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par Po-Han HSU et Peng-Jie WONG

Résumé. Dans cet article, nous présentons une nouvelle preuve du théorème central limite de Selberg pour les fonctions L de Dirichlet, basée sur une méthode de Radziwiłł et Soundararajan. De plus, nous étudions la propriété d'indépendance pour les variables aléatoires apparaissant dans ce théoréme central limite.

Abstract. In this article, based on a method of Radziwiłł and Soundararajan, we present a new proof of Selberg's central limit theorem for Dirichlet L-functions. Also, we study the independence property for the random variables arising from such a central limit theorem.

1. Introduction

In light of the Riemann hypothesis and the Lindelöf hypothesis, the value distribution of L-functions (and their moments) over the critical line $\Re(s) = \frac{1}{2}$ has attracted many mathematicians. More than eighty years ago, Selberg [14] proved that for T sufficiently large, as t varies in $[T, 2T]$, $\log\zeta(\frac{1}{2}+it)$, the logarithm of the Riemann zeta function over the critical line, is normally distributed with mean 0 and variance $\frac{1}{2}\log\log|t|$. This has been further extended by Selberg himself to all L-functions belonging to the "Selberg class" (see [15]). In particular, for any (primitive) Dirichlet character χ, $\log L(\frac{1}{2}+it,\chi)$ is "approximately" normally distributed with mean 0 and variance $\frac{1}{2}\log\log|t|$. (Here, as later, $L(s,\chi)$ denotes the Dirichlet L-function attached to χ.)

Recently, Radziwiłł and Soundararajan [12] gave a new and elegant proof of Selberg's central limit theorem for $\log|\zeta(\frac{1}{2}+it)|$ and remarked that their argument may be applied to general L-functions in the t-aspect. The primary object of this article is to present a proof of Selberg's central limit theorem for Dirichlet L-functions, stated formally below, by adapting the method of [12].

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Theorem 1.1 (Selberg)**.** *Let χ be a primitive Dirichlet character and V a fixed positive real number. Then as $T \to \infty$, one has*

$$\frac{1}{T}\mathfrak{L}\left\{t \in [T, 2T] : \log\left|L\left(\frac{1}{2}+it,\chi\right)\right| \geq v\sqrt{\frac{1}{2}\log\log T}\right\} \sim \frac{1}{\sqrt{2\pi}}\int_v^{\infty} e^{-\frac{x^2}{2}}\,\mathrm{d}x,$$

uniformly for $v \in [-V, V]$, where, as later, $\mathfrak{L}$ denotes the usual Lebesgue measure.

After deriving his central limit theorem, Selberg remarked that subject to his orthogonality conjecture, [15, Conjecture 1.2], primitive L-functions in the Selberg class are "statistically independent" (see [15, p. 53]). However, Selberg did not give any precise description for the "independence" involved.[1] The second object of this article is to prove the following explicit version of the independence of Dirichlet L-functions.

Theorem 1.2. *Let χ_1 and χ_2 be distinct primitive Dirichlet characters. For T sufficiently large and $t \in [T, 2T]$, the random vector*

$$\left(\log\left|L\left(\frac{1}{2}+it,\chi_1\right)\right|, \log\left|L\left(\frac{1}{2}+it,\chi_2\right)\right|\right)$$

is, approximately, a bivariate normal distribution with mean vector 0_2 and covariance matrix $\frac{1}{2}(\log\log T)\,\mathrm{I}_2$.

Consequently, $\log|L(\frac{1}{2}+it,\chi_1)|$ and $\log|L(\frac{1}{2}+it,\chi_2)|$ are approximately independent.

Moreover, in light of this independence theorem, we will prove that the logarithms of the absolute values of the Dirichlet L-functions form a Gaussian process by studying their joint distribution as stated formally in the following theorem.

Theorem 1.3. *Let $(\chi_j)_{j=1}^N$ be a sequence of distinct primitive Dirichlet characters. Then, for T sufficiently large and $t \in [T, 2T]$, the random vector*

$$\left(\log\left|L\left(\frac{1}{2}+it,\chi_1\right)\right|, \ldots, \log\left|L\left(\frac{1}{2}+it,\chi_N\right)\right|\right)$$

is approximately an N-variate normal distribution with mean vector 0_N and covariance matrix $\frac{1}{2}(\log\log T)\,\mathrm{I}_N$.

Consequently, the random variables $\log|L(\frac{1}{2}+it,\chi_j)|$ are approximately independent, and $(\log|L(\frac{1}{2}+it,\chi)|)_{\chi\in J}$ forms a Gaussian process for any totally ordered set J of (distinct) primitive Dirichlet characters.[2]

[1] It seems that (at least, according to the argument sketched in [15] and the language of modern probability theory) Selberg's assertion is most likely the "uncorrelatedness" among random variables, which is a consequence of the "independence."

[2] A stochastic process $(X_j)_{j\in J}$ is called an Gaussian process if every finite subsequence of $(X_j)_{j\in J}$ has a multivariate normal distribution (see, e.g., [8]).

Remark 1.4. It may be of interest in further studying the " large deviations" for the value distribution of L-functions in the consideration (cf. [11]). However, for the sake of conceptual clarity, we shall focus on Selberg's central limit theorem and independence property of Dirichlet L-functions in this article.

This article is arranged as follows. In the next section, we will collect some notation and preliminaries from number theory and probability theory. Section 3 will be devoted to giving an outline of the argument for proving Theorem 1.1 while the detailed proofs are given thorough Sections 4-7. In Section 8, we will prove Theorems 1.2 and 1.3.

2. Notation and Preliminaries

Throughout this article, we employ the Landau–Vinogradov notation. We write $f \sim g$ (resp., $f = o(g)$) if the ratio $f(x)/g(x)$ tends to 1 (resp., 0) as $x \to \infty$. Both $f = O(g)$ and $f \ll g$ mean that there is a constant M such that $|f(x)| \le Mg(x)$ for x sufficiently large. Also, we let $\Omega(n)$ denote the number of prime divisors of n counted with multiplicity. The Möbius function is defined by

$$\mu(n) = \begin{cases} (-1)^{\Omega(n)} & \text{if } n \text{ is square-free;} \\ 0 & \text{otherwise.} \end{cases}$$

The von Mangoldt function $\Lambda(n)$ is given by $\Lambda(n) = \log p$ if n is a power of a prime p, and $\Lambda(n) = 0$ otherwise.

Let $q > 1$ be a natural number. A Dirichlet character χ modulo q is a homomorphism from $(\mathbb{Z}/q\mathbb{Z})^\times$ to $\mathbb{C}^\times$, extended to $\mathbb{N}$ by setting $\chi(n) = 0$ for $(n, q) > 1$; χ is called primitive if it is not induced from a Dirichlet character χ^* modulo q^* for any $q^* \mid q$. The Dirichlet L-function attached to χ is defined by

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p (1 - \chi(p)p^{-s})^{-1}$$

for $\Re(s) > 1$ (we indicate the interested reader to [3, 10] for a detailed discussion of Dirichlet L-functions).

We recall Stirling's formula states that for any fixed σ,

$$|\Gamma(\sigma + it)| \sim e^{-\frac{1}{2}\pi|t|}|t|^{\sigma-\frac{1}{2}}\sqrt{2\pi}, \tag{2.1}$$

as $|t| \to \infty$. Moreover, as an application of the Stirling approximation (see, e.g., [5] or [7, Ch. 1, Problem 7]), it can be shown that for fixed $\delta > 0$ (which is sufficiently small),

$$\frac{\Gamma(\mathfrak{z} + \alpha)}{\Gamma(\mathfrak{z} + \beta)} = \mathfrak{z}^{\alpha-\beta}\left(1 + O\Big(\frac{|(\alpha-\beta)(\alpha+\beta-1)|}{|\mathfrak{z}|}\Big)\right), \tag{2.2}$$

where α, β are arbitrary constants and $|\arg(\mathfrak{z})| \leq \pi - \delta$.

We also require the following estimates (see, e.g., [12, Sec. 3]). For any $m, n \in \mathbb{N}$, one has

$$\int_T^{2T} \left(\frac{m}{n}\right)^{it} \mathrm{d}t = \begin{cases} T & \text{if } m = n; \\ O\Big(\min\Big\{T, \frac{1}{|\log(m/n)|}\Big\}\Big) & \text{if } m \neq n. \end{cases} \tag{2.3}$$

For $m \neq n$, one further has

$$\frac{1}{|\log(m/n)|} \ll \begin{cases} 1 & \text{if } m \geq 2n, \text{ or } m \leq n/2; \\ m/|m-n| & \text{if } n/2 < m < 2n; \\ \sqrt{mn} & \text{for all } m \neq n. \end{cases} \tag{2.4}$$

We recall some facts regarding normal random variables. Let $\mathcal{N}$ be a normal random variable with mean 0 and variance σ^2. The n-th moment of $\mathcal{N}$ satisfies

$$\mathbb{E}[\mathcal{N}^n] = \begin{cases} 0 & \text{if } n \text{ is odd;} \\ (n-1)!!\sigma^n & \text{if } n \text{ is even,} \end{cases} \tag{2.5}$$

where $\mathbb{E}[\mathcal{N}^n]$ denotes the mean of $\mathcal{N}^n$, and for a positive integer m, $m!!$ stands for the double factorial which is defined by

$$m!! := \prod_{k=0}^{\lceil m/2 \rceil - 1} (m - 2k).$$

Let X_1 and X_2 be random variables. The covariance $\mathrm{Cov}(X_1, X_2)$ between X_1 and X_2 is defined as

$$\mathrm{Cov}(X_1, X_2) := \mathbb{E}[(X_1 - \mathbb{E}(X_1))(X_2 - \mathbb{E}(X_2))].$$

If $\mathrm{Var}(X_1)$ and $\mathrm{Var}(X_2)$ are positive, then $\rho(X_1, X_2)$, the correlation of X_1 and X_2, is defined by

$$\rho(X_1, X_2) := \frac{\mathrm{Cov}(X_1, X_2)}{\sqrt{\mathrm{Var}(X_1)\,\mathrm{Var}(X_2)}}.$$

We recall that X_1 and X_2 are said to be uncorrelated if $\rho(X_1, X_2) = 0$. It is known that X_1, X_2 are uncorrelated if and only if

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) \tag{2.6}$$

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be a random vector in $\mathbb{R}^n$. For each j, denote $\mathbb{E}(X_j) = m_j$ and $\mathrm{Var}(X_j) = \sigma_j^2$. We call $\mathbf{X}$ an n-variate normal distribution if its probability density function $f_{\mathbf{X}}(x)$ is given by

$$f_{\mathbf{X}}(x) = \frac{1}{(2\pi)^{\frac{n}{2}}\sqrt{\det(\mathfrak{K})}} e^{-\frac{1}{2}(x-\mathfrak{m})^T \mathfrak{K}^{-1}(x-\mathfrak{m})}$$

where $\mathfrak{m} = (m_1, \ldots, m_n)$, $(x - \mathfrak{m})^T$ denotes the transpose of the vector $(x - \mathfrak{m})$, and $\mathfrak{K} = (\sigma_{ij})$ is an $n \times n$ symmetric positive definite matrix of real numbers with $\sigma_{ii} = \sigma_i^2$ and $\sigma_{ij} = \rho(X_i, X_j)\sigma_j\sigma_j$. We shall call 2-variate normal distributions bivariate normal distributions. We have the following properties of n-variate normal distributions (see, e.g., [4, Theorems 5.3.18, 5.3.25, and 5.5.33] for details).

Proposition 2.1. *Let $(X_j)_{j=1}^n$ be a sequence of normal distributions. Then $(X_j)_{j=1}^n$ is an n-variate normal distribution if and only if any linear combination of X_j is a normal distribution.*

Suppose, further, that (X_k, X_ℓ) is a bivariate normal distribution. Then X_k and X_ℓ are independent if and only if they are uncorrelated.

For the sake of convenience, we will say "X_0 is $\mathcal{AN}(m, \sigma^2)$" if X_0 is approximately normally distributed with mean m and variance σ^2. More precisely, for any fixed positive real number V, as $T \to \infty$, we have

$$\frac{1}{T}\mathfrak{L}\left\{t \in [T, 2T] : \frac{X_0(t) - m}{\sigma} \geq v\right\} \sim \frac{1}{\sqrt{2\pi}} \int_v^\infty e^{-\frac{x^2}{2}} \mathrm{d}x$$

uniformly for $v \in [-V, V]$.

3. Strategy of the Proof of Theorem 1.1

Following [12], throughout this article, parameters W, X, and Y, are set to be

$$W = (\log\log\log T)^4, \quad X = T^{(\log\log\log T)^{-2}}, \quad Y = T^{(\log\log T)^{-2}},$$

where $T > 0$ is sufficiently large so that $W \geq 3$, and σ_0 is defined by $\sigma_0 = \frac{1}{2} + \frac{W}{\log T}$. The proof of Theorem 1.1 consists of four parts. First of all, we shall show that for σ "close" to $\frac{1}{2}$, the distributions of $\log|L(\frac{1}{2} + it, \chi)|$ and $\log|L(\sigma + it, \chi)|$ are "approximately the same" as stated formally below. (Here, as later, χ is a fixed Dirichlet character modulo $q > 1$.)

Proposition 3.1. *For T sufficiently large, $t \in [T, 2T]$, and any $\sigma > \frac{1}{2}$, one has*

$$\int_{t-1}^{t+1} \left|\log\left|L\left(\frac{1}{2} + iy, \chi\right)\right| - \log|L(\sigma + iy, \chi)|\right| \mathrm{d}y \ll \left(\sigma - \frac{1}{2}\right) \log T,$$

where the implied constant depends on q.

As shall be seen, the proof of Proposition 3.1 is the only place requiring the information of the zeros of Dirichlet L-functions, and the purpose of this proposition is to allow one to study the problem away from the critical line.

For the second step, we will show that the auxiliary series

$$\mathcal{P}(s) = \mathcal{P}(s, \chi) = \mathcal{P}(s, \chi; X) = \sum_{2 \leq n \leq X} \frac{\Lambda(n)\chi(n)}{n^s \log n} \tag{3.1}$$

has the cumulative distribution equal to $\mathcal{AN}(0, \frac{1}{2}\log\log T)$.

Proposition 3.2. *If V is a fixed positive real number, then as $T \to \infty$,*

$$\frac{1}{T}\mathfrak{L}\left\{t \in [T, 2T] : \Re(\mathcal{P}(\sigma_0 + it)) \geq v\sqrt{\frac{1}{2}\log\log T}\right\} \sim \frac{1}{\sqrt{2\pi}}\int_v^\infty e^{-\frac{x^2}{2}}\,\mathrm{d}x$$

uniformly for all $v \in [-V, V]$.

Thus, to show that $\log|L(\frac{1}{2} + it, \chi)|$ is approximately normally distributed, it suffices to study the connection between $\Re(\mathcal{P}(\sigma_0 + it))$ and $\log|L(\frac{1}{2} + it, \chi)|$. Following [12], we further introduce an auxiliary series as follows. If n does not admit any prime divisor greater X, and n has at most $100\log\log T$ primes below Y and at most $100\log\log\log T$ primes between Y and X, then we set $a(n) = 1$. Otherwise, we set $a(n) = 0$. We then define

$$M(s) = M(s, \chi) := \sum_n \frac{\mu(n)a(n)\chi(n)}{n^s}. \tag{3.2}$$

As $a(n) = 0$ unless $n \leq Y^{100\log\log T}X^{100\log\log\log T} < T^\epsilon$ for any $\epsilon > 0$, the series $M(s)$ is, in fact, a Dirichlet polynomial. Our goal is to show $M(s)$ can be approximated by $\mathcal{P}(s)$ as follows.

Proposition 3.3. *In the same notation as above, for $t \in [T, 2T]$,*

$$M(\sigma_0 + it) = (1 + o(1))\exp(-\mathcal{P}(\sigma_0 + it)),$$

except possibly on a set of measure $o(T)$, where the $o(1)$-term may be taken as $O((\log\log T)^{-20})$.

To summarise, we know that $M(s)$ is approximately equal to $e^{-\mathcal{P}(s)}$ and that the distribution of $\mathcal{P}(s)$ is $\mathcal{AN}(0, \frac{1}{2}\log\log T)$. Thus, in order to prove Theorem 1.1, it remains to show that $L(s, \chi)$ and $M(s)^{-1}$ are approximately the same as stated rigorously below.

Proposition 3.4. *In the same notation as above, one has*

$$\frac{1}{T}\int_T^{2T} |1 - L(\sigma_0 + it, \chi)M(\sigma_0 + it)|^2\,\mathrm{d}t = o(1).$$

In particular, for $t \in [T, 2T]$, one has

$$L(\sigma_0 + it, \chi)M(\sigma_0 + it) = 1 + o(1)$$

except possibly on a set of measure $o(T)$.

Assuming the validity of the four propositions above, we now give a proof of Theorem 1.1.

Proof of Theorem 1.1. From Proposition 3.4, it follows that for all $t \in [T, 2T]$, except possibly on a set of measure $o(T)$, one has

$$L(\sigma_0 + it, \chi) = (1 + o(1))M(\sigma_0 + it)^{-1},$$

which together with Proposition 3.3 yields that except possibly on a set of measure $o(T)$,

$$|L(\sigma_0+it,\chi)| = (1+o(1))\exp(\Re(\mathcal{P}(\sigma_0+it)))$$

except possibly on a set of measure $o(T)$. Thus, Proposition 3.2 implies that the distribution of $\log|L(\sigma_0+it,\chi)|$ is $\mathcal{AN}(0,\frac{1}{2}\log\log T)$.

Recalling Proposition 3.1 and our choices of W and σ_0, we deduce

$$\int_T^{2T}\left|\log\left|L\left(\frac{1}{2}+it,\chi\right)\right| - \log|L(\sigma_0+it,\chi)|\right|\mathrm{d}t \ll T\left(\sigma_0-\frac{1}{2}\right)\log T = WT,$$

and herein, outside a subset in $[T,2T]$ of measure $O(\frac{T}{W}) = o(T)$, we have

$$\log\left|L\left(\frac{1}{2}+it,\chi\right)\right| = \log|L(\sigma_0+it,\chi)| + O(W^2).$$

Finally, we conclude by noting that the estimate $W^2 = o(\sqrt{\log\log T})$ implies that $\log|L(\frac{1}{2}+it,\chi)|$ and $\log|L(\sigma_0+it,\chi)|$ have the same distribution. □

4. Proof of Proposition 3.1

Let $q>1$. Let χ be a primitive Dirichlet character modulo q and set

$$G(s,\chi) = \left(\frac{\pi}{q}\right)^{-(s+a)/2}\Gamma\left(\frac{s+a}{2}\right), \quad a = a(\chi) = \begin{cases} 0 & \text{if } \chi(-1)=1; \\ 1 & \text{if } \chi(-1)=-1. \end{cases}$$

We let $\xi(s,\chi) = G(s,\chi)L(s,\chi)$ stand for the complete Dirichlet L-function attached to χ (see, e.g., [10, Sec. 5.4]). By (2.1), it follows that for t sufficiently large and $y\in[t-1,t+1]$,

$$\log\left|\frac{G(\sigma+iy,\chi)}{G(\frac{1}{2}+iy,\chi)}\right| \ll \left(\sigma-\frac{1}{2}\right)\log t.$$

Therefore, it suffices to show that

$$\int_{t-1}^{t+1}\left|\log\left|\frac{\xi(\frac{1}{2}+iy,\chi)}{\xi(\sigma+iy,\chi)}\right|\right|\mathrm{d}y \ll \left(\sigma-\frac{1}{2}\right)\log T.$$

Recall that the Hadamard factorisation of $\xi(s,\chi)$ is

$$\xi(s,\chi) = e^{A+Bs}\prod_{\rho\in\mathcal{S}}\left(1-\frac{s}{\rho}\right)e^{s/\rho},$$

where $e^A = \xi(0,\chi)$, $\Re(B) = -\sum_{\rho\in\mathcal{S}}\Re(\frac{1}{\rho})$, and $\mathcal{S}$ denotes the set of nontrivial zeros of $L(s,\chi)$. Thus, for y that is not the ordinate of a zero of $L(s,\chi)$, we have

$$\log\left|\frac{\xi(\frac{1}{2}+iy,\chi)}{\xi(\sigma+iy,\chi)}\right| = \sum_{\rho\in\mathcal{S}}\log\left|\frac{\frac{1}{2}+iy-\rho}{\sigma+iy-\rho}\right|,$$

which implies that

$$\int_{t-1}^{t+1}\left|\log\left|\frac{\xi(\frac{1}{2}+iy,\chi)}{\xi(\sigma+iy,\chi)}\right|\right|\mathrm{d}y \le \sum_{\rho\in\mathcal{S}}\int_{t-1}^{t+1}\left|\log\left|\frac{\frac{1}{2}+iy-\rho}{\sigma+iy-\rho}\right|\right|\mathrm{d}y. \tag{4.1}$$

Suppose $\rho=\beta+i\gamma$ is a zero of $L(s,\chi)$. As argued in [12, Sec. 2], it may be checked that

$$\int_{t-1}^{t+1}\left|\log\left|\frac{\frac{1}{2}+iy-\rho}{\sigma+iy-\rho}\right|\right|\mathrm{d}y \ll \frac{\sigma-\frac{1}{2}}{1+(t-\gamma)^2}.$$

This combined with (4.1) and the fact that the number of zeros of $L(s,\chi)$ in the box $k\le|t-\gamma|\le k+1$ is at most $O(\log(t+k))$ completes the proof.

5. Proof of Proposition 3.2

To prove Proposition 3.2, we will apply the "method of moments" (see, e.g., [2, p. 19] or [1, Sec. 30]). Although the method and results are well-known by experts, for the sake of conceptual clarity, we shall still list a precise assertion throughout our discussion.

Let $(\mathfrak{S},\mathfrak{F},\mathfrak{P})$ be a (complete) probability space. We say that (X_n) converge to X in distribution if $\lim_{n\to\infty}\mathfrak{P}(s\in\mathfrak{S}:X_n(s)\le x)=\mathfrak{P}(s\in\mathfrak{S}:X(s)\le x)$ for every x such that $\mathfrak{P}(s\in\mathfrak{S}:X(s)=x)=0$.

Now we are in a position to introduce the theorem of Fréchet and Shohat [6].

Proposition 5.1. *Suppose that the distribution of a random variable X is determined by its moments, that the (X_n) have moments of all orders, and that $\mathbb{E}(X^r)=\mathbb{E}(X_n^r)+o(\mathbb{E}(X_n^r))$, as $n\to\infty$, for all r. Then X_n converges to X in distribution.*

In order to prove Proposition 3.2, we further consider the auxiliary series

$$\mathcal{P}_0(\sigma_0+it)=\mathcal{P}_0(\sigma_0+it,\chi;X)=\sum_{p\le X}\frac{\chi(p)}{p^{\sigma_0+it}}.$$

As shall be seen later, the moments of $\mathcal{P}(\sigma_0+it)$ is basically contributed by primes. Hence, we will first study the moments of $\mathcal{P}_0(\sigma_0+it)$.

Lemma 5.2. *Assume that $k,\ell\in\mathbb{Z}$ are non-negative and with $X^{k+\ell}\le T$. Then*

$$\int_T^{2T}\mathcal{P}_0(\sigma_0+it)^k\overline{\mathcal{P}_0(\sigma_0+it)^\ell}\mathrm{d}t\ll T$$

whenever $k\ne\ell$. If $k=\ell$, we have

$$\int_T^{2T}\left|\mathcal{P}_0(\sigma_0+it)\right|^{2k}\mathrm{d}t=k!T(\log\log T)^k+O_k(T(\log\log T)^{k-1+\epsilon}).$$

Proof. Write $\mathcal{P}_0(s)^k = \sum_n a_k(n)\chi(n)n^{-s}$, where

$$(5.1)\quad a_k(n) = \begin{cases} \frac{k!}{\alpha_1!\ldots\alpha_r!} & \text{if } n=\prod_{j=1}^r p_j^{\alpha_j},\ p_1<\ldots<p_r\le X,\ \sum_{j=1}^r \alpha_j = k; \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, $\int_T^{2T} \mathcal{P}_0(\sigma_0+it)^k\overline{\mathcal{P}_0(\sigma_0+it)^\ell}\mathrm{d}t$ is

$$T\sum_n \frac{a_k(n)a_\ell(n)}{n^{2\sigma_0}} + O\Bigg(\sum_{m\neq n} \frac{a_k(n)a_\ell(m)}{(mn)^{\sigma_0}}\frac{1}{|\log\frac{m}{n}|}\Bigg),$$

where the sum runs over positive integers n co-prime to q. For $m\neq n$, from the third estimate in (2.4), it follows that the off-diagonal terms contribute at most

$$\sum_{m\neq n} a_k(n)a_\ell(m) \ll X^{k+\ell} \ll T.$$

Noticing that if $k\neq \ell$, then $a_k(n)a_\ell(n)$ is zero by the construction, we conclude the first part of this lemma. For the case $k=\ell$, we have

$$(5.2)\qquad \int_T^{2T} |\mathcal{P}_0(\sigma_0+it)|^{2k}\mathrm{d}t = T\sum_n \frac{a_k(n)^2}{n^{2\sigma_0}} + O(T).$$

As argued in [12, Sec. 3], the non-square-free n in the first sum on the right of (5.2) is at most of order $O((\log\log T)^{k-1})$. For the square-free n in the sum, we have

$$k! \sum_{\substack{p_1,\ldots,p_k\le X \\ p_j \text{ distinct and } (p_j,q)=1}} \frac{1}{(p_1\ldots p_k)^{2\sigma_0}} = k!\Bigg(\sum_{\substack{p\le X\\(p,q)=1}} \frac{1}{p^{2\sigma_0}}\Bigg)^k + O_k((\log\log T)^{k-1}).$$

Recalling the definition of X, we conclude the proof. □

Now we are in a position to prove Proposition 3.2.

Proof of Proposition 3.2. The contribution of p^k, with $k\ge 3$, in $\mathcal{P}(s)$, defined in (3.1), is

$$(5.3)\qquad \Bigg|\sum_{\substack{2\le p^k\le X,\\ k\ge 3}} \frac{(\log p)\chi(p^k)}{p^{ks}(k\log p)}\Bigg| \le \sum_{\substack{2\le p^k\le X,\\ k\ge 3}} \frac{1}{3p^{k\sigma_0}} = O(1),$$

where $\Re(s)=\sigma_0>1/2$. For the contribution of squares of primes in $\mathcal{P}(s)$, we consider

$$\int_T^{2T}\Bigg|\sum_{2\le p^2\le X}\frac{\chi(p^2)}{p^{2(\sigma_0+it)}\cdot 2}\Bigg|^2\mathrm{d}t = \frac{1}{4}\sum_{p_1,p_2\le\sqrt{X}}\int_T^{2T}\frac{\chi(p_1^2)\overline{\chi}(p_2^2)}{p_1^{2(\sigma_0+it)}p_2^{2(\sigma_0-it)}}\mathrm{d}t,$$

which by (2.3) and (2.4), is

$$\ll T \sum_{p\le\sqrt{X}} \frac{1}{p^{4\sigma_0}} + \sum_{\substack{p_1,p_2\le\sqrt{X}\\ p_1\neq p_2}} \frac{1}{p_1^{2\sigma_0}p_2^{2\sigma_0}}\sqrt{p_1p_2} \ll T.$$

Hence, by denoting $A(t)=A(t;X):=\sum_{2\le p^2\le X}\frac{\chi(p^2)}{2p^{2(\sigma_0+it)}}$, we conclude from the estimate above and Chebyshev's inequality that

$$\mathfrak{L}\Big\{t\in[T,2T]:|A(t)|>R\Big\}\le\frac{1}{R^2}\int_T^{2T}|A(t)|^2\mathrm{d}t\ll\frac{T}{R^2}, \tag{5.4}$$

for any $R\in\mathbb{R}$. In other words, the square of primes in $\mathcal{P}(s)$ contribute a measure at most $O(T/R^2)$.

With these analyses in mind, we shall complete the proof by showing that the distribution of $\Re(\mathcal{P}_0(\sigma_0+it))$ is $\mathcal{AN}(0,\frac{1}{2}\log\log T)$. According to Lemma 5.2, for $X^k\le T$ and any odd k,

$$\begin{aligned}\int_T^{2T}\big(\Re(\mathcal{P}_0(\sigma_0+it))\big)^k\mathrm{d}t &= \int_T^{2T}\frac{1}{2^k}\big(\mathcal{P}_0(\sigma_0+it)+\overline{\mathcal{P}_0(\sigma_0+it)}\big)^k\mathrm{d}t\\ &=\frac{1}{2^k}\sum_{\ell=0}^{k}\binom{k}{\ell}\int_T^{2T}\mathcal{P}_0(\sigma_0+it)^{\ell}\overline{\mathcal{P}_0(\sigma_0+it)^{k-\ell}}\mathrm{d}t\\ &\ll T\end{aligned}$$

as it is impossible to have $\ell=k-\ell$ for any odd k. If k is even, then we apply Lemma 5.2 (with $\ell=k-\ell=k/2$) to obtain

$$\begin{aligned}\frac{1}{T}\int_T^{2T}\big(\Re(\mathcal{P}_0(\sigma_0+it))\big)^k\mathrm{d}t &= 2^{-k}\binom{k}{k/2}\Big(\frac{k}{2}\Big)!(\log\log T)^{\frac{k}{2}}\\ &\quad+O_k((\log\log T)^{\frac{k}{2}-1+\epsilon})\\ &=\Big(\frac{k}{2}\Big)!!\Big(\frac{1}{2}\log\log T\Big)^{\frac{k}{2}}+O_k((\log\log T)^{\frac{k}{2}-1+\epsilon}).\end{aligned}$$

Applying Proposition 5.1 with (2.5), we complete the proof. □

6. Proof of Proposition 3.3

We shall write $\mathcal{P}(s)=\mathcal{P}_1(s)+\mathcal{P}_2(s)$, where

$$\mathcal{P}_1(s)=\sum_{2\le n\le Y}\frac{\Lambda(n)\chi(n)}{n^s\log n},\quad \mathcal{P}_2(s)=\sum_{Y<n\le X}\frac{\Lambda(n)\chi(n)}{n^s\log n}.$$

We further set

$$\mathcal{M}_1(s) = \sum_{0 \le k \le 100 \log\log T} \frac{(-1)^k}{k!} \mathcal{P}_1(s)^k,$$

$$\mathcal{M}_2(s) = \sum_{0 \le k \le 100 \log\log\log T} \frac{(-1)^k}{k!} \mathcal{P}_2(s)^k.$$

(Note that from the choices of X and Y, for any $\epsilon > 0$, each $\mathcal{M}_j(s)$ is a Dirichlet polynomial of length at most $O_\epsilon(T^\epsilon)$.) Now our goal is to connect $\mathcal{P}_j$ with $\mathcal{M}_j$.

Lemma 6.1. *For $t \in [T, 2T]$, we have*

$$|\mathcal{P}_1(\sigma_0 + it)| \le \log\log T, \quad |\mathcal{P}_2(\sigma_0 + it)| \le \log\log\log T, \tag{6.1}$$

except possibly on a set of measure at most $O(T/(\log\log\log T))$. Furthermore,

$$\mathcal{M}_1(\sigma_0 + it) = \exp(-\mathcal{P}_1(\sigma_0 + it))(1 + O((\log T)^{-99})), \tag{6.2}$$

$$\mathcal{M}_2(\sigma_0 + it) = \exp(-\mathcal{P}_2(\sigma_0 + it))(1 + O((\log\log T)^{-99})). \tag{6.3}$$

Proof. Separating the sum into diagonal and off-diagonal terms, and then using (2.3) and (2.4), we obtain

$$\int_T^{2T} |\mathcal{P}_1(\sigma_0 + it)|^2 \mathrm{d}t \ll T \sum_{2 \le n_1 = n_2 \le Y} \frac{\Lambda(n_1)\Lambda(n_2)}{(n_1 n_2)^{\sigma_0} \log n_1 \log n_2} + \sum_{2 \le n_1 \ne n_2 \le Y} \frac{\Lambda(n_1)\Lambda(n_2)}{(n_1 n_2)^{\sigma_0} \log n_1 \log n_2} \sqrt{n_1 n_2},$$

which is $\ll T \log\log T$. Similarly, we have

$$\int_T^{2T} |\mathcal{P}_2(\sigma_0 + it)|^2 \mathrm{d}t \ll T \log\log\log T.$$

Assume $K \ge 1$. If $|z| \le K$, then it may be checked (cf. [12, Lemma 2]) that

$$\sum_{0 \le k \le 100K} \frac{z^k}{k!} = e^z (1 + O(e^{-99K})).$$

The estimate (6.2) holds by taking $z = -\mathcal{P}_1(\sigma_0 + it)$ and $K = \log\log T$, and (6.3) follows similarly. □

Define

$$M_1(s) = \sum_n \frac{\mu(n) a_1(n) \chi(n)}{n^s}, \quad M_2(s) = \sum_n \frac{\mu(n) a_2(n) \chi(n)}{n^s},$$

where $a_1(n) = 1$ if n is composed of at most $100 \log\log T$ primes that all are below Y, and zero otherwise; $a_2(n) = 1$ if n is composed of at most $100 \log\log\log T$ primes that all are between Y and X, and zero otherwise. From the definition (3.2) of $M(s)$, it is not hard to see that $M(s) = M_1(s)M_2(s)$. We shall prove the next lemma, which roughly states that for each j, the difference between $\mathcal{M}_j(s)$ and $M_j(s)$ is small on average.

Lemma 6.2. *In the notation as above, we have*

$$\int_T^{2T} |\mathcal{M}_1(\sigma_0 + it) - M_1(\sigma_0 + it)|^2 \mathrm{d}t \ll T(\log T)^{-60},$$
$$\int_T^{2T} |\mathcal{M}_2(\sigma_0 + it) - M_2(\sigma_0 + it)|^2 \mathrm{d}t \ll T(\log\log T)^{-60}.$$

Proof. Writing $\mathcal{M}_1(s) = \sum_n b(n)\chi(n)n^{-s}$, as in the proof [12, Lemma 3], $b(n)$ possesses the following properties:

(1) $|b(n)| \le 1$ for all n,
(2) $b(n) = 0$ unless $n \le Y^{100\log\log T}$ has only primes factors below Y, and
(3) $b(n) = \mu(n)a_1(n)$ unless either $\Omega(n) > 100\log\log T$, or there is a prime $p \le Y$ such that $p^k \mid n$ with $p^k > Y$.

We note that the verification of such an assertion was omitted in [12]. For the sake of completeness, we give a sketch of the proof for it.

Proof of the properties of $b(n)$. From the definition of $\mathcal{M}_1(s)$ and $\mathcal{P}_1(s)$, we have

$$\mathcal{M}_1(s) = \sum_n \frac{b(n)\chi(n)}{n^s} = \sum_{0\le k\le 100\log\log T} \frac{(-1)^k}{k!}\Bigg(\sum_{p\le Y}\sum_{\substack{j\\ p^j\le Y}} \frac{\chi(p^j)}{jp^{js}}\Bigg)^k.$$

It shall be clear that the second property (of $b(n)$) and "unless part" of the third property follows immediately from the range of k and p^j. (Indeed, $b(n) = 0$ if $\Omega(n) > 100\log\log T$, or there is a prime $p \le Y$ such that $p^k \mid n$ with $p^k > Y$.)

Now we consider $L_Y(s,\chi) = \prod_{p\le Y}(1-\chi(p)p^{-s})^{-1}$. As $\log(1-z)^{-1} = \sum_{j\ge1} z^j/j$ for $|z|<1$, we have

$$\log L_Y(s,\chi) = \sum_{p\le Y}\sum_{j=1}^{\infty} \frac{\chi(p^j)}{jp^{js}} = \mathcal{P}_1(s) + \sum_{p\le Y}\sum_{\substack{j\\ p^j > Y}} \frac{\chi(p^j)}{jp^{js}},$$

which implies

$$\begin{aligned} L_Y(s,\chi)^{-1} &= \exp\left(-\log L_Y(s,\chi)\right)\\ &= \exp(-\mathcal{P}_1(s))\exp\Bigg(-\sum_{p\le Y}\sum_{\substack{j\\ p^j>Y}} \frac{\chi(p^j)}{jp^{js}}\Bigg). \end{aligned}$$

By the Taylor expansion of e^{-x}, we deduce

$$\prod_{p\le Y}\left(1-\frac{\chi(p)}{p^{-s}}\right)=\left(\mathcal{M}_1(s)+\sum_{k>100\log\log T}\frac{(-1)^k}{k!}\mathcal{P}_1(s)^k\right)$$
$$\times\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\left(\sum_{p\le Y}\sum_{\substack{j\\ p^j>Y}}\frac{\chi(p^j)}{jp^{js}}\right)^k\right)$$
$$=\mathcal{M}_1(s)+\sum_{k>100\log\log T}\frac{(-1)^k}{k!}\mathcal{P}_1(s)^k$$
$$+\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\mathcal{P}_1(s)^k\right)\left(\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\left(\sum_{p\le Y}\sum_{\substack{j\\ p^j>Y}}\frac{\chi(p^j)}{jp^{js}}\right)^k\right).$$

Writing the last part above by

$$\mathcal{M}_1(s)+\sum_n b'(n)\chi(n)n^{-s}+\sum_n b''(n)\chi(n)n^{-s},$$

we see that $b'(n)=0$ if $\Omega(n)\le 100\log\log T$ and that $b''(n)=0$ if there is $p\mid n$ with $p^k\le Y$. Finally, since

$$\prod_{p\le Y}\left(1-\frac{\chi(p)}{p^{-s}}\right)=\sum_n\frac{\mu(n)\chi(n)}{n^s}$$
$$=\sum_n\frac{b(n)\chi(n)}{n^s}+\sum_n\frac{b'(n)\chi(n)}{n^s}+\sum_n\frac{b''(n)\chi(n)}{n^s},$$

where the second sum is over n for which if $p\mid n$, then $p\le Y$, comparing the coefficients in the Dirichlet series on the both sides proves properties (1) and (3). □

Set $c(n)=(b(n)-\mu(n)a_1(n))\chi(n)$. We note that our $a_1(n)$, $a_2(n)$, and $b(n)$ are exactly the same as in the proof [12, Lemma 3]. The only the difference is that now we have a further twisting by $\chi(n)$. Nevertheless, as $|\chi(n)|\le 1$ for all n, the desired upper bounds follow from the estimates derived in the proof [12, Lemma 3] immediately. Indeed, by employing (2.3) and (2.4), we have

$$\int_T^{2T}|\mathcal{M}_1(\sigma_0+it)-M_1(\sigma_0+it)|^2\mathrm{d}t$$
$$\ll T\sum_{n_1=n_2}\frac{|c(n_1)\overline{c(n_2)}|}{(n_1n_2)^{\sigma_0}}+\sum_{n_1\ne n_2}\frac{|c(n_1)\overline{c(n_2)}|}{(n_1n_2)^{\sigma_0}}\sqrt{n_1n_2}.$$

Thus, as done in the proof of [12, Lemma 3], this is at most $O(T(\log T)^{-60})$. The second part of the lemma follows similarly. □

To end this section, we shall prove Proposition 3.3.

Proof of Proposition 3.3. It follows from (6.2) that

$$\mathcal{M}_1(\sigma_0+it)=\exp(-\mathcal{P}_1(\sigma_0+it))(1+O((\log T)^{-99})),$$

except for a possible set of measure $o(T)$. Also, one may see from (6.1) that

$$(\log T)^{-1}\ll|\mathcal{M}_1(\sigma_0+it)|\ll\log T,$$

except for a possible set of measure $o(T)$. Therefore, we conclude from Lemma 6.2 that, except on a set of measure $o(T)$,

$$\begin{aligned}M_1(\sigma_0+it)&=\mathcal{M}_1(\sigma_0+it)+O((\log T)^{-25})\\&=\exp(-\mathcal{P}_1(\sigma_0+it))(1+O((\log T)^{-20})).\end{aligned}$$

By a similar reasoning, except for a set of measure $o(T)$, we have

$$M_2(\sigma_0+it)=\exp(-\mathcal{P}_2(\sigma_0+it))(1+O((\log\log T)^{-20})).$$

Recalling that $M(s)=M_1(s)M_2(s)$ and $\mathcal{P}(s)=\mathcal{P}_1(s)+\mathcal{P}_2(s)$, we then deduce from the above estimates that except possibly on a set of measure $o(T)$,

$$M(\sigma_0+it)=\exp(-\mathcal{P}(\sigma_0+it))(1+O((\log\log T)^{-20})),$$

which concludes the proof. □

7. Proof of Proposition 3.4

To prove Proposition 3.4, we need the following two lemmata. We shall sketch the proof of Lemma 7.1 as it is standard; we will emphasise Lemma 7.2. As shall be seen, we adapt Radziwiłł-Soundararajan's approach to proving Lemma 7.2 by utilising the (complete) multiplicity of Dirichlet characters.

Recall the complete Dirichlet L-function $\xi(s,\chi)=G(s,\chi)L(s,\chi)$ satisfies the functional equation

$$\xi(s,\chi)=\omega_\chi\xi(1-s,\overline{\chi}),\quad \omega_\chi:=\begin{cases}\tau(\chi)/\sqrt{q} & \text{if } \chi(-1)=1;\\ \tau(\chi)/i\sqrt{q} & \text{if } \chi(-1)=-1,\end{cases}\tag{7.1}$$

where $\tau(\chi)$ denotes the Gauß sum of χ.

Lemma 7.1. *For $c>0$, define*

$$\begin{aligned}I(s,\chi)&:=\frac{1}{2\pi i}\int_{(c)}\xi(z+s,\chi)\xi(z+\bar{s},\overline{\chi})e^{z^2}\frac{\mathrm{d}z}{z},\\ I(1-s,\overline{\chi})&:=\frac{1}{2\pi i}\int_{(c)}\xi(z+(1-s),\overline{\chi})\xi(z+(1-\bar{s}),\chi)e^{z^2}\frac{\mathrm{d}z}{z},\end{aligned}$$

where (c) *denotes the line from* $c - i\infty$ *to* $c + i\infty$. *Then we have*

$$|L(s,\chi)|^2 = \frac{1}{|G(s,\chi)|^2}\Big(I(s,\chi) + I(1-s,\overline{\chi})\Big), \tag{7.2}$$

where $G(s,\chi)$ *and* $\xi(s,\chi)$ *are defined in Section 4.*

Proof. First, we claim that $I(s,\chi)$ is independent of the choice of c. Let $\mathcal{R}$ be a rectangle on $\Re(s) > 0$. We have

$$\int_{\mathcal{R}} \xi(z+s,\chi)\xi(z+\bar{s},\overline{\chi})e^{z^2}\frac{\mathrm{d}z}{z} = 0.$$

Letting the length of the sides that parallel to imaginary axes to infinity, we conclude that $I(s,\chi)$ is independent of the choice of c. Using (7.1), we deduce

$$\xi(z+s,\chi)\xi(z+\bar{s},\overline{\chi}) = \omega_\chi \xi(-z+(1-s),\overline{\chi})\omega_{\overline{\chi}}\xi(-z+(1-\bar{s}),\chi). \tag{7.3}$$

Denoting $\langle -c\rangle$ the line from the line from $-c+i\infty$ to $-c-i\infty$, we obtain

$$\int_{\langle -c\rangle} \xi(z+s,\chi)\xi(z+\bar{s},\overline{\chi})e^{z^2}\frac{\mathrm{d}z}{z} = \int_{(c)} \xi(-z+s,\chi)\xi(-z+\bar{s},\overline{\chi})e^{z^2}\frac{\mathrm{d}z}{z}, \tag{7.4}$$

which is independent of the choice of c. Let S be a square centred at the origin and equipped with positive orientation. A standard contour integral together with the Cauchy integral formula implies

$$\xi(s,\chi)\xi(\bar{s},\overline{\chi}) = \frac{1}{2\pi i}\int_S \xi(z+s,\chi)\xi(z+\bar{s},\overline{\chi})e^{z^2}\frac{\mathrm{d}z}{z}. \tag{7.5}$$

By letting the length of the sides that parallel to imaginary axis to infinity, it can be shown that (7.5) equals

$$\frac{1}{2\pi i}\left(\int_{(c)} \xi(z+s,\chi)\xi(z+\bar{s},\overline{\chi})e^{z^2}\frac{\mathrm{d}z}{z} + \int_{\langle -c\rangle} \xi(z+s,\chi)\xi(z+\bar{s},\overline{\chi})e^{z^2}\frac{\mathrm{d}z}{z}\right).$$

Noticing that the first term is $I(s,\chi)$, and that by (7.4) and (7.3) the second term is $I(1-s,\overline{\chi})$, we complete the proof. □

Lemma 7.2. *Assume* h, k *are non-negative integers such that* $h,k \le T$ *and* $(hk, q)=1$. *Then for any* $\frac{1}{2} < \sigma \le 1$,

$$\begin{aligned}
&\int_T^{2T} \left(\frac{h}{k}\right)^{it} |L(\sigma+it,\chi)|\mathrm{d}t \\
&\qquad = \chi(h)\overline{\chi}(k)\int_T^{2T} L(2\sigma,\chi_0)\left(\frac{(h,k)^2}{hk}\right)^{\sigma}\mathrm{d}t \\
&\qquad\quad + \chi(h)\overline{\chi}(k)\int_T^{2T} \left(\frac{qt}{2\pi}\right)^{1-2\sigma} L(2-2\sigma,\chi_0)\left(\frac{(h,k)^2}{hk}\right)^{1-\sigma}\mathrm{d}t \\
&\qquad\quad + O\Big(T^{1-\sigma+\epsilon}\min\{h,k\}\Big).
\end{aligned}$$

Proof. Note that

$$\frac{G(z+s,\chi)G(z+\bar{s},\bar{\chi})}{|G(s,\chi)|^2} = \frac{G(z+s,\chi)}{G(s,\chi)}\frac{G(z+\bar{s},\bar{\chi})}{\overline{G(s,\chi)}} = \left(\frac{q}{\pi}\right)^z \frac{\Gamma(\frac{z+s+a}{2})}{\Gamma(\frac{s+a}{2})}\frac{\Gamma(\frac{z+\bar{s}+a}{2})}{\overline{\Gamma(\frac{s+a}{2})}}.$$

Denote $s = \sigma + it$ with $T \le t \le 2T$ and $\frac{1}{2} \le \sigma \le 1$. If z is a complex number with real part $c = 1 - \sigma + \frac{1}{\log T}$, then recalling that $\overline{\Gamma(s)} = \Gamma(\bar{s})$ and applying (2.2) (with $\mathfrak{z} = \pm\frac{it}{2}$, $\alpha = \frac{1}{2}(1 + a + \frac{1}{\log T} + i\Im(z))$, $\beta = \frac{\sigma+a}{2}$), we can write the right of the equation as

$$\left(\frac{q}{\pi}\right)^z \frac{\Gamma(\frac{it}{2}+\alpha)}{\Gamma(\frac{it}{2}+\beta)}\frac{\Gamma(\frac{-it}{2}+\alpha)}{\Gamma(\frac{-it}{2}+\beta)} = \left(\frac{qt}{2\pi}\right)^z \left(1 + O\left(\frac{|z|^2}{T}\right)\right)$$

since in our consideration $\alpha - \beta = \frac{z}{2}$ and $|\mathfrak{z}| \gg T$. Hence, we have

$$\frac{I(s,\chi)}{|G(s,\chi)|^2} = \frac{1}{2\pi i}\int_{(1-\sigma+\frac{1}{\log T})} \frac{e^{z^2}}{z} L(z+s,\chi)L(z+\bar{s},\bar{\chi})\left(\frac{qt}{2\pi}\right)^z \mathrm{d}z + O(T^{-\sigma+\epsilon}).$$

The above estimate together with $\xi(s,\chi) = G(s,\chi)L(s,\chi)$ then yields

$$\int_T^{2T}\left(\frac{h}{k}\right)^{it}\frac{I(s,\chi)}{|G(s,\chi)|^2}\mathrm{d}t = \frac{1}{2\pi i}\int_{(1-\sigma+\frac{1}{\log T})}\frac{e^{z^2}}{z}\sum_{m,n=1}^{\infty}\frac{\bar{\chi}(m)\chi(n)}{(mn)^{z+\sigma}}\left(\int_T^{2T}\left(\frac{hm}{kn}\right)^{it}\left(\frac{qt}{2\pi}\right)^z \mathrm{d}t\right)\mathrm{d}z + O(T^{1-\sigma+\epsilon}).$$

(Note that we can interchange the order of summations and integrals since we are in the region where both $L(z+s,\chi)$ and $L(z+\bar{s},\bar{\chi})$ are absolutely convergent.) Now we shall split the sum on the right into diagonal and off-diagonal terms, say $kn = hm$ and $kn \neq hm$, respectively. As the diagonal terms may be parametrised by $m = Nk/(h,k)$ and $n = Nh/(h,k)$, the inner sum above can be simplified as

$$\sum_{m,n=1}^{\infty}\frac{\bar{\chi}(m)\chi(n)}{(mn)^{z+\sigma}} = L(2z+2\sigma,\chi_0)\chi(h)\bar{\chi}(k)\Big(\frac{(h,k)^2}{hk}\Big)^{z+\sigma}$$

(We recall that h, k are natural numbers such that $(hk,q) = 1$.) Hence, the diagonal terms contribute

$$\frac{1}{2\pi i}\int_{(c)}\frac{e^{z^2}}{z}L(2z+2\sigma,\chi_0)\chi(h)\bar{\chi}(k)\left(\frac{(h,k)^2}{hk}\right)^{z+\sigma}\left(\int_T^{2T}\left(\frac{qt}{2\pi}\right)^z \mathrm{d}t\right)\mathrm{d}z,$$

where $c = 1 - \sigma + \frac{1}{\log T}$.

For the off-diagonal terms, similar to the reasoning of [12, Eq. (17)], by recalling that $c = 1 - \sigma + \frac{1}{\log T}$ is the real part of z, one can deduce that the contribution of off-diagonal terms is bounded by

$$T^{1-\sigma} \sum_{\substack{m,n=1,\\ hm \neq kn}}^{\infty} \frac{1}{(mn)^{1+\frac{1}{\log T}}} \min\left\{T, \frac{1}{|\log \frac{hm}{kn}|}\right\} \ll T^{1-\sigma+\epsilon} \min\{h,k\}.$$

Then we have

$$\begin{aligned} (7.6)\quad & \int_T^{2T} \left(\frac{h}{k}\right)^{it} \frac{I(s,\chi)}{|G(s,\chi)|^2} \mathrm{d}t \\ &= \frac{1}{2\pi i} \int_{(c)} \frac{e^{z^2}}{z} L(2z+2\sigma, \chi_0)\chi(h)\overline{\chi}(k) \left(\frac{(h,k)^2}{hk}\right)^{z+\sigma} \left(\int_T^{2T} \left(\frac{qt}{2\pi}\right)^z \mathrm{d}t\right) \mathrm{d}z \\ &\qquad + O(T^{1-\sigma+\epsilon}\min\{h,k\}), \end{aligned}$$

where $c = 1 - \sigma + \frac{1}{\log T}$.

Again, an application of (2.2) yields

$$\frac{G(z+(1-s),\overline{\chi})G(z+(1-\bar{s}),\chi)}{|G(s,\chi)|^2} = \left(\frac{qt}{2\pi}\right)^{1+z-2\sigma} \left(1 + O\left(\frac{|z|^2}{T}\right)\right).$$

Similarly, defining

$$\mathcal{I}(s) := \int_T^{2T} \left(\frac{qt}{2\pi}\right)^s \mathrm{d}t,$$

we obtain

$$\begin{aligned} (7.7)\quad & \int_T^{2T} \left(\frac{h}{k}\right)^{it} \frac{I(1-s,\overline{\chi})}{|G(s,\chi)|^2} \mathrm{d}t \\ &= \frac{1}{2\pi i} \int_{(c')} \frac{e^{z^2}}{z} L(2+2z-2\sigma, \chi_0)\chi(h)\overline{\chi}(k) \left(\frac{(h,k)^2}{hk}\right)^{1+z-\sigma} \mathcal{I}(1+z-2\sigma) \mathrm{d}z \\ &\qquad + O(T^{1-\sigma+\epsilon}\min\{h,k\}), \end{aligned}$$

where $c' = \sigma + \frac{1}{\log T}$. Via the change of variables $z + \sigma \mapsto z$ in (7.6) and $1 + z - \sigma \mapsto z$ in (7.7), respectively, recalling (7.2), we have

$$\begin{aligned} & \int_T^{2T} \left(\frac{h}{k}\right)^{it} |L(\sigma+it,\chi)|^2 \mathrm{d}t \\ &= \frac{\chi(h)\overline{\chi}(k)}{2\pi i} \int_{(c'')} L(2z,\chi_0) \left(\frac{(h,k)^2}{hk}\right)^z \mathcal{I}(z-\sigma) \left(\frac{e^{(z-\sigma)^2}}{z-\sigma} + \frac{e^{(z-1+\sigma)^2}}{z-1+\sigma}\right) \mathrm{d}z \\ &\qquad + O(T^{1-\sigma+\epsilon}\min\{h,k\}) \end{aligned}$$

where $c'' = 1 + \frac{1}{\log T}$.

Finally, we compute the residues of the poles at $z = \sigma$ and $z = 1 - \sigma$ of the right of integral above to complete the proof. □

With the above lemmata in hand, we are in a position to prove Proposition 3.4.

Proof of Proposition 3.4. For T sufficiently large and $t \in [T, 2T]$, by [13, Corollary], we have an approximate functional equation

$$L(\sigma_0 + it, \chi) = \sum_{n \le T} \frac{\chi(n)}{n^{\sigma_0 + it}} + O(T^{-\frac{1}{2}}). \tag{7.8}$$

Using (7.8) and recalling from (3.2) that $a(n) = 0$ unless $n < T^{\epsilon}$, we have

$$\begin{aligned}
&\int_T^{2T} L(\sigma_0 + it, \chi) M(\sigma_0 + it)\, \mathrm{d}t \\
&\quad = \sum_{1 \le n \le T} \frac{\chi(n)}{n^{\sigma_0}} \sum_{1 \le m < T^{\epsilon}} \frac{\mu(m) a(m) \chi(m)}{m^{\sigma_0}} \int_T^{2T} (mn)^{-it} \mathrm{d}t + O(T^{\frac{1}{2}+\epsilon}) \\
&\quad = T + O(T^{\frac{1}{2}+\epsilon}),
\end{aligned}$$

where the last estimate follows from (3.2). Thus, we have

$$\begin{aligned}
&\int_T^{2T} |1 - L(\sigma_0 + it, \chi) M(\sigma_0 + it)|^2 \mathrm{d}t \\
&\quad = \int_T^{2T} |L(\sigma_0 + it, \chi) M(\sigma_0 + it)|^2 \mathrm{d}t - T + O(T^{\frac{1}{2}+\epsilon}) \\
&\quad = \sum_{h,k} \frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{\sigma_0}} \chi(k)\overline{\chi}(h) \int_T^{2T} \left(\frac{h}{k}\right)^{it} |L(\sigma_0 + it, \chi)|^2 \mathrm{d}t \\
&\qquad\qquad - T + O(T^{\frac{1}{2}+\epsilon}).
\end{aligned} \tag{7.9}$$

Applying Lemma 7.2 to (7.9) and recalling $|a(n)| \le 1$ for any n and $a(n) = 0$ unless $n < T^{\epsilon}$, we see that the first integral in (7.9) equals

$$\begin{aligned}
&- T + O(T^{\frac{1}{2}+\epsilon}) + o(T) \\
&\quad + \sum_{h,k} \frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{\sigma_0}} \int_T^{2T} L(2\sigma_0, \chi_0) \left(\frac{(h,k)^2}{hk}\right)^{\sigma_0} \mathrm{d}t \\
&\quad + \sum_{h,k} \frac{\mu(h)\mu(k)a(h)a(k)}{(hk)^{\sigma_0}} \int_T^{2T} \left(\frac{qt}{2\pi}\right)^{1-2\sigma_0} L(2-2\sigma_0, \chi_0) \left(\frac{(h,k)^2}{hk}\right)^{1-\sigma_0} \mathrm{d}t,
\end{aligned} \tag{7.10}$$

where the sums are over h, k such that $(hk, q) = 1$. Recalling that

$$L(s, \chi_0) = \zeta(s) \prod_{p \mid q} \left(1 - \frac{1}{p^s}\right)$$

and processing the argument as in [12, pp. 13-14], one may deduce that the first main term in (7.10) is $\sim T$. Moreover, as argued in [12, p. 14], the contribution of the second term is $o(T)$, which completes the proof. $\square$

8. Independence of Dirichlet *L*-Functions

8.1. Pairs of Dirichlet *L*-functions. As mentioned in the introduction, after deriving his central limit theorem for L-functions belonging to the Selberg class, Selberg further remarked an independence property for these L-functions would follow from his orthogonality conjecture.[3] In this section, we shall apply the method developed in previous sections (à la Radziwiłł et Soundararajan) to give the following explicit version of Selberg's independence property for pairs of Dirichlet L-functions.

Theorem 8.1. *Let χ_1 and χ_2 be distinct primitive Dirichlet characters. Let V be a fixed positive real number. As $T \to \infty$, we have, for any $a_1, a_2 \in \mathbb{R}$,*

$$\frac{1}{T}\mathfrak{L}\left\{t \in [T, 2T] : \log\left|\mathcal{L}_{a_1,a_2}\left(\frac{1}{2} + it\right)\right| \geq v\sqrt{\frac{a_1^2 + a_2^2}{2}\log\log T}\right\} \sim G(v),$$

uniformly in $v \in [-V, V]$, where $G(v) := \frac{1}{\sqrt{2\pi}}\int_v^\infty e^{-\frac{x^2}{2}}\,\mathrm{d}x$ and

$$\mathcal{L}_{a_1,a_2}(s) = \mathcal{L}(s, \chi_1, \chi_2; a_1, a_2) := |L(s, \chi_1)|^{a_1}|L(s, \chi_2)|^{a_2}.$$

In other words, $\log|\mathcal{L}_{a_1,a_2}(\frac{1}{2} + it)|$ is $\mathcal{AN}(0, \frac{a_1^2+a_2^2}{2}\log\log T)$.

Denote, for $a_1, a_2 \in \mathbb{R}$,

$$\mathcal{P}_{a_1,a_2,0}(s) = \mathcal{P}_0(s, \chi_1, \chi_2; a_1, a_2; X) := \sum_{p \leq X} \frac{a_1\chi_1(p) + a_2\chi_2(p)}{p^s}. \tag{8.1}$$

Similar to the case of a single Dirichlet L-function, we require the following moment calculation for $\mathcal{P}_{a_1,a_2,0}(s)$.

Lemma 8.2. *Let χ_1 and χ_2 be distinct primitive Dirichlet characters. Assume that $k, \ell \in \mathbb{Z}$ are non-negative and with $X^{k+\ell} \leq T$. Then for any real numbers a_1, a_2, we have*

$$\int_T^{2T} \mathcal{P}_{a_1,a_2,0}(\sigma_0 + it)^k\overline{\mathcal{P}_{a_1,a_2,0}(\sigma_0 + it)^\ell} \ll T, \tag{8.2}$$

---

[3]However, Selberg did not indicate the independence involved precisely. Nonetheless, in the view of Selberg's orthogonality conjecture, Selberg's argument, at least, implies the validity of Theorem 8.1 and Lemma 8.2 for $a_1 = a_2 = 1$, which yields the uncorrelatedness among Dirichlet L-Functions.

for $k \neq \ell$, and

$$\int_T^{2T} \left|\mathcal{P}_{a_1,a_2,0}(\sigma_0 + it)\right|^{2k} \mathrm{d}t = k! T((a_1^2 + a_2^2) \log\log T)^k + O_k(T(\log\log T)^{k-1+\epsilon}).$$

Proof. For the sake of simplicity, we assume χ_1 and χ_2 are distinct primitive Dirichlet characters *modulo* q. Setting $\psi(p) = \psi_{a_1,a_2}(p) := a_1\chi_1(p) + a_2\chi_2(p)$ and $\Psi_k(n) := \prod_{j=1}^r \psi(p_j)^{\alpha_j}$, whenever $n = p_1^{\alpha_1} \dots p_r^{\alpha_r}$ with $\alpha_1 + \dots + \alpha_r = k$, we can write

$$\mathcal{P}_{a_1,a_2,0}(\sigma_0 + it)^k = \sum_n \frac{a_k(n)\Psi_k(n)}{n^{\sigma_0 + it}},$$

where $a_k(n)$ is defined in (5.1). Hence, we have

$$\begin{aligned} &\int_T^{2T} \mathcal{P}_{a_1,a_2,0}(\sigma_0 + it)^k \overline{\mathcal{P}_{a_1,a_2,0}(\sigma_0 + it)^\ell} \mathrm{d}t \\ &= T \sum_n \frac{a_k(n)a_\ell(n)\Psi_k(n)\overline{\Psi_\ell(n)}}{n^{2\sigma_0}} + O\Bigg(\sum_{n\neq m} \frac{a_k(n)a_\ell(m)|\Psi_k(n)\overline{\Psi_\ell(m)}|}{(nm)^{\sigma_0 - \frac{1}{2}}}\Bigg). \end{aligned} \tag{8.3}$$

It follows from the definition of $\Psi_k(n)$ that writing $n = \prod_{j=1}^r p_j^{\alpha_j}$, one has

$$|\Psi_k(n)| = \prod_{j=1}^r |\psi(p_j)|^{\alpha_j} \leq \prod_{j=1}^r (|a_1| + |a_2|)^{\alpha_j} = (|a_1| + |a_2|)^k.$$

Thus, the big-O term in (8.3) is at most

$$\sum_{n\neq m} a_k(n)a_\ell(m)(|a_1| + |a_2|)^{k+\ell} \ll X^{k+\ell} \ll T.$$

As $a_k(n)a_\ell(n) = 0$ if $k \neq \ell$, we conclude (8.2) immediately.

It remains to consider the case $k = \ell$. For $n = \prod_{j=1}^r p_j^{\alpha_j}$, we write

$$\Psi_k(n)\overline{\Psi_k(n)} = \prod_{j=1}^r \Big((a_1^2 + a_2^2)\chi_0(p_j) + a_1a_2\chi(p_j) + a_1a_2\overline{\chi}(p_j)\Big)^{\alpha_j}, \tag{8.4}$$

where χ_0 denotes the trivial character and $\chi := \chi_1\overline{\chi_2}$. By (8.4) and the fact that $\sum_j \alpha_j = k$, we have

$$\Bigg|\sum_{n \text{ non-square-free}} \frac{a_k(n)a_k(n)\Psi_k(n)\overline{\Psi_k(n)}}{n^{2\sigma_0}}\Bigg| \leq (|a_1| + |a_2|)^{2k} \sum_{n \text{ non-square-free}} \frac{a_k(n)^2}{n^{2\sigma_0}},$$

Therefore, the non-square-free n in the first sum of (8.3) contribute a quantity of order $O((\log\log T)^{k-1})$. For square-free n, we express $\Psi_k(n)\overline{\Psi_k(n)}$ as

$$\sum (a_1^2 + a_2^2)^\beta (a_1a_2)^{k-\beta} \chi_0(n')\chi(m)\overline{\chi}(m'), \tag{8.5}$$

where the sum is over $\beta + \Omega(m) + \Omega(m') = k$ such that $0 \le \beta \le k$ and $n = n'mm'$ (so n', m, m' are composed of primes in $\{p_1, \dots, p_k\}$). By (8.5), we may express the first sum in (8.3) as

$$\sum_{0\le\beta\le k} (a_1^2 + a_2^2)^\beta (a_1 a_2)^{k-\beta} \sum_n \frac{a_k(n)^2 \chi_0(n')\chi(m)\overline{\chi}(m')}{n^{2\sigma_0}}, \tag{8.6}$$

where the inner sum is over $n = n'mm'$ with n', m, and m' pairwise coprime and $\beta + \Omega(m) + \Omega(m') = k$. By the mulitiplicativity of χ_0 and χ, we may further write the inner sum in (8.6) as

$$k! \sum_{0\le\gamma\le k-\beta} \frac{k!}{\beta!\gamma!(k-\beta-\gamma)!} \cdot \sum_{\substack{p_1,\dots,p_k\le X \\ p_j \text{ distinct}}} \frac{\chi_0(p_1\dots p_\beta)\chi(p_{\beta+1}\dots p_{\beta+\gamma})\overline{\chi}(p_{\beta+\gamma+1}\dots p_k)}{(p_1\dots p_k)^{2\sigma_0}},$$

which is

$$k! \sum_{0\le\gamma\le k-\beta} \frac{k!}{\beta!\gamma!(k-\beta-\gamma)!} \left(\sum_{p\le X} \frac{\chi_0(p)}{p^{2\sigma_0}}\right)^\beta \left(\sum_{p\le X} \frac{\chi(p)}{p^{2\sigma_0}}\right)^\gamma \left(\sum_{p\le X} \frac{\overline{\chi}(p)}{p^{2\sigma_0}}\right)^{k-\beta-\gamma}.$$

Recall that as χ_0 is the trivial Dirichlet character modulo q, we have

$$\sum_{p\le x} \frac{\chi_0(p)}{p} = \log\log x + O_q(1) \tag{8.7}$$

(see [9, p. 126]). Also, if χ is a non-trivial Dirichlet character, then we have

$$\sum_{p\le x} \frac{\chi(p)}{p} = O_\chi(1) \tag{8.8}$$

(see [9, Theorem 4.11]). Thus, we conclude that the first sum in (8.3) is mainly contributed by $\beta = k$, which is

$$k!((a_1^2 + a_2^2)\log\log T)^k + O_k((\log\log T)^{k-1+\epsilon}).$$

Hence, we complete the proof. □

Denoting $X_j = \log|L(\sigma_0 + it, \chi_j)|$ for $j = 1, 2$, we are now in a position to prove Theorem 8.1.

Proof of Theorem 8.1. Observing that $a_1X_1 + a_2X_2 = \log|\mathcal{L}_{a_1,a_2}(s)|$, we have

$$\int_{t-1}^{t+1} \left| \log\left|\mathcal{L}_{a_1,a_2}\left(\frac{1}{2}+iy\right)\right| - \log|\mathcal{L}_{a_1,a_2}(\sigma_0+iy)| \right| \mathrm{d}y \ll \left(\sigma_0 - \frac{1}{2}\right)\log T \tag{8.9}$$

by Proposition 3.1. Therefore, we may study the function $\log|\mathcal{L}_{a_1,a_2}(\sigma_0+it)|$ away from the critical line $\Re(s) = \frac{1}{2}$.

Define $\mathcal{P}_{a_1,a_2}(s) := a_1\mathcal{P}(s,\chi_1) + a_2\mathcal{P}(s,\chi_2)$, where $\mathcal{P}(s,\chi_j)$ are defined as in (3.1). It can be derived from (5.3) that the contribution of p^k, $k \geq 3$, in $\mathcal{P}_{a_1,a_2}(s)$ is at most $O(1)$; the contribution of p^2, with $p > R$, is at most $O(T/R^2)$ by (5.4). Therefore, we can instead consider $\mathcal{P}_{a_1,a_2,0}(\sigma_0 + it)$, defined in (8.1).

Now by Lemma 8.2, for $X^k \leq T$ and k odd, $\int_T^{2T} \left(\Re(\mathcal{P}_{a_1,a_2,0}(\sigma_0+it))\right)^k \mathrm{d}t$ is

$$\frac{1}{2^k}\sum_{\ell=0}^{k}\binom{k}{\ell}\int_T^{2T}\mathcal{P}_{a_1,a_2,0}(\sigma_0+it)^{\ell}\overline{\mathcal{P}_{a_1,a_2,0}(\sigma_0+it)^{k-\ell}}\mathrm{d}t \ll T.$$

Also, by Lemma 8.2, for $X^k \leq T$ and k even, $\frac{1}{T}\int_T^{2T} \left(\Re(\mathcal{P}_{a_1,a_2,0}(\sigma_0+it))\right)^k \mathrm{d}t$ is

$$\left(\frac{k}{2}\right)!!\left(\frac{a_1^2+a_2^2}{2}\log\log T\right)^{\frac{k}{2}} + O_k((\log\log T)^{\frac{k}{2}-1+\epsilon}).$$

This shows that the real part of $\mathcal{P}_{a_1,a_2,0}(\sigma_0 + it)$ is $\mathcal{AN}(0, \frac{a_1^2+a_2^2}{2}\log\log T)$, and so is $\Re(\mathcal{P}_{a_1,a_2}(\sigma_0 + it))$.

Finally, we shall connect $\mathcal{L}_{a_1,a_2}(\sigma_0 + it)$ with $\mathcal{P}_{a_1,a_2}(\sigma_0 + it)$. By Proposition 3.3 and 3.4, $|L(\sigma_0 + it, \chi_j)| = (1 + o(1))\exp(\Re(\mathcal{P}(\sigma_0 + it, \chi_j)))$, except for a possible set of measure $o(T)$, for each j. Since $\mathcal{L}_{a_1a_2}(s) = |L(s,\chi_1)|^{a_1}|L(s,\chi_2)|^{a_2}$ and $\mathcal{P}_{a_1,a_2}(s) = a_1\mathcal{P}(s,\chi_1) + a_2\mathcal{P}(s,\chi_2)$, we see that

$$\mathcal{L}_{a_1,a_2}(\sigma_0 + it) = (1 + o(1))\exp(\Re(\mathcal{P}_{a_1,a_2}(\sigma_0 + it)))$$

Thus, $\log|\mathcal{L}_{a_1,a_2}(\sigma_0 + it)|$ is $\mathcal{AN}(0, \frac{a_1^2+a_2^2}{2}\log\log T)$ (as $\Re(\mathcal{P}_{a_1,a_2}(\sigma_0 + it))$ is), which combined with (8.9) concludes the proof □

To end this section, we shall prove Theorem 1.2.

Proof of Theorem 1.2. Let $\mathbf{X} := (X_j)_{j=1}^2 = (\log|L(\frac{1}{2} + it, \chi_j)|)_{j=1}^2$. Theorem 8.1 and Proposition 2.1 imply that $\mathbf{X}$ is an approximate bivariate normal distribution. Finally, applying Theorem 8.1 with $a_1 = a_2 = 1$, we deduce from (2.6) that X_1, X_2 are uncorrelated. Hence, X_1, X_2 are independent by Proposition 2.1. □

8.2. Gaussian process for Dirichlet *L*-functions. For $(a_j)_{j=1}^N \subset \mathbb{R}^N$, we consider

$$\mathcal{P}_{a_1,\ldots,a_N,0}(s) := \sum_{p\leq X}\frac{a_1\chi_1(p) + \cdots + a_N\chi_N(p)}{p^s}.$$

To prove Theorem 1.3, we shall require the following lemma.

Lemma 8.3. *Let $(\chi_j)_{j=1}^N$ be a sequence of distinct primitive Dirichlet characters. Assume that $k, \ell \in \mathbb{Z}$ are non-negative and with $X^{k+\ell} \leq T$. Then for any real numbers $(a_j)_{j=1}^N$, we have, for $k \neq \ell$*

$$\int_T^{2T} \mathcal{P}_{a_1,\ldots,a_N,0}(\sigma_0+it)^k \overline{\mathcal{P}_{a_1,\ldots,a_N,0}(\sigma_0+it)^\ell} \ll T,$$

and for $k = \ell$,

$$\int_T^{2T} \left|\mathcal{P}_{a_1,\ldots,a_N,0}(\sigma_0+it)\right|^{2k} \mathrm{d}t = k! T \left(\left(\sum_{j=1}^N a_j^2 \right) \log\log T \right)^k + O_k(T(\log\log T)^{k-1+\epsilon}).$$

Proof. Setting $\psi(p) := \sum_{j=1}^N a_j \chi_j(p)$ and $\Psi_k(n) := \prod_{j=1}^r \psi(p_j)^{\alpha_j}$, whenever $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ with $\alpha_1 + \cdots + \alpha_r = k$, we can write

$$\mathcal{P}_{a_1,\ldots,a_N,0}(\sigma_0+it)^k = \sum_n \frac{a_k(n)\Psi_k(n)}{n^{\sigma_0+it}},$$

where $a_k(n)$ is defined in (5.1). Hence, we obtain

$$\begin{aligned} (8.10) \quad & \int_T^{2T} \mathcal{P}_{a_1,\ldots,a_N,0}(\sigma_0+it)^k \overline{\mathcal{P}_{a_1,\ldots,a_N,0}(\sigma_0+it)^\ell} \mathrm{d}t \\ & = T \sum_n \frac{a_k(n)a_\ell(n)\Psi_k(n)\overline{\Psi_\ell(n)}}{n^{2\sigma_0}} + O\left(\sum_{n\neq m} \frac{a_k(n)a_\ell(m)|\Psi_k(n)\overline{\Psi_\ell(m)}|}{(nm)^{\sigma_0-\frac{1}{2}}} \right). \end{aligned}$$

It follows from the definition of $\Psi_k(n)$ that for $n = \prod_{j=1}^r p_j^{\alpha_j}$, one has

$$|\Psi_k(n)| = \prod_{j=1}^r |\psi(p_j)|^{\alpha_j} \leq \prod_{j=1}^r (|a_1| + \cdots + |a_N|)^{\alpha_j} = (|a_1| + \cdots + |a_N|)^k.$$

Thus, the big-O term in (8.10) is at most

$$\sum_{n\neq m} a_k(n)a_\ell(m)(|a_1| + \cdots + |a_N|)^{k+\ell} \ll X^{k+\ell} \ll T.$$

As $a_k(n)a_\ell(n) = 0$ if $k \neq \ell$, we conclude the first part of the lemma.

It remains to consider the case $k = \ell$. For $n = \prod_{j=1}^r p_j^{\alpha_j}$, we write

$$\begin{aligned} (8.11) \quad & \Psi_k(n)\overline{\Psi_k(n)} \\ & = \prod_{j=1}^r \left((a_1^2 + \cdots + a_N^2)\chi_0(p_j) \sum_{i=1}^N \sum_{i'\neq i} a_i a_{i'} (\chi_i \overline{\chi}_{i'}(p_j) + \chi_{i'}\overline{\chi}_i(p_j)) \right)^{\alpha_j} \end{aligned}$$

By (8.11) and the fact that $\sum_j \alpha_j = k$, we have

$$\left| \sum_{n \text{ non-square-free}} \frac{a_k(n)a_k(n)\Psi_k(n)\overline{\Psi_k(n)}}{n^{2\sigma_0}} \right| \leq \left(\sum_{j=1}^{N} |a_j| \right)^{2k} \sum_{n \text{ non-square-free}} \frac{a_k(n)^2}{n^{2\sigma_0}}.$$

Therefore, the non-square-free n in the first sum of (8.10) contribute a quantity of order $O((\log\log T)^{k-1})$. For square-free n, we express $\Psi_k(n)\overline{\Psi_k(n)}$ as

$$\sum_{0\leq\beta\leq k} A_N^{\beta}(a_1a_2)^{\Omega(m_1)+\Omega(m_1')} \dots (a_{N-1}a_N)^{\Omega(m_{(N-1)N/2})+\Omega(m'_{(N-1)N/2})} \cdot \sum_{n} \frac{a_k(n)^2\chi_0(n')\chi_1\overline{\chi}_2(m_1)\chi_2\overline{\chi}_1(m_1')\dots\chi_N\overline{\chi}_{N-1}(m'_{(N-1)N/2})}{n^{2\sigma_0}}, \tag{8.12}$$

where $A_N = \sum_{j=1}^{N} a_j^2$ and the second sum is over $n = n' \prod_{j=1}^{(N-1)N/2} m_j m_j'$ with pairwise co-prime n', m_i, and m_j' such that $\beta + \sum_{j=1}^{(N-1)N/2}(\Omega(m_j) + \Omega(m_j')) = k$, and χ_0 denotes the trivial character. By the mulitiplicativity of Dirichlet characters and the change of variables $\Omega(m_j) \mapsto \gamma_{2j-1}, \Omega(m_j') \mapsto \gamma_{2j}$, we may further write the second sum in (8.12) as

$$\begin{aligned} &k! \sum_{0\leq\gamma_1,\dots,\gamma_{(N-1)N}\leq k-\beta} \frac{k!}{\beta!\gamma_1!\cdots\gamma_{(N-1)N}!(k-\beta-\gamma_1-\cdots-\gamma_{(N-1)N})!} \\ &\cdot \sum_{\substack{p_1,\dots,p_k\leq X \\ p_j \text{ distinct}}} \frac{\chi_0(p_1\cdots p_\beta)\cdots\chi_N\overline{\chi}_{N-1}(p_{\beta+\gamma_1+\cdots+\gamma_{(N-1)N-1}+1}\cdots p_{\beta+\delta_N})}{(p_1\dots p_k)^{2\sigma_0}} \\ &= k! \sum_{0\leq\gamma_1,\dots,\gamma_{(N-1)N}\leq k-\beta} \frac{k!}{\beta!\gamma_1!\cdots\gamma_{(N-1)N}!(k-\beta-\gamma_1-\cdots-\gamma_{(N-1)N})!} \\ &\cdot \left(\sum_{p\leq X}\frac{\chi_0(p)}{p^{2\sigma_0}}\right)^{\beta} \left(\sum_{p\leq X}\frac{\chi_1\overline{\chi}_2(p)}{p^{2\sigma_0}}\right)^{\gamma_1} \cdots \left(\sum_{p\leq X}\frac{\chi_N\overline{\chi}_{N-1}(p)}{p^{2\sigma_0}}\right)^{\gamma_{(N-1)N}}, \end{aligned} \tag{8.13}$$

where $\delta_N = \gamma_1 + \dots \gamma_{(N-1)N}$.

By (8.7), (8.8), and (8.13) we see that the first sum in (8.10) is mainly contributed by $\beta = k$, which is

$$k!\left(\left(\sum_{j=1}^{N} a_j^2\right)\log\log T\right)^{k} + O_k((\log\log T)^{k-1+\epsilon}).$$

Hence, we complete the proof. □

With this moment calculation in hand, we have the following theorem regarding the joint distribution for Dirichlet L-functions.

Theorem 8.4. *Let* $(\chi_j)_{j=1}^N$ *be a sequence of distinct primitive Dirichlet characters. Let* V *be a fixed positive real number. For any* $(a_j)_{j=1}^N \subset \mathbb{R}^N$, $\log|\mathcal{L}_{a_1,\dots,a_N}(\frac{1}{2}+it)|$ *is* $\mathcal{AN}(0, \frac{a_1^2+\cdots+a_N^2}{2}\log\log T)$*, as* $T \to \infty$*, where*

$$\mathcal{L}_{a_1,\dots,a_N}(s) = \mathcal{L}(s, \chi_1, \dots, \chi_N; a_1, \dots, a_N) := |L(s,\chi_1)|^{a_1} \dots |L(s,\chi_N)|^{a_N}.$$

Proof. As the proof of Theorem 8.4 is essentially the same as Theorem 8.1 (upon applying Lemma 8.3 instead), we shall omit it. □

We will conclude this section by proving Theorem 1.3.

Proof of Theorem 1.3. Let J denote a totally ordered set of (distinct) primitive Dirichlet characters. For any finite ordered subset $\{\chi_1, \dots, \chi_N\}$ of J, we consider $\mathbf{X} := (\log|L(\frac{1}{2}+it,\chi_j)|)_{j=1}^N$. From Theorem 8.4 and Proposition 2.1, it follows that $\mathbf{X}$ is an approximate N-variate normal distribution. Thus, we see that any finite linear combination of elements in $(\log|L(\frac{1}{2}+it,\chi)|)_{\chi\in J}$ is a multivariate normal distribution. Hence, $(\log|L(\frac{1}{2}+it,\chi)|)_{\chi\in J}$ forms a Gaussian process.

In addition, the components in $\mathbf{X}$ are mutually independent since they are pairwisely independent by Theorem 1.2. □

9. Concluding Remarks

As remarked in [12, Sec. 7], analogues of their Propositions 1-3 may be established for automorphic L-functions. From this point of view, our Propositions 3.1-3.3 present an "abelian" instance. As may be noticed throughout our argument, we heavily rely on the bound $|\chi(n)| \le 1$, and thus we expect the Ramanujan–Petersson conjecture would be required for general automorphic L-functions. Also, it seems that by present-day techniques, one can only extend Proposition 3.4 for automorphic L-functions of degree 2.

Furthermore, in light of Theorems 1.2 and 1.3 (together with Selberg's remark and his orthogonality conjecture), for any subclass $\mathcal{C}$ of the Selberg class, the logarithms of primitive L-functions in $\mathcal{C}$ shall form a Gaussian process. As the Ramanujan–Petersson conjecture for holomorphic modular forms has been established by Deligne, we expect that, at least, some partial results could be attained for "modular" L-functions.

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