhood $\mathcal{V}$ such that any $g \in \mathcal{V} \cap \mathscr{C}^{\mathrm{cl}}$ corresponds to a newform of tame level N. Moreover, $\mathscr{C}$ and $\mathscr{C}^{\mathrm{full}}$ are isomorphic locally at f, in particular $\mathscr{C}^{\mathrm{full}}$ is reduced locally at f.*

Proof. As ρ_f is irreducible, by standard arguments (see for example [6, Propositions 5.1]) there exists an affinoid neighborhood $\mathcal{V}$ of f in $\mathscr{C}$ and a continuous representation $\rho_{\mathcal{V}} : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathcal{O}(\mathcal{V}))$ whose trace equals the pseudo-character $\tau_{\mathcal{V}} : \mathrm{G}_{\mathbb{Q}} \to \mathcal{O}(\mathcal{V})$ obtained from (1.1).

Fix a prime ℓ dividing N and recall that f is new at ℓ. As the local and global Langlands correspondences for GL(2) are compatible, in order to show that $g \in \mathcal{V} \cap \mathscr{C}^{\rm cl}$ is new at ℓ as well, it suffices to show that the restrictions of ρ_f and ρ_g to the inertia subgroup I_ℓ at ℓ are isomorphic. To perform this part of the argument we may restrict our study to the irreducible component of $\mathcal{V}$ containing g, i.e. we may and do temporarily assume that $\mathcal{V}$ is irreducible. Let $(r_{\mathcal{V}}, \mathcal{N}_{\mathcal{V}})$ be the Weil–Deligne representation attached to $\rho_{\mathcal{V}|\mathrm{G}_{\mathbb{Q}_\ell}}$ in [5, Lemma 7.8.14], and similarly let $(r_g, \mathcal{N}_g)$ denote the Weil–Deligne representation attached to $\rho_{g|\mathrm{G}_{\mathbb{Q}_\ell}}$. By [5, Lemmas 7.8.17] one knows that $\tau_{\mathcal{V}|\mathrm{I}_\ell}$ and $r_{\mathcal{V}|\mathrm{I}_\ell}$ are constant over $\mathcal{V}$, isomorphic to $\mathrm{tr}(\rho_f)_{|\mathrm{I}_\ell}$ and $r_{f|\mathrm{I}_\ell}$, respectively.

If $\mathcal{N}_{\mathcal{V}} = 0$ then $\rho_{\mathcal{V}}(\mathrm{I}_\ell) = r_{\mathcal{V}}(\mathrm{I}_\ell)$ is finite and isomorphic to $\rho_f(\mathrm{I}_\ell)$.

If $\mathcal{N}_{\mathcal{V}} \neq 0$ then, after possibly shrinking the irreducible affinoid $\mathcal{V}$, one can assume that $\mathcal{N}_{\mathcal{V}}$ does not vanish over $\mathcal{V}\backslash\{f\}$. It follows that any classical $g \neq f$ in $\mathcal{V}$ is given at ℓ by the Steinberg representation twisted by a character χ. One deduces then from $\mathrm{tr}(\rho_f)_{|\mathrm{I}_\ell} = 2\chi_{|\mathrm{I}_\ell}$ that either

- f at ℓ is the Steinberg representation twisted by a character having the same restriction to I_ℓ as χ, in which case $\rho_{g|\mathrm{I}_\ell}$ and $\rho_{f|\mathrm{I}_\ell}$ are isomorphic, or
- f at ℓ is a principal series attached to two characters χ_1 and χ_2 having the same restriction to I_ℓ as χ. By continuity of $\rho_{\mathcal{V}|\mathrm{G}_{\mathbb{Q}_\ell}}$ and using the density of such g in $\mathcal{V}$, one deduces that $(\chi_1/\chi_2)(\mathrm{Frob}_\ell) = \ell^{\pm 1}$. This is impossible as the cuspform f satisfies the Ramanujan Conjecture, proved by P. Deligne.

So far we have proven that the restriction $r_{g|\mathrm{I}_\ell}$ and the rank of $\mathcal{N}_g$ are both constant as g varies over the classical points in a neighborhood $\mathcal{V}$ of f in $\mathscr{C}$. In particular, all points of $\mathcal{V}^{\rm cl}$ are new at all primes $\ell \neq p$. By Coleman's Control Theorem and the Strong Multiplicity One Theorem for GL(2) it follows that for any $g \in \mathcal{V} \cap \mathscr{C}^{\rm cl}$ of non-critical slope, the corresponding generalized eigenspace in $M^{\dagger,\leqslant s}_{\mathsf{w}(g)}$ is one dimensional, generated by g.

We consider the affinoid neighborhood $\mathcal{U} = \mathsf{w}(\mathcal{V})$ of $\mathsf{w}(f)$ in $\mathcal{W}$, and the affinoid neighborhood $\mathcal{V}^{\rm full}$ of f in $\mathscr{C}^{\rm full}$ obtained by taking inverse image of $\mathcal{V}$ under the natural projection $\mathscr{C}^{\rm full} \to \mathscr{C}$. As the $\mathcal{O}(\mathcal{U})$-algebra $\mathcal{O}(\mathcal{V}^{\rm full})$ acts faithfully on the $M^{\dagger,\leqslant s}_{\mathcal{U}}$, a projective $\mathcal{O}(\mathcal{U})$-module of finite rank, it follows that $\mathcal{O}(\mathcal{V}^{\rm full})$ is also projective as $\mathcal{O}(\mathcal{U})$-module. It follows that $\bigcap_{i\in\mathbb{Z}} \mathfrak{m}_i \mathcal{O}(\mathcal{V}^{\rm full}) = \{0\}$, where $(\mathfrak{m}_i)_{i\in\mathbb{Z}}$ is any Zariski dense set of maximal ideals of $\mathcal{O}(\mathcal{U})$. Letting the $\mathfrak{m}_i$'s correspond to classical weights k_i which are large enough (with respect to the slope s), one deduces that $\mathcal{V}^{\rm full}$ is reduced. To see that any nilpotent element of $\mathcal{O}(\mathcal{V}^{\rm full})$ belongs to $\mathfrak{m}_i\mathcal{O}(\mathcal{V}^{\rm full})$ we recall that $\mathcal{O}(\mathcal{V}^{\rm full})/\mathfrak{m}_i\mathcal{O}(\mathcal{V}^{\rm full})$ is a product of fields indexed by $\mathcal{V} \cap \mathsf{w}^{-1}(k_i)$.

It remains to show that the natural inclusion of reduced $\mathcal{O}(\mathcal{U})$-algebras $\mathcal{O}(\mathcal{V}) \to \mathcal{O}(\mathcal{V}^{\mathrm{full}})$ is an isomorphism, i.e. that $U_\ell \in \mathcal{O}(\mathcal{V})$ for all ℓ dividing N. One proceeds exactly as in [6, Proposition 7.1] using $\rho_{\mathcal{V}}$ to construct an element of $\mathcal{O}(\mathcal{V})$ whose value at each $g \in \mathcal{V} \cap \mathscr{C}^{\mathrm{cl}}$ is given by $U_\ell(g)$. □

In view of Proposition 1.4, the irreducible components of $\mathscr{C}$ containing a p-ordinary newform f of tame level N, are in bijection with the minimal primes of the p-ordinary Hida Hecke algebra of tame level N which are contained in the height one prime attached to f (see [27] for more details).

Remark 1.5. Analogues of Proposition 1.4 have also been studied for $f \in \mathscr{C}^{\mathrm{cl}}$ which are not cuspidal, but still are cuspidal-overconvergent, i.e. belong to the cuspidal eigencurve $\mathscr{C}^{\mathrm{cusp}}$ defined in Section 1.6. If f has weight 1 then it corresponds to a p-irregular Eisenstein points and an exact analogue is proven in [10, Proposition 4.4]. If f has weight $k \geqslant 2$ then corresponds to a critical stabilization of an Eisenstein series, and one can argue similarly using the Galois representation $\rho_{\mathcal{V}}$ constructed by J. Bellaïche and G. Chenevier [4]. The only potential problem is when the family $\mathcal{V}$ is generically Steinberg at ℓ, while ρ_f is unramified at ℓ, and the study of the Weil–Deligne representation at ℓ then implies that $k = 2$ (Eisenstein series do not satisfy the Ramanujan conjecture!). This case was studied in detail in the PhD thesis of D. Majumdar [42]. A case which is particularly piquant is that of the unique weight 2 level $\Gamma_0(\ell)$ Eisenstein series, which is *not* old at ℓ, despite of ρ_f being unramified at ℓ, as the weight 2 level 1 Einsenstein series is not classical.

1.6. Cuspidal Hida duality. The $\mathcal{O}(\mathcal{U})$-submodule of $M_{\mathcal{U}}^{\dagger,\leqslant s}$ of cuspidal Coleman families is defined as

$$S_{\mathcal{U}}^{\dagger,\leqslant s} = \varinjlim_{v>0} \mathrm{H}^0(X(v) \times \mathcal{U}, \omega_{\mathcal{U}}(-D))^{\leqslant s},$$

where D is the cuspidal divisor of the ordinary locus X^{ord}. Note that it is $\mathcal{T}_{\mathcal{U}}^{\leqslant s}$-stable and one defines the cuspidal Hecke $\mathcal{O}(\mathcal{U})$-algebra $\mathcal{T}_{\mathcal{U}}^{\mathrm{cusp},\leqslant s}$ as the quotient of $\mathcal{T}_{\mathcal{U}}^{\leqslant s}$ acting faithfully on it. Using $\mathcal{T}_{\mathcal{U}}^{\mathrm{cusp},\leqslant s}$ instead of $\mathcal{T}_{\mathcal{U}}^{\leqslant s}$ one defines the cuspidal eigencurve $\mathscr{C}^{\mathrm{cusp}}$ which is endowed with a closed immersion $\mathscr{C}^{\mathrm{cusp}} \hookrightarrow \mathscr{C}$ of reduced flat rigid curves over $\mathcal{W}$.

Let f be a classical cuspidal point in $\mathscr{C}$. As recalled in Section 1.1, $\mathsf{w}(f)$ has a neighborhood $\mathcal{U}$ such that $S_{\mathcal{U}}^{\dagger,\leqslant s}$ and $\mathcal{T}_{\mathcal{U}}^{\mathrm{cusp},\leqslant s}$ are both free of finite rank as $\mathcal{O}(\mathcal{U})$-modules.

Proposition 1.6. *Hida's $\mathcal{O}(\mathcal{U})$-linear pairing*

$$\langle \cdot, \cdot \rangle : \mathcal{T}_{\mathcal{U}}^{\mathrm{cusp},\leqslant s} \times S_{\mathcal{U}}^{\dagger,\leqslant s} \longrightarrow \mathcal{O}(\mathcal{U})$$

sending $(T, \mathscr{G}) \in \mathcal{T}_{\mathcal{U}}^{\mathrm{cusp},\leqslant s} \times S_{\mathcal{U}}^{\dagger,\leqslant s}$ to $\langle T, \mathscr{G}\rangle = a_1(T(\mathscr{G}))$ is a perfect duality.

Proof. By Proposition 1.4 and the abstract recurrence relations between Hecke operators, one know that for all $n \in \mathbb{Z}_{\geqslant 1}$ one has $T_n \in \mathcal{T}_{\mathcal{U}}^{\mathrm{cusp},\leqslant s}$. As

$$\langle T, T_n(\mathscr{G})\rangle = \langle T_n T, \mathscr{G}\rangle = a_1(T_n T(\mathscr{G})) = a_n(T(\mathscr{G}))$$

for any $n \geqslant 1$, the q-expansion Principle from Proposition 1.2 shows that the following natural $\mathcal{O}(\mathcal{U})$-linear maps are injective

$$S_{\mathcal{U}}^{\dagger,\leqslant s} \longrightarrow \mathrm{Hom}_{\mathcal{O}(\mathcal{U})\text{-mod}}(\mathcal{T}_{\mathcal{U}}^{\mathrm{cusp},\leqslant s}, \mathcal{O}(\mathcal{U})), \quad \mathscr{G} \longmapsto (T \longmapsto \langle T, \mathscr{G}\rangle) \tag{1.2}$$

$$\mathcal{T}_{\mathcal{U}}^{\mathrm{cusp},\leqslant s} \longrightarrow \mathrm{Hom}_{\mathcal{O}(\mathcal{U})\text{-mod}}(S_{\mathcal{U}}^{\dagger,\leqslant s}, \mathcal{O}(\mathcal{U})), \qquad T \longmapsto (\mathscr{G} \longmapsto \langle T, \mathscr{G}\rangle). \tag{1.3}$$

In particular, if $\mathcal{U} = \{\mathsf{w}(f)\}$ then the above maps are isomorphisms, as $S_{\mathsf{w}(f)}^{\dagger,\leqslant s}$ and $\mathcal{T}_{\mathsf{w}(f)}^{\mathrm{cusp},\leqslant s}$ are vector spaces of finite dimension. For general $\mathcal{U}$, as $S_{\mathcal{U}}^{\dagger,\leqslant s}$ and $\mathcal{T}_{\mathcal{U}}^{\mathrm{cusp},\leqslant s}$ are free $\mathcal{O}(\mathcal{U})$-modules of finite rank, we proceed by localization at $\mathsf{w}(f)$. By Nakayama's lemma, it suffices to show that the cokernel of (1.2) vanishes residually, which follows from the surjectivity in the particular case $\mathcal{U} = \{\mathsf{w}(f)\}$ that we already considered. ☐

Remark 1.7. Similar techniques were used in [10] to establish a perfect duality between $M_{\mathcal{U}}^{\dagger,\leqslant s}$ and $\mathcal{T}_{\mathcal{U}}^{\leqslant s}$ locally at a point corresponding to a weight 1 Eisenstein series irregular at p, the issue being the control of the constant terms. To that effect, one introduces the notion of evaluation for Hida families at all cusps of X^{ord} and one establishes a "fundamental exact sequence".

2. Generalized eigenforms at classical points of the eigencurve

2.1. Geometry of $\mathscr{C}$ at classical points of weight at least 2. Recast in rigid geometry, Hida's famous Control Theorem [39] states that any point of $\mathscr{C}^{\mathrm{ord}}$ having classical weight $k \geqslant 2$ is classical. Note that ordinary forms of weight with $k \geqslant 2$ are always regular at p and, when a second p-stabilization exists, it necessarily has critical slope. Coleman's Control Theorem generalizing Hida's result, states that any non-critical slope point of $\mathscr{C}$ having weight in $\mathcal{W}^{\mathrm{cl}}$ is classical (see [18]). Those results imply that $\mathscr{C}$ is étale over $\mathcal{W}$ at any non-critical slope, p-regular classical point of weight $k \geqslant 2$, hence it is also smooth at these points. There are no known examples of classical forms of weight $k \geqslant 2$ which are p-irregular, i.e., for which the Hecke polynomial at p has a double root. There are only three cases in which w could fail to be étale at a classical point f of weight $k \geqslant 2$ and regular at p, potentially providing a cuspidal-overconvergent generalized eigenform (see [3, §7.6]).

(i) **Critical (a.k.a. evil) Eisenstein points.** Bellaïche and Chenevier proved in [4] that $\mathscr{C}$ is smooth and is conjecturally étale over $\mathcal{W}$ at such points.
(ii) **Critical CM points.** Bellaïche showed in [1] that the eigencurve is smooth, although ramified over the weight space, at such points.

He further showed that Jannsen's conjecture in Galois cohomology implies that the ramification degree is precisely 2.

(iii) **Critical non-CM points.** Breuil and Emerton proved in [13] (see also [30] for a partial result and [7] for a different proof) that w ramifies at a classical weight $k \geqslant 2$ point of critical slope if, and only if, there exists an ordinary companion form, in which case the restriction to $\mathrm{G}_{\mathbb{Q}_p}$ of the attached p-adic Galois representation splits. It is a folklore conjecture, attributed to R. Coleman and to R. Greenberg, that such non-CM points should not exist (see for example [16]).

Note that if a p-irregular classical weight $k \geqslant 2$ points were to exist, they would be of non-critical slope and Coleman's Classicality Theorem [18] would imply that the corresponding generalized eigenspace consists only of classical forms (see [12]). Thus the second case above is (conjecturally) the only one yielding cuspidal-overconvergent generalized eigenforms, and their q-expansions have recently been computed by Hsu [40] (note that the technical condition preceding Theorem 1.1 in *loc. cit.* appears to be superfluous).

2.2. Geometry of $\mathscr{C}$ at classical points of weight 1 and Hida theory. Classical weight 1 points in $\mathscr{C}$ all belong to $\mathscr{C}^{\mathrm{ord}}$ and all have critical slope, hence Hida's Control Theorem cannot be applied them and a specializations of a Hida family in weight 1 need not a classical eigenform. According to a result of E. Ghate and V. Vatsal [32], only Hida families with complex multiplication (CM) admit infinitely many classical weight 1 specializations. An explicit bound for the number of classical weight 1 specializations of a non-CM family can be found in [28].

Deligne and Serre attached in [25, Proposition 4.1] to any weight 1 newform f an irreducible Galois representation $\rho_f : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\overline{\mathbb{Q}}) \xrightarrow{\iota_p} \mathrm{GL}_2(\overline{\mathbb{Q}}_p)$ having finite image. As well-known (see for example [28]), the projective image is either dihedral, or else is isomorphic to one of the groups A_4, S_4 or A_5 in which case the form (or the corresponding point on $\mathscr{C}$) is referred to as exotic. When the projective image is dihedral, then f has real multiplication (RM) or complex multiplication (CM), depending on whether the corresponding quadratic extension of $\mathbb{Q}$ is real or imaginary. Note that, when the projective image is the Klein four group, then f has multiplication by three quadratic fields (two imaginary and one real).

One motivation for studying the geometry of the eigencurve at weight 1 points arises from the question of determining whether there is a unique, up to Galois conjugacy, Hida family specializing to a given weight 1 eigenform f. By Proposition 1.4 the minimal primes in Hida's p-ordinary Hecke algebra which are contained in the height 1 prime attached to f are in bijection with the irreducible components of $\mathscr{C}$ containing f. It follows that if

$\mathscr{C}$ is smooth at f, then there exists a unique, up to Galois conjugacy, Hida family specializing to f. It is observed in [28] that any classical weight 1 form of Klein type and irregular at p is contained in two Hida families having CM by different imaginary quadratic fields, for which reason they cannot be Galois conjugates. It turns out that this is part of a more general phenomenon. Indeed, it is proven in [9] that any weight 1 form irregular at p and having CM by K belongs to exactly 3 or 4 irreducible components of $\mathscr{C}$, exactly 2 out of which have CM by K. In particular the uniqueness, up to Galois conjugacy, of the Hida family systematically fails at such weight 1 classical points.

While conjecturally all classical points of weight $k \geqslant 2$ are expected to be p-regular (see [19]), the Chebotarev Density theorem applied to ρ_f, shows that for each classical weight 1 point f there are infinitely many irregular primes p, providing many classical points at which w is not étale (and not even smooth if f has CM).

The main result of [6] asserts that $\mathscr{C}$ is smooth at any classical weight 1 point f which is regular at p. Furthermore, the weight map w is not étale at f if, and only if, f has real multiplication (RM) by a quadratic field in which p splits. The next subsection is devoted to this case.

2.3. Regular RM case. Let f be a p-stabilization of a p-regular weight 1 newform of level N having multiplication by a real quadratic field in which p splits, and let $\mathfrak{m}$ denote the corresponding maximal ideal of the Hecke algebra. In this case the classical subspace of the generalized eigenspace $S^{\dagger}_{\mathsf{w}(f)}[\![f]\!] \subset S^{\dagger}_{\mathsf{w}(f)}$ is given by the line $\overline{\mathbb{Q}}_p \cdot f = S_1(Np)[\mathfrak{m}]$. Moreover a natural supplement of $\overline{\mathbb{Q}}_p \cdot f$ in $S^{\dagger}_{\mathsf{w}(f)}[\mathfrak{m}^2]$ is given by the subspace $S^{\dagger}_{\mathsf{w}(f)}[\mathfrak{m}^2]_0$ of cuspforms whose first Fourier coefficient vanishes ($a_1 = 0$). As $S^{\dagger}_{\mathsf{w}(f)}[\mathfrak{m}^2]_0$ is naturally isomorphic to the relative tangent space of $\mathscr{C}$ over $\mathcal{W}$ at f (i.e., the tangent space of the fiber of $\mathsf{w}^{-1}(\mathsf{w}(f))$ at f), the results of [6] show that $S^{\dagger}_{\mathsf{w}(f)}[\mathfrak{m}^2]_0$ is a line, having a basis $f^{\dagger}$. Using a cohomological computations from [6], H. Darmon, A. Lauder and V. Rotger determine in [21] the precise q-expansion of this genuine overconvergent generalized eigenform $f^{\dagger}$. They also draw some parallels with the famous Hilbert's twelfth problem which remains unsolved for real quadratic fields.

We use the present opportunity to observe that, contrarily to what was claimed in [6], based on a misinterpretation of a result by Cho and Vatsal [17], the expected equality $S^{\dagger}_{\mathsf{w}(f)}[\![f]\!] = S^{\dagger}_{\mathsf{w}(f)}[\mathfrak{m}^2]$ remains an open question. It is equivalent to showing that the ramification index of w at f equals 2 (see [8] for a thorough study of this case).

The geometry of the eigencurve is expected to be more intricate at a weight 1 points irregular at p and to have fascinating applications in Iwasawa theory.

2.4. Eisenstein case. We refer to [10] for a detailed study of this case. A p-regular weight 1 Eisenstein point belongs to a unique irreducible component of $\mathscr{C}$ which is Eisenstein and étale over the weight space. A p-irregular weight 1 Eisenstein point f is attached to an odd primitive Dirichlet character $\phi : (\mathbb{Z}/N\mathbb{Z})^\times \to \overline{\mathbb{Q}}_p^\times$ such that $\phi(p) = 1$. In contrast with the p-regular case, f belongs to exactly two Eisenstein components. Moreover f is cuspidal-overconvergent, i.e. vanishes at all cusps of the multiplicative ordinary locus of the modular curve $X(\Gamma_0(p)\cap\Gamma_1(N))$ corresponding to the $\Gamma_0(p)$-orbit of ∞, hence f belongs to the cuspidal eigencurve $\mathscr{C}^{\mathrm{cusp}}$. It is shown in *loc. cit.* that $\mathscr{C}^{\mathrm{cusp}}$ is étale at f over the weight space, hence there exists a unique, up to a Galois conjugacy, cuspidal Hida family $\mathscr{F}$ specializing to f. It is also shown that $\mathscr{C}^{\mathrm{cusp}}$ intersects transversally each of the two Eisenstein components containing f (note that the evil weight Eisenstein series of weight $\geqslant 2$ do not belong to Eisenstein components). Furthermore, one computes in *loc. cit.* the q-expansions of a basis $\{f^\dagger_{1,\phi}, f^\dagger_{\phi,1}\}$ of the space of genuine overconvergent generalized eigenforms in terms of p-adic logarithms of algebraic numbers, and one remarks that these forms are *not* cuspidal-overconvergent. Finally, the expression of the constant term $a_0(f^\dagger_{1,\phi}) = (\mathscr{L}(\phi) + \mathscr{L}(\phi^{-1}))\frac{L(\phi,0)}{2}$, where $\mathscr{L}(\phi)$ denotes the cyclotomic $\mathscr{L}$-invariant appearing in the derivative at a trivial zero of the Kubota–Leopoldt p-adic L-function $L_p(\phi\omega_p, s)$, allows to give a geometric flavored proof of Gross' formula $L'_p(\phi\omega_p, 0) = \mathscr{L}(\phi)\cdot L(\phi, 0)$ (see [10, Section 5]).

2.5. The *p*-irregular weight one case. Let us first say that this case is still the subject of active research. A weight one newform which is irregular at p yields a unique point f on $\mathscr{C}$.

Let $\mathcal{T}$ be the completed local ring of $\mathscr{C}$ at f and $S^\dagger_{\mathfrak{m}}$ be the $\mathcal{T}$-module obtained by localizing and completing $S^{\dagger,\leqslant s}_{\mathcal{U}}$ at the maximal ideal $\mathfrak{m}$ of $\mathcal{T}^{\leqslant s}_{\mathcal{U}}$ corresponding to the system of Hecke eigenvalues of f, where $\mathcal{U}$ is an admissible affinoid of $\mathcal{W}$ containing $\mathsf{w}(f)$. The localization at $\mathfrak{m}$ of the pairing defined in Proposition 1.6 gives rises to a perfect pairing

$$\mathcal{T}\times S^\dagger_{\mathfrak{m}} \longrightarrow \Lambda,\ \text{given by } \langle h,\mathscr{G}\rangle = a_1(T\cdot\mathscr{G}) \in \Lambda, \tag{2.1}$$

where Λ is the completed local ring of $\mathcal{W}$ at $\mathsf{w}(f)$. Specializing in weight $\mathsf{w}(f)$, corresponding to the maximal ideal (X) of Λ, yields a natural isomorphism

$$S^\dagger_{\mathsf{w}(f)}[\![f]\!] = S^\dagger_{\mathfrak{m}}/\mathfrak{m}_\Lambda S^\dagger_{\mathfrak{m}} \xrightarrow{\sim} \mathrm{Hom}_{\bar{\mathbb{Q}}_p}\left(\mathcal{T}/(X\cdot\mathcal{T}), \overline{\mathbb{Q}}_p\right). \tag{2.2}$$

Since $\mathcal{T}/X\cdot\mathcal{T}$ is an Artinian $\overline{\mathbb{Q}}_p$-algebra, the space $S^{\dagger}_{\mathsf{w}(f)}[\![f]\!]$ of overconvergent weight $\mathsf{w}(f)$ generalized eigenforms is by definition the union over all $i\geqslant 1$ of its subspaces $S^{\dagger}_{\mathsf{w}(f)}[\mathfrak{m}^i]$ annihilated by the ideal $\mathfrak{m}^i$.

Let us observe that the classical subspace of the generalized eigenspace $S^{\dagger}_{\mathsf{w}(f)}[\![f]\!]$ is given by the plane $\overline{\mathbb{Q}}_p\cdot f(z)\oplus\overline{\mathbb{Q}}_p\cdot f(pz)=S_1(\Gamma_0(p)\cap\Gamma_1(N))[\mathfrak{m}^2]$ which has a natural supplement $S^{\dagger}_{\mathsf{w}(f)}[\mathfrak{m}^2]_0$ in $S^{\dagger}_{\mathsf{w}(f)}[\mathfrak{m}^2]$, consisting of cuspforms whose first and p-th Fourier coefficients both vanish ($a_1=a_p=0$). As in Section 2.3 one can reasonably conjecture that $S^{\dagger}_{\mathsf{w}(f)}[\![f]\!]=S^{\dagger}_{\mathsf{w}(f)}[\mathfrak{m}^2]$.

In [22] Darmon, Lauder and Rotger constructed a map

$$S^{\dagger}_{\mathsf{w}(f)}[\mathfrak{m}^2]_0\longrightarrow \mathrm{H}^1(\mathbb{Q},\mathrm{ad}^0\rho) \tag{2.3}$$

and conjectured that it is an isomorphism. A first evidence was found by Hao Lee [41] when f is of Klein type. The only other case where a full study of the local geometry has been successfully completed is the case of a p-irregular weight 1 CM form, presented in the following subsection.

2.6. The *p*-irregular CM case. We will use without recalling the notations and assumptions introduced in the paragraph preceding Theorem A. Let $-D$ be the fundamental discriminant of the imaginary quadratic field K.

Since $\psi\neq\bar{\psi}$, the reducibility $\rho_{f|\mathrm{G}_K}=\psi\oplus\bar{\psi}$ allows to choose a basis of eigenvectors (e_1,e_2) which is uniquely defined up to individual scaling. Using the complex conjugation τ to further impose that $e_2=\rho(\tau)e_1$ determines projectively uniquely this basis and one has:

$$\rho_{f|\mathrm{G}_K}=\begin{pmatrix}\psi & 0\\ 0 & \bar{\psi}\end{pmatrix}\qquad \rho_{f|\mathrm{G}_{\mathbb{Q}}\setminus\mathrm{G}_K}=\begin{pmatrix}0 & \psi(\cdot\tau)\\ \psi(\tau\cdot) & 0\end{pmatrix}. \tag{2.4}$$

The local-global compatibility for $\rho_f=\mathrm{Ind}_K^{\mathbb{Q}}\psi$ yields that

$$a_\ell=\begin{cases}\psi(\mathfrak{l})+\psi(\bar{\mathfrak{l}}), & \text{if } (\ell)=\mathfrak{l}\bar{\mathfrak{l}}\neq(p) \text{ splits in } K,\\ 0, & \text{if } \ell \text{ is inert in } K,\\ \psi(\mathfrak{l}), & \text{if } \mathfrak{l}|\ell|pD,\end{cases} \tag{2.5}$$

By the q-expansion Principle f is uniquely determined by (2.5) together with the usual recurrence relations $a_{\ell^{r+1}}=a_\ell\cdot a_{\ell^r}-\varepsilon(\ell)a_{\ell^{r-1}}$, for $r\in\mathbb{Z}_{\geqslant 1}$, and $a_{mn}=a_ma_n$, for m,n relatively prime. The above formulas are understood with the convention that $\psi(\mathfrak{l})=0$, if $\mathfrak{l}$ divides the conductor of ψ, and similarly $\varepsilon(\ell)=0$, if ℓ divides the conductor of ρ_f.

It has been shown in [9] that in addition to belonging to two components of $\mathscr{C}$ having CM by K, f also belongs to one or two other components, i.e., $\mathcal{O}_{\mathscr{C},f}$, as well as the Hida Hecke algebra localized at f, have exactly three or four minimal primes. According to Hida's work in Iwasawa theory, points lying at the intersection of CM and non-CM families are expected to correspond to zeros of anti-cyclotomic Katz p-adic L-functions. In our situation, we are in the presence of a so-called "trivial" zero and, prior to *loc. cit.*, one ignored whether this zero was simple or not. Indeed, there is no an "anti-cyclotomic" analogue of the famous Ferrero–Greenberg Theorem on the Kubota–Leopoldt p-adic L-functions of a Dirichlet character encountered in the Eisenstein case described in Section 2.4. By using p-adic geometry, commutative algebra and Galois theoretic tools together, it is shown in *loc. cit.* is that these "anti-cyclotomic" trivial zeros are simple whenever a certain $\mathscr{L}$-invariant does not vanish, as predicted by the Strong Four Exponentials Conjecture.

A corollary of the main results of [9] is a proof of the Darmon–Lauder–Rotger Conjecture on (2.3) being an isomorphism, for all p-irregular weight 1 CM forms. In the next section we will compute the q-expansions of a basis $\{f^\dagger_{\mathscr{F}}, f^\dagger_{\Theta}\}$ of the genuine generalized eigenspace $S^\dagger[\mathfrak{m}^2]_0$.

3. Overconvergent q-expansions at p-irregular CM forms

The purpose of this section is to prove Theorems A and B which are a natural extension of the results of [9]. We keep the setting and notations from Section 2.6 and we let $\nu = 2$ if $p = 2$, and $\nu = 1$ otherwise. Define

$$\mathscr{L} = \frac{\mathscr{L}_{\text{-}}(\varphi)}{\log_p(1+p^\nu)\cdot(\mathscr{L}_{\text{-}}(\varphi)+\mathscr{L}_{\text{-}}(\bar\varphi))},\ \ \overline{\mathscr{L}} = \frac{\mathscr{L}_{\text{-}}(\bar\varphi)}{\log_p(1+p^\nu)\cdot(\mathscr{L}_{\text{-}}(\varphi)+\mathscr{L}_{\text{-}}(\bar\varphi))}.$$

3.1. Infinitesimal non-CM Hida families. Under the assumption (0.1), the results of [9] recalled in Section 2.6 imply that f belongs to exactly four irreducible components of $\mathscr{C}$, all étale over Λ, two having CM by K, while the other two corresponding to a Hida family $\mathscr{F} = \sum_{n\geqslant 1} a_n(\mathscr{F})q^n \in \Lambda[\![q]\!]$ without CM by K, and to its quadratic twist $\mathscr{F}\otimes\varepsilon_K$.

Consider the unique cochains $\eta_\varphi : \mathrm{G}_K \to \overline{\mathbb{Q}}_p$ representing the cocycles $[\eta_\varphi] \in \mathrm{H}^1(K,\varphi)$, normalized so that $\mathrm{res}_{\mathfrak{p}}([\eta_\varphi]) = \log_p \in \mathrm{H}^1(\mathbb{Q}_p, \overline{\mathbb{Q}}_p)$ and $\eta_\varphi(\gamma_0) = 0$ for a fixed element $\gamma_0 \in \mathrm{G}_K$ such that $\varphi(\gamma_0) \neq 1$ (see [9, Section 1]). The slope $\mathscr{S}_\varphi$ is a non-zero p-adic number defined by $\mathrm{res}_{\bar{\mathfrak{p}}}([\eta_\varphi]) = \mathscr{S}_\varphi \cdot [\log_p]$. As explained in *loc. cit.* one has the following formula

$$\mathscr{S}_\varphi = -\frac{\log_p(u_\varphi)}{\log_p(\tau(u_\varphi))}, \tag{3.1}$$

where $\overline{\mathbb{Q}}\cdot u_\varphi = (\overline{\mathbb{Q}}\otimes\mathcal{O}_H^\times)[\varphi]$, and moreover $\mathscr{S}_{\bar\varphi} = \mathscr{S}_\varphi^{-1}$.

Proposition 3.1 ([9, Section 2])**.** *Any $\mathrm{G}_{K/\mathbb{Q}}$-stable ordinary infinitesimal deformation of $\rho = \begin{pmatrix}\psi & \psi\eta_\varphi\\ 0 & \bar\psi\end{pmatrix}$ is reducible, i.e., of the form $\begin{pmatrix}1+\bar dX & bX\\ 0 & 1+dX\end{pmatrix}\cdot\rho$ mod X^2 and one has a natural isomorphism:*

$$t_\rho^{\mathrm{ord}} \xrightarrow{\ \sim\ } \mathrm{H}^1(K,\overline{\mathbb{Q}}_p),\ \ \left[\begin{pmatrix}\bar d & b\\ 0 & d\end{pmatrix}\right] \longmapsto d.$$

Recall the basis $\{\eta_{\mathfrak{p}}, \eta_{\bar{\mathfrak{p}}}\}$ of $\mathrm{H}^1(K, \overline{\mathbb{Q}}_p)$ where $\eta_{\mathfrak{p}}$, resp. $\eta_{\bar{\mathfrak{p}}}$, is unramified outside $\mathfrak{p}$, resp. $\bar{\mathfrak{p}}$, and $\mathrm{res}_{\mathfrak{p}}(\eta_{\mathfrak{p}}) = \mathrm{res}_{\bar{\mathfrak{p}}}(\eta_{\bar{\mathfrak{p}}}) = \log_p$. One has $\eta_{\mathfrak{p}} + \eta_{\bar{\mathfrak{p}}} = \eta_{p|\mathrm{G}_K}$, where $\eta_p \in \mathrm{H}^1(\mathbb{Q}, \overline{\mathbb{Q}}_p)$ is such that $\mathrm{res}_p(\eta_p) = \log_p$. Writing $d = x \cdot \eta_{\bar{\mathfrak{p}}} + y \cdot \eta_{\mathfrak{p}}$ Figure 3.1 represents t_ρ^{ord} and its relevant subspaces.

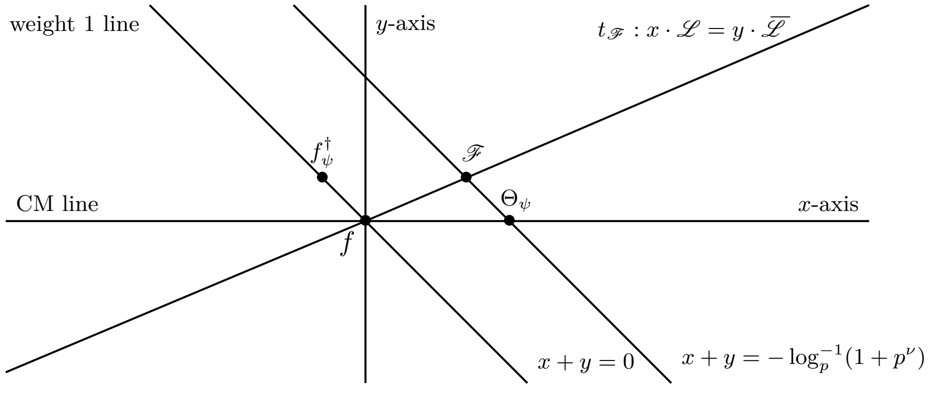


Figure 3.1. Ordinary tangent space of ρ

We note that $t_{\mathscr{F}}$ corresponds to the tangent space of a Λ-adic deformations of ρ having no CM by K, whereas the line $x + y = -\log_p^{-1}(1 + p^\nu)$ corresponds to the deformations whose determinant differs from $\det(\rho)$ by the infinitesimal universal cyclotomic character $\left(1 - \frac{\eta_p}{\log_p(1+p^\nu)} X\right)$ (see [10, Section 2.4]) We refer to the closing Remark 3.7 for a definition of the genuine overconvergent generalized eigenform $f_\psi^\dagger$.

Proposition 3.2. *There exists a basis where* $\rho_{\mathscr{F}}(\tau) \equiv \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \mod X^2$, *and*

$$\rho_{\mathscr{F}|\mathrm{G}_K} \equiv \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \overline{\mathscr{L}}\eta_{\mathfrak{p}} + \mathscr{L}\eta_{\bar{\mathfrak{p}}} & \xi \mathscr{L} \eta_\varphi \\ \xi^{-1} \overline{\mathscr{L}} \mathscr{S}_{\bar{\varphi}} \overline{\eta}_\varphi & \mathscr{L}\eta_{\mathfrak{p}} + \overline{\mathscr{L}}\eta_{\bar{\mathfrak{p}}} \end{pmatrix} \cdot X \right) \begin{pmatrix} \psi & \\ & \bar{\psi} \end{pmatrix} \mod X^2,$$

where $\xi \in \overline{\mathbb{Q}}_p$ *is such that* $\xi^2 = \mathscr{S}_{\bar{\varphi}} \frac{\bar{\mathscr{L}}}{\mathscr{L}}$. *Moreover, the* $\mathrm{G}_{K_{\mathfrak{p}}}$*-stable line of* $\rho_{\mathscr{F}}$ *is generated residually by* $\xi \cdot e_1 + e_2$ *and the corresponding unramified quotient is given by the character*

$$\chi_{\mathscr{F}} \equiv \psi \cdot \Big(1 - (\mathscr{L}(\eta_{\mathfrak{p}} - \eta_\varphi) + \overline{\mathscr{L}}\eta_{\bar{\mathfrak{p}}}) \cdot X\Big) \mod X^2.$$

The expression $\rho_{\mathscr{F} \otimes \varepsilon_K}$ *is obtained by replacing* ξ *with* $-\xi$.

Proof. The claim about $\rho_{\mathscr{F}|\mathrm{G}_K}$ follows from Proposition 3.1, Figure 3.1, the equality $(\xi\mathscr{L})^2 = \mathscr{S}_{\bar{\varphi}} \overline{\mathscr{L}} \mathscr{L}$ and the fact, proved in [9, Lemma 3.10], that its reducibility ideal equals (X^2). Note that while the cocycles $\mathscr{S}_{\bar{\varphi}} \overline{\eta}_\varphi$ and $\eta_{\bar{\varphi}}$ from *ibid.* might differ by a coboundary, this coboundary necessarily vanishes on $\mathrm{G}_{K_{\mathfrak{p}}}$, facilitating the computation of the ordinary filtration.

It follows from *loc. cit.* that $\rho_{\mathscr{F}}(\tau) \equiv \left(\begin{smallmatrix} 0 & \mu^{-1} \\ \mu & 0 \end{smallmatrix}\right)$ mod X^2, for some $\mu \in \overline{\mathbb{Q}}_p^\times$. By rescaling the basis by an element of $1 + \overline{\mathbb{Q}}_p X$ (which would not alter the expression of $\rho_{\mathscr{F}|\mathrm{G}_K}$ modulo X^2), one can find $\mu' \in \overline{\mathbb{Q}}_p$ such that

$$\rho_{\mathscr{F}}(\tau) \equiv \left(\begin{smallmatrix} -\mu' X & \mu^{-1} \\ \mu & \mu' X \end{smallmatrix}\right) \quad \bmod X^2.$$

Computing $\rho_{\mathscr{F}|\mathrm{G}_K}(\tau \cdot \tau)$ one finds that:

$$\left(\begin{smallmatrix} \mathscr{L}\eta_{\mathfrak{p}} + \bar{\mathscr{L}}\eta_{\bar{\mathfrak{p}}} & \xi \mathscr{L}\bar{\eta}_\varphi \\ \xi^{-1}\bar{\mathscr{L}}\mathscr{S}_{\bar{\varphi}}\eta_\varphi & \bar{\mathscr{L}}\eta_{\mathfrak{p}} + \mathscr{L}\eta_{\bar{\mathfrak{p}}} \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & \\ & \varphi \end{smallmatrix}\right)$$
$$= \left(\begin{smallmatrix} 0 & \mu^{-1} \\ \mu & 0 \end{smallmatrix}\right) \left(\begin{smallmatrix} \bar{\mathscr{L}}\eta_{\mathfrak{p}} + \mathscr{L}\eta_{\bar{\mathfrak{p}}} & \xi \mathscr{L}\eta_\varphi \\ \xi^{-1}\bar{\mathscr{L}}\mathscr{S}_{\bar{\varphi}}\bar{\eta}_\varphi & \mathscr{L}\eta_{\mathfrak{p}} + \bar{\mathscr{L}}\eta_{\bar{\mathfrak{p}}} \end{smallmatrix}\right) \left(\begin{smallmatrix} \varphi & \\ & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} 0 & \mu^{-1} \\ \mu & 0 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & \mu^{-1}\mu'(\varphi-1) \\ \mu\mu'(\varphi-1) & 0 \end{smallmatrix}\right).$$

Since φ is non-trivial the above equality implies that $\mu' = 0$ and $\mu^2 = 1$. If $\mu = -1$, then one can change the signs of μ and ξ simultaneously, to obtain $\rho_{\mathscr{F}}(\tau) \equiv \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \bmod X^2$ in all cases. ☐

3.2. Some ℓ-units. Recall the $\mathscr{L}$-invariant $\mathscr{L}_{\mathfrak{p}} = -\frac{\log_p(u_{\mathfrak{p}})}{\mathrm{ord}_{\mathfrak{p}}(u_{\mathfrak{p}})}$ defined in [9], where $u_{\mathfrak{p}} \in \mathcal{O}_K[\frac{1}{\mathfrak{p}}]^\times$ is any element having non-zero $\mathfrak{p}$-adic valuation. Analogously given a prime $\mathfrak{l}$ of K, relatively prime to p, we let

$$\mathscr{L}_{\mathfrak{l}} = -\frac{\log_p(u_{\mathfrak{l}})}{\mathrm{ord}_{\mathfrak{l}}(u_{\mathfrak{l}})}, \tag{3.2}$$

where $u_{\mathfrak{l}} \in \mathcal{O}_K[\frac{1}{\mathfrak{l}}]^\times$ is any element whose $\mathfrak{l}$-adic valuation $\mathrm{ord}_{\mathfrak{l}}(u_{\mathfrak{l}})$ is non-zero. Clearly, $\mathscr{L}_{\mathfrak{l}}$ only depends on $\mathfrak{l}$ (and not on the choice of $u_{\mathfrak{l}}$), and it equals $-\log_p(\ell)$ (resp. $-\frac{1}{2}\log_p(\ell)$) when ℓ is inert (resp. ramified) in K.

We let H denote the splitting field of the anti-cyclotomic character φ.

Assume for the rest of Section 3.2 that ℓ is inert or ramified in K, and that $\ell D \nmid N$. Then ψ is unramified at the unique prime $\mathfrak{l}$ of K above ℓ and

$$\varphi(\mathfrak{l}) = \psi(\mathfrak{l})/\psi(\bar{\mathfrak{l}}) = 1, \tag{3.3}$$

i.e. $\mathfrak{l}$ splits completely in H. Let λ be a prime of H above $\mathfrak{l}$ and choose an element $u_\lambda \in \mathcal{O}_H[\frac{1}{\lambda}]^\times$ whose λ-adic valuation $\mathrm{ord}_\lambda(u_\lambda)$ is non-zero. Using (3.3) one can prove that $(\overline{\mathbb{Q}} \otimes \mathcal{O}_H[\frac{1}{\ell}]^\times)[\varphi]$ is a $\overline{\mathbb{Q}}$-plane with a basis consisting of u_φ and

$$u_{\varphi,\lambda} = \sum_{h \in \mathrm{Gal}(H/K)} \varphi^{-1}(h) \otimes h(u_\lambda) \in (\overline{\mathbb{Q}} \otimes \mathcal{O}_H[\tfrac{1}{\ell}]^\times)[\varphi]. \tag{3.4}$$

By (3.1) the following does not depend on the particular choice of u_λ

$$\mathscr{L}_{\varphi,\lambda} = -\frac{\log_p(u_{\varphi,\lambda}) + \mathscr{S}_\varphi \log_p(\tau(u_{\varphi,\lambda}))}{\mathrm{ord}_\lambda(u_\lambda)}. \tag{3.5}$$

Let us now investigate how $\mathscr{L}_{\varphi,\lambda}$ depends on λ. Since $\mathrm{Gal}(H/K)$ acts transitively on the primes λ of H above ℓ, any such prime is of the form $h(\lambda)$ for some $h \in \mathrm{G}_K$. As $h(u_\lambda) = u_{h(\lambda)}$, one has $u_{\varphi,h(\lambda)} = \varphi(h) \cdot u_{\varphi,\lambda}$ and

$$\mathscr{L}_{\varphi,h(\lambda)} = \varphi(h) \cdot \mathscr{L}_{\varphi,\lambda}.$$

On the other hand, given any element $\gamma \in \mathrm{G}_{\mathbb{Q}} \backslash \mathrm{G}_K$, one has

$$\psi(\tau h \gamma h^{-1}) = \psi(\tau h \tau \cdot \tau\gamma \cdot h^{-1}) = \varphi(h)^{-1}\psi(\tau\gamma),$$

hence $\psi(\tau \cdot h\gamma h^{-1}) \cdot \mathscr{L}_{\varphi,h(\lambda)} = \psi(\tau\gamma) \cdot \mathscr{L}_{\varphi,\lambda}$.

We will now make the following specific choice for γ:

• if ℓ is inert in K, we let γ be any Frobenius element $\mathrm{Frob}_\ell \in \mathrm{G}_{\mathbb{Q}}$ whose image in $\mathrm{Gal}(H/\mathbb{Q})$ equals the Frobenius at λ;

• if ℓ is ramified in K, we let γ be any element of an inertia subgroup of $\mathrm{G}_{\mathbb{Q}}$ at ℓ whose image in $\mathrm{Gal}(H/\mathbb{Q})$ generates the order two subgroup $\mathrm{Gal}(H_\lambda/\mathbb{Q}_\ell) = \mathrm{I}(H_\lambda/\mathbb{Q}_\ell)$.

Definition 3.3. Let ℓ be a prime which is inert or ramified in K and such that $\ell D \nmid N$. For γ chosen as above we define

$$\mathscr{L}_{\psi,\ell} = \psi(\tau\gamma) \cdot \mathscr{L}_{\varphi,\lambda} \in \overline{\mathbb{Q}}_p. \tag{3.6}$$

As the restriction of γ to $\overline{\mathbb{Q}}^{\ker(\rho_f)}$ is uniquely determined, up to G_H-conjugacy, it follows that $\mathscr{L}_{\psi,\ell}$ only depends on ℓ and on ψ, and not on the particular choices of λ and of γ as above.

3.3. Computation at split primes. Assume that the prime ℓ splits in K, and write $(\ell) = \mathfrak{l}\bar{\mathfrak{l}}$ with $\mathfrak{l} \neq \bar{\mathfrak{l}}$. We recall that $a_\ell = \psi(\mathfrak{l}) + \psi(\bar{\mathfrak{l}})$, for $\ell \neq p$ and $a_p = \psi(\mathfrak{p}) = \psi(\bar{\mathfrak{p}})$. We will now determine $a_\ell(\mathscr{F})$ infinitesimally.

Proposition 3.4. *Let $\ell \neq p$ be a prime splitting in K as $\mathfrak{l} \cdot \bar{\mathfrak{l}}$.*

One has $\eta_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{l}}) = \mathscr{L}_{\mathfrak{l}}$, $\eta_{\bar{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{l}}) = \mathscr{L}_{\bar{\mathfrak{l}}}$ and

$$\tfrac{\mathrm{d}}{\mathrm{d}X}\big|_{X=0} a_\ell(\mathscr{F}) = -\psi(\mathfrak{l}) \cdot (\overline{\mathscr{L}}\mathscr{L}_{\mathfrak{l}} + \mathscr{L}\mathscr{L}_{\bar{\mathfrak{l}}}) - \psi(\bar{\mathfrak{l}}) \cdot (\overline{\mathscr{L}}\mathscr{L}_{\bar{\mathfrak{l}}} + \mathscr{L}\mathscr{L}_{\mathfrak{l}}).$$

Moreover $\tfrac{\mathrm{d}}{\mathrm{d}X}\big|_{X=0} a_p(\mathscr{F}) = \frac{\psi(\mathfrak{p})}{\log_p(1+p^\nu)} \cdot \left(\frac{\mathscr{L}\text{-}(\varphi)\cdot\mathscr{L}\text{-}(\bar{\varphi})}{\mathscr{L}\text{-}(\varphi)+\mathscr{L}\text{-}(\bar{\varphi})} + \mathscr{L}_{\mathfrak{p}} \right)$.

Proof. By Class Field Theory one has an exact sequence

$$0 \to \mathrm{Hom}(\mathrm{G}_K, \overline{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathcal{O}_{K,\mathfrak{p}}^\times \times \mathcal{O}_{K,\bar{\mathfrak{p}}}^\times \times K_{\mathfrak{l}}^\times, \overline{\mathbb{Q}}_p) \to \mathrm{Hom}(\mathcal{O}_K[\tfrac{1}{\mathfrak{l}}]^\times, \overline{\mathbb{Q}}_p)$$

whose first map sends $\eta_{\mathfrak{p}}$ to $(\log_p, 0, \eta_{\bar{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{l}}) \cdot \mathrm{ord}_{\mathfrak{l}})$. Hence

$$\log_p(u_{\mathfrak{l}}) + \eta_{\mathfrak{p}}(\mathrm{Frob}_{\mathfrak{l}})\,\mathrm{ord}_{\mathfrak{l}}(u_{\mathfrak{l}}) = 0$$

as claimed (see (3.2)). Similarly $\eta_{\bar{\mathfrak{p}}}(\mathrm{Frob}_{\mathfrak{l}}) = \mathscr{L}_{\bar{\mathfrak{l}}}$.

One can associate to $\mathscr{F}$ a p-ordinary Λ-adic representation $\rho_{\mathscr{F}} : \mathrm{G}_{\mathbb{Q}} \to \mathrm{GL}_2(\Lambda)$ whose trace is given by the pushforward of $\tau_{\mathscr{C}}$ introduced in (1.1). It follows that for any $\ell \neq p$

$$a_\ell(\mathscr{F}) = \mathrm{tr}\,\rho_{\mathscr{F}}^{\mathrm{I}_\ell}(\mathrm{Frob}_\ell), \text{ and } a_p(\mathscr{F}) = \chi_{\mathscr{F}}(\mathrm{Frob}_p),$$

where $\chi_{\mathscr{F}}$ is the unramified character acting on the unramified $\mathrm{G}_{K_\mathfrak{p}}$-quotient of $\rho_{\mathscr{F}}$. The computation of $\frac{\mathrm{d}}{\mathrm{d}X}\big|_{X=0} a_\ell(\mathscr{F})$ then follows directly from the infinitesimal expression for $\rho_{\mathscr{F}|\mathrm{G}_K}$ given in Proposition 3.2. The value $\frac{\mathrm{d}}{\mathrm{d}X}\big|_{X=0} a_p(\mathscr{F})$ is computed similarly using the formulas $(\eta_\varphi - \eta_\mathfrak{p})(\mathrm{Frob}_\mathfrak{p}) = \mathscr{L}_-(\bar\varphi) + \mathscr{L}_\mathfrak{p}$ and $\eta_{\bar{\mathfrak{p}}}(\mathrm{Frob}_\mathfrak{p}) = -\mathscr{L}_\mathfrak{p}$ established in [9]. □

3.4. Computation at inert primes. We now turn to the case of a prime ℓ which is inert in K. If $\ell \nmid N$, then we let $\gamma = \mathrm{Frob}_\ell \in \mathrm{G}_\mathbb{Q}\backslash\mathrm{G}_K$ denote a Frobenius element and let λ denote the resulting place of H above ℓ, i.e. $\mathrm{Frob}_\ell^2 \in \mathrm{G}_H$ defines a Frobenius element at λ, denoted Frob_λ. We have seen that $a_\ell = 0$ and the aim of this section is to express infinitesimally $a_\ell(\mathscr{F})$ in terms of logarithms of ℓ-units.

Proposition 3.5. *Assume that ℓ is inert in K. If $\ell | N$, then $a_\ell(\mathscr{F}) = 0$. If $\ell \nmid N$ then*

(i) $\eta_\varphi(\mathrm{Frob}_\lambda) = \mathscr{L}_{\varphi,\lambda}$, *and*

(ii) $\frac{\mathrm{d}}{\mathrm{d}X}\big|_{X=0} a_\ell(\mathscr{F}) = -\xi\mathscr{L}\cdot\mathscr{L}_{\psi,\ell}$.

Proof. If ℓ divides N then it also divides the conductor of ψ, hence $\rho_f^{\mathrm{I}_\ell} = \{0\}$ and $a_\ell = 0$. Then $a_\ell(\mathscr{F}) = 0$ as well, since by [27, Section 6] all classical specializations of a Hida family $\mathscr{F}$ share the same local type at ℓ.

(i). Letting $\pi|\mathfrak{p}$ be the prime of H given by ι_p, Class Field Theory yields an exact sequence

$$0 \longrightarrow \mathrm{Hom}(\mathrm{G}_H, \overline{\mathbb{Q}}_p) \longrightarrow \mathrm{Hom}\left(\prod_{h\in\mathrm{Gal}(H/K)} (\mathcal{O}_{H,h(\pi)}^\times \times \mathcal{O}_{H,h(\bar\pi)}^\times) \times H_\lambda^\times, \overline{\mathbb{Q}}_p\right) \longrightarrow \mathrm{Hom}(\mathcal{O}_H[\tfrac{1}{\lambda}]^\times, \overline{\mathbb{Q}}_p),$$

sending $\eta_{\varphi|\mathrm{G}_H}$ to $\Big((\varphi^{-1}(h)(1,\mathscr{S}_\varphi)\log_p)_{h\in\mathrm{Gal}(H/K)}, \eta_\varphi(\mathrm{Frob}_\lambda)\cdot\mathrm{ord}_\lambda\Big)$. The triviality on $u_\lambda \in \mathcal{O}_H[\frac{1}{\lambda}]^\times$ yields by (3.4) and (3.5) the desired equality

$$\log_p(u_{\varphi,\lambda}) + \mathscr{S}_\varphi \log_p(\tau(u_{\varphi,\lambda})) + \eta_\varphi(\mathrm{Frob}_\lambda)\,\mathrm{ord}_\lambda(u_\lambda) = 0.$$

(ii). As $\rho_f(g) = \begin{pmatrix} 0 & \psi(g\tau) \\ \psi(\tau g) & 0 \end{pmatrix}$, for all $g \in \mathrm{G}_\mathbb{Q}\backslash\mathrm{G}_K$, one has

$$\rho_{\mathscr{F}}(g^2) = \rho_{\mathscr{F}}(g)\rho_{\mathscr{F}}(g) \equiv \begin{pmatrix} * & \psi(g\tau)\cdot\mathrm{tr}(\rho_{\mathscr{F}}(g)) \\ \psi(\tau g)\,\mathrm{tr}(\rho_{\mathscr{F}}(g)) & * \end{pmatrix} \mod X^2.$$

Comparing the expression of the upper right coefficient of $\rho_{\mathscr{F}}(g^2)$ with Proposition 3.2, one finds

$$\mathrm{tr}\,\rho_{\mathscr{F}}(g) = -\xi\mathscr{L}\psi(\tau g)\eta_\varphi(g^2)X \pmod{X}^2, \text{ for all } g \in \mathrm{G}_\mathbb{Q}\backslash\mathrm{G}_K. \tag{3.7}$$

Applying this to $g = \mathrm{Frob}_\ell$, so that $g^2 = \mathrm{Frob}_\lambda$, one deduces that

$$\frac{\mathrm{d}}{\mathrm{d}X}\Big|_{X=0} a_\ell(\mathscr{F}) = -\xi\mathscr{L}\psi(\tau\,\mathrm{Frob}_\ell)\eta_\varphi(\mathrm{Frob}_\lambda).$$

The claim then follows from (i), in view of (3.6). □

3.5. Computation at ramified primes. Finally, we turn to the case when $(\ell) = \mathfrak{l}^2$ ramifies in K. We have seen that $a_\ell = \psi(\mathfrak{l})$ and the aim of this section is to express infinitesimally $a_\ell(\mathscr{F})$ in terms of logarithms of ℓ-units. If $\ell D | N$, applying the same argument as in Proposition 3.5 yields $a_\ell(\mathscr{F}) = 0$. Henceforth we assume that $\ell D \nmid N$, and recall that $\gamma \in \mathrm{G}_\mathbb{Q} \backslash \mathrm{G}_K$ is an element of an inertia subgroup of $\mathrm{G}_\mathbb{Q}$ at ℓ whose image in $\mathrm{Gal}(H/\mathbb{Q})$ generates the order two subgroup $\mathrm{Gal}(H_\lambda/\mathbb{Q}_\ell) = \mathrm{I}(H_\lambda/\mathbb{Q}_\ell)$ (see Section 3.2).

Proposition 3.6. *Assume that $\ell | D$, but $\ell D \nmid N$. Then*

(i) $\eta_\mathfrak{p}(\mathrm{Frob}_\mathfrak{l}) = \eta_{\bar{\mathfrak{p}}}(\mathrm{Frob}_\mathfrak{l}) = \mathscr{L}_\mathfrak{l} = -\frac{1}{2}\log_p(\ell)$ *and* $\eta_\varphi(\mathrm{Frob}_\ell) = \mathscr{L}_{\varphi,\lambda}$,

(ii) $\frac{\mathrm{d}}{\mathrm{d}X}\big|_{X=0} a_\ell(\mathscr{F}) = \psi(\mathfrak{l}) \cdot \left(\frac{\log_p(\ell)}{2\log_p(1+p^\nu)} - \xi\mathscr{L} \cdot \mathscr{L}_{\psi,\ell}\right)$.

Proof. (i). The restrictions of $\eta_\mathfrak{p}$, $\eta_{\bar{\mathfrak{p}}}$ and η_φ (see (3.3)) to the inertia subgroup at $\mathfrak{l}$ belong to $\mathrm{Hom}(\mathcal{O}_{K_\mathfrak{l}}^\times, \overline{\mathbb{Q}}_p) = \{0\}$. Therefore their values at $\mathrm{Frob}_\mathfrak{l}$ are well defined (depending on the choice of $\lambda | \mathfrak{l}$ for η_φ) and can be computed using Class Field Theory exactly as in Propositions 3.4 and 3.5.

(ii). By (i) and Proposition 3.2, $\rho_\mathscr{F}(\gamma) \mod X^2 = \rho_f(\gamma) = \begin{pmatrix} 0 & \psi(\gamma\tau) \\ \psi(\tau\gamma) & 0 \end{pmatrix}$ has a fixed line generated by the vector $e_1 + \psi(\tau\gamma)e_2$. Using again (i) and Proposition 3.2, together with (3.6), one finds

$$\begin{aligned} a_\ell(\mathscr{F}) &= (1,0)\rho_\mathscr{F}(\mathrm{Frob}_\ell)\begin{pmatrix} 1 \\ \psi(\tau\gamma) \end{pmatrix} \\ &\equiv \psi(\mathfrak{l})(1,0)\left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} (\bar{\mathscr{L}}\eta_\mathfrak{p} + \mathscr{L}\eta_{\bar{\mathfrak{p}}})(\mathrm{Frob}_\mathfrak{l}) & \xi\mathscr{L}\eta_\varphi(\mathrm{Frob}_\ell) \\ \xi^{-1}\bar{\mathscr{L}}\mathscr{S}_{\bar{\varphi}}\bar{\eta}_\varphi(\mathrm{Frob}_\ell) & (\mathscr{L}\eta_\mathfrak{p} + \bar{\mathscr{L}}\eta_{\bar{\mathfrak{p}}})(\mathrm{Frob}_\mathfrak{l}) \end{pmatrix} X\right)\begin{pmatrix} 1 \\ \psi(\tau\gamma) \end{pmatrix} \\ &\equiv \psi(\mathfrak{l}) - \psi(\mathfrak{l}) \cdot \left((\mathscr{L} + \overline{\mathscr{L}}) \cdot \mathscr{L}_\mathfrak{l} + \xi\psi(\tau\gamma)\mathscr{L} \cdot \mathscr{L}_{\varphi,\lambda}\right) \cdot X \pmod{X^2}. \quad \square \end{aligned}$$

As $\mathscr{F}$ has local type $1 \oplus \varepsilon_K$ at ℓ, one deduces that $\frac{a_\ell(\mathscr{F}) \cdot a_\ell(\mathscr{F} \otimes \varepsilon_K)}{\psi(\mathfrak{l})^2} = \left(\frac{\det \rho_\mathscr{F}}{\det \rho_f}\right)(\mathrm{Frob}_\ell)$ which concords with Proposition 3.6 (ii).

3.6. On the q-expansion of CM families specializing to f. Denote by $\mathscr{C}\ell_K^{(p)}(\mathfrak{p}^\infty)$ the p-primary part of the ray class group of K of conductor $\mathfrak{p}^\infty$. The torsion-free quotient $\mathscr{C}\ell_K^{(p)}(\mathfrak{p}^\infty)_{/\,\mathrm{tor}}$ is the Galois group of the unique $\mathfrak{p}$-ramified $\mathbb{Z}_p$-extension of K. We introduced in [9, Section 3] a p-adic avatar $\mathrm{G}_K \twoheadrightarrow \mathscr{C}\ell_K^{(p)}(\mathfrak{p}^\infty)_{/\,\mathrm{tor}} \to \overline{\mathbb{Z}}_p^\times$ of a Hecke character of infinity type $(1,0)$, further used to define of a universal character

$$\mathrm{G}_K \twoheadrightarrow \mathscr{C}\ell_K^{(p)}(\mathfrak{p}^\infty)_{/\,\mathrm{tor}} \hookrightarrow \overline{\mathbb{Z}}_p^\times[\![\mathscr{C}\ell_K^{(p)}(\mathfrak{p}^\infty)_{/\,\mathrm{tor}}]\!]^\times$$

interpolating p-adically Hecke characters of K whose infinity type belongs to $\{(k,0); k \in \mathbb{Z}\}$. Its localization yields a character $\chi_\mathfrak{p} : \mathrm{G}_K \to \Lambda^\times$ such that

$$\chi_{\mathfrak{p}} \equiv 1 - \frac{\eta_{\mathfrak{p}}}{\log_p(1+p^\nu)} X \pmod{X^2}. \tag{3.8}$$

It is easy to check that $\chi_{\mathfrak{p}}\bar{\chi}_{\mathfrak{p}}$ extends to $\mathrm{G}_{\mathbb{Q}}$ and equals the universal cyclotomic character.

There exist two CM families Θ_ψ and $\Theta_{\bar\psi}$ specializing to f in weight one and such that their attached Λ-adic representation is given by $\mathrm{Ind}_K^{\mathbb{Q}}\,\psi\chi_{\mathfrak{p}}$ and $\mathrm{Ind}_K^{\mathbb{Q}}\,\bar\psi\chi_{\mathfrak{p}}$ respectively, hence in particular their Fourier coefficients vanish at all primes ℓ inert in K. Since the CM-line given by $y=0$ in Figure 3.1 corresponds to the tangent space of Θ_ψ, if one lets $\mathscr{L}=0$ in Proposition 3.4, then one obtains that for any prime $\ell \neq p$ splitting in K

$$\tfrac{\mathrm{d}}{\mathrm{d}X}\big|_{X=0} a_\ell(\Theta_\psi) = -\frac{\psi(\mathfrak{l})\mathscr{L}_{\mathfrak{l}} + \psi(\bar{\mathfrak{l}})\mathscr{L}_{\bar{\mathfrak{l}}}}{\log_p(1+p^\nu)}, \text{ and } \tfrac{\mathrm{d}}{\mathrm{d}X}\big|_{X=0} a_p(\Theta_\psi) = \frac{\psi(\mathfrak{p})\mathscr{L}_{\mathfrak{p}}}{\log_p(1+p^\nu)}.$$

Finally, for any $\ell | D$, one has (by letting $\mathscr{L}=0$ in Proposition 3.6):

$$\tfrac{\mathrm{d}}{\mathrm{d}X}\big|_{X=0} a_\ell(\Theta_\psi) = \tfrac{\psi(\mathfrak{l})\log_p(\ell)}{2\log_p(1+p^\nu)}.$$

3.7. Generalized eigenforms at weight 1 CM points of the eigencurve. Recall from Section 2.5 that one can attach to f a generalized eigenspace $S^\dagger_{\mathsf{w}(f)}[\![f]\!] = S^\dagger_{\mathsf{w}(f)}[\![\mathfrak{m}]\!]$. One clearly has $S^\dagger_{\mathsf{w}(f)}[\mathfrak{m}] = \overline{\mathbb{Q}}_p \cdot f$ and we have already observed that classical subspace of $S^\dagger_{\mathsf{w}(f)}[\![f]\!]$ has a basis $\{f, \theta_\psi\}$ whose elements belong to $S^\dagger_{\mathsf{w}(f)}[\mathfrak{m}^2]$.

Under the running assumption (0.1), it is shown in [9] that $\dim S^\dagger_{\mathsf{w}(f)}[\![f]\!] = \dim S^\dagger_{\mathsf{w}(f)}[\mathfrak{m}^2] = 4$ and hence the space $S^\dagger_{\mathsf{w}(f)}[\![f]\!]_0 = S^\dagger_{\mathsf{w}(f)}[\mathfrak{m}^2]_0$ of genuine overconvergent generalized eigenforms defined in Section 2.5 is two-dimensional. We consider the following forms in $S^\dagger_{\mathsf{w}(f)}[\![f]\!]$:

$$f^\dagger_{\mathscr{F}} = \log_p(1+p^\nu)\cdot \tfrac{\mathrm{d}}{\mathrm{d}X}\big|_{X=0}\left(\mathscr{F}\otimes\varepsilon_K - \mathscr{F}\right), \text{ and} \tag{3.9}$$

$$f^\dagger_{\Theta} = \log_p(1+p^\nu)\cdot \tfrac{\mathrm{d}}{\mathrm{d}X}\big|_{X=0}\left(\Theta_{\bar\psi} - \Theta_\psi\right). \tag{3.10}$$

A computation based on the q-expansions Principle (see Proposition 1.2) and Proposition 1.4, implies the following linear relation in $S^\dagger_{\mathsf{w}(f)}[\![f]\!]$:

$$\log_p(1+p^\nu)^2 \cdot \tfrac{\mathrm{d}}{\mathrm{d}X}\big|_{X=0}\left(\bar{\mathscr{L}}\Theta_\psi + \mathscr{L}\Theta_{\bar\psi} - \tfrac{(\mathscr{L}+\bar{\mathscr{L}})}{2}(\mathscr{F} + \mathscr{F}\otimes\varepsilon_K)\right) = \frac{\mathscr{L}_{\text{-}}(\varphi)\cdot\mathscr{L}_{\text{-}}(\bar\varphi)}{\mathscr{L}_{\text{-}}(\varphi)+\mathscr{L}_{\text{-}}(\bar\varphi)}(f-\theta_\psi).$$

Proof of Theorem A. Parts (i)–(iii) are direct consequence of Propositions 3.4, 3.5 and 3.6, in view of the q-expansion Principle (see Definition 3.3 for the definition of $\mathscr{L}_{\psi,\ell}$). Part (iv) results from the well-known relations between the abstract Hecke operators. $\square$

Proof of Theorem B. In the basis (e_1, e_2) from Section 2.6, $\rho_f(\tau)$ fixes the vector e_1+e_2, while the ordinary line of Θ_ψ, resp. $\Theta_{\bar\psi}$ is spanned by e_1, resp. e_2. Finally by Proposition 3.2 the ordinary line of $\mathscr{F}$ is spanned residually by $\xi \cdot e_1 + e_2$, allowing us to compute the desired cross-ratio as follows:

$$[e_1 + e_2, \xi \cdot e_1 + e_2; e_1, e_2] = \frac{\left|\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right| \cdot \left|\begin{smallmatrix} \xi & 0 \\ 1 & 1 \end{smallmatrix}\right|}{\left|\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right| \cdot \left|\begin{smallmatrix} \xi & 1 \\ 1 & 0 \end{smallmatrix}\right|} = \xi. \qquad \square$$

Remark 3.7 (The point $f_\psi^\dagger$ in Figure 3.1)**.** The Galois representation ρ from Proposition 3.1, whose semi-simplification is given by $\rho_{f|\mathrm{G}_K}$, has a unique ordinary line and the tangent space of the corresponding deformation problem is represented in Figure 3.1. It is clear that it has a unique CM deformation given by Θ_ψ and that both $\mathscr{F}$ and its twist $\mathscr{F} \otimes \varepsilon_K$ yield the same non-CM deformation. Then $f_\psi^\dagger = \frac{\mathrm{d}}{\mathrm{d}X}\big|_{X=0} (\mathscr{F} - \Theta_\psi)$ is a genuine overconvergent generalized eigenform, and its Fourier coefficient at a prime ℓ is given by

$$a_\ell(f_\psi^\dagger) = \mathscr{L} \cdot \begin{cases} (\psi(\mathfrak{l}) - \psi(\bar{\mathfrak{l}})) \cdot (\mathscr{L}_{\mathfrak{l}} - \mathscr{L}_{\bar{\mathfrak{l}}}), & \text{if } \ell \neq p \text{ splits as } \mathfrak{l} \cdot \bar{\mathfrak{l}}, \\ -\xi \cdot \mathscr{L}_{\psi,\ell}, & \text{if } \ell \nmid N \text{ is inert} \\ 0, & \text{if } \ell | N \text{ is inert} \\ \psi(\mathfrak{p}) \cdot \mathscr{L}_{\text{-}}(\bar\varphi), & \text{if } \ell = p, \\ -\psi(\mathfrak{l}) \cdot \xi \cdot \mathscr{L}_{\psi,\ell}, & \text{if } \ell | D. \end{cases}$$

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