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Quartic Salem numbers which are Mahler measures of non-reciprocal 2-Pisot numbers

par Toufik ZAÏMI

Résumé. Motivé par une question de M. J. Bertin, on obtient des paramétrisations des polynômes minimaux des nombres de Salem quartiques, disons α, qui sont des mesures de Mahler des 2 -nombres de Pisot non-réciproques. Cela nous permet de déterminer de tels nombres α, de trace donnée, et de déduire que pour tout entier naturel t (resp. $t \geq 2$), il y a un nombre de Salem quartique, de trace t, qui est (resp. qui n'est pas) une mesure de Mahler d'un 2 -nombre de Pisot non-réciproque.

Abstract. Motivated by a question of M. J. Bertin, we obtain parametrizations of minimal polynomials of quartic Salem numbers, say α, which are Mahler measures of non-reciprocal 2-Pisot numbers. This allows us to determine all such numbers α with a given trace, and to deduce that for any natural number t (resp. $t \geq 2$) there is a quartic Salem number of trace t which is (resp. which is not) a Mahler measure of a non-reciprocal 2-Pisot number.

1. Introduction

A Salem number, named after R. Salem [12, 13], is a real algebraic integer greater than 1 whose other conjugates are of modulus at most 1, with at least one conjugate of modulus 1; the set of such numbers is traditionally denoted by $\mathcal{T}$ [1]. An algebraic integer θ is said to be a j-Pisot number if it has j conjugates, including θ, with modulus greater than 1, and no conjugate with modulus 1. This is a generalization of the classical notion of Pisot numbers, where $j = 1$ and $\theta \in \mathbb{R}$ is positive. It seems it was Cantor [6] who came up with a similar definition (he called such numbers k-PV "tuples"). Cases where $j = 2$ and $\theta \in \mathbb{C} \setminus \mathbb{R}$ are called complex Pisot numbers; they were considered by Kelly [11] and then investigated in more detail by Chamfy [7]. Some results on complex Pisot numbers in a given algebraic number field may be found in [2]. Imaginary Gaussian Pisot numbers are examples of complex Pisot numbers. A Gaussian Pisot number is an algebraic integer with modulus greater than 1 whose other conjugates, over the Gaussian field $\mathbb{Q}(i)$, where $i := \sqrt{-1}$, are of modulus less than 1. Some properties of the set of Gaussian Pisot numbers may be

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found in [16, 17], and [18, Proposition 1] says that a complex Pisot number θ such that $i \in \mathbb{Q}(\theta)$ is an imaginary Gaussian Pisot.

Recall also that the Mahler measure $M(\theta)$ of a non–zero algebraic integer θ is the absolute value of the product of the conjugates of θ with modulus at least 1, and the number θ is said to be reciprocal whenever $1/\theta$ is a conjugate of θ. In particular, a reciprocal algebraic integer is a unit, and a Salem number, say α, is reciprocal and has $(\deg(\alpha) - 2) \geq 2$ conjugates with modulus 1.

Throughout, when we speak about conjugates, the minimal polynomial, the trace and the degree of an algebraic number θ, without mentioning the basic field, this is meant over $\mathbb{Q}$. As usual, we denote by $\deg(\theta)$, Γ_θ, G_θ, $\mathrm{tr}(\theta)$, and $\mathrm{Irr}(\theta, K, x)$ the degree of θ, the normal closure of the extension $\mathbb{Q} \subset \mathbb{Q}(\theta)$, the Galois group of Γ_θ realized as a subgroup of the symmetric group $S_{\deg(\theta)}$, the trace of θ, and the minimal polynomial of θ over a number field K, respectively. Also, if we designate by $\mathcal{P}_j$ and $\mathcal{U}$ the sets of non-reciprocal j-Pisot numbers and non-reciprocal totally imaginary quartic units with modulus greater than 1, respectively, then $\mathcal{U} \subset \mathcal{P}_2$ and any element of $\mathcal{U}$ is a complex Pisot number.

Initiated by Boyd [3], several authors studied the question of whether a given algebraic integer is a Mahler measure of an algebraic number, and some related results may be found in [4], [8] and [9]. To answer a more specific question raised also by Boyd, Dubickas showed in [10] that there are families of Salem numbers for every degree of the form $2 + 4n$ ($n \in \mathbb{N}$) and also for degree 4 which are Mahler measures of non-reciprocal algebraic integers.

In this context, we have recently investigated Salem numbers which are Mahler measures of non-reciprocal 2-Pisot numbers [15, 19]. We proved, in particular, that if $\alpha \in \mathcal{T}$ satisfies $\alpha = M(\theta)$ for some $\theta \in \mathcal{P}_2$, then $\deg(\alpha) \in \{4, 6\}$ and $\theta \in \mathcal{U}$ [19]. Also, we obtained a characterization of such numbers α with $\deg(\alpha) = 6$, implying that G_α is isomorphic to S_4 or to A_4 [15].

The above mentioned result of [19] may be completed as follows.

Theorem 1.1. *Let $\theta \in \mathbb{C}$. Then, $\theta \in \mathcal{P}_2$ and $M(\theta) \in \mathcal{T} \Leftrightarrow \theta \in \mathcal{U}$ and one of the following equivalent criteria holds:*

(i) $\deg(M(\theta)) \in \{4, 6\}$;
(ii) $\bar{\theta} \notin \mathbb{Q}(\theta)$;
(iii) $\mathbb{Q}(\theta)$ *is non-normal;*
(iv) G_θ *is the symmetric group S_4, the alternating group A_4, or the dihedral group D_4.*

Whenever we write $G_\theta = D_4$ we mean, with an abuse of notation, that $G_\theta = D_4$ for an appropriate labelling of the conjugates of θ. In fact the first equivalence in Theorem 1.1, namely

$$(1.1)\qquad (\theta \in \mathcal{P}_2 \text{ and } M(\theta) \in \mathcal{T}) \Longleftrightarrow (\theta \in \mathcal{U} \text{ and } \deg(M(\theta)) \in \{4, 6\}),$$

is contained in [19, Theorem 1.1]. This theorem also yields

$$(1.2)\quad \theta \in \mathcal{U} \Longrightarrow (\deg(M(\theta)) \in \{2, 4, 6\}, \text{ and } \deg(M(\theta)) = 6 \Leftrightarrow G_\theta \in \{S_4, A_4\}).$$

From (1.2) and Theorem 1.1 we easily deduce:

Corollary 1.2. *Let $\theta \in \mathcal{U}$. Then we have the following.*

(i) *$M(\theta)$ is a Salem number of degree* $6 \Leftrightarrow \deg(M(\theta)) = 6 \Leftrightarrow G_\theta \in \{S_4, A_4\}$.

(ii) *$M(\theta)$ is a Salem number of degree* $4 \Leftrightarrow \deg(M(\theta)) = 4 \Leftrightarrow G_\theta = D_4$.

(iii) *$M(\theta)$ is not a Salem number* $\Leftrightarrow \deg(M(\theta)) = 2 \Leftrightarrow \mathbb{Q}(\theta)$ *is normal.*

Proof. The first (resp. second, last) assertion follows immediately from the relations (1.1) and (1.2) (resp. the relations (1.1), (1.2) and the equivalence (i) $\Leftrightarrow$ (iv) in Theorem 1.1, the relations (1.1), (1.2) and the equivalence (i) $\Leftrightarrow$ (iii) in Theorem 1.1). □

Explicit examples of numbers $\theta \in \mathcal{U}$ satisfying $G_\theta = D_4$ (and so by Corollary 1.2 (ii), $M(\theta) \in \mathcal{T}$ with $\deg(M(\theta)) = 4$) are given in [19, Theorem 1.2] and in [19, Theorem 1.3]. These theorems describe all non-reciprocal Gaussian Pisot numbers θ satisfying $M(\theta) \in \mathcal{T}$, and in a private communication M. J. Bertin proposed me to determine the minimal polynomials of such numbers $M(\theta)$. In fact Theorem 3.4, below, gives parametrizations of the minimal polynomials of all elements of

$$\mathbf{T} := \{\alpha \in \mathcal{T} \mid \deg(\alpha) = 4 \text{ and } \alpha = M(\theta) \text{ for some } \theta \in \mathcal{P}_2\},$$

yielding a simple algorithm to determine all $\alpha \in \mathbf{T}$ with a given trace. This is the major result of the present note from which we obtain the following.

Theorem 1.3. *For any natural number t (resp. $t \geq 2$) there is a quartic Salem number of trace t which is (resp. which is not) a Mahler measure of a non-reciprocal 2-Pisot number. Furthermore, if m_t (resp. n_t) denotes the number of quartic Salem numbers of trace t which are (resp. which are not) Mahler measures of non-reciprocal 2-Pisot numbers, then* $\lim_{t\to\infty} m_t = \lim_{t\to\infty} n_t/m_t = \infty$.

The proofs of Theorem 1.1 and Theorem 1.3 are presented in the next and the last section, respectively. As mentioned above Section 3 contains also parametrizations of the minimal polynomials of all quartic Salem numbers which are Mahler measures of non-reciprocal 2-Pisot numbers. All computations are performed using PARI [14].

To end this section let us show the following simple generalization of the direct implication in the third equivalence in Theorem 1.1.

Proposition 1.4. *If $\theta \in \mathcal{P}_j$ and $M(\theta) \in \mathcal{T}$, then θ is a unit,* $\deg(\theta) = 2j \geq 4$*, and $\mathbb{Q}(\theta)$ is non-normal whenever j is odd or $j = 2$.*

Proof. Let $\theta \in \mathcal{P}_j$ such that $M(\theta) \in \mathcal{T}$. If the conjugates $\theta_1, \dots, \theta_d$ of θ are labelled so that

$$|\theta_1| \geq \cdots \geq |\theta_j| > 1 > |\theta_{j+1}| \geq \cdots \geq |\theta_d|,$$

then $M(\theta) = \varepsilon\theta_1 \dots \theta_j$ for some $\varepsilon \in \{-1, 1\}$, $j \geq 2$, $d \geq j+1$ ($d = j \Rightarrow M(\theta) \in \mathbb{N}$), and $M(\theta)$ is a unit, as $M(\theta)$ is reciprocal; thus θ is a unit. Writing $1/M(\theta) = \varepsilon\theta_{k_1} \dots \theta_{k_j}$ for some $1 \leq k_1 < \cdots < k_j \leq d$ we see that $\theta_1 \cdots \theta_j \theta_{k_1} \dots \theta_{k_j} = 1 = |\theta_1 \dots \theta_j \theta_{j+1} \dots \theta_d|$,

$$|\theta_{k_1} \dots \theta_{k_j}| = |\theta_{j+1} \dots \theta_d|,$$

and so $d = 2j$, since otherwise the inequalities $|\theta_{k_j}| \geq |\theta_d|$, $|\theta_{k_{j-1}}| \geq |\theta_{d-1}|, \dots, |\theta_{k_1}| \geq |\theta_{d-j+1}|$ imply $|\theta_{k_1} \dots \theta_{k_j}| \geq |\theta_{d-j+1} \dots \theta_d| > |\theta_{j+1} \dots \theta_d|$. Also, if $\mathbb{Q}(\theta)$ is normal, then θ is totally imaginary (recall that $M(\theta)$ has a non-real conjugate with modulus 1), and so the number of conjugates of θ with modulus greater than 1, namely j, must be even. Finally, we have by Theorem 1.1 that $\mathbb{Q}(\theta)$ is non–normal when $j = 2$. □

2. Proof of Theorem 1.1

As mentioned above the relation (1.1) follows from [19, Theorem 1.1]. Hence we have to show that the assertions (i)–(iv) are equivalent for any $\theta \in \mathcal{U}$. Using (1.2) and the fact that the transitive subgroups of S_4 with cardinality greater than 4 are S_4, A_4 and the three isomorphic copies of D_4, it is enough to prove

(2.1) $\deg(M(\theta)) = 2 \Longleftrightarrow \bar{\theta} \in \mathbb{Q}(\theta) \Longleftrightarrow \mathbb{Q}(\theta)$ is normal $\Longleftrightarrow \mathrm{Card}(G_\theta) = 4$,

i.e.,

$$\deg(M(\theta)) = 2 \Longleftrightarrow \bar{\theta} \in \mathbb{Q}(\theta) \tag{2.2}$$

and

$$\mathbb{Q}(\theta) \text{ is normal} \Longleftrightarrow \bar{\theta} \in \mathbb{Q}(\theta), \tag{2.3}$$

$\forall\, \theta \in \mathcal{U}$, as the last equivalence in (2.1) is always true.

Let $\theta, \bar{\theta}, \theta_2, \bar{\theta}_2$ be the conjugates of θ. Then, the conjugates of $M(\theta) = \theta\bar{\theta}$ are among the algebraic integers

$$\theta\bar{\theta},\ \theta\theta_2,\ \theta\bar{\theta}_2, \bar{\theta}\theta_2,\ \overline{\theta\theta_2},\ \theta_2\bar{\theta}_2 = 1/M(\theta), \tag{2.4}$$

and $1/M(\theta)$ is necessarily one of them, since $M(\theta) > 1 = |\theta||\theta_2| > 1/M(\theta) > 0$; thus $\deg(M(\theta)) \geq 2$, and $\deg(M(\theta)) = 2 \Leftrightarrow P(x) := (x - \theta\bar{\theta})(x - 1/\theta\bar{\theta})$ is the minimal polynomial of $M(\theta)$.

Clearly, if $\bar{\theta} \in \mathbb{Q}(\theta)$, then $\theta\bar{\theta} \in \mathbb{Q}(\theta) \cap \mathbb{R}$ and so $\deg(M(\theta)) = 2$. To complete the proof of (2.2) suppose $\deg(M(\theta)) = 2$ and on the contrary $\bar{\theta} \notin$

$\mathbb{Q}(\theta)$. Then, $M(\theta) \notin \mathbb{Q}(\theta)$ and so $P(x) = \mathrm{Irr}(M(\theta), \mathbb{Q}(\theta), x)$. By considering the embedding, say σ, of the field $\mathbb{Q}(\theta, M(\theta))$ into $\mathbb{C}$, which sends $\theta\bar{\theta}$ to $1/\theta\bar{\theta}$, and whose restriction to $\mathbb{Q}(\theta)$ is the identity, we see that $\sigma(\bar{\theta}) = \sigma(\theta\bar{\theta}/\theta) = \sigma(\theta\bar{\theta})/\sigma(\theta) = 1/\theta^2\bar{\theta}$ and this leads immediately to a contradiction, because $|1/\bar{\theta}\theta^2| = 1/|\theta|^3 < 1/|\theta|$ and θ has no conjugate with modulus less than $1/|\theta|$.

Similarly, the direct implication in (2.3) is trivial. To prove the converse, suppose $\bar{\theta} \in \mathbb{Q}(\theta)$. Then, the identity $id_{\mathbb{Q}(\theta)}$ of $\mathbb{Q}(\theta)$ and the restriction $c_{\mathbb{Q}(\theta)}$ of the complex conjugation to $\mathbb{Q}(\theta)$ send the pair $(\theta, \bar{\theta})$ to $(\theta, \bar{\theta})$ and $(\bar{\theta}, \theta)$, respectively. Also, the image of $\bar{\theta}$ under the action of the embedding of $\mathbb{Q}(\theta)$ into $\mathbb{C}$ sending θ to θ_2 (resp. to $\bar{\theta}_2$) is $\bar{\theta}_2$ (resp. is θ_2), because this embedding is distinct from $id_{\mathbb{Q}(\theta)}$ and $c_{\mathbb{Q}(\theta)}$. Therefore the conjugates the algebraic integer $\theta/\bar{\theta} \in \mathbb{Q}(\theta)$, namely $\theta/\bar{\theta}$, $\bar{\theta}/\theta$, $\theta_2/\bar{\theta}_2$, $\bar{\theta}_2/\theta_2$, are all of modulus 1, and so $\theta/\bar{\theta}$ is a root of unity.

It follows, when $\deg(\theta/\bar{\theta}) = 4$, that the field $\mathbb{Q}(\theta) = \mathbb{Q}(\theta/\bar{\theta})$ is cyclotomic, and hence $\mathbb{Q}(\theta)$ is normal. Also, if $\deg(\theta/\bar{\theta}) = 1$, then $\bar{\theta} = -\theta$, $\bar{\theta}_2 = -\theta_2$, and so there are two real numbers y and z such that $\theta = iy$ and $\theta_2 = iz$. As $|\theta_2| = 1/|\theta|$ there is $\varepsilon \in \{-1, 1\}$ such that $yz = \varepsilon$; thus $\theta_2 = i\varepsilon/y = -\varepsilon/iy = -\varepsilon/\theta \in \mathbb{Q}(\theta)$, and $\mathbb{Q}(\theta)$ is normal. Finally, suppose $\deg(\theta/\bar{\theta}) = 2$. Then, $\mathbb{Q}(\theta/\bar{\theta})$ is a non-real quadratic subfield of $\mathbb{Q}(\theta)$ (in fact $\mathbb{Q}(\theta/\bar{\theta}) \in \{\mathbb{Q}(i), \mathbb{Q}(i\sqrt{3})\}$). Moreover, as $\theta\bar{\theta} \in \mathbb{Q}(\theta) \cap \mathbb{R}$ and $\deg(\theta\bar{\theta}) = 2$, we have that $\mathbb{Q}(\theta\bar{\theta})$ is a real quadratic subfield of $\mathbb{Q}(\theta)$, $\mathbb{Q}(\theta) = \mathbb{Q}(\theta/\bar{\theta}, \theta\bar{\theta})$ is a composite of two quadratic fields, and so $\mathbb{Q}(\theta)$ is normal. $\square$

3. Quartic Salem numbers which are Mahler measures of elements of $\mathcal{U}$

As it was observed, by Boyd [5], a polynomial

$$S^{(a,b)}(x) := x^4 - ax^3 + bx^2 - ax + 1 \in \mathbb{Z}[x]$$

is the minimal polynomial of a quartic Salem number $\alpha^{(a,b)}$ if and only if

$$|b + 2| < 2a \quad \text{and} \quad b \notin \{1 - a, 2, 1 + a\}.$$

Therefore, the trace of a quartic Salem is a natural number, and for each $a \in \mathbb{N}$ with $a \geq 4$ (resp. with $a \leq 3$) there are $4(a-1)$ (resp. $3a$) quartic Salem numbers α with $\mathrm{tr}(\alpha) = a$.

Also, notice that if

$$\alpha_1 := \alpha, \quad \alpha_2, \quad \alpha_3 := 1/\alpha_1, \quad \alpha_4 := 1/\alpha_2 = \bar{\alpha}_2 \tag{3.1}$$

designate the conjugates of a quartic Salem number α, then the conjugates of the unit $\alpha_1\alpha_2$ are $\beta_1 := \alpha_1\alpha_2$, $\beta_2 := \alpha_1\bar{\alpha}_2$, $\beta_3 := \bar{\alpha}_2/\alpha_1 = 1/\beta_1$, $\beta_4 := \alpha_2/\alpha_1 = 1/\beta_2$, since $\Gamma_\alpha = \mathbb{Q}(\alpha_1, \alpha_2)$ and $G_\alpha = D_4$ (for more details see the proof of Lemma 3.1 below), $\mathrm{tr}(1/\beta_1) = \mathrm{tr}(\beta_1) = \sum_{1 \leq j < k \leq 4} \alpha_j\alpha_k - 2 = b - 2$,

$\beta_1\beta_2\beta_3\beta_4 = 1$, $\sum_{1\le j<k\le 4}\beta_j\beta_k = \operatorname{tr}(\alpha^2)+2 = tr^2(\alpha) - 2\operatorname{tr}(\beta_1) - 2 = a^2-2b+2$, and so

$$\begin{aligned}(3.2)\quad & S^{(a,b)}(x) = \operatorname{Irr}(\alpha,\mathbb{Q},x)\\ & \Longrightarrow \operatorname{Irr}(\alpha_1\alpha_2,\mathbb{Q},x) = x^4-(b-2)x^3+(a^2-2b+2)x^2-(b-2)x+1.\end{aligned}$$

Now, consider an element $\theta\in\mathcal{U}$ with conjugates θ, $\bar\theta$, θ_2, $\bar\theta_2$. As it was indicated in the proof of Theorem 1.1, $1/\theta\bar\theta$ is a conjugate of $M(\theta)=\theta\bar\theta$, and the conjugates of $M(\theta)$ are among the numbers given by (2.4). It follows, when $\deg(M(\theta))=4$, that one the two numbers $\theta\theta_2$ and $\theta\bar\theta_2$ is a conjugate of $M(\theta)$, and the other is a root of unity with degree at most 2, since the set $\{\theta\bar\theta,\theta\theta_2,\theta\bar\theta_2,\bar\theta\theta_2,\overline{\theta\theta_2},\theta_2\bar\theta_2\}$ is closed under complex conjugation.

Replacing, if necessary, θ by $\bar\theta$ we may assume without loss of generality that $\theta\theta_2$ is a conjugate of $M(\theta)$. Therefore,

$$\zeta := \theta\bar\theta_2 \in \{-1,\pm i, e^{\pm i2\pi/3}, e^{\pm i\pi/3}\},$$

$\bar\theta_2\in\{-1/\theta,\pm i/\theta, e^{\pm i2\pi/3}/\theta, e^{\pm i\pi/3}/\theta\}$, and this leads to a partition of set

$$V=\{\theta\in\mathcal{U}\mid M(\theta)\in\mathbf{T}\}=\{\theta\in\mathcal{U}\mid \deg(M(\theta))=4\}$$

as stated by the following lemma.

Lemma 3.1. *Let V_1 (resp. V_2, V_3, V_4) be the collection of elements θ of $\mathcal{U}$ such that the set of conjugates of θ is $\{\theta,\bar\theta,-1/\theta,-1/\bar\theta\}$ (resp. $\{\theta,\bar\theta,i/\theta,-i/\bar\theta\}$, $\{\theta,\bar\theta,e^{i2\pi/3}/\theta,e^{-i2\pi/3}/\bar\theta\}$, $\{\theta,\bar\theta,e^{i\pi/3}/\theta,e^{-i\pi/3}/\bar\theta\}$) and $\deg(M(\theta))=4$. Then*

$$V=V_1\cup V_2\cup V_3\cup V_4.$$

Furthermore, if $\alpha\in\mathcal{T}$ and $\theta\in V$ such that $\alpha=M(\theta)$, then $G_\theta=D_4$ and $\Gamma_\theta=\mathbb{Q}(\theta,\bar\theta)=\mathbb{Q}(\alpha_1,\alpha_2)=\Gamma_\alpha$.

Proof. The first statement in Lemma 3.1 follows trivially from the computation above, Theorem 1.1 and Corollary 1.2. It is also clear that (for any quartic Salem number α) $\Gamma_\alpha=\mathbb{Q}(\alpha_1,\alpha_2)$, $\alpha_2\notin\mathbb{Q}(\alpha)$, $\operatorname{Irr}(\alpha_2,\mathbb{Q}(\alpha),x) = (x-\alpha_2)(x-1/\alpha_2)$, $[\Gamma_\alpha:\mathbb{Q}]$ =8, and $G_\alpha=D_4$ (for the ordering of the conjugates of α given by (3.1)). Also, Corollary 1.2 (ii) yields $G_\theta=D_4$ and $[\Gamma_\theta:\mathbb{Q}]$ =8; thus $\Gamma_\alpha=\Gamma_\theta$, since $\{\alpha_1,\alpha_2\}\subset\{\theta\bar\theta,\theta\theta_2,\overline{\theta\theta_2}\}\subset\Gamma_\theta$ (recall that $\zeta=\theta\bar\theta_2$ and $\bar\zeta=\bar\theta\theta_2$). Finally, we have by the implication (i) $\Rightarrow$ (ii) in Theorem 1.1 that $\bar\theta\notin\mathbb{Q}(\theta)$, $[\mathbb{Q}(\theta,\bar\theta):\mathbb{Q}]$ ≥8 and so $\mathbb{Q}(\theta,\bar\theta)=\Gamma_\theta$. □

The lemma below is the main tool in the proof of Theorem 3.4.

Lemma 3.2. *The polynomial $S^{(a,b)}$ is the minimal polynomial of a Salem number α satisfying $\alpha=M(\theta)$ for some $\theta\in V_1$ if and only if $b\notin\{1-a, 2, 1+a\}$ and there is a quadratic algebraic integer s such that $\operatorname{Im}(s^2)\neq 0$ and $(a,b)=(|s|^2,-s^2-\bar s^2-2)$. If one of these two assertions is true, then*

$$(3.3)\qquad \operatorname{Irr}(\theta,\mathbb{Q},x)=x^4-(s+\bar s)x^3+(|s|^2-2)x^2+(s+\bar s)x+1.$$

Similarly, the polynomial $S^{(a,b)}$ is the minimal polynomial of a Salem number α satisfying $\alpha = M(\theta)$ for some $\theta \in V_2$ (resp. $\theta \in V_3$, $\theta \in V_4$) if and only if $b \notin \{1-a, 2, 1+a\}$ and there is an algebraic integer $s \in \mathbb{Q}(\zeta)$, where $\zeta := i$ (resp. $\zeta := e^{i2\pi/3}$, $\zeta := e^{i\pi/3}$), such that $\mathrm{Im}(\bar{\zeta}s^2) \neq 0$ and $(a,b) = (|s|^2, \bar{\zeta}s^2 + \zeta\bar{s}^2 - 2)$. If one of these two assertions is true, then

$$\mathrm{Irr}(\theta, \mathbb{Q}, x) = x^4 - (s+\bar{s})x^3 + (|s|^2 + \zeta + \bar{\zeta})x^2 - (s\bar{\zeta} + \bar{s}\zeta)x + 1. \tag{3.4}$$

Proof. To show the direct implication in Lemma 3.2 consider an element $\alpha \in \mathcal{T}$ such that $\alpha = M(\theta)$ for some $\theta \in V$. Then, Lemma 3.1 says that the conjugates of θ are θ, $\bar{\theta}$, ζ/θ, $\bar{\zeta}/\bar{\theta}$, where $\zeta := -1$ when $\theta \in V_1$ and ζ is as in the second statement of Lemma 3.2 for $\theta \notin V_1$. Hence, $\{\theta\bar{\theta}, \theta\bar{\zeta}/\bar{\theta}, \bar{\theta}\zeta/\theta, 1/\theta\bar{\theta}\}$ is the set of conjugates of $\alpha = \theta\bar{\theta}$, $\alpha_2 \in \{\theta\bar{\zeta}/\bar{\theta}, \bar{\theta}\zeta/\theta\}$ and

$$\alpha_1\alpha_2 \in \{\bar{\zeta}\theta^2, \zeta\bar{\theta}^2\}.$$

It is clear when $\theta \in V_1$ that the conjugates of the algebraic integer

$$s := \theta - 1/\theta \in \mathbb{Q}(\theta)$$

are $\theta - 1/\theta$ and $\bar{\theta} - 1/\bar{\theta} = \bar{s}$, and so $[\mathbb{Q}(s) : \mathbb{Q}] \leq 2$. Because θ is a root of the polynomial $x^2 - sx - 1 \in \mathbb{Q}(s)[x]$, we have $[\mathbb{Q}(s)(\theta) : \mathbb{Q}(s)] \leq 2$. It follows by the equations $\mathbb{Q}(\theta) = \mathbb{Q}(\theta, s)$ and $[\mathbb{Q}(\theta, s) : \mathbb{Q}(s)][\mathbb{Q}(s) : \mathbb{Q}] = 4$ that $[\mathbb{Q}(\theta, s) : \mathbb{Q}(s)] = [\mathbb{Q}(s) : \mathbb{Q}] = 2$, s is a quadratic imaginary algebraic integer and

$$\mathrm{Irr}(\theta, \mathbb{Q}(s), x) = (x-\theta)(x-1/\theta) = x^2 - sx - 1.$$

Suppose now $\theta \notin V_1$. If $\zeta \notin \mathbb{Q}(\theta)$, then $\bar{\zeta}$ is the other conjugate of ζ, over $\mathbb{Q}(\theta)$, i.e., $\mathrm{Irr}(\zeta, \mathbb{Q}(\theta), x) = \mathrm{Irr}(\zeta, \mathbb{Q}, x)$, and by considering the embedding, say σ, of $\mathbb{Q}(\theta, \zeta)$ into $\mathbb{C}$ which sends the ordered pair (θ, ζ) to $(\theta, \bar{\zeta})$ we obtain the contradiction $\sigma(\zeta/\theta) = \bar{\zeta}/\theta$, since $\bar{\zeta}/\theta$ is not a conjugate of θ. Therefore, $\zeta \in \mathbb{Q}(\theta)$, $\mathbb{Q}(\theta) = \mathbb{Q}(\theta, \zeta)$, $[\mathbb{Q}(\theta, \zeta) : \mathbb{Q}(\zeta)] = 4/[\mathbb{Q}(\zeta) : \mathbb{Q}] = 2$, and so θ is quadratic over $\mathbb{Q}(\zeta)$. It is also easy to see that the other conjugate, say θ', of θ, over $\mathbb{Q}(\zeta)$, is ζ/θ. If not, $\theta' \in \{\bar{\theta}, \bar{\zeta}/\bar{\theta}\}$, $\theta\theta' \in \{\theta\bar{\theta}, \theta\bar{\zeta}/\bar{\theta}\}$, $\deg(\theta\theta') = 4$ (recall that $\theta\bar{\zeta}/\bar{\theta}$ is a conjugate of the quartic number $\bar{\theta}\theta$) and so $\theta\theta' \notin \mathbb{Q}(\zeta)$; hence

$$\mathrm{Irr}(\theta, \mathbb{Q}(\zeta), x) = (x-\theta)(x-\zeta/\theta) = x^2 - sx + \zeta \in \mathbb{Q}(\zeta)[x],$$

where

$$s := \theta + \zeta/\theta \in \mathbb{Q}(\zeta).$$

Consequently, we have (for $\zeta = -1$ or ζ imaginary)

$$\mathrm{Irr}(\theta, \mathbb{Q}, x) = (x^2 - sx + \zeta)(x^2 - \bar{s}x + \bar{\zeta}),$$

and so (3.3) and (3.4) are true.

Setting $K := \mathbb{Q}(\theta - 1/\theta)$ for $\zeta = -1$, and $K := \mathbb{Q}(\zeta)$ otherwise, we see that the conjugates of $\bar{\zeta}\theta^2$, over K, are $\bar{\zeta}\theta^2$ and ζ/θ^2 (recall that the conjugates of θ, over K, are θ and ζ/θ), $\mathrm{Irr}(\bar{\zeta}\theta^2, K, x) = x^2 - (\bar{\zeta}s^2 - 2)x + 1$, since $\bar{\zeta}\theta^2 \neq \zeta/\theta^2$,

$$\mathrm{Irr}(\bar{\zeta}\theta^2, \mathbb{Q}, x) = (x^2 - (\bar{\zeta}s^2 - 2)x + 1)(x^2 - (\zeta\bar{s}^2 - 2)x + 1),$$

and so

$$\mathrm{Irr}(\alpha_1\alpha_2, \mathbb{Q}, x) = x^4 - (\bar{\zeta}s^2 + \zeta\bar{s}^2 - 4)x^3 + (2 + |\bar{\zeta}s^2 - 2|^2)x^2 - (\bar{\zeta}s^2 + \zeta\bar{s}^2 - 4)x + 1,$$

as (it was mentioned above) $\alpha_1\alpha_2 \in \{\bar{\zeta}\theta^2, \zeta\bar{\theta}^2\}$. Finally, if $S^{(a,b)}$ denotes the minimal polynomial of α, then $b \notin \{1-a, 2, 1+a\}$, the relation (3.2) yields

$$b = \bar{\zeta}s^2 + \zeta\bar{s}^2 - 2 \quad \text{and} \quad a^2 = |\bar{\zeta}s^2 - 2|^2 + 2b = |\bar{\zeta}s^2 - 2|^2 + 2\bar{\zeta}s^2 + 2\zeta\bar{s}^2 - 4 = |s|^4,$$

and the inequality $\mathrm{Im}(\bar{\zeta}s^2) \neq 0$ follows from the fact that $|b+2| < 2a \Leftrightarrow |\bar{\zeta}s^2 + \zeta\bar{s}^2| < 2|s|^2 = 2|\bar{\zeta}s^2|$.

To unify the notation in the proof of the "if" part of the two equivalences in Lemma 3.2, set again $\zeta := -1$ and $K := \mathbb{Q}(s)$ (resp. $\zeta := i$ and $K := \mathbb{Q}(i)$, $\zeta := e^{i2\pi/3}$ and $K := \mathbb{Q}(e^{i2\pi/3})$, $\zeta := e^{i\pi/3}$ and $K := \mathbb{Q}(e^{i\pi/3})$). Then, K is an imaginary quadratic field. Because $b \notin \{1-a, 2, 1+a\}$, $(a, b) = (|s|^2, \bar{\zeta}s^2 + \zeta\bar{s}^2 - 2)$ and

$$|b+2| < 2a \Longleftrightarrow |\bar{\zeta}s^2 + \zeta\bar{s}^2| < 2|s|^2 \Longleftrightarrow |\bar{\zeta}s^2 + \zeta\bar{s}^2| \neq 2|s|^2 \Longleftrightarrow \mathrm{Im}(\bar{\zeta}s^2) \neq 0,$$

we see that $S^{(a,b)}$ is the minimal polynomial of a Salem number α.

To show that $\alpha = M(\theta)$ for some $\theta \in V_1$ (resp. V_2, V_3, V_4) consider a root, say again θ, of the polynomial $x^2 - sx + \zeta \in K[x]$. Then, $x^2 - sx + \zeta = (x-\theta)(x-\zeta/\theta)$,

$$s = \theta + \zeta/\theta,$$

$x^2 - \bar{s}x + \bar{\zeta} = (x - \bar{\theta})(x - \overline{\zeta/\theta})$, θ is a root of

$$\begin{aligned} C(x) &:= (x^2 - sx + \zeta)(x^2 - \bar{s}x + \bar{\zeta}) \\ &= x^4 - (s + \bar{s})x^3 + (|s|^2 + \zeta + \bar{\zeta})x^2 - (s\bar{\zeta} + \bar{s}\zeta)x + 1, \end{aligned}$$

and so $\deg(\theta) \leq 4$, as $C(x) \in \mathbb{Z}[x]$.

Assume, without loss of generality, that $|\theta| \geq 1$. Then, $|\theta| > 1$, since otherwise $1/\theta = \bar{\theta}$, $s = \theta + \zeta\bar{\theta}$, $\bar{s} = \bar{\theta} + \bar{\zeta}\theta = \bar{\zeta}s$, $\bar{\zeta}s^2 = \bar{s}s \in \mathbb{R}$ and $\mathrm{Im}(\bar{\zeta}s^2) = 0$. Therefore, $0 < |\zeta/\theta| < 1$, $x^2 - sx + \zeta$ is irreducible over K, ζ/θ is a conjugate of θ over K, $\deg(\theta) \geq 2$ and $\overline{\zeta/\theta}$ is a conjugate of $\bar{\theta}$ over K. Moreover, because $(s, \zeta) \neq (\bar{s}, \bar{\zeta})$, we have $x^2 - sx + \zeta \neq x^2 - \bar{s}x + \bar{\zeta}$, $\theta \neq \bar{\theta}$ and so $\deg(\theta) = 4$. Hence, the polynomial C is the minimal polynomial of θ, the conjugates of θ are θ, ζ/θ, $\bar{\theta}$, $\bar{\zeta}/\bar{\theta}$, and $\theta \in V_1$ (resp. V_2, V_3, V_4). Also, $K \subset \mathbb{Q}(\theta)$, as $\mathbb{Q}(\theta) \subset K(\theta)$ and $[K(\theta) : \mathbb{Q}] = [K(\theta) : K][K : \mathbb{Q}] = 4$, and similarly as in the proof of direct implication, we see that the conjugates of $\bar{\zeta}\theta^2$, over K, are $\bar{\zeta}\theta^2$ and ζ/θ^2, $\operatorname{Irr}(\bar{\zeta}\theta^2, K, x) = x^2 - (\bar{\zeta}s^2 - 2)x + 1$, and so $\bar{\zeta}\theta^2$ is a root of

$$\begin{aligned} D(x) &:= (x^2-(\bar{\zeta}s^2-2)x+1)(x^2-(\zeta\bar{s}^2-2)x+1) \\ &= x^4-(\bar{\zeta}s^2+\zeta\bar{s}^2-4)x^3+(2+|\bar{\zeta}s^2-2|^2)x^2-(\bar{\zeta}s^2+\zeta\bar{s}^2-4)x+1. \end{aligned}$$

It follows by (3.2) that

$$D(x) = x^4 - (b-2)x^3 + (a^2 - 2b + 2)x^2 - (b-2)x + 1 = \operatorname{Irr}(\alpha_1\alpha_2, \mathbb{Q}, x),$$

$\bar{\zeta}\theta^2 \in \{\alpha_1\alpha_2, \alpha_1\bar{\alpha}_2\}$, $M(\theta)^2 = (\theta\bar{\theta})^2 = (\bar{\zeta}\theta^2)(\zeta\bar{\theta}^2) = \alpha_1\alpha_2\alpha_1\bar{\alpha}_2 = \alpha^2$ and $M(\theta) = \alpha$. □

Remark 3.3. It is easy to see from the proof of Lemma 3.2 that if $\alpha \in \mathcal{T}$ and $\alpha = M(\theta)$ for some $\theta \in V_1$, then $\theta^2 \in \{-\alpha_1\alpha_2, -\alpha_1\bar{\alpha}_2\}$. It follows when $M(\theta') = \alpha$ for some $\theta' \in V_1$ that $\theta' \in \{\pm\theta, \pm\bar{\theta}\}$. It is also worth noting that it may happen that $\alpha = M(\theta) = M(\theta'')$ for some $\theta'' \in V_j$, where $j \geq 2$ (see Remark 3.5 below).

Theorem 3.4. *The polynomial $S^{(a,b)}$ is the minimal polynomial of a Salem number α, satisfying $\alpha = M(\theta)$ for some $\theta \in V_1$ (resp. V_2, V_3, V_4) if and only if $b \notin \{1-a, 2, 1+a\}$ and there is $c \in \mathbb{N}$ such that*

$$c < 2\sqrt{a} \quad \text{and} \quad 2(a-1) - b = c^2$$

(resp. there is $(k,l) \in \mathbb{Z}^2$ such that

$$(a,b) = (k^2 + l^2, 4kl - 2) \quad \text{and} \quad l \neq \pm k,$$

there is $(k,l) \in \mathbb{Z}^2$ such that

$$(a,b) = \left(\frac{(2k+l)^2 + 3l^2}{4}, 2l^2 + 2kl - k^2 - 2\right) \quad \text{and} \quad k(k+2l) \neq 0,$$

there is $(k,l) \in \mathbb{Z}^2$ such that

$$(a,b) = \left(\frac{(2k+l)^2 + 3l^2}{4}, l^2 + 4kl + k^2 - 2\right) \quad \text{and} \quad l \neq \pm k).$$

Furthermore, if $m_{(1,a)}$ (resp. $m_{(2,a)}$, $m_{(3,a)}$, $m_{(4,a)}$) designates the number of quartic Salem numbers α with $\operatorname{tr}(\alpha) = a \in \mathbb{N}$ which are Mahler measures of elements of V_1 (resp. V_2, V_3, V_4), then $m_{(1,4)} = 0$, and

$$\max\{1, [\sqrt{4a-1}] - 3\} \leq m_{(1,a)} \leq [\sqrt{4a-1}], \tag{3.5}$$

(where $[\cdot]$ is the integer part function) for all $a \neq 4$ (resp. then

$$m_{(2,a)} \leq 1 + \sqrt{2a-1}, \quad m_{(3,a)} < 4\sqrt{\frac{a}{3}} + 2, \quad m_{(4,a)} < 4\sqrt{\frac{a}{3}} + 2).$$

Proof. To consider the case corresponding to the set V_1 it is enough, by Lemma 3.2, to prove that the two assertions below are equivalent for any $(a,b) \in \mathbb{Z}^2$.

(i) There is a quadratic integer s such that $(a,b) = (|s|^2, -s^2 - \bar{s}^2 - 2)$ and $\mathrm{Im}(s^2) \neq 0$.

(ii) There is a natural number c such that $c < 2\sqrt{a}$ and $2(a-1)-b = c^2$.

Clearly, the direct implication (i) $\Rightarrow$ (ii) holds with $c := 2|\mathrm{Re}(s)|$, because $2(a-1)-b = 2|s|^2 + s^2 + \bar{s}^2 = (2\,\mathrm{Re}(s))^2$, $\mathrm{Im}(s^2) \neq 0 \Rightarrow (\mathrm{Re}(s) \neq 0$ and $\mathrm{Im}(s) \neq 0) \Rightarrow |\mathrm{Re}(s)| < |s| = \sqrt{a}$, and $\bar{s}$ is the other conjugate of s so that $2\,\mathrm{Re}(s) = s + \bar{s} \in \mathbb{Z}$. To prove the converse, notice first that there is a unique pair (d,m), where $d \in \mathbb{N}$ and m is a squarefree negative rational integer, such that $4a = c^2 - md^2$, as $c \in \mathbb{N} \cap [1, 2\sqrt{a})$. It follows, when c is odd, that $md^2 \equiv 1 \bmod 4$, $d^2 \equiv 1 \bmod 4$, $m \equiv 1 \bmod 4$, and if we set

$$s := \frac{c + d\sqrt{m}}{2},$$

then s is a quadratic algebraic integer ($c \equiv d \equiv 1 \bmod 2$), $dc \neq 0 \Rightarrow \mathrm{Im}(s^2) \neq 0$, $a = (c^2 - md^2)/4 = |s|^2$, and the assumption $2(a-1)-b = c^2$ implies that $b = 2a-2-c^2 = -(c^2+md^2)/2-2 = -s^2-\bar{s}^2-2$. Similarly, we obtain, for c being even, that $md^2 \equiv 0 \bmod 4$, d is even, $s := (c + d\sqrt{m})/2$ is a quadratic algebraic integer, $\mathrm{Im}(s^2) \neq 0$, and the pair (a,b) satisfies the required conditions.

To show the relation (3.5) suppose that a is a fixed natural number. From the above we may define a bijection f from $\{b \in \mathbb{Z} \,|(a,b)$ satisfies (ii)$\}$ to $\mathbb{N} \cap [1, 2\sqrt{a})$, as follows:

$$f(b) = \sqrt{2(a-1)-b}. \tag{3.6}$$

Consequently, the cardinality of $\{b \in \mathbb{Z} \,|(a,b)$ satisfies (ii)$\}$ is $[\sqrt{4a-1}]$ and so $[\sqrt{4a-1}] - 3 \leq m_{(1,a)} \leq [\sqrt{4a-1}]$, as $b \notin \{1-a, 2, 1+a\}$, leading to (3.5), when $a \geq 5$. Also, we have, for $a = 4$ (resp. $a = 3$, $a = 2$, $a = 1$) that $f(b) \in \mathbb{N} \cap [1, 2\sqrt{a}) = \{1,2,3\}$ (resp. $\{1,2,3\}$, $\{1,2\}$, $\{1\}$) and so, by (3.6), $m_{(1,4)} = 0$ as $b \in \{-3 = 1-a, 2, 5 = 1+a\}$ (resp. $m_{(1,3)} = 3$ as $b \in \{-5,0,3\}$, $m_{(1,2)} = 2$ as $b \in \{-2,1\}$, $m_{(1,1)} = 1$ as $b \in \{-1\}$). Finally, notice that the unique Salem number α obtained for $a = 1$, namely $\alpha = \alpha^{(1,-1)} = 1.722\ldots$ (root of $x^4 - x^3 - x^2 - x + 1$) is the smallest quartic Salem number, and the number $\theta \in V_1$, satisfying $M(\theta) = \alpha$, is a root of the polynomial $x^4 - x^3 - x^2 + x + 1$ (defined by (3.4) with $s = (1+i\sqrt{3})/2$ and $\zeta = -1$).

The proof of the remaining part of Theorem 3.4 follows immediately from the second statement in Lemma 3.2. Indeed, a short computation shows that $S^{(a,b)}$ is the minimal polynomial of a Salem number α satisfying $\alpha = M(\theta)$ for some $\theta \in V_2$ (resp. V_3, V_4) if and only if $b \notin \{1-a, 2, 1+a\}$

and there is a pair $(k,l) \in \mathbb{Z}^2$ such that $(a,b) = (k^2+l^2, 2(2kl-1))$ and $l \neq \pm k$ (resp. $(4a,b) = ((2k+l)^2+3l^2, 2l^2+2kl-k^2-2)$ and $k(k+2l) \neq 0$, $(4a,b) = ((2k+l)^2+3l^2, l^2+4kl+k^2-2)$ and $l \neq \pm k$).

Moreover, since the pairs (k,l), (l,k) and $(-k,-l)$ (resp. (k,l) and $(-k,-l)$, (k,l) and $(-k,-l)$) yield the same values of (a,b) we may assume, without loss of generality, that $0 \leq l < |k|$; thus $(l+1)^2+l^2 \leq k^2+l^2 = a$, $l \leq (-1+\sqrt{2a-1})/2$ and so $m_{(2,a)} \leq 1+\sqrt{2a-1}$, as k takes at most the values $\pm\sqrt{a-l^2}$ (resp. that $0 \leq l \leq 2\sqrt{a/3}$; thus $m_{(3,a)} < 4\sqrt{a/3}+2$, that $0 \leq l \leq 2\sqrt{a/3}$; thus $m_{(4,a)} < 4\sqrt{a/3}+2$, as $(2k+l)$ takes at most the values $\pm\sqrt{4a-3l^2}$), when (a,l) is fixed. □

Proof of Theorem 1.3. Recall, by Theorem 1.1, that if a quartic Salem number is a Mahler measure of a non-reciprocal 2-Pisot number θ, then $\theta \in V$. Let q_a be the number of quartic Salem numbers α with $\mathrm{tr}(\alpha) = a$. Then, $q_a = n_a + m_a$, where m_a is (also) the cardinality of $\mathbf{T}$, and the above mentioned remark of Boyd says that $q_a = 4(a-1)$ whenever $a \geq 4$.

From Lemma 3.1 we have

$$m_{(1,a)} \leq m_a \leq m_{(1,a)} + m_{(2,a)} + m_{(3,a)} + m_{(4,a)}, \tag{3.7}$$

and it follows by (3.5) that $\lim_{a\to\infty} m_a = \lim_{a\to\infty} m_{(1,a)} = \infty$, and $m_a \geq m_{(1,a)} \geq 1$ when $a \neq 4$. For $a = 4$ the table below gives that $m_{(2,4)} = 1$ (and also $m_{(3,4)} = m_{(4,4)} = 1$); hence for any natural number a there is $\alpha \in \mathbf{T}$ with $\mathrm{tr}(\alpha) = a$.

Using the relation (3.7) and the upper bounds of $m_{(1,a)}, \dots, m_{(4,a)}$, given in Theorem 3.4, a simple calculation gives

$$m_a < 16\sqrt{a/3} + 8, \tag{3.8}$$

for a being sufficiently large, and

$$a \geq 8 \Rightarrow m_a < 4(a-1) = q_a. \tag{3.9}$$

From the last column in the table below we see that $m_a < q_a$ for $a \in \{2,3,\dots,7\}$, and it follows by (3.9) that $n_a \geq 1$ for all $a \geq 2$. Finally, (3.8) yields $n_a > 4(a-1) - 16\sqrt{a/3} - 8$ when a is sufficiently large, and so $\lim_{a\to\infty} n_a/m_a = \infty$. □

The following table gives for each $a \in \{1,2,\dots,7\}$ the possible values of b, so that $\alpha^{(a,b)} = M(\theta)$ for some $\theta \in V$. The corresponding values of b are exhibited in the second column (resp. the third column, the forth column, the fifth column) when $\theta \in V_1$ (resp. $\theta \in V_2$, $\theta \in V_3, \theta \in V_4$).

To explain how to determine the content of the table, consider for example the case $a = 3$. We know, from the above mentioned observation of Boyd that $q_3 = 9$, i.e., there are 9 quartic Salem numbers of the form $\alpha^{(3,b)}$, where $b \in \{-7,-6,\dots,3\} \setminus \{-2,2\}$. To determine the values of b so that

$\alpha^{(3,b)} = M(\theta)$ for some $\theta \in V_1$, we use Theorem 3.4 and solve the equation $4 - b = c^2$, where $c \in \mathbb{N} \cap [1, 2\sqrt{3}) = \{1,2,3\}$, yielding $b \in \{-5,0,3\}$; thus $\alpha^{(3,-5)}, \alpha^{(3,0)}$ and $\alpha^{(3,3)}$ are Mahler measures of elements of V_1 and so $m_{(1,3)} = 3$. Similarly, to find the values of b that make $\alpha^{(3,b)} = M(\theta)$ for some $\theta \in V_2$, we use the related parametrization in Theorem 3.4 and solve the equation $3 = k^2 + l^2$. Since this equation has no solution $(k,l) \in \mathbb{Z}^2$, the corresponding set of values of b is empty and $m_{(2,3)} = 0$. In a similar manner we treat the case $\alpha^{(3,b)} = M(\theta)$, where $\theta \in V_3$ (resp. $\theta \in V_4$), giving $b = 1$ and $m_{(3,3)} = 1$ (resp. $b = -5$ and $m_{(3,4)} = 1$). Consequently, $b \in \{-5,-3,0,1\}$ and $m_3 = 4$.

a	$\{b\} \hookrightarrow V_1$	$\{b\} \hookrightarrow V_2$	$\{b\} \hookrightarrow V_3$	$\{b\} \hookrightarrow V_4$	m_a/q_a
1	$\{-1\}$	$\{-2\}$	$\{-3\}$	$\{-1\}$	3/3
2	$\{-2,1\}$	$\emptyset$	$\emptyset$	$\emptyset$	2/6
3	$\{-5,0,3\}$	$\emptyset$	$\{1\}$	$\{-5\}$	4/9
4	$\emptyset$	$\{-2\}$	$\{-6\}$	$\{-10\}$	3/12
5	$\{-8,-1,4,7\}$	$\{-10\}$	$\emptyset$	$\emptyset$	5/16
6	$\{-6,1,6,9\}$	$\emptyset$	$\emptyset$	$\emptyset$	4/20
7	$\{-13,-4,3,11\}$	$\emptyset$	$\{-15,0,9\}$	$\{-13,-4,11\}$	7/24

Remark 3.5. A short computation gives that there are eight quartic Salem numbers less than 3 :

$$\alpha^{(1,-1)} < \alpha^{(2,1)} \simeq 1.88 < \alpha^{(1,-2)} \simeq 2.08 < \cdots < \alpha^{(1,-3)} \simeq 2.36 < \alpha^{(2,-2)} < \alpha^{(3,1)}.$$

From the table above we see that among these numbers two, namely $\alpha^{(3,3)} \simeq 2.15$ and $\alpha^{(2,0)} \simeq 2.29$, are not Mahler measures of non-reciprocal 2-Pisot numbers. On the contrary, $\alpha^{(1,-1)}$ is simultaneously a Mahler measure of an element of V_1 and of an element of V_4 (the same property holds for $\alpha^{(3,-5)}$, $\alpha^{(7,-13)}$, $\alpha^{(7,-4)}$ and $\alpha^{(7,11)}$).

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