JOURNAL
de Théorie des Nombres
de BORDEAUX

anciennement Séminaire de Théorie des Nombres de Bordeaux

Ben HEUER

Rigid τ-crystals

Tome 29, nº 3 (2017), p. 1059-1082.

<http://jtnb.cedram.org/item?id=JTNB_2017__29_3_1059_0>

Journal de Théorie des Nombres de Bordeaux **29** (2017), 1059–1082

Rigid τ-crystals

par Ben HEUER

In memory of David Goss

Résumé. Nous présentons un analogue en égale charactéristique des F-isocristaux sur un anneau parfait, que nous appelons τ-crystaux rigides. Nous introduisons des polygones de Newton pour les τ-crystaux rigides, et nous montrons que ceux-ci peuvent être étudiés au moyen des τ-crystaux formels, qui sont analogues aux F-crystaux. Ainsi, nous démontrons un analogue du théorème de Grothendieck–Katz pour les τ-crystaux rigides qui proviennent d'un modèle formel.

Abstract. We present an equicharacteristic analogue of F-isocrystals over perfect rings, which we call rigid τ-crystals. We introduce Newton polygons for rigid τ-crystals and show how these can be studied via formal τ-crystals, the natural analogue of F-crystals. This leads to an analogue of the Grothendieck–Katz theorem for rigid τ-crystals that admit a formal model.

1. Introduction

F-crystals are objects in function field arithmetic that are closely related to p-divisible groups: Let k be a field of characteristic $p > 0$. For an elliptic curve E over k, instead of the Tate module $T_l(E)$ for $l = p$ one studies the p-divisible group of E. These are categorically anti-equivalent to so-called Dieudonné-modules, which are examples of F-crystals: locally free modules with a certain Frobenius structure over the Witt vectors of k. There is a good theory of studying F-crystals up to isogeny via so-called F-isocrystals, which have deep connections to p-adic representations of étale fundamental groups of varieties.

In this article we present an equicharacteristic analogue of F-crystals and F-isocrystals: The Witt vectors $W(A)$ over an $\mathbb{F}_p$-algebra A behave in many ways like a mixed characteristic analogue of the ring of formal power series $A[\![t]\!]$ over A. We will consider analogues of F-crystals with $W(k)$ replaced by $k[\![t]\!]$, which we shall call formal τ-crystals and which are vector bundles over a formal scheme together with a Frobenius structure.

Manuscrit reçu le 2 novembre 2016, révisé le 26 juin 2017, accepté le 4 juillet 2017.

2010 *Mathematics Subject Classification.* 14F30, 14G22.

Mots-clefs. F-crystals, equicharacteristic, Grothendieck–Katz theorem, rigid geometry.

There is a natural functor from F-crystals to F-isocrystals which is essentially tensoring over $\mathbb{Z}_p$ with $\mathbb{Q}_p$. The analogue of this in equicharacteristic is passing from $k[\![t]\!]$ to the field $k((t))$ of formal Laurent series. More precisely, by a well-known procedure of creating a rigid analytic $k((t))$-space from a topologically finite type formal $k[\![t]\!]$-scheme, we obtain a rigid analytic object which we call a rigid τ-crystal. We hope that ultimately these rigid τ-crystals can be used to study representations of étale fundamental groups with coefficients in $\overline{\mathbb{F}}_p((t))$, as outlined by Pál in [13].

This article is organised as follows: In the first chapter we introduce rigid τ-crystals. We discuss results for F-crystals that can be adopted to rigid τ-crystals immediately, and on the other hand illustrate some arising difficulties. In order to surpass these, we then introduce the natural analogue of F-crystals in this setting, namely formal τ-crystals. For these the theory of F-crystals carries over without problems. In particular, for formal τ-crystals we can define Hodge and Newton polygons and these behave as expected. The goal will be to prove the theorem of Grothendieck–Katz in this setting: For a formal τ-crystal $\mathfrak{T}$ on X and $\lambda \in \mathbb{R}_{\geq 0}$, the locus of points of X at which the Newton slope of $\mathfrak{T}$ is $\geq \lambda$ is Zariski-closed in X. Finally, we discuss how our results can be applied to rigid τ-crystals, and we present some questions this raises about the existence of formal models of rigid τ-crystals.

Acknowledgements. We would like to thank Ambrus Pál for the supervision of the project which lead to this article, as well as for many interesting discussions. We also thank the referee for several useful suggestions and comments.

This work was supported by the Engineering and Physical Sciences Research Council [EP/L015234/1] and by the EPSRC Centre for Doctoral Training in Geometry and Number Theory (The London School of Geometry and Number Theory), University College London.

2. Rigid τ-crystals

Throughout let k be a perfect field of characteristic $p > 0$. Let $R = k[\![t]\!]$ be the ring of formal power series, which is a discrete valuation ring with field of fractions $K = k((t))$, the field of formal Laurent series over k. We fix an algebraic closure $\overline{k}$ of k. All unadorned fibre products will be over $\operatorname{Spec} k$.

2.1. Preliminaries in formal and rigid geometry.

Definition 2.1. Let X be a Noetherian k-scheme. Then the fibre product $X \times \operatorname{Spec} k[t]$ is again Noetherian. By base change along the morphism $\operatorname{Spec} k \to \operatorname{Spec} k[t]$ that corresponds to the reduction $t \mapsto 0$, we get a closed immersion

$$X = X \times \operatorname{Spec} k \hookrightarrow X \times \operatorname{Spec} k[t]\,. \tag{2.1}$$

We call this map the *special fibre*. Via the special fibre, we can consider X as a closed subscheme of $X \times \operatorname{Spec} k[t]$. Completion along this subscheme gives a formal $k[\![t]\!]$-scheme

$$X_{\mathrm{for}} = \widehat{X \times \operatorname{Spec} k[t]}\,.$$

We collect some properties of X_{for} in the following two lemmas.

Lemma 2.2. *There is a canonical morphism $i_X : X_{\mathrm{for}} \to X \times \operatorname{Spec} k[t]$ of locally ringed spaces. The underlying continuous map identifies the topological space underlying X_{for} with the image of the map $X \hookrightarrow X \times \operatorname{Spec} k[t]$ from equation* (2.1)*.*

Lemma 2.3. *There is a functor $X \mapsto X_{\mathrm{for}}$ from Noetherian k-schemes to formal $k[\![t]\!]$-schemes that sends a morphism $f : X \to Y$ to a morphism f_{for} with the following properties:*

(1) *When we identify the underlying topological spaces of X_{for} and X as well as of Y_{for} and Y, then as a map of topological spaces, $f_{\mathrm{for}} : X_{\mathrm{for}} \to Y_{\mathrm{for}}$ coincides with $f : X \to Y$.*

(2) *We have a commutative diagram of locally ringed spaces*

$$\begin{array}{ccc} X_{\mathrm{for}} & \xrightarrow{\quad f_{\mathrm{for}} \quad} & Y_{\mathrm{for}} \\ \downarrow{\scriptstyle i_X} & & \downarrow{\scriptstyle i_Y} \\ X \times \operatorname{Spec} k[t] & \xrightarrow{f \times \mathrm{id}} & Y \times \operatorname{Spec} k[t]\,. \end{array}$$

Proof. This is a special case of results in Sections 10.9.1 and 10.9.2 of [5]. □

Lemma 2.4. *In the affine case, the $-_{\mathrm{for}}$-functor has the following form:*

(1) *Let A be a Noetherian k-algebra, then $(\operatorname{Spec} A)_{\mathrm{for}} = \operatorname{Spf} A[\![t]\!]$ where we have endowed $A[\![t]\!]$ with the t-adic topology.*

(2) *Let $f : \operatorname{Spec} A \to \operatorname{Spec} B$ be a morphism of Noetherian k-schemes induced by $f^\sharp : B \to A$, then $f_{\mathrm{for}} : (\operatorname{Spec} A)_{\mathrm{for}} \to (\operatorname{Spec} B)_{\mathrm{for}}$ is induced by the homomorphism of $k[\![t]\!]$-algebras*

$$B[\![t]\!] \to A[\![t]\!], \quad \sum_{n=0}^{\infty} b_n t^n \mapsto \sum_{n=0}^{\infty} f^\sharp(b_n) t^n.$$

Proof. For the first part, we just need to go through the steps of the $-_{\mathrm{for}}$-construction: We have

$$X \times \operatorname{Spec}(k[t]) = \operatorname{Spec}(A[t])\,.$$

The subscheme along which we then complete corresponds to the morphism $A[t] \to A$, $t \mapsto 0$. The completion of $\operatorname{Spec}(A[t])$ along the special fibre will therefore on rings correspond to t-adic completion. We conclude that $X_{\text{for}} = \operatorname{Spf}(A[\![t]\!])$.

For the second claim, we first note that the functor $- \times \operatorname{Spec} k[t]$ sends f to the map which is induced by the morphism $B[t] \to A[t]$ that sends $\sum_{n=0}^{N} b_n t^n$ to $\sum_{n=0}^{N} f^\sharp(b_n) t^n$. The t-adic completion extends this to a map $B[\![t]\!] \to A[\![t]\!]$ described in the Lemma. □

In order to make the notation consistent, we also refer to $A[\![t]\!]$ as A_{for}, so that $(\operatorname{Spec} A)_{\text{for}} = \operatorname{Spf} A_{\text{for}}$. The following Lemma shows that we can use Lemma 2.4 to describe X_{for} locally. This way we can deduce geometric properties of X_{for} from algebraic properties of $A[\![t]\!]$.

Lemma 2.5. *Let X be a finite type k-scheme and let $\cup_i U_i$ be a cover of X by affine open subschemes. Then each $(U_i)_{\text{for}}$ can be interpreted as an affine open formal subscheme of X_{for}. We thus obtain a cover $X_{\text{for}} = \cup_i (U_i)_{\text{for}}$ of affine open formal subschemes.*

Proof. When we follow through the steps of the $-_{\text{for}}$-construction, we first obtain a cover of $X \times \operatorname{Spec} k[t]$ by affine open subsets $U_i \times \operatorname{Spec} k[t]$. After formal completion, we can interpret $(U_i)_{\text{for}}$ as an affine formal subscheme of X_{for} by Proposition 10.8.5 in [5]. On the underlying topological subspaces, this corresponds to the inclusion of the open subset $U_i \subseteq X$. Since the U_i cover X, this shows that we indeed get a cover $X_{\text{for}} = \cup (U_i)_{\text{for}}$. □

The way that rigid geometry comes in relies on the fact that the resulting scheme X_{for} is topologically of finite type over $k[\![t]\!]$. This is a consequence of the following identity:

Proposition 2.6. *Let $A = k[X_1, \ldots, X_n]$. Then A_{for} is the Tate algebra over $k[\![t]\!]$ in the variables X_i with respect to the t-adic norm:*

$$A_{\text{for}} = k[X_1, \ldots, X_n][\![t]\!] = k[\![t]\!]\langle X_1, \ldots, X_n \rangle .$$

Proof. This is a matter of rearranging variables and the proof in fact works for polynomial rings over any k-algebra R. Therefore, by induction, it is enough to prove the case $n = 1$. So let $f \in R[X][\![t]\!]$, $f = \sum f_n t^n$. We can write each $f_n \in R[X]$ as a power series $f_n = \sum_{m=0}^{\infty} a_{m,n} X^m$ with $a_{m,n} \in R$ such that for all n we have $a_{m,n} = 0$ for m large enough. Now rearrange:

$$\begin{aligned} f &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{\infty} a_{m,n} X^m \right) t^n = \sum_{m=0}^{\infty} \left(\sum_{n=0}^{\infty} a_{m,n} t^n \right) X^m \\ &= \sum_{m=0}^{\infty} g_m X^m \qquad \text{where we set } g_m := \sum_{n=0}^{\infty} a_{m,n} t^n \in R[\![t]\!] . \end{aligned}$$

This is a power series in the variable X with coefficients in $R[\![t]\!]$. The condition that for any n there are only finitely many m with non-trivial $a_{m,n}$ is now equivalent to the t-adic absolute value of the coefficients g_m converging to 0. This proves that f is an element of $R[\![t]\!]\langle X\rangle$. The converse follows by reversing this argument. □

Corollary 2.7. *If A is a finite type k-algebra, then A_{for} is an algebra topologically of finite type over $k[\![t]\!]$.*

Proof. Let $\mathfrak{a} \subseteq A = k[X_1, \ldots, X_n]$ be an ideal and consider the map $A[\![t]\!] \to (A/\mathfrak{a})[\![t]\!]$. It is continuous, surjective and moreover open as one can see from the projective limit topology. This shows that $(A/\mathfrak{a})[\![t]\!]$ is isomorphic as a topological algebra to a quotient of $A[\![t]\!] = k[\![t]\!]\langle X_1, \ldots, X_r\rangle$ and hence is topologically of finite type over $k[\![t]\!]$. □

Corollary 2.8. *Let X be a finite type k-scheme. Then the formal $k[\![t]\!]$-scheme X_{for} is topologically of finite type, Noetherian and separated.*

Proof. It is clear from Lemma 2.2 that X_{for} is quasi-compact. Since $X \times \operatorname{Spec} k[t]$ is Noetherian, it follows that X_{for} is Noetherian by Corollaire 10.8.6 in [5]. Lemma 2.5 gives a finite cover of X_{for} by affine formal schemes which by Corollary 2.7 are topologically of finite type. This shows that X_{for} is topologically of finite type. Finally, $X \times \operatorname{Spec} k[t]$ is separated and thus its completion X_{for} is separated by Proposition 10.15.7 in [5]. □

Proposition 2.9. *Let $A = k[X_1, \ldots, X_n]$, then $A((t))$ is the Tate algebra*

$$A((t)) = k[X_1, \ldots, X_n]((t)) = k((t))\langle X_1, \ldots, X_n\rangle\,.$$

Proof. We can proceed like we did in the proof of Corollary 2.7, but now allowing the index n of $a_{m,n}$ to be negative. Alternatively, one can also use Corollary 2.7 and use that $k[\![t]\!]\langle X_1, \ldots, X_n\rangle \otimes_{k[\![t]\!]} k((t)) = k((t))\langle X_1, \ldots, X_n\rangle$. See [3, §7.4] for more details. □

Corollary 2.10. *Let A be a finite type k-algebra. Then $A((t))$ is an affinoid $k((t))$-algebra.*

Proof. We can write $A((t))$ as the localisation $S^{-1}A[\![t]\!]$ at the multiplicative set $S = \{t^n | n \in \mathbb{N}\}$. By Corollary 2.7 we have $A[\![t]\!] = k[\![t]\!]\langle X_1, \ldots, X_n\rangle/\mathfrak{a}$ for some ideal $\mathfrak{a} \subseteq k[\![t]\!]\langle X_1, \ldots, X_n\rangle$. Since localisation is exact, this shows that $A((t))$ is a quotient of $S^{-1}(k[X_1, \ldots, X_n][\![t]\!])$. The latter equals $k((t))\langle X_1, \ldots, X_n\rangle$ by Proposition 2.9. □

The above result generalizes as follows. Recall that $R = k[\![t]\!]$ is a discrete valuation ring with fraction field $K = k((t))$. In this situation, we have the following construction:

Proposition 2.11. *There is a functor $-_{\mathrm{rig}}$ from topologically finite type formal R-schemes $\mathfrak{X}$ to rigid K-spaces which assigns to an affine formal* *scheme* $\operatorname{Spf} B$ *the affinoid* K*-space* $(\operatorname{Spf} B)_{\mathrm{rig}} = \operatorname{Sp}(B \otimes_R K)$*. The rigid space* $\mathfrak{X}_{\mathrm{rig}}$ *is called the* generic fibre *of* $\mathfrak{X}$*.*

Proof. One first constructs this functor on affinoid spaces by showing that $B \otimes_R K$ is an affinoid K-algebra. In the case we are interested in, this is Corollary 2.10. Since in our case all formal R-schemes of interest are separated by Corollary 2.8, we may proceed as follows: Choose an affine cover $\mathfrak{X} = \cup_i U_i$. Then since $\mathfrak{X}$ is separated, the intersection $U_i \cap U_j$ is affine again. We can now define $\mathfrak{X}_{\mathrm{rig}}$ by glueing the affinoid spaces $(U_i)_{\mathrm{rig}}$ along the affinoid subspaces $(U_i \cap U_j)_{\mathrm{rig}}$. Finally, one shows that this space does not depend on the affine cover up to canonical isomorphism. See [3, §7.4] for details of the construction in general. □

Corollary 2.12. *We obtain a functor from finite type* k*-schemes to rigid* K*-spaces*

$$X \mapsto X_{\mathrm{for}} \mapsto (X_{\mathrm{for}})_{\mathrm{rig}}\,.$$

We also denote this functor by $-_{\mathrm{rig}}$*.*

Proof. For a finite type k-scheme X, Corollary 2.8 says that X_{for} is a topologically finite type $k[\![t]\!]$-scheme. Therefore the $-_{\mathrm{rig}}$-construction applies to X_{for}. □

Corollary 2.13. *In the affine case, the* $-_{\mathrm{rig}}$*-functor can be described as follows:*

(1) *Let* $\operatorname{Spec} A$ *be a finite type* k*-scheme, then* $(\operatorname{Spec} A)_{\mathrm{rig}} = \operatorname{Sp} A((t))$*.*
(2) *Let* $f : \operatorname{Spec} A \to \operatorname{Spec} B$ *be a morphism of finite type* k*-schemes induced by* $f^\sharp : B \to A$*, then* $f_{\mathrm{rig}} : (\operatorname{Spec} A)_{\mathrm{rig}} \to (\operatorname{Spec} B)_{\mathrm{rig}}$ *corresponds to the morphism of* K*-algebras*

$$f^\sharp_{\mathrm{rig}} : B((t)) \to A((t)), \quad \sum b_n t^n \mapsto \sum f^\sharp(b_n) t^n.$$

Proof. We know from Lemma 2.4 that $(\operatorname{Spec} A)_{\mathrm{for}} = \operatorname{Spf} A[\![t]\!]$. Therefore, by construction of the $-_{\mathrm{rig}}$-functor, the space $(\operatorname{Spec} A)_{\mathrm{rig}} = (\operatorname{Spf} A[\![t]\!])_{\mathrm{rig}}$ is the affinoid K-space associated to $A((t))$. Since in the affine case we have $f^\sharp_{\mathrm{rig}} = f^\sharp_{\mathrm{for}} \otimes_R \mathrm{id} : B_{\mathrm{for}} \otimes_R K \to A_{\mathrm{for}} \otimes_R K$, the claimed description of $f^\sharp_{\mathrm{rig}}$ follows from Lemma 2.4. □

For consistency of notation, we also refer to $A((t))$ as A_{rig}, so that $(\operatorname{Spec} A)_{\mathrm{rig}} = \operatorname{Sp} A_{\mathrm{rig}}$.

Lemma 2.14. *Let* $\mathfrak{X}$ *be a formal scheme and let* $\mathfrak{X} = \cup_i U_i$ *be a cover by affine open formal subschemes. Then we can identify each* $(U_i)_{\mathrm{rig}}$ *with an affinoid open subspace of* $\mathfrak{X}_{\mathrm{rig}}$ *and we obtain an admissible cover* $\mathfrak{X}_{\mathrm{rig}} = \cup_i (U_i)_{\mathrm{rig}}$ *by affinoid open subspaces.*

Proof. We know from Corollary 2.13 that the $(U_i)_{\mathrm{rig}}$ are going to be affinoid. The claim then follows from the fact that $\mathfrak{X}_{\mathrm{rig}}$ is constructed by glueing the affinoid spaces $(U_i)_{\mathrm{rig}}$. See [3, Proposition 5.3.5] for more detail. □

Proposition 2.15. *If* $\operatorname{Spec} A \subseteq X$ *is an affine open subset, we can identify* $(\operatorname{Spec} A)_{\mathrm{rig}} = \operatorname{Sp}(A_{\mathrm{rig}})$ *with an affinoid open subspace of* X_{rig}. *Moreover, if* $(\operatorname{Spec} A_i)_{i\in I}$ *is an affine open cover of* X, *then* $(\operatorname{Sp}((A_i)_{\mathrm{rig}}))_{i\in I}$ *is an admissible affinoid cover of* X_{rig}.

Proof. This is immediate from Lemma 2.5 and Lemma 2.14. □

2.2. Lifting morphisms. By functoriality, a morphism of finite type k-schemes $f : X \to Y$ gives rise to a morphism $f_{\mathrm{rig}} : X_{\mathrm{rig}} \to Y_{\mathrm{rig}}$ of rigid K-spaces. We would like to produce a similar result for the absolute Frobenius morphism $\sigma : X \to X$ as well as for geometric points $\varphi : \operatorname{Spec} \overline{k} \to X$. However, in general we cannot expect any morphism $f : X \to Y$ to induce a map $f_{\mathrm{rig}} : X_{\mathrm{rig}} \to Y_{\mathrm{rig}}$ of locally G-ringed spaces. To illustrate a possible difficulty, let A and B be k-algebras and let $f : B \to A$ be any ring homomorphism. While we do have a natural map

$$f \widehat{\otimes} \mathrm{id} : B((t)) \to A((t)),$$

it is not even clear that maximal ideals pull back to maximal ideals under this map if f is not k-linear or if A or B are not of finite type.

2.2.1. ***Lifting the absolute Frobenius morphism.***

Lemma 2.16. *Let* X *be a finite type* k*-scheme. Then the absolute Frobenius on* X *extends to a morphism* $\tau : X_{\mathrm{for}} \to X_{\mathrm{for}}$ *of locally ringed spaces which on affine open subsets* $\operatorname{Spec} A \subseteq X$ *restricts to the morphism* $\tau : \operatorname{Spf} A_{\mathrm{for}} \to \operatorname{Spf} A_{\mathrm{for}}$ *corresponding to the map*

$$\tau^\sharp : A[\![t]\!] \to A[\![t]\!], \quad \sum a_n t^n \mapsto \sum a_n^p t^n.$$

Proof. Consider X as a Noetherian $\mathbb{F}_p$-scheme. Then since $X \times_k \operatorname{Spec} k[t] = X \times_{\mathbb{F}_p} \operatorname{Spec} \mathbb{F}_p[t]$ as $\mathbb{F}_p$-schemes, we obtain the same formal $\mathbb{F}_p[\![t]\!]$-scheme X_{for} from both $-_{\mathrm{for}}$ as a functor on k-schemes, as well as from $-_{\mathrm{for}}$ as a functor on $\mathbb{F}_p$-schemes. When we consider $\sigma : X \to X$ as a morphism of $\mathbb{F}_p$-schemes, we obtain a map $\tau : X_{\mathrm{for}} \to X_{\mathrm{for}}$ of formal $\mathbb{F}_p[\![t]\!]$-schemes. On affine X, Lemma 2.4 then shows that $\tau^\sharp$ is of the desired form. □

An analogous result holds for the $-_{\mathrm{rig}}$-functor:

Proposition 2.17. *Let* X *be a finite type* k*-scheme. Then the absolute Frobenius on* X *lifts to a morphism* $\tau : X_{\mathrm{rig}} \to X_{\mathrm{rig}}$ *of locally G-ringed spaces which for an affine open subset* $\operatorname{Spec} A \subseteq X$ *restricts to the map* $\tau : \operatorname{Spec} A_{\mathrm{rig}} \to \operatorname{Spec} A_{\mathrm{rig}}$ *that corresponds to the morphism*

$$\tau^\sharp : A((t)) \to A((t)), \quad \sum a_n t^n \mapsto \sum a_n^p t^n.$$

As with the Frobenius morphism, we also simply write τ for $\tau^\sharp$. Since the $-_{\mathrm{rig}}$-construction only works for morphisms of k-schemes, the approach for the $-_{\mathrm{for}}$-functor fails in this case. Instead we use the following Lemma:

Lemma 2.18. *Let X be an affine finite type k-scheme and let σ denote the Frobenius morphism on k. Denote by ${}_\sigma X$ the k-scheme X with structure map $X \to \operatorname{Spec} k \xrightarrow{\sigma} \operatorname{Spec} k$. Then $({}_\sigma X)_{\mathrm{rig}}$ is the rigid K-space with the same underlying locally G-ringed space as X_{rig}, but with a K-space structure given by the structure map $X_{\mathrm{rig}} \to \operatorname{Sp} K \xrightarrow{\tau} \operatorname{Sp} K$.*

Proof. Let $X = \operatorname{Spec} A$, then ${}_\sigma X = \operatorname{Spec}({}_\sigma A)$ where ${}_\sigma A$ is the k-algebra with structure map $k \xrightarrow{\sigma} k \to A$. We then have $X_{\mathrm{rig}} = \operatorname{Sp} A_{\mathrm{rig}}$ and $({}_\sigma X)_{\mathrm{rig}} = \operatorname{Sp}({}_\tau A_{\mathrm{rig}})$ where ${}_\tau A_{\mathrm{rig}}$ denotes the ring A_{rig} with K-algebra structure $K \xrightarrow{\tau} K \to A((t))$. Since ${}_\tau A_{\mathrm{rig}}$ and A_{rig} have the same ring structure, it is clear that the underlying sets of $({}_\sigma X)_{\mathrm{rig}}$ and X_{rig} coincide. Furthermore, by our running assumption that k is perfect, the functor ${}_\tau-$ is a category equivalence from affinoid K-algebras to affinoid K-algebras. Therefore, when we apply ${}_\tau-$ to the diagram describing the universal property of affinoid subdomains, we see that a subspace $U \subseteq X_{\mathrm{rig}}$ is an affinoid subdomain if and only if $U \subseteq ({}_\sigma X)_{\mathrm{rig}}$ is an affinoid subdomain. Moreover, if B is the affinoid algebra of the affinoid space representing morphisms with image in $U \subseteq X_{\mathrm{rig}}$, then the same argument shows that ${}_\tau B$ is the affinoid algebra of the affinoid space representing morphisms with image in $U \subseteq ({}_\sigma X)_{\mathrm{rig}}$.

This shows that the structure G-sheaf of $({}_\sigma X)_{\mathrm{rig}}$ is just the structure G-sheaf of X_{rig} but with the K-algebra structure changed via the map $\tau : K \to K$ as claimed. □

Proof of Proposition 2.17. Choose an affine cover $\cup_i U_i$ of X. Then as we have seen in Proposition 2.11, the space X_{rig} can be constructed from glueing the affinoid spaces $(U_i)_{\mathrm{rig}}$ along the affinoid spaces $(U_i \cap U_j)_{\mathrm{rig}}$ and we get an admissible cover $X_{\mathrm{rig}} = \cup_i (U_i)_{\mathrm{rig}}$ by Lemma 2.15. When we construct the desired morphism on each $(U_i)_{\mathrm{rig}}$, the morphisms will agree on intersections $(U_i \cap U_j)_{\mathrm{rig}}$. Indeed, the $U_i \cap U_j$ are affine and thus Lemma 2.18 implies that the morphisms are uniquely determined on $(U_i \cap U_j)_{\mathrm{rig}}$ by the description of $\tau^\sharp$ in Proposition 2.17. Therefore, we can glue all morphisms to give the desired map $X_{\mathrm{rig}} \to X_{\mathrm{rig}}$.

We may therefore assume without loss of generality that X is affine. The absolute Frobenius morphism of X gives a commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\quad\sigma\quad} & X \\ \downarrow & \searrow & \downarrow \\ \operatorname{Spec} k & \xrightarrow{\ \sigma\ } & \operatorname{Spec} k\,. \end{array}$$

When we interpret σ as a morphism ${}_\sigma X \to X$ of finite type k-schemes, we can apply the $-_{\mathrm{rig}}$-functor to it and obtain a morphism $\tau : ({}_\sigma X)_{\mathrm{rig}} \to X_{\mathrm{rig}}$ of rigid K-spaces. When we then forget the K-space structure and just consider τ as a morphism of locally G-ringed spaces, we can identify the underlying G-ringed spaces of $({}_\sigma X)_{\mathrm{rig}}$ and X_{rig} by Lemma 2.18 to get the desired morphism of locally G-ringed spaces $\tau : X_{\mathrm{rig}} \to X_{\mathrm{rig}}$.

Finally, to see that the map $\tau^\sharp$ is as described in the proposition, we just need to apply Corollary 2.13 to the morphism $\tau : ({}_\sigma X) \to X$ of affine k-schemes. □

2.2.2. *Lifting geometric points.* Another type of morphism that can be lifted from k-schemes to give a morphism of the corresponding locally G-ringed spaces are closed $\overline{k}$-points, where $\overline{k}$ is an algebraic closure of k. While every such point factors through $\operatorname{Spec} k'$ for some finite extension k' of k, it will be desirable to pull back rigid τ-crystals to $\operatorname{Sp} \overline{k}((t))$ due to Dieudonné's Theorem (Corollary 2.37).

Lemma 2.19. *Let X be a finite type k-scheme and let $\phi : \operatorname{Spec} \overline{k} \to X$ be a geometric point of X. Then ϕ can be canonically lifted to a morphism of locally G-ringed K-spaces*

$$\phi_{\mathrm{rig}} : \operatorname{Sp} \overline{k}((t)) \to X_{\mathrm{rig}} .$$

Proof. When we show the result for affine X, then in general we can restrict the codomain of ϕ to an affine subset $U \subseteq X$ and thus obtain a map $\operatorname{Sp} \overline{k}((t)) \to U_{\mathrm{rig}} \subseteq X_{\mathrm{rig}}$.

So assume that $X = \operatorname{Spec} A$ is affine. Then ϕ on global sections gives a morphism $\phi^\sharp : A \to \overline{k}$. Since A is a finite type k-algebra, we can find a finite extension k' of k for which we can decompose $\phi^\sharp$ into a map $A \to k'$ followed by an embedding $k' \hookrightarrow \overline{k}$.

Since the rigid $\overline{k}((t))$-space $\operatorname{Sp} \overline{k}((t))$ and the rigid $k((t))$-space $\operatorname{Sp} k'((t))$ are each just a point, the embedding $k'((t)) \hookrightarrow \overline{k}((t))$ gives a canonical map of locally G-ringed K-spaces $\operatorname{Sp} \overline{k}((t)) \to \operatorname{Sp} k'((t))$. The composition $\operatorname{Sp} \overline{k}((t)) \to \operatorname{Sp} k'((t)) \to \operatorname{Sp} A((t))$ then gives the desired morphism of locally G-ringed K-spaces. □

2.3. Definition of rigid $\boldsymbol{\tau}$-crystals. Let X be a finite type k-scheme and let $\mathcal{F}$ be a vector bundle on the rigid space X_{rig}. Here by a vector bundle we mean a coherent locally free $\mathcal{O}_{X_{\mathrm{rig}}}$-module. We have seen in Proposition 2.17 that the absolute Frobenius morphism $\sigma : X \to X$ lifts to a morphism of locally G-ringed spaces $\tau : X_{\mathrm{rig}} \to X_{\mathrm{rig}}$. So we can pull-back $\mathcal{F}$ along τ to obtain a coherent $\mathcal{O}_{X_{\mathrm{rig}}}$-module $\tau^*\mathcal{F}$ on X_{rig}, which is again locally free and thus a vector bundle on X_{rig}.

Definition 2.20. Let X be a finite type k-scheme. A *rigid τ-crystal* on X is a pair $(\mathcal{F}, F)$ consisting of the following data:

(1) a vector bundle $\mathcal{F}$ on X_{rig},
(2) an isomorphism $F : \tau^*\mathcal{F} \to \mathcal{F}$ of vector bundles on X_{rig}.

A morphism of τ-crystals $f : (\mathcal{F}, F) \to (\mathcal{F}', F')$ on X is a morphism $f : \mathcal{F} \to \mathcal{F}'$ of the underlying vector bundles such that pullback along τ induces a commutative diagram

$$\begin{array}{ccc} \tau^*\mathcal{F} & \xrightarrow{F} & \mathcal{F} \\ \downarrow{\scriptstyle \tau^* f} & & \downarrow{\scriptstyle f} \\ \tau^*\mathcal{F}' & \xrightarrow{F'} & \mathcal{F}'. \end{array} \tag{2.2}$$

Definition 2.21. Let $a \in \mathbb{N}$. By iterating τ we obtain a morphism of locally G-ringed spaces $\tau^a = \tau \circ \cdots \circ \tau : X_{\text{rig}} \to X_{\text{rig}}$. We define a *rigid τ^a-crystal* to be a vector bundle $\mathcal{F}$ on X_{rig} with an isomorphism $(\tau^a)^*\mathcal{F} \to \mathcal{F}$ of vector bundles on X_{rig}.

Lemma 2.22. *Let $\mathcal{T} = (\mathcal{F}, F)$ be a rigid τ-crystal on X. We denote by $F^a = F \circ \cdots \circ F$ the a-th iteration, that is the composition of a times F with itself. Then we get a rigid τ^a-crystal $(\mathcal{F}, F^a)$ on X which is called a-th* iteration *of $\mathcal{T}$.*

Proof. For any $b \in \mathbb{Z}_{\geq 0}$, the isomorphism $\tau^*\mathcal{F} \to \mathcal{F}$ pulls back along τ^b to an isomorphism $(\tau^{b+1})^*\mathcal{F} \to (\tau^b)^*\mathcal{F}$. When we compose these maps for $b = 0, \ldots, a-1$ we obtain the desired isomorphism $(\tau^a)^*\mathcal{F} \to \mathcal{F}$. ☐

Remark 2.23. For simplicity of the exposition we will often only treat the case of $\tau^{a=1}$-crystals, but everything we present will go through verbatim for rigid τ^a-crystals.

Remark 2.24 (Rigid τ-crystals in the literature)**.** The idea of rigid τ-crystals as presented in this article is due to Ambrus Pál [13]. Other authors have defined objects which are similar to rigid τ-crystals:

To some extent, rigid τ-crystals resemble Anderson $k[t]$-motives as studied by Böckle [1]. These are vector bundles on $X \times \operatorname{Spec} k[t]$ whereas rigid τ-crystals live on the generic fibre of the completion of this scheme (see Remark 3.5 below for more details).

Another related concept are locally free rigid analytic τ-sheaves as introduced by Böckle and Hartl in [2]. These arise from algebraic τ-sheaves via rigid GAGA after base changing from k to $k((t))$. So compared to the rigid τ-crystals studied here, these sheaves have a different Frobenius structure and an additional variable. The same is true for the σ-modules of Hartl in [7]. However, σ-modules contain rigid τ-crystals on $k[X]$ as a special case.

Finally, Hartl in [6] introduces objects which he calls Dieudonné-$\mathbb{F}_p((t))$-modules, which are a generalisation of rigid τ-crystals in a more general setting. However, in this more general setting the objects don't have a rigid analytic structure, because Proposition 2.10 only holds when X is of finite type.

Example 2.25. When $X = \operatorname{Spec} A$ is an affine k-scheme of finite type, then a vector bundle on X_{rig} is associated to a locally free $A((t))$-module ([4, Proposition 4.7.2]). The Frobenius structure then amounts to the following: The sheaf $\tau^*\widetilde{M}$ will be the one associated to $M \otimes_\tau A((t))$ (this can be proved exactly like in the scheme case using right-exactness of τ^* on a sequence representing $\widetilde{M}$ as a coherent sheaf). A rigid τ-crystal is thus the same data as a locally free $A((t))$-module M with an isomorphism of $A((t))$-modules

$$M \otimes_\tau A((t)) \xrightarrow{\sim} M\,.$$

If A is perfect, that is if τ is an isomorphism, this is the same as a τ-linear isomorphism

$$F : M \xrightarrow{\sim} M \text{ such that } F(am) = \tau(a)F(m) \text{ for } a \in A((t)) \text{ and } m \in M\,.$$

This situation is analogous to the one for F-isocrystals on a perfect ring A, where the role of $A((t))$ is played by $W(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

To give an explicit example, for $k = \mathbb{F}_p$, any pair $(\mathbb{F}_p((t))^n, F)$ where $F \in \mathrm{GL}_n(\mathbb{F}_p((t)))$ defines a rigid τ-crystal.

Example 2.26. An important class of rigid τ-crystals are the *standard Dieudonné-modules* on $\operatorname{Spec} k$, as defined in [6]. For coprime $m \in \mathbb{Z}$, $n \in \mathbb{N}$ consider the $K = k((t))$-vector space $M = k((t))^n$ and the τ-linear map F determined on the standard basis $e_1, \dots, e_n$ of M by the matrix:

$$\begin{pmatrix} 0 & \cdots & 0 & t^m \\ 1 & \ddots & & 0 \\ & \ddots & \ddots & \vdots \\ 0 & & 1 & 0 \end{pmatrix} : M \to M, \quad \sum_{i=1}^n a_i e_i \mapsto \sum_{i=1}^n \tau(a_i) F(e_i)\,.$$

It is not hard to see that this map is a τ-linear isomorphism. Thus by Example 2.25, this indeed defines a rigid τ-crystal on X which we denote by $\mathcal{V}(m/n)$. The same construction also works for rigid τ^a-crystals when we choose the same images for the standard basis but extend τ^a-linearly.

Lemma 2.27. *Let $\phi : Y \to X$ be a morphism of finite type k-schemes and let $\mathcal{T} = (\mathcal{F}, F)$ be a rigid τ-crystal on X. Then the pair $\phi^*\mathcal{T} = (\phi_{\mathrm{rig}}^*\mathcal{F}, \phi_{\mathrm{rig}}^* F)$ is naturally a rigid τ-crystal on Y. We obtain a functor ϕ^*- from rigid τ-crystals on X to rigid τ-crystals on Y.*

Proof. Since Frobenius commutes with any map, we get a commutative diagram

$$\begin{array}{ccc} Y_{\mathrm{rig}} & \xrightarrow{\tau_Y} & Y_{\mathrm{rig}} \\ \downarrow{\scriptstyle \phi_{\mathrm{rig}}} & & \downarrow{\scriptstyle \phi_{\mathrm{rig}}} \\ X_{\mathrm{rig}} & \xrightarrow{\tau_X} & X_{\mathrm{rig}}\,. \end{array}$$

Therefore, if $(\mathcal{F}, F)$ is a rigid τ-crystal on X, pullback along ϕ gives an isomorphism

$$\tau_Y^* \phi_{\mathrm{rig}}^* \mathcal{F} = \phi_{\mathrm{rig}}^* \tau_X^* \mathcal{F} \xrightarrow{\phi_{\mathrm{rig}}^* F} \phi_{\mathrm{rig}}^* \mathcal{F}\,.$$

We see from this that $(\phi_{\mathrm{rig}}^* \mathcal{F}, \phi_{\mathrm{rig}}^* F)$ has the natural structure of a rigid τ-crystal on Y. This construction gives a functor from τ-crystals on X to τ-crystals on Y as one can see from applying the functor $\phi_{\mathrm{rig}}^* -$ to diagram (2.2) in Definition 2.20. □

Remark 2.28. Note that in the proof of Lemma 2.27 we have only used that ϕ_{rig} is a morphism of locally G-ringed K-spaces. Since a closed point $\phi : \operatorname{Spec} \overline{k} \to X$ by Lemma 2.19 induces a morphism of locally G-ringed K-spaces $\operatorname{Sp} \overline{k}((t)) \to X_{\mathrm{rig}}$ as well, the same construction gives a functor $\phi^* -$ from rigid τ-crystals on X to rigid τ-crystals on $\operatorname{Spec} \overline{k}$.

Definition 2.29. Let $\mathcal{T}$ be a τ-crystal on X and let $i : U \hookrightarrow X$ be an open immersion. We define the restriction $\mathcal{T}|_U$ to be the τ-crystal $i^* \mathcal{T}$ on U.

2.4. Rigid τ-crystals over algebraically closed fields. Recall that k denotes a perfect field of characteristic p. We can better understand rigid τ-crystals on $\operatorname{Spec} k$ when we shift perspective as follows (we follow Kedlaya, [10, §14]):

Definition 2.30. A *difference field* is a pair (L, φ) consisting of a field L and an endomorphism φ of L. A difference field is called inversive if φ is an automorphism.

The difference field we have in mind is the pair $(L, \varphi) = (k((t)), \tau)$ with τ the map defined in Proposition 2.17. This difference field is inversive because k is perfect.

Definition 2.31. Let (L, φ) be a difference field. Then a pair (V, ϕ) of an L-vector space V together with a φ-linear group homomorphism $\phi : V \to V$ is called a *difference module* over (L, φ). Here by φ-linearity we mean that $\phi(cv) = \varphi(c)\phi(v)$ for all $c \in L$ and $v \in V$.

Example 2.32. By Example 2.25, a rigid τ-crystal on $\operatorname{Spec} k$ is the same as a difference module (M, F) over the difference field $(k((t)), \tau)$ for which F is an isomorphism.

Definition 2.33. We denote by $V(m/n)$ the difference module (M, F) corresponding to the standard Dieudonné-module $\mathcal{V}(m/n)$.

Definition 2.34. Let (L, φ) be a difference field. Then the *twisted polynomial ring* $L\{T\}$ is the non-commutative polynomial ring in the variable T over L subject to

$$Tc = \varphi(c)T \text{ for } c \in L\,.$$

For an $L\{T\}$-module V denote by $T\cdot : V \to V$ the map which is left-multiplication by T. Then the functor $V \mapsto (V, T\cdot)$ from $L\{T\}$-left modules to difference modules over (L, φ) is an equivalence of categories.

Example 2.35. For any $f \in L\{T\}$, the cokernel $L\{T\}/L\{T\}f$ of multiplication by f is a finite $L\{T\}$-module. When we look at the definition of $\mathcal{V}(m/n)$ in Example 2.26, we see that the corresponding difference module is precisely of this form. More precisely, the $K\{T\}$-module corresponding to the difference module $V(m/n)$ is $K\{T\}/K\{T\}(T^n - t^m)$.

Now assume that k is algebraically closed and consider the inversive difference field $(k((t)), \tau)$. In this situation, there is a version of Dieudonné's Theorem for finite difference modules which in the particular case of $k((t))$ was proved by Laumon [11, Theorem 2.4.5]. We present it here in a more general adaptation due to Kedlaya [10]:

Theorem 2.36 (Dieudonné–Manin Theorem for rigid τ-crystals)**.** *If k is algebraically closed, then every finite difference module of $(k((t)), \tau^a)$ splits into the direct sum of difference modules of the form $V(m_i/n_i)$ for some uniquely determined sequence of rational numbers $m_1/n_1 \leq \cdots \leq m_k/n_k$ written in lowest terms.*

Proof. This is Dieudonné's theorem applied to $(k((t)), \tau^a)$ (see [10, Theorem 14.6.3 and Corollary 14.6.4]. Note that this part of §14 of [10] does apply to our situation despite the characteristic zero assumption in other sections). The theorem applies because τ^a induces a power of the absolute Frobenius endomorphism on the residue field k of $k((t))$. The proof of Dieudonné's theorem relies on a decomposition of M into isoclinic submodules ([10, Theorem 14.4.13]), which in turn is proved by showing that $\mathrm{Ext}^1_{L\{T\}}(V_1, V_2)$ vanishes for $L\{T\}$-modules V_1 and V_2 of different slopes. Uniqueness follows from the fact that the $\mathcal{V}(m/n)$ are irreducible and non-isomorphic (for example because the slope can be recovered from the valuation of the determinant). $\square$

Corollary 2.37 (Dieudonné's theorem for rigid τ-crystals)**.** *When k is an algebraically closed field, every rigid τ^a-crystal $\mathcal{T}$ on k factors into a direct sum*

$$\mathcal{T} \cong \bigoplus_{i=1}^{k} \mathcal{V}(m_i/n_i)$$

for some uniquely determined sequence of rational numbers $m_1/n_1 \leq \cdots \leq m_k/n_k$ written in lowest terms.

This result is also stated in [6, Theorem 7.6] and [7, Theorem 1.2.9].

Definition 2.38. We call the ascending sequence $\lambda_1, \ldots, \lambda_r$ of the numbers m_i/n_i, each one occuring with multiplicity n_i, the *Newton slopes* of $\mathcal{T}$. The

Newton polygon associated to these numbers is defined to be the graph of the following piecewise linear real function on the interval $0 \leq i \leq r$: On integers, it is given by $i \mapsto \lambda_1 + \cdots + \lambda_i$ (where we set $0 \mapsto 0$) and the function is extended linearly between integers. In other words, λ_i is the slope of the Newton polygon on the interval $(i-1, i)$.

Example 2.39. Recall that a rigid τ-crystal on $\overline{k}$ is the same as a $K\{T\}$-module M, finite over $K = \overline{k}((t))$, for which multiplication by T is an isomorphism. In the special case that $\mathcal{T}$ is given by $M = K\{T\}/K\{T\}P$ for some monic polynomial $P \in K\{T\}$, the Newton polygon coincides with the Newton polygon of P ([10, Proposition 14.5.7], which as before applies to our situation despite the characteristic zero assumption in other sections). Note that this statement can be seen as a generalisation of Example 2.35.

2.5. Geometric points of $\boldsymbol{\tau}$-crystals. Let X be a k-scheme of finite type, let $\mathcal{T}$ be a τ-crystal on X and let $\phi : \mathrm{Spec}(\overline{k}) \to X$ be a geometric point of X. By Lemma 2.19 we get a morphism

$$\phi_{\mathrm{rig}} : \mathrm{Sp}\,\overline{k}((t)) \to X_{\mathrm{rig}},$$

along which we can pull back $\mathcal{T}$ by Remark 2.28. This way we get a τ-crystal $\phi^*\mathcal{T}$ on $\mathrm{Sp}\,\overline{k}((t))$, which by Theorem 2.37 is determined by its slopes up to isomorphism.

Definition 2.40. We call the Newton slopes of $\phi^*\mathcal{T}$ the *Newton slopes of $\mathcal{T}$ at ϕ.*

Example 2.41. Recall from Example 2.25 that a rigid τ-crystal over $k = \mathbb{F}_p$ corresponds to a finite $K = \mathbb{F}_p((t))$-vector space M and an isomorphism F of M. Let χ_F be the characteristic polynomial of F. Since $K\{T\} = K[T]$ is a principal ideal domain in this case, we see that $M = K[T]/(d_1^{e_1}) \oplus \ldots \oplus K[T]/(d_n^{e_n})$ for some monic polynomials $d_1, \ldots, d_n$ such that $d_1^{e_1} \cdots d_n^{e_n} = \chi_F$. After tensoring with $K'\{T\}$ where $K' = \overline{\mathbb{F}}_p((t))$, Example 2.39 implies that the Newton slopes at $\mathrm{Spec}\,\overline{k} \to \mathrm{Spec}\,k$ are the Newton slopes of χ_F, which are precisely the t-adic valuations of the eigenvalues of F in some extension of K, with multiplicities.

Remark 2.42. There is no good notion of Hodge slopes for rigid τ-crystals as the following example illustrates: Let $k = \mathbb{F}_p$ be like in Example 2.25, let $M = \mathbb{F}_p((t))^2$ and let F be represented with respect to the standard basis by the matrix $\left(\begin{smallmatrix} t & 1 \\ 0 & t \end{smallmatrix}\right)$. Then as discussed in 2.41, the Newton slopes are the valuations of the zeros of the characteristic polynomial. Here both slopes are equal to 1. In order to define the Hodge slope of (M, F), one would have to choose an $\mathbb{F}_p[\![t]\!]$-sublattice of M and define the Hodge slopes to be the valuations of the elementary divisors of any matrix representing F. However, even if lattices stable under F exist, this depends on the choice of lattice: While the lattice spanned by the standard basis has minimal Hodge slope 0 as we can read from the matrix, the basis $(1,0)$, $(0,t)$ of M represents F as $\left(\begin{smallmatrix} t & t \\ 0 & t \end{smallmatrix}\right)$, which has Hodge slopes $1, 1$.

2.6. Local freeness on the base. When working with τ-crystals, we would often like to assume without loss of generality that the underlying vector bundle $\mathcal{F}$ is trivial. For example, one might ask if there is always a covering $X = X_1 \cup \cdots \cup X_n$ such that the τ-crystal $\mathcal{T}|_{X_i}$ that we get from restricting the base from X to X_i is trivial.

Definition 2.43. We call a rigid τ-crystal $\mathcal{T} = (\mathcal{F}, F)$ on X *locally free on the base* if there is a covering $X = X_1 \cup \cdots \cup X_n$ such that $\mathcal{F}$ is trivial when restricted to $(X_i)_{\mathrm{rig}} \subseteq X_{\mathrm{rig}}$.

Remark 2.44. The analogous notion for F-crystals over a perfect base is always satisfied. This is immediate when working with vector bundles over the formal scheme $\operatorname{Spf} W(A)$, which as a topological space can be identified with $\operatorname{Spec} A$. But it is also easy to see algebraically: Given a locally free $W(A)$-module M with associated vector bundle $\mathcal{F}$, one can always find a cover of $\operatorname{Spec} A$ by affine subsets $\operatorname{Spec} B$ such that the restriction of $\mathcal{F}$ to $\operatorname{Spec}(W(B))$ is free: There are $f_1, \ldots, f_r \in W(A)$ generating the unit ideal such that M_{f_i} is a free $W(A)_{f_i}$-module. Then if $a_1, \ldots, a_r$ are the projections of the f_i to A, these will generate the unit ideal in A. We thus get a cover of $\operatorname{Spec} A$ by the $\operatorname{Spec} A_{a_i}$. The natural map $W(A)_{f_i} \to W(A_{a_i})$ (which is p-adic completion) shows that $\mathcal{F}$ is trivial when restricted to $\operatorname{Spec}(W(A_{a_i}))$.

$$\begin{array}{ccc} \operatorname{Spec}(W(A)) & \leftarrow & \operatorname{Spec}(W(A_a)) \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \longleftarrow & \operatorname{Spec} A_a\,. \end{array}$$

In the rigid setting, the topological situation is more subtle: The issue is that there are many admissible open sets $U \subset X_{\mathrm{rig}}$, few of which arise from open sets on X via rigidification:

$$\begin{array}{ccc} X_{\mathrm{rig}} & \xleftarrow{\;w\;} & U \\ \downarrow & & \vdots \\ X & \leftarrow\cdots\cdots & ? \end{array}$$

Algebraically, the problem in adapting the situation from the Witt vectors is that in $A((t))$ we are missing information at the ideal (t) of $A[\![t]\!]$, by which we mean the following: Given elements $f_1, \ldots, f_r \in A[\![t]\!]$ such that $(f_1, \ldots, f_r) = A((t))$, we can only conclude that

$$\sum r_k f_k = t^m \text{ for some } m \in \mathbb{N} \text{ and some } r_k \in A[\![t]\!]\,.$$

So the constant terms a_k of f_k do not necessarily generate the unit ideal in A. Consequently, $\cup_i D(a_i)$ might not be a cover of X and therefore it is not clear that the cover $X_{\mathrm{rig}} = \cup D(f_k)$ can be refined by a cover $\cup_i D(a_i)_{\mathrm{rig}} = \cup_i X_{\mathrm{rig}}|_{D(a_i)}$ of restrictions.

Remark 2.45. In some situations the above difficulties do not arise because all τ-crystals are actually free. For example, the analogue of the Quillen–Suslin theorem holds for Tate algebras [12]. This shows that any vector bundle on $(\mathbb{A}_k^n)_{\mathrm{rig}}$ is free. Also the existence of a Frobenius structure is a significant additional information: For example, for the punctured open unit disc, all vector bundles with Frobenius structure are trivial [8].

3. Formal τ-crystals

In the analogy between F-crystals and τ-crystals, the right replacement of the Witt vectors $W(A)$ is the ring $A[\![t]\!]$ of formal power series: Both are adic rings with residue ring A whose projection map has a multiplicative section. In this chapter, we will show that F-crystals have a natural analogue over $A[\![t]\!]$, which we call formal τ-crystals. For these much of the F-crystal theory carries over. Furthermore, formal τ-crystals are to rigid τ-crystals what F-crystals are to F-isocrystals: There is a natural rigidification functor which transforms formal τ-crystals into rigid τ-crystals.

Let X be a scheme of finite type over k. Let $\mathfrak{X} = X_{\mathrm{for}}$ be the formal scheme from Definition 2.1. Recall from Lemma 2.16 that the Frobenius on X lifts to a morphism $\tau : \mathfrak{X} \to \mathfrak{X}$.

Definition 3.1. A *formal τ-crystal* on X is a pair $(\mathfrak{F}, F)$ consisting of

(1) a vector bundle $\mathfrak{F}$ on $\mathfrak{X}$ (whereby we mean a locally free coherent $\mathcal{O}_{\mathfrak{X}}$-module),
(2) a morphism $F : \tau^*\mathfrak{F} \to \mathfrak{F}$ of vector bundles such that $\operatorname{coker} F$ is a t-torsion $\mathcal{O}_{\mathfrak{X}}$-module.

Definition 3.2. We define a *formal τ^a-crystal* to be a vector bundle $\mathfrak{F}$ on $\mathfrak{X}$ with a map $F : (\tau^a)^*\mathfrak{F} \to \mathfrak{F}$ of vector bundles such that $\operatorname{coker} F$ is a t-torsion $\mathcal{O}_{\mathfrak{X}}$-module.

Remark 3.3. As for rigid τ^a-crystals, everything we show for formal τ-crystals will go through analogously for formal τ^a-crystals. Iterates will prove useful in the proof of 3.26.

Lemma 3.4. *Let $a \in \mathbb{N}$. Let $\mathfrak{T} = (\mathfrak{F}, F)$ be a formal τ-crystal on X and let $F^a = F \circ \cdots \circ F$ be the a-th iteration, that is the composition of a times F with itself. Then we get a formal τ^a-crystal $\mathfrak{T}^a := (\mathfrak{F}, F^a)$ on $\mathfrak{X}$, which is called a-th* iteration *of $\mathfrak{T}$.*

Remark 3.5. Formal τ-crystals are a special case of Hartl's Dieudonné-$\mathbb{F}_p[\![t]\!]$-modules as defined in [6]. They are also very closely related to π-adic φ-sheaves as introduced by Taguchi and Wan [14], although formal τ-crystals also require $\operatorname{coker} F$ to be t-torsion.

Moreover, formal τ-crystals are related to Anderson A-motives: If X is a finite type k-scheme, then there is a natural functor from Anderson $k[t]$-motives to formal τ-crystals. Here by an Anderson $k[t]$-motive we mean a pair $(\mathcal{M}, F)$ of a vector bundle $\mathcal{M}$ on $X \times \operatorname{Spec} k[t]$ and a morphism $F : (\sigma \times \mathrm{id})^*\mathcal{M} \to \mathcal{M}$ such that $\operatorname{coker} F$ is supported on the subscheme $X \hookrightarrow X \times \operatorname{Spec} k[t]$ induced by the reduction $k[t] \to k$. The latter condition implies that $\operatorname{coker} F$ is t-torsion.

As in Definition 2.1, we can complete $X \times \operatorname{Spec} k[t]$ along X. This also gives a completion functor $\mathcal{F} \mapsto \widehat{\mathcal{F}}$ from coherent sheaves on $X \times \operatorname{Spec} k[t]$ to coherent sheaves on X_{for} which sends $\mathcal{M}$ to a vector bundle $\widehat{\mathcal{M}}$ and $(\sigma \times \mathrm{id})^*\mathcal{M}$ to $\tau^*\widehat{\mathcal{M}}$. By functoriality, F gives a map

$$\widehat{F} : \tau^*\widehat{\mathcal{M}} \to \widehat{\mathcal{M}}.$$

The cokernel of $\widehat{F}$ will be $\widehat{\operatorname{coker} F}$ since completion of coherent sheaves is an exact functor. In particular, $\operatorname{coker} \tau$ is a t-torsion $\mathcal{O}_{X_{\mathrm{for}}}$-module. This shows that $(\widehat{\mathcal{M}}, \widehat{F})$ is a formal τ-crystal.

Lemma 3.6. *In the case that $X = \operatorname{Spec} A$ is an affine k-scheme of finite type, we can identify the underlying topological space of $X_{\mathrm{for}} = \operatorname{Spf} A[\![t]\!]$ with* $\operatorname{Spec} A$. *Under this identification, there is a finitely generated $A[\![t]\!]$-module M such that for any $a \in A$, we have $\mathfrak{F}(D(a)) = M \otimes_{A[\![t]\!]} A_a[\![t]\!]$.*

Proof. This is Theorem A for topologically finite type formal schemes with the appropriate notion of associated sheaves (see [3, §8.1]). ☐

Remark 3.7. More precisely, in the situation of Lemma 3.6, vector bundles on X_{for} correspond to locally free $A[\![t]\!]$-modules: When $f \in A[\![t]\!]$ is such that M_f is free, and a is the constant term of f, then $M \otimes A_a[\![t]\!]$ is a free $A_a[\![t]\!]$-module. This shows that locally free modules give rise to vector bundles on X_{for}. Conversely, use that for any $(a_1, \ldots, a_k) = A$ the map $A[\![t]\!] \to \prod A_{a_i}[\![t]\!]$ is faithfully flat.

Remark 3.8. Consequently, in the case that $X = \operatorname{Spec} A$ is an affine k-scheme of finite type, a formal τ-crystal is a locally free $A[\![t]\!]$-module M together with a τ-linear morphism $F : M \to M$ which becomes an isomorphism after tensoring with $- \otimes_{\mathbb{F}_p[\![t]\!]} \mathbb{F}_p((t))$. This is analogous to the definition of F-crystals over a perfect ring (cf. [9]), with $- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ replaced by $- \otimes_{\mathbb{F}_p[\![t]\!]} \mathbb{F}_p((t))$.

Example 3.9. Recall the rigid τ-crystal $\mathcal{V}(m/n)$ on $\overline{k}((t))$ from Example 2.25. When we take the $\overline{k}[\![t]\!]$-span of the standard basis, then F maps the resulting lattice into itself as long as $m \geq 0$. So by Remark 3.8, any $m/n \in \mathbb{Q}_{\geq 0}$ gives a formal τ-crystal $\mathfrak{V}(m/n)$ on $\overline{k}[\![t]\!]$.

Remark 3.10. The pullback of a formal τ-crystals can be defined like for rigid τ-crystals in Lemma 2.27. In particular, we can study a formal

τ-crystal $(\mathfrak{F}, F)$ on X at a geometric point of X. Also we can define the restriction $(\mathfrak{F}, F)|_U$ to an open subset $U \subseteq X$. Lemma 3.6 tells us that on an affine base $X = \operatorname{Spec} A$, the restriction $(\mathfrak{F}, F)|_{D(a)}$ to an affine open subset $D(a) \subseteq \operatorname{Spec} A$ on modules is given by passing from $\mathfrak{F}(X) = M$ to $\mathfrak{F}(D(a)) = M \otimes A_a[\![t]\!]$.

One of the technical advantages of formal τ-crystals over rigid τ-crystals is that formal τ-crystals are always locally free on the base:

Proposition 3.11. *Let $(\mathfrak{F}, F)$ be a formal τ-crystal on* $\operatorname{Spec} A$*. Then we can find a cover* $\operatorname{Spec} A = \cup_i D(a_i)$ *such that $(\mathfrak{F}, F)|_{D(a_i)}$ is a trivial bundle.*

Proof. This is a consequence of Remarks 3.7 and 3.10. □

If we were only interested in affine X, we could equivalently define formal τ-crystals via locally free $A[\![t]\!]$-modules to emphasise the analogy with F-crystals. With this definition, we could copy the proof of local triviality from Remark 2.44.

3.1. The rigidification functor. The condition on F is of course chosen in such a way that given a formal τ-crystal $(\mathcal{F}, F)$ on X, the generic fibre from Lemma 2.11 gives a rigid τ crystal $(\mathcal{F}_{\mathrm{rig}}, F_{\mathrm{rig}})$.

Lemma 3.12. *Rigidification of the underlying spaces $\mathfrak{X} \mapsto \mathfrak{X}_{\mathrm{rig}}$ gives rise to a rigidification functor that assigns to a formal τ-crystal $\mathfrak{T}$ on X a rigid τ-crystal $\mathfrak{T}_{\mathrm{rig}}$ on X.*

Proof. By definition, for any finite type k-scheme X, the $-_{\mathrm{rig}}$-functor sends X_{for} to X_{rig}. We can define a similar functor on vector bundles, which on an affine formal scheme $\operatorname{Spf} A[\![t]\!]$ sends the vector bundle associated to the locally free $A[\![t]\!]$-module M to the locally free $A((t))$-module $M \otimes_{A[\![t]\!]} A((t)) = M \otimes_{\mathbb{F}_p[\![t]\!]} \mathbb{F}_p((t))$. The morphism $F : M \to M$ is sent to the map $F \otimes \mathrm{id}$ on $M \otimes_{\mathbb{F}_p[\![t]\!]} \mathbb{F}_p((t))$. This is an isomorphism since $- \otimes_{\mathbb{F}_p[\![t]\!]} \mathbb{F}_p((t))$ kills all t-torsion. We conclude that (M, F) corresponds to a rigid τ-crystal. □

Definition 3.13. A morphism of formal τ-crystals is called an *isogeny* if it becomes an isomorphism after rigidification.

3.2. Newton and Hodge slopes. By the Dieudonné–Manin classification of τ-crystals on $\overline{k}$ (Corollary 2.37), we can classify formal τ-crystals on $\overline{k}$ up to isogeny:

Corollary 3.14. *Every formal τ-crystal $\mathfrak{T}$ on $\overline{k}$ is isogeneous to a direct sum of formal crystals of the form $\mathfrak{V}(m_i/n_i)$ from Example 3.9 for some unique ascending sequence of non-negative rational numbers $m_1/n_1 \leq \cdots \leq m_k/n_k$ written in lowest term.*

Proof. By 2.37, the rigid τ-crystal $\mathfrak{T}_{\mathrm{rig}}$ splits into a direct sum of crystals of the form $\mathcal{V}(m_i/n_i)$ for some uniquely determined sequence of rational numbers $m_1/n_1 \leq \cdots \leq m_k/n_k$. Each of these have to be non-negative: If a/b is negative, then $\mathcal{V}(a/b)$ has no $\overline{k}[\![t]\!]$-sublattice that is fixed by F and therefore can't come from a formal τ-crystal.

It remains to find an isomorphism $\psi : \mathfrak{T}_{\mathrm{rig}} \to \oplus_{i=1}^{k} \mathcal{V}(m_i/n_i)$ that restricts to a morphism $\mathfrak{T} \to \oplus_{i=1}^{k} \mathfrak{V}(m_i/n_i)$. We begin by choosing any isomorphism ψ of the rigid τ-crystals. The vector bundle underlying $\mathfrak{T}$ corresponds to a free module over $\overline{k}[\![t]\!]$, and we choose a basis to identify it with $\overline{k}[\![t]\!]^r$. Using the canonical basis for the module $V(m_i/n_i)$, the morphism ψ is then a map $\overline{k}((t))^r \to \overline{k}((t))^r$ commuting with the respective τ-semilinear maps on both sides. It is represented by a matrix U with coefficients in $\overline{k}((t))$.

Since $\tau(t) = t$, the τ-semilinear morphisms are in fact $\mathbb{F}_p((t))$-linear. Thus if we multiply ψ by a power t^d such that $t^d U$ has coefficients in $\overline{k}[\![t]\!]$, the morphism $t^d\psi$ still commutes with the τ-semilinear maps but now restricts to a morphism $\overline{k}[\![t]\!]^r \to \overline{k}[\![t]\!]^r$. This induces the desired isogeny $\mathfrak{T} \to \oplus_{i=1}^{k} \mathfrak{V}(m_i/n_i)$. ☐

Definition 3.15. Given a geometric point $\phi : \operatorname{Spec} \overline{k} \to X$, we define the *Newton slopes of* $\mathfrak{T}$ *at* ϕ to be the unique sequence of numbers $m_1/n_1 \leq \cdots \leq m_k/n_k$ that we get from applying Proposition 3.14 to $\phi^*\mathfrak{T}$, each one counted with multiplicity n_i.

Lemma 3.16. *Let X be a finite type k-scheme and let $\mathfrak{T}$ be a formal τ-crystal on X. Let $\phi : \operatorname{Spec} \overline{k} \to X$ be a geometric point. Then when we rigidify $\phi^*\mathfrak{T}$ over $\overline{k}[\![t]\!]$, the result will be the same as when we first rigidify $\mathfrak{T}$ and then pull back along $\phi_{\mathrm{rig}} : \operatorname{Sp} \overline{k}((t)) \to X_{\mathrm{rig}}$:*

$$(\phi^*\mathfrak{T})_{\mathrm{rig}} = \phi^*(\mathfrak{T}_{\mathrm{rig}})\,.$$

In particular, the Newton slopes of $\mathfrak{T}$ coincide with those of $\mathfrak{T}_{\mathrm{rig}}$.

Proof. By choosing an affine open subset of X containing the image of ϕ, we can without loss of generality assume that $X = \operatorname{Spec} A$ is affine. Then we get a commutative diagram

$$\begin{array}{ccc} \overline{k}((t)) & \xleftarrow[\phi^\sharp_{\mathrm{rig}}]{\hspace{3em}} & A((t)) \\ \uparrow & & \uparrow \\ \overline{k}[\![t]\!] & \xleftarrow[\phi^\sharp_{\mathrm{for}}]{\hspace{3em}} & A[\![t]\!]\,. \end{array}$$

Since pull back of sheaves corresponds to push forward of modules, this proves the claim. ☐

One way that formal τ-crystals help to understand rigid τ-crystals is that they provide a setting to study the analogue of Hodge slopes of an F-crystal, as we shall now explain.

Let $\phi : \mathrm{Spec}(\overline{k}) \to X$ be a closed point of X and let $R = \overline{k}[\![t]\!]$. If $\mathfrak{T} = (\mathfrak{F}, F)$ is a formal τ-crystal on X, then the τ-crystal $\phi^*\mathfrak{T} = (\phi_{\mathrm{for}}^*\mathfrak{F}, \phi_{\mathrm{for}}^* F)$ corresponds to a locally free R-module M, which is necessarily free since R is a principal ideal domain. In particular, we can consider M as an R-sublattice of the K-vector space $M \otimes_R K$ corresponding to the underlying vector bundle of $\phi^*\mathfrak{T}_{\mathrm{rig}}$.

Definition 3.17. If A is any matrix over $\overline{k}[\![t]\!]$ representing $\phi_{\mathrm{for}}^* F$, we define the *Hodge slopes* $\sigma_1 \leq \cdots \leq \sigma_r$ of $\mathfrak{T}$ at ϕ to be the t-valuations of the elementary divisors of A, counted with multiplicity. We associate a *Hodge polygon* to these numbers like in Definition 2.38.

Remark 3.18. The theory of elementary divisors tells us that the Hodge slopes are independent of the choice of $\overline{k}[\![t]\!]$-basis. An alternative characterisation of Hodge slopes is that $\sigma_1 + \cdots + \sigma_m$ is the minimal valuation of all $m \times m$-minors of A (see [10, 14.5.1]). In particular, the smallest Hodge slope σ_1 is the minimal valuation of all entries of A.

We are now ready to adopt some results of Katz from §1 of [9] for formal τ-crystals. In what follows we fix a formal τ-crystal $\mathfrak{T} = (\mathfrak{F}, F)$ of rank r on $\mathrm{Spec}\,\overline{k}$. By Corollary 2.37, the rigid τ-crystal $\mathcal{T} = \mathfrak{T}_{\mathrm{rig}}$ factors into a direct sum of standard Dieudonné-modules $\mathcal{V}(m/n)$.

In the F-crystal setting, one can use this to "diagonalize" F, which can be adapted as follows. Let $N = r!$ and consider the finite extension $K' = \overline{k}((t^{1/N}))$ of K. This is a valued field with valuation ring $R' = \overline{k}[\![t^{1/N}]\!]$. We can base change $\mathfrak{T}$ to R' by pulling back along $\mathrm{Spf}\,R' \to \mathrm{Spf}\,R$. The resulting formal τ-crystal $\mathfrak{T}'$ corresponds to the R'-module $M' = M \otimes_R R'$ with morphism $F' = F \otimes \tau$. The slopes of $\mathfrak{T}$ and $\mathfrak{T}'$ are related as follows:

Lemma 3.19. *If the rigid τ-crystal $\mathcal{T}$ has Newton slopes $\lambda_1, \dots, \lambda_r$, then after base change along $\varphi : \overline{k}((t)) \to \overline{k}((t^{1/N}))$, the τ-crystal $\varphi^*\mathcal{T}$ has integral slopes $N\lambda_1, \dots, N\lambda_r$.*

Proof. It suffices to prove this for $\mathcal{T} = \mathcal{V}(m/n)$. By Example 2.35,

$$M = K\{T\}/K\{T\}P \quad \text{for } P = T^n - t^m\,.$$

After base change to K', we still have $M' = K'\{T\}/K'\{T\}P$, but the Newton slopes have changed: By Example 2.39, they are the Newton slopes of P with respect to K'. We have $v_{K'}|_K = N \cdot v_K$, so the Newton slopes are now $N \cdot m/n$ with multiplicity n. This is integral because we have set $N = r!$ and because n is smaller or equal to the rank r of $\mathcal{T}$. ☐

Remark 3.20. In other words, Lemma 3.19 says that the Dieudonné-module M decomposes over K' into a direct sum of one-dimensional standard Dieudonné-modules. In particular, we can find a K'-basis of M' of "eigenvectors" of F, for which $F'(v_i) = t^{\lambda_i} v_i$.

Proposition 3.21. *The Newton slopes of a formal τ-crystal on* $\operatorname{Spec} \overline{k}$ *can be equivalently characterised as the unique rational numbers $\lambda_1 \leq \cdots \leq \lambda_r \in \frac{1}{N}\mathbb{Z}$ for which there is an R'-basis of M' with respect to which F' can be represented by a matrix of the following form:*

$$\begin{pmatrix} t^{\lambda_1} & * & * \\ & \ddots & * \\ 0 & & t^{\lambda_r} \end{pmatrix}.$$

Proof. By Remark 3.20, we can find $v \in M' \otimes_{R'} K$ such that $F(v) = t^{\lambda_1} v$. After rescaling v by the minimal power of t such that $v \in M'$, use that R' is a discrete valuation ring to extend v to an R'-basis of M'. We then proceed inductively with the free R'-module $M'/R'v$. Since $(M'/R'v) \otimes_{R'} K' = (M' \otimes_{R'} K')/K'v$, the slopes of $M'/R'v$ will be $\lambda_2, \ldots, \lambda_r$. □

Corollary 3.22. *If $\mathfrak{T}$ is a formal τ-crystal on X and ϕ is a closed point with Newton slopes $\lambda_1 \leq \cdots \leq \lambda_r$ then the formal τ^a-crystal $\mathfrak{T}^a$ has Newton slopes $a\lambda_1 \leq \cdots \leq a\lambda_r$ at ϕ.*

Remark 3.23. In general it is not clear from the Hodge slopes of $\mathfrak{T}$ what the Hodge slopes of $\mathfrak{T}^a$ are. But our later results will imply that $1/a$ times the Hodge slopes of $\mathfrak{T}^a$ converges to the Newton slopes of $\mathfrak{T}$ (see also [10, Proposition 14.5.8]). This is one way that iterates of crystals are useful.

Proposition 3.24 ("Hodge above Newton", Mazur)**.** *The Hodge-polygon lies above the Newton polygon. Both have the same start and end point.*

Proof. The start point coincides by definition. That both polygons have the same endpoint follows from Lemma 3.21: For both polygons, the height of the end point will be the valuation of the determinant. Finally, it is clear from Remark 3.18 and Proposition 3.21 that $\sigma_1 \geq \lambda_1$. The general case follows from this one using symmetric powers, see [10, Corollary 14.5.4]. □

Proposition 3.25 (Katz)**.** *Let $\mathfrak{T} = (\mathfrak{F}, F)$ be a formal τ-crystal of rank r on X and let $\lambda \geq 0$ be a real number. Then for n large enough, the following are equivalent for $\phi : \operatorname{Spec} \overline{k} \to X$:*

(1) *all Newton slopes of $\mathfrak{T}$ at ϕ are $\geq \lambda$.*
(2) *all Hodge slopes of $\mathfrak{T}^{n+r-1}$ at ϕ are $\geq n\lambda$.*

Proof. In light of Proposition 3.21, Corollary 3.22 and Proposition 3.24, Katz' argument in [9] (split into §1.4.3 and the first paragraph of §2.3.1) goes through unchanged. The proof is not hard but a bit technical, so we choose not to reproduce it here. □

3.3. Grothendieck–Katz for formal *τ*-crystals. We finally come to the main theorem of this article, an analogue of §2.3 in [9] for rigid τ-crystals, which says something about how the local information of slopes behaves globally. We first prove the result for formal τ-crystals and then deduce that it holds for rigid τ-crystals that admit a formal model.

Theorem 3.26 (Grothendieck–Katz for formal τ-crystals)**.** *Let $\mathfrak{T} = (\mathfrak{F}, F)$ be a formal τ-crystal on X and let $\lambda \geq 0$. Then the set of closed points of X at which all Hodge slopes are $\geq \lambda$ is Zariski-closed in X. The same holds true for Newton slopes.*

Proof. With the preparations of the last section at hand, Katz' argument carries over with only minor modifications, as we shall now demonstrate.

By Proposition 3.25, the theorem follows for Newton slopes when we show it for Hodge slopes. Since the statement of the theorem is local on X, we may assume without loss of generality that $X = \operatorname{Spec} A$ is affine. Furthermore, by Proposition 3.11, we can cover $X = \cup_i U_i$ by affine open subsets such that on restrictions $\mathfrak{T}|_{U_i}$ the underlying vector bundle is trivial. So we can further assume that $\mathfrak{F}$ is a trivial vector bundle.

Let M be the free $A[\![t]\!]$-module corresponding to $\mathfrak{F}$. Choose any $A[\![t]\!]$-basis of M and let $T = (T_{i,j})_{1\leq,i,j,\leq r}$ be the matrix representing F with respect to this basis. If $\phi : A \to \overline{k}$ is any closed point of A, then ϕ^*F will be represented by the matrix $\phi_{\mathrm{for}}(T) = (\phi(T_{i,j})_{\mathrm{for}})_{1\leq i,j\leq r}$.

By Remark 3.18, the minimal Hodge slope is now the minimal valuation of all entries of $\phi(T)$. By Lemma 2.4, the lift $\phi_{\mathrm{for}} : A[\![t]\!] \to \overline{k}[\![t]\!]$ of ϕ to X_{for} can be described as follows:

$$T_{i,j} = \sum_{k=0}^{\infty} T_{i,j}^{(k)} t^k \mapsto \phi_{\mathrm{for}}(T_{i,j}) = \sum_{k=0}^{\infty} \phi(T_{i,j}^{(k)}) t^k .$$

In particular, by Remark 3.18, the minimal Hodge slope at ϕ will be $\geq \lambda$ if and only if

$$\phi(T_{i,j}^{(k)}) = 0 \text{ for all } 1 \leq i, j \leq n \text{ and for all } 0 \leq k < \lambda .$$

Equivalently, ϕ is in the Zariski-closed subset $V((T_{i,j}^{(k)})_{i,j,k})$ of $\operatorname{Spec} A$. □

Theorem 3.27. *Let $\mathfrak{T}$ be a formal τ-crystal on X and let $\mathcal{T} = \mathfrak{T}_{\mathrm{rig}}$ be the associated rigid τ-crystal. Let $\lambda \geq 0$. Then the set of closed points of X at which all Newton slopes of $\mathcal{T}$ are $\geq \lambda$ is Zariski-closed in X.*

Proof. Since pulling back along points commutes with $-_{\mathrm{rig}}$ by Lemma 3.16, this is an immediate consequence of the theorem of Grothendieck–Katz for formal τ-crystals. □

4. Open questions

In light of Theorem 3.27 we now wonder which rigid τ-crystals arise from a formal τ-crystal. It is clear that not every rigid τ-crystal is the rigidification of a formal τ-crystal: Formal τ-crystals always have non-negative

Newton slopes. If a rigid τ-crystals $\mathcal{T}$ has some closed point at which the Newton polygon has a negative slope, for example $\mathcal{T} = \mathcal{V}(m/n)$ for $m < 0$, it is clear that $\mathcal{T}$ cannot arise directly from rigidification of a formal τ-crystal. However, this can be resolved by rescaling the τ-crystal in the following sense:

Definition 4.1. A rigid τ-crystal $\mathcal{T} = (\mathcal{F}, F)$ has a *formal model* if there is a formal τ-crystal $\mathfrak{T}$ for which $\mathfrak{T}_{\mathrm{rig}}$ is isomorphic to $\mathcal{T}$ after possibly multiplying F by a power of t.

Since multiplying F by t^n means shifting all Newton slopes of $\mathcal{T}$ by n, the Grothendieck–Katz Theorem 3.27 more generally holds for rigid τ-crystals that admit a formal model.

Remark 4.2. There is a weaker assumption one can put in place under which Grothendieck–Katz for rigid τ-crystals still follows from the analogue for formal τ-crystals, namely that the τ-crystal locally on the base admits a formal model. Since formal models for free τ-crystals always exist, this is equivalent to the rigid τ-crystal being locally free on the base. Of course one could ask whether both conditions are actually equivalent:

Question 4.3. Does a τ-crystal always have a formal model if it is locally free on the base?

The above is essentially a question of when local formal models glue together. More generally, we ask the following question:

Question 4.4. Which rigid τ-crystals admit a formal model?

References

[1] G. Böckle, "Cohomological theory of crystals over function fields and applications", in *Arithmetic Geometry over Global Function Fields*, Advanced Courses in Mathematics - CRM Barcelona, Springer, 2014, p. 1-118.

[2] G. Böckle & U. Hartl, "Uniformizable families of t-motives", *Trans. Am. Math. Soc.* **359** (2007), no. 8, p. 3933-3972.

[3] S. Bosch, *Lectures on formal and rigid geometry*, Lecture Notes in Mathematics, vol. 2105, Springer, 2014, viii+254 pages.

[4] J. Fresnel & M. van der Put, *Rigid analytic geometry and its applications*, Progress in Mathematics, vol. 218, Birkhäuser, 2004, xii+296 pages.

[5] A. Grothendieck & J. A. Dieudonné, "Éléments de géométrie algébrique. I. Le langage des schémas", *Publ. Math., Inst. Hautes Étud. Sci.* **4** (1960), p. 1-228.

[6] U. Hartl, "Uniformizing the Stacks of Abelian Sheaves", in *Number fields and function fields – two parallel worlds*, Progress in Mathematics, vol. 239, Birkhäuser, 2005, p. 167-222.

[7] ———, "Period spaces for Hodge structures in equal characteristic", *Ann. Math.* **173** (2011), no. 3, p. 1241-1358.

[8] U. Hartl & R. Pink, "Vector bundles with a Frobenius structure on the punctured unit disc", *Compos. Math.* **140** (2004), no. 3, p. 389-716.

[9] N. M. Katz, "Slope filtration of F-crystals", in *Journées de Géométrie Algébrique de Rennes*, Astérisque, vol. 63, Société Mathématique de France, 1979, p. 113-164.

[10] K. S. KEDLAYA, *p-adic differential equations*, Cambridge Studies in Advanced Mathematics, vol. 125, Cambridge University Press, 2010, xvii+380 pages.
[11] G. LAUMON, *Cohomology of Drinfeld modular varieties. Part 1: Geometry, counting of points and local harmonic analysis*, Cambridge Studies in Advanced Mathematics, vol. 41, Cambridge University Press, 1996, xiii+344 pages.
[12] W. LÜTKEBOHMERT, "Vektorraumbündel über nichtarchimedischen holomorphen Räumen", *Math. Z.* **152** (1977), p. 127-143.
[13] A. PÁL, "Images of Galois representations", preprint, 2016.
[14] Y. TAGUCHI & D. WAN, "L-functions of φ-sheaves and Drinfeld modules", *J. Am. Math. Soc.* **9** (1996), no. 3, p. 755-781.

Ben HEUER
The London School of Geometry and Number Theory
Department of Mathematics
University College London
Gower Street, London WC1E 6BT, UK
E-mail: `ben.heuer@kcl.ac.uk`