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Bruno ANGLÈS, Tuan NGO DAC et Floric TAVARES RIBEIRO

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Special functions and twisted *L*-series

par Bruno ANGLÈS, Tuan NGO DAC et Floric TAVARES RIBEIRO

To the memory of David Goss

Résumé. Nous donnons une généralisation de la fonction spéciale d'Anderson–Thakur et nous prouvons un théorème de rationalité pour les séries L à plusieurs variables associées aux fonctions chtoucas.

Abstract. We present a generalization of the Anderson–Thakur special function, and we prove a rationality result for several variable twisted L-series associated to shtuka functions.

1. Introduction

Let $X = \mathbb{P}^1/\mathbb{F}_q$ be the projective line over a finite field $\mathbb{F}_q$ having q elements and let K be its function field. Let ∞ be a closed point of X of degree $d_\infty = 1$. Then $K = \mathbb{F}_q(\theta)$ for some $\theta \in K$ such that θ has a pole of order one at ∞. We set $A = \mathbb{F}_q[\theta]$. Following Anderson ([1], see also [23]), we consider:

$$Y = K \otimes_{\mathbb{F}_q} X\,.$$

Let $\mathbb{K} = \mathrm{Frac}(K \otimes_{\mathbb{F}_q} K)$ be the function field of Y. We identify K with $K \otimes 1 \subset \mathbb{K}$. If we set $t = 1 \otimes \theta$, then $\mathbb{K} = K(t)$. Let $\tau : \mathbb{K} \to \mathbb{K}$ be the homomorphism of $\mathbb{F}_q(t)$-algebras such that:

$$\forall\, x \in K, \quad \tau(x) = x^q\,.$$

Let $\bar{\infty} \in Y(K)$ be the pole of t, and let $\xi \in Y(K)$ be the point corresponding to the kernel of the homomorphism of K-algebras $K \otimes_{\mathbb{F}_q} K \to K$ which sends t to θ. Then the divisor of $f := t - \theta$ is equal to $(\xi) - (\bar{\infty})$. The function $t - \theta$ is a shtuka function, and in particular:

$$\forall\, a \in A, \quad a(t) = \sum_{k=0}^{\deg_\theta a} C_{a,i} f \ldots f^{(i-1)}, \text{ with } C_{a,i} \in A\,.$$

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The map $C : A \to A\{\tau\}, a \mapsto C_a := \sum_{k=0}^{\deg_\theta a} C_{a,i}\tau^i$ is a homomorphism of $\mathbb{F}_q$-algebras ([23, §0.3.5 and 2.1]) called the Carlitz module. Note that:

$$C_\theta = \theta + \tau\,.$$

There exists a unique element $\exp_C \in K\{\{\tau\}\}$ such that $\exp_C \equiv 1 \pmod{\tau}$ and:

$$\forall\ a \in A, \quad \exp_C a = C_a \exp_C\,.$$

Let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of $K_\infty := \mathbb{F}_q((\frac{1}{\theta}))$. Then $\exp_C$ defines an entire function on $\mathbb{C}_\infty$, and:

$$\operatorname{Ker}\exp_C = \widetilde{\pi}A\,,$$

for some $\widetilde{\pi} \in \mathbb{C}_\infty^\times$ (well-defined modulo $\mathbb{F}_q^\times$) called the Carlitz period. We consider $\mathbb{T}$ the Tate algebra in the variable t with coefficients in $\mathbb{C}_\infty$, i.e. $\mathbb{T} := \mathbb{C}_\infty \widehat{\otimes}_{\mathbb{F}_q} A$. Let $\tau : \mathbb{T} \to \mathbb{T}$ be the continuous homomorphism of $\mathbb{F}_q[t]$-algebras such that $\forall\ x \in \mathbb{C}_\infty, \tau(x) = x^q$. Anderson and Thakur ([3]) showed that:

$$\{x \in \mathbb{T}, \tau(x) = fx\} = \omega\mathbb{F}_q[t]\,,$$

where $\omega \in \mathbb{T}^\times$ is such that:

$$f\omega|_\xi = \widetilde{\pi}\,.$$

The function ω is called the Anderson–Thakur special function attached to the Carlitz module C. This function is intimately connected to Thakur–Gauss sums ([7]).

In 2012, Pellarin ([19]) initiated the study of a twist of the Carlitz module by the shtuka function f. Let's consider the following homomorphism of $\mathbb{F}_q$-algebras $\varphi : A \to A[t]\{\tau\}, \theta \mapsto \theta + f\tau$. Then, one observes that C and φ are isomorphic over $\mathbb{T}$, i.e. we have the following equality in $\mathbb{T}\{\tau\}$:

$$\forall\ a \in A, \quad C_a\omega = \omega\varphi_a\,.$$

To such an object, one can associate the special value of some twisted L-function (see [9]):

$$\mathcal{L} = \sum_{a \in A, a \text{ monic}} \frac{a(t)}{a} \in \mathbb{T}^\times.$$

Then, using the Anderson log-algebraicity Theorem for the Carlitz module ([2], see also [8, 18]), Pellarin proved the following remarkable rationality result:

$$\frac{\mathcal{L}\omega}{\widetilde{\pi}} = \frac{1}{f} \in \mathbb{K}\,.$$

This result has been extended to the case of "several variables" ([9, 12]) using methods developed by Taelman ([10, 13, 14, 15, 20, 21]). This kind of rationality results leads to new advances in the arithmetic of function fields (see [4, 9, 11]).

The aim of this paper is to extend the previous results to the general context, i.e. for any smooth projective geometrically irreducible curve $X/\mathbb{F}_q$ of genus g and any closed point ∞ of degree d_∞ of X. In particular, we obtain a rationality result similar to that of Pellarin (Theorem 5.3). Our result involves twisted L-series (see [5]) and a generalization of the Anderson–Thakur special function. The involved techniques are based on ideas developed in [4] where an analogue of Stark Conjectures is proved for sign-normalized rank one Drinfeld modules.

We should mention that Green and Papanikolas ([17]) have recently studied the particular case $g=1$ and $d_\infty=1$ and, in this case, they have obtained explicit formulas similar to that obtained by Pellarin (in the case $g=0$ and $d_\infty=1$).

2. Notation and background

2.1. Notation. Let $X/\mathbb{F}_q$ be a smooth projective geometrically irreducible curve of genus g, and ∞ be a closed point of degree d_∞ of X. Denote by K the function field of X, and by A the ring of elements of K which are regular outside ∞. The completion K_∞ of K at the place ∞ has residue field $\mathbb{F}_\infty$. We fix an algebraic closure $\overline{K}_\infty$ of K_∞ and denote by $\mathbb{C}_\infty$ the completion of $\overline{K}_\infty$.

We will fix a sign function $\operatorname{sgn}: K_\infty^\times \to \mathbb{F}_\infty^\times$ which is a group homomorphism such that $\operatorname{sgn}|_{\mathbb{F}_\infty^\times} = \operatorname{Id}|_{\mathbb{F}_\infty^\times}$. We fix $\pi \in K \cap \operatorname{Ker}(\operatorname{sgn})$ and such that $K_\infty = \mathbb{F}_\infty((\pi))$. Let $v_\infty : \mathbb{C}_\infty \to \mathbb{Q} \cup \{+\infty\}$ be the valuation on $\mathbb{C}_\infty$ normalized such that $v_\infty(\pi)=1$. Observe that:

$$\forall\, x \in K^\times, \quad \deg(xA) = -d_\infty v_\infty(x)\,.$$

Let $\overline{K}$ be the algebraic closure of K in $\mathbb{C}_\infty$.

Let $\mathcal{I}(A)$ be the group of non-zero fractional ideals of A. We have a natural surjective group homomorphism $\deg : \mathcal{I}(A) \to \mathbb{Z}$, such that for $I \in \mathcal{I}(A), I \subset A$, we have:

$$\deg I = \dim_{\mathbb{F}_q} A/I\,.$$

Let $\mathcal{P}(A) = \{xA, x \in K^\times\}$, then $\operatorname{Pic}(A) = \frac{\mathcal{I}(A)}{\mathcal{P}(A)}$ is a finite abelian group.

Let I_K be the group of idèles of K, and H/K be the finite abelian extension of K, $H \subset \mathbb{C}_\infty$, corresponding via class field theory to the following subgroup of I_K:

$$K^\times \operatorname{Ker}\operatorname{sgn} \prod_{v \neq \infty} O_v^\times\,,$$

where for a place $v \neq \infty$ of K, $O_v^\times$ denotes the group of units of the v-adic completion of K. Then H/K is a finite extension of degree $|\operatorname{Pic}(A)|\frac{q^{d_\infty}-1}{q-1}$, unramified outside ∞, and the decomposition group of ∞ in H/K is equal to its inertia group and is isomorphic to $\frac{\mathbb{F}_\infty^\times}{\mathbb{F}_q^\times}$. Set $G = \mathrm{Gal}(H/K)$. If we define $\mathcal{P}_+(A) = \{xA, x \in K^\times, \mathrm{sgn}(x) = 1\}$, then the Artin map

$$(\,\cdot\,, H/K) : \mathcal{I}(A) \longrightarrow G\,.$$

induces a group isomorphism:

$$\frac{\mathcal{I}(A)}{\mathcal{P}_+(A)} \simeq G\,.$$

For $I \in \mathcal{I}(A)$, we set:

$$\sigma_I = (I, H/K) \in G\,.$$

Let H_A be the Hilbert class field of A, i.e. H_A/K corresponds to the following subgroup of the idèles of K:

$$K^\times K_\infty^\times \prod_{v \neq \infty} O_v^\times\,.$$

Then H/H_A is totally ramified at the places of H_A above ∞. Furthermore:

$$\mathrm{Gal}(H/H_A) \simeq \frac{\mathbb{F}_\infty^\times}{\mathbb{F}_q^\times}\,.$$

We denote by B the integral closure of A in H and B' the integral closure of A in H_A. Observe that $\mathbb{F}_\infty \subset B$.

2.2. Sign-normalized rank one Drinfeld modules. We define the map $\tau : \mathbb{C}_\infty \to \mathbb{C}_\infty, x \mapsto x^q$. By definition, a sign-normalized rank one Drinfeld module is a homomorphism of $\mathbb{F}_q$-algebras $\phi : A \to \mathbb{C}_\infty\{\tau\}$ such that there exists $n(\phi) \in \{0, \ldots, d_\infty - 1\}$ with the following property:

$$\forall\ a \in A, \quad \phi_a = a + \cdots + \mathrm{sgn}(a)^{q^{n(\phi)}} \tau^{\deg a}\,.$$

Let $n \in \{0, \ldots, d_\infty - 1\}$. We denote by Drin_n the set of sign-normalized rank one Drinfeld modules ϕ with $n(\phi) = n$, and by $\mathrm{Drin} = \cup_{n=0}^{d_\infty - 1} \mathrm{Drin}_n$ the set of sign-normalized rank one Drinfeld modules. By [16, Cor. 7.2.17], Drin is a finite set and we have:

$$|\,\mathrm{Drin}\,| = |\,\mathrm{Pic}(A)| \frac{q^{d_\infty} - 1}{q - 1}\,.$$

Let $\phi \in \mathrm{Drin}$ be a sign-normalized rank one Drinfeld module, we say that ϕ is standard if $\mathrm{Ker}\exp_\phi$ is a free A-module, where $\exp_\phi : \mathbb{C}_\infty \to \mathbb{C}_\infty$ is the exponential map attached to ϕ (see for example [16, §4.6]).

Lemma 2.1. *Let $n \in \{0, \ldots, d_\infty - 1\}$. We have:*

$$|\,\mathrm{Drin}_n\,| = \frac{1}{d_\infty} |\,\mathrm{Pic}(A)| \frac{q^{d_\infty} - 1}{q - 1}\,.$$

Let ϕ *in* Drin_n *and let* $[\phi]$ *denote the set of the* ϕ' *in* Drin_n *which are isomorphic to* ϕ*. Then:*

$$\forall\ \phi \in \mathrm{Drin}_n\,, \quad |[\phi]| = \frac{q^{d_\infty}-1}{q-1}\,.$$

In particular, if $[\mathrm{Drin}_n] = \{[\phi], \phi \in \mathrm{Drin}_n\}$*, we have:*

$$|[\mathrm{Drin}_n]| = \frac{1}{d_\infty}|\operatorname{Pic}(A)|\,.$$

Proof. Let $\psi : A \to H\{\tau\}$ be a sign-normalized rank one Drinfeld module (see [16, Ch. 7]). Let $n(\psi) \in \mathbb{Z}$ be such that:

$$\forall\ a \in A, \quad \psi_a = a + \cdots + \operatorname{sgn}(a)^{q^{n(\psi)}}\tau^{\deg a}\,.$$

Then the set of sign-normalized rank one Drinfeld modules is exactly $\mathrm{Drin} = \{\psi^\sigma, \sigma \in G\}$. Let $\sigma \in G$ and write $\sigma = (I, H/K)$ for some $I \in \mathcal{I}(A)$. We have:

$$\forall\ a \in A, \quad \psi^\sigma_a = a + \cdots + \operatorname{sgn}(a)^{q^{n(\psi)+\deg(I)}}\tau^{\deg a}\,.$$

Note that $\deg : \mathcal{I}(A) \to \mathbb{Z}$ induces a surjective homomorphism of finite abelian groups:

$$\deg : \frac{\mathcal{I}(A)}{\mathcal{P}_+(A)} \to \frac{\mathbb{Z}}{d_\infty\mathbb{Z}}\,.$$

Since there are exactly $|\operatorname{Pic}(A)|\frac{q^{d_\infty}-1}{q-1}$ sign-normalized rank one Drinfeld modules and d_∞ divides $|\operatorname{Pic}(A)|$, we get the first assertion.

Let $\phi \in \mathrm{Drin}_n$ and let $\phi' \in [\phi]$. Then there exists $\alpha \in \mathbb{C}_\infty^\times$ such that:

$$\forall\ a \in A, \quad \alpha\phi_a = \phi'_a\alpha\,.$$

Thus, $\alpha \in \mathbb{F}_\infty^\times$. Since $\operatorname{End}_{\mathbb{C}_\infty}(\phi) = \{\phi_a, a \in A\}$, we obtain:

$$\operatorname{End}_{\mathbb{C}_\infty}(\phi) \cap \mathbb{F}_\infty = \mathbb{F}_q\,.$$

Hence,

$$|[\phi]| = \frac{q^{d_\infty}-1}{q-1}\,. \qquad \square$$

Lemma 2.2. *There are exactly* $\frac{q^{d_\infty}-1}{q-1}$ *standard elements in* Drin*. Furthermore, if* ϕ *is such a Drinfeld module, then* $[\phi]$ *is the set of standard elements in* Drin*.*

Proof. By [16, Cor. 4.9.5 and Thm. 7.4.8], there exists $\phi \in \mathrm{Drin}$ such that ϕ is standard. In particular, $\mathrm{Drin} = \{\phi^\sigma, \sigma \in G\}$. Again, by [16, Cor. 4.9.5 and Thm. 7.4.8], the Drinfeld module ϕ^σ is standard if and only if $\sigma|_{H_A} = \mathrm{Id}_{H_A}$. The Lemma follows. $\square$

2.3. Shtuka functions. The results of this section are originally due to D. Thakur (see [23]). Let $\bar{X} = \mathbb{C}_\infty \otimes_{\mathbb{F}_q} X$, $\bar{A} = \mathbb{C}_\infty \otimes_{\mathbb{F}_q} A$, and let F be the function field of $\bar{X}$, i.e. $F = \mathrm{Frac}(\bar{A})$. We will identify $\mathbb{C}_\infty$ with its image $\mathbb{C}_\infty \otimes 1$ in F. There are d_∞ points in $\bar{X}(\mathbb{C}_\infty)$ above ∞, and we denote the set of such points by S_∞. Observe that $\bar{A}$ is the set of elements of $F/\mathbb{C}_\infty$ which are "regular outside ∞". We denote by $\tau : F \to F$ the homomorphism of K-algebras such that:

$$\tau|_{\bar{A}} = \tau \otimes 1\,.$$

For $m \in \mathbb{Z}$, we also set:

$$\forall\, x \in F\,, \quad x^{(m)} = \tau^m(x)\,.$$

Let P be a point of $\bar{X}(\mathbb{C}_\infty)$. We denote by $P^{(i)}$ the point of $\bar{X}(\overline{K})$ obtained by applying τ^i to the coordinates of P. If $D = \sum_{j=1}^n n_{P_j} P_j \in \mathrm{Div}(\bar{X})$, with $P_j \in \bar{X}(\mathbb{C}_\infty)$, and $n_{P_j} \in \mathbb{Z}$, we set:

$$D^{(i)} = \sum_{j=1}^n n_{P_j} P_j^{(i)}\,.$$

If $D = (x)$, $x \in F^\times$, then:

$$D^{(i)} = (x^{(i)})\,.$$

We consider $\xi \in \bar{X}(\mathbb{C}_\infty)$ the point corresponding to the kernel of the map:

$$\bar{A} \to \mathbb{C}_\infty\,, \quad \sum_i x_i \otimes a_i \mapsto \sum x_i a_i\,.$$

Let $\rho : K \to F, x \mapsto 1 \otimes x$ and set $t = \rho(\pi^{-1})$.

Let $\bar{\infty} \in S_\infty$. We identify the $\bar{\infty}$-adic completion of F to

$$\mathbb{C}_\infty\left(\left(\frac{1}{t}\right)\right).$$

Let $\mathrm{sgn}_{\bar{\infty}} : \mathbb{C}_\infty((\frac{1}{t}))^\times \to \mathbb{C}_\infty^\times$ be the group homomorphism such that $\mathrm{Ker}(\mathrm{sgn}_{\bar{\infty}}) = t^{\mathbb{Z}} \times (1 + \frac{1}{t}\mathbb{C}_\infty[\![\frac{1}{t}]\!])$, and $\mathrm{sgn}_{\bar{\infty}}|_{\mathbb{C}_\infty^\times} = \mathrm{Id}|_{\mathbb{C}_\infty^\times}$.

Let $\phi \in \mathrm{Drin}$. For $a \in A$, we write $\phi_a = \sum_{i=0}^{\deg a} \phi_{a,i}\tau^i$, $\phi_{a,i} \in H$. By [16, Ch. 6 and Prop. 7.11.4], there exist $\bar{\infty} \in S_\infty$ and $f_\phi \in F^\times$ such that:

$$\forall\, a \in A\,, \quad \rho(a) = \sum_{i=0}^{\deg a} \phi_{a,i} f_\phi \dots f_\phi^{(i-1)}\,,$$

and the divisor of f_ϕ is of the form:

$$(f_\phi) = V^{(1)} - V + (\xi) - (\bar{\infty})\,,$$

where V is some effective divisor of degree g. Let $(\infty) = \sum_{\bar{\infty}' \in S_\infty} (\bar{\infty}')$. Set

$$W(\mathbb{C}_\infty) = \cup_{m \geq 0} L(V + m(\infty))\,,$$

and

$$L(V+m(\infty))=\{x\in F^\times,(x)+V+m(\infty)\geq 0\}\cup\{0\}.$$

We have:

$$W(\mathbb{C}_\infty)=\oplus_{i\geq 0}\mathbb{C}_\infty f_\phi\dots f_\phi^{(i-1)}.$$

The function f_ϕ is called the shtuka function attached to ϕ, and we say that ϕ is the sign-normalized rank one Drinfeld module associated to f_ϕ. We define the set of shtuka functions to be:

$$\mathfrak{F}=\{f_\phi,\phi\in\mathrm{Drin}\}.$$

Then, the map $\mathrm{Drin}\to\mathfrak{F},\phi\to f_\phi$ is a bijection called the Drinfeld correspondence.

Remark 2.3. There is a misprint in [16, p. 229]. In fact, as we will see in the proof of Lemma 3.3, when $d_\infty>1$, we do not have:

$$\mathrm{sgn}_{\bar{\infty}^{(-1)}}(f_\phi)^{\frac{q^{d_\infty}-1}{q-1}}=1$$

as stated in [16].

3. Special functions attached to shtuka functions

3.1. Basic properties of a shtuka function. Let $\mathbb{H}=\mathrm{Frac}(H\otimes_{\mathbb{F}_q}A)$, and $\mathbb{K}=\mathrm{Frac}(K\otimes_{\mathbb{F}_q}A)$. Recall that $G=\mathrm{Gal}(H/K)$ and we will identify G with the Galois group of $\mathbb{H}/\mathbb{K}$. Let $f\in\mathfrak{F}$, and let $\phi\in\mathrm{Drin}_{n(\phi)}$ be the sign-normalized rank one Drinfeld module attached to f for some $n(\phi)\in\{0,\dots,d_\infty-1\}$. Then $\phi:A\to B\{\tau\}$ is a homomorphism of $\mathbb{F}_q$-algebras such that:

$$\forall\ a\in A,\quad \phi_a=\sum_{i=0}^{\deg a}\phi_{a,i}\tau^i,$$

where $\phi_{a,0}=a$, $\phi_{a,\deg a}=\mathrm{sgn}(a)^{q^{n(\phi)}}$, and $\rho(a)=\sum_{i=0}^{\deg a}\phi_{a,i}f\dots f^{(i-1)}$. Recall that there exists an effective $\mathbb{H}$-divisor V ([16, Ch. 6]) of degree g such that the divisor of f is:

$$(f)=V^{(1)}-V+(\xi)-(\bar{\infty}),$$

for some $\bar{\infty}\in S_\infty$. By [16, Lem. 7.1.3], $\xi,\bar{\infty}^{(-1)}$ do not belong to the support of V. Let $v_{\bar{\infty}}$ be the normalized valuation on $\mathbb{H}$ attached to $\bar{\infty}$ ($v_{\bar{\infty}}(t)=-1$). Note that $v_{\bar{\infty}}(f)\leq -1$ and, when $d_\infty>1$, $\bar{\infty}$ can a priori belong to the support of V. We identify the $\bar{\infty}$-adic completion of $\mathbb{H}$ with $H((\frac{1}{t}))$. Therefore we deduce that:

$$f=\frac{\alpha(f)}{t^k}+\sum_{i\geq k+1}f_i\frac{1}{t^i},\quad k\leq -1$$

where $\alpha(f)\in H^\times$, and $f_i\in H$, for all $i\geq k+1$.

Let $\exp_\phi$ be the unique element in $H\{\{\tau\}\}$ such that $\exp_\phi \equiv 1 \pmod{\tau}$ and:

$$\forall\ a \in A, \quad \exp_\phi a = \phi_a \exp_\phi .$$

Write $\exp_\phi = \sum_{i\geq 0} e_i(\phi)\tau^i$, then by [16, Cor. 7.4.9], we obtain:

$$H = K(e_i(\phi), i \geq 0).$$

Observe that $\exp_\phi$ induces an entire function on $\mathbb{C}_\infty$, and there exist $\alpha \in \mathbb{C}_\infty^\times$ and $I \in \mathcal{I}(A)$ such that:

$$\forall\ z \in \mathbb{C}_\infty, \quad \exp_\phi(z) = \sum_{i\geq 0} e_i(\phi) z^{q^i} = z \prod_{a\in I\setminus\{0\}} \left(1 - \frac{z}{\alpha a}\right).$$

Furthermore, we have (see for example [23, Prop. 0.3.6]):

$$\forall\ i \geq 0, \quad e_i(\phi) = \frac{1}{f \dots f^{(i-1)}|_{\xi^{(i)}}}.$$

Thakur proved that if $e_n(\phi) = 0$, then $n \in \{2, \dots, g-1\}$ ([23, proof of Thm. 3.2]), and if K has a place of degree one then $\forall\ n \geq 0, e_n(\phi) \neq 0$.

Let $W(B) = \oplus_{i\geq 0} Bf \dots f^{(i-1)}$. Then $W(B)$ is a finitely generated $B \otimes_{\mathbb{F}_q} A = B[\rho(A)]$-module of rank one (see for example [6, Lem. 4.4]). Furthermore,

$$\forall\ x \in W(B), \quad fx^{(1)} \in W(B).$$

Let $I \in \mathcal{I}(A)$. Let $\phi_I \in H\{\tau\}$ such that the coefficient of its term of highest degree in τ is one , and such that:

$$\sum_{a\in I} H\{\tau\}\phi_a = H\{\tau\}\phi_I .$$

Then, we get:

$$\deg_\tau \phi_I = \deg I,$$
$$\operatorname{Ker} \phi_I|_{\mathbb{C}_\infty} = \cap_{a\in I} \operatorname{Ker} \phi_a|_{\mathbb{C}_\infty},$$
$$\phi_I \in B\{\tau\}.$$

We denote by $\psi_\phi(I) \in B \setminus \{0\}$ the constant term of ϕ_I. We set:

$$u_I = \sum_{j=0}^{\deg I} \phi_{I,j} f \dots f^{(j-1)} \in W(B),$$

where $\phi_I = \sum_{j=0}^{\deg I} \phi_{I,j}\tau^j$.

Lemma 3.1. *Let I, J be two non-zero ideals of A. We have:*

$$u_I|_\xi = \psi_\phi(I),$$
$$\sigma_I(f)u_I = fu_I^{(1)},$$
$$u_{IJ} = \sigma_I(u_J)u_I .$$

Proof. In [6, Lem. 4.6], we only gave a sketch of the proof of the above results. We give here a detailed proof for the convenience of the reader.

Observe that:

$$\forall\ i \geq 1\,, \quad (f \ldots f^{(i-1)}) = V^{(i)} - V + \sum_{k=0}^{i-1} \left(\xi^{(k)}\right) - \sum_{k=0}^{i-1} \left(\bar{\infty}^{(k)}\right).$$

Since ξ does not belong to the support of V, we deduce that:

$$u_I|_\xi = \psi_\phi(I)\,.$$

Note that we have a natural isomorphism of B-modules:

$$\gamma : \begin{cases} W(B) & \xrightarrow{\sim} \quad B\{\tau\} \\ \forall\ i \geq 0\,,\ f \ldots f^{(i-1)} & \longmapsto \quad \tau^i\,. \end{cases}$$

For all $x \in W(B)$ and for all $a \in A$, we have:

$$\gamma(fx^{(1)}) = \tau\gamma(x)\,,$$
$$\gamma(\rho(a)x) = \gamma(x)\phi_a\,.$$

In particular γ is an isomorphism of $B[\rho(A)]$-modules, and since $W(B)$ is a finitely generated $B[\rho(A)]$-module of rank one, this is also the case of $B\{\tau\}$. Write $f = \frac{\sum_i \rho(a_i)b_i}{\sum_k \rho(c_k)d_k}$, for some $a_i, c_k \in A$, $b_i, d_k \in B$, we have the following equality in $B\{\tau\}$:

$$\sum_i b_i\phi_{a_i} = \sum_k d_k\tau\phi_{c_k}\,.$$

For $\sigma \in G$, we set:

$$W_\sigma(B) = \oplus_{i\geq 0} B\sigma(f) \ldots \sigma(f)^{(i-1)}\,.$$

We have again an isomorphism of $B[\rho(A)]$-modules:

$$\gamma_\sigma : W_\sigma(B) \simeq B\{\tau\}\,.$$

Again,

$$\forall\ x \in W_\sigma(B)\,, \forall\ a \in A\,, \quad \gamma_\sigma(\rho(a)x) = \gamma_\sigma(x)\phi_a^\sigma\,.$$

Let I be a non-zero ideal of A, and let $\sigma = \sigma_I \in G$. We start from the relation:

$$\sum_i b_i^\sigma \phi_{a_i}^\sigma = \sum_k d_k^\sigma \tau\phi_{c_k}^\sigma\,.$$

We multiply on the right by ϕ_I, to obtain (see [16, Thm. 7.4.8]):

$$\sum_i b_i^\sigma \phi_I\phi_{a_i} = \sum_k d_k^\sigma \tau\phi_I\phi_{c_k}\,.$$

Since $\gamma(fu_I^{(1)}) = \tau\phi_I$, we get:

$$\left(\sum_i \rho(a_i)b_i^\sigma\right) \cdot \gamma(u_I) = \left(\sum_k d_k^\sigma \rho(c_k)\right) \cdot \gamma(fu_I^{(1)}) .$$

In other words, we have proved:

$$\sigma(f)u_I = fu_I^{(1)} .$$

Now, let J be a non-zero ideal of A. We have:

$$\gamma(u_{IJ}) = \phi_{IJ} = \phi_J^\sigma \phi_I .$$

Since $\forall\ i \geq 0, \sigma(f \dots f^{(i-1)})u_I = f \dots f^{(i-1)} u_I^{(i)}$, we get:

$$\gamma(u_J^\sigma u_I) = \phi_J^\sigma \phi_I .$$

It implies:

$$u_{IJ} = \sigma(u_J)u_I .$$

□

Corollary 3.2. *We have:*

$$\mathfrak{F} = \{\sigma(f), \sigma \in G\} .$$

Furthermore, for $\sigma \in G$, ϕ^σ is the Drinfeld module associated to the shtuka function $\sigma(f)$.

Proof. Let $\sigma \in G$ and let $g \in \mathfrak{F}$ be the shtuka function associated to ϕ^σ. By the proof of Lemma 3.1, if $a'_i, c'_k \in A$, $b'_i, d'_k \in B$ are such that $\sum_i b'_i \phi^\sigma_{a'_i} = \sum_k d'_k \tau \phi^\sigma_{c'_k}$, then:

$$g = \frac{\sum_i \rho(a'_i)b'_i}{\sum_k \rho(c'_k)d'_k} .$$

Again, by the proof of Lemma 3.1, we get:

$$g = \sigma(f) .$$

□

Lemma 3.3. *Let $\iota_{\bar\infty} : \mathbb{H} \to H((\frac{1}{t}))$ be a homomorphism of $\mathbb{K}$-algebras corresponding to $\bar\infty$. Write $\iota_{\bar\infty}(f) = \frac{\alpha(f)}{t^k} + \sum_{i\geq k+1} f_i \frac{1}{t^i} \in H((\frac{1}{t})), \alpha(f) \in H^\times, f_i \in H, i \geq 0, k \leq -1$. Then:*

$$H = K(\mathbb{F}_\infty, \alpha(f), f_i, i \geq k+1) .$$

Furthermore:

$$H_A = K\left(\mathbb{F}_\infty, \frac{f_i}{\alpha(f)}, i \geq k+1\right).$$

In particular, there exists $u(f) \in B^\times$ such that:

- $H = H_A(u(f))$,
- $\alpha(f) \equiv \iota_{\bar\infty}(u(f)) \pmod{H_A^\times}$,
- $\mathbb{K}(\frac{f}{u(f)}) = \mathrm{Frac}(H_A \otimes_{\mathbb{F}_q} A)$.

Proof. By Corollary 3.2, since $|G| = |\mathfrak{F}|$, we have:

$$\mathbb{H} = \mathbb{K}(f)\,.$$

Recall that $H((\frac{1}{t}))$ is isomorphic to the completion of $\mathbb{H}$ at $\bar{\infty}$. Since ∞ splits totally in $K(\mathbb{F}_\infty)$ in d_∞ places, we deduce that the natural map $\iota_{\bar{\infty}} : \mathbb{H} \hookrightarrow H((\frac{1}{t}))$ is $\mathrm{Gal}(H/K(\mathbb{F}_\infty))$-equivariant. Thus:

$$H = K(\mathbb{F}_\infty, \alpha(f), f_i, i \geq k+1)\,.$$

If $I = aA, a \in A \setminus \{0\}$, then $u_I = \frac{\rho(a)}{sgn(a)^{q^{n(\phi)}}}$, so that we have by Lemma 3.1:

$$\sigma_I(f) = \mathrm{sgn}(a)^{q^{n(\phi)} - q^{n(\phi)+1}} f\,.$$

In particular:

$$\mathrm{sgn}_{\bar{\infty}(-1)}(\iota_{\bar{\infty}(-1)}(f)) \notin \mathbb{F}_\infty^\times\,.$$

We have $\alpha(f)^{\frac{q^{d_\infty}-1}{q-1}} \in H_A$, and $\frac{f}{\alpha'(f)} \in \mathrm{Frac}(H_A \otimes_{\mathbb{F}_q} A)$, where $\alpha'(f) \in H^\times$ is such that $\iota_{\bar{\infty}}(\alpha'(f)) = \alpha(f)$ (observe that $\iota_{\bar{\infty}}|_H \in G$). Since $\mathbb{H} = \mathbb{K}(f)$, we get the second assertion.

Since H/H_A is totally ramified at each place of H_A above ∞, $\frac{B^\times}{(B)'^\times}$ is a finite abelian group, where we recall that B' is the integral closure of A in H_A. Now recall that H/H_A is a cyclic extension of degree $\frac{q^{d_\infty}-1}{q-1}$, and $\mathbb{F}_\infty \subset H_A$. Let $\langle \sigma \rangle = \mathrm{Gal}(H_A((B)^\times)/H_A)$. Then we have an injective homomorphism:

$$\frac{B^\times}{(B')^\times} \hookrightarrow \mathbb{F}_\infty^\times, \quad x \mapsto \frac{x}{\sigma(x)}\,.$$

The image of this homomorphism is a cyclic group of order dividing $\frac{q^{d_\infty}-1}{q-1}$. By the proof [16, Thm. 7.6.4], there exists $\zeta \in \mathbb{C}_\infty^\times, \zeta^{q-1} \in H$, such that:

$$\forall\ a \in A \setminus \{0\},\ \zeta\phi_a\zeta^{-1} \in B'\{\tau\} \text{ and its highest coefficient is in } (B')^\times.$$

Thus $\zeta^{q-1} \in B^\times$ and $H = H_A(\zeta^{q-1})$. In particular, there exists a group isomorphism:

$$\frac{B^\times}{(B')^\times} \simeq \frac{\mathbb{F}_\infty^\times}{\mathbb{F}_q^\times}\,.$$

This implies by Kummer Theory that:

$$\alpha(f) \equiv u'(f) \pmod{H_A^\times},$$

for some $u'(f) \in B^\times$ that generates the cyclic group $\frac{B^\times}{(B')^\times}$. Now define $u(f)$ to be the element in $B^\times$ such that $\iota_{\bar{\infty}}(u(f)) = u'(f)$. $\square$

3.2. Special functions. We fix $\sqrt[q^{d_\infty}-1]{-\pi} \in \mathbb{C}_\infty$ a root of the polynomial $X^{q^{d_\infty}-1} + \pi = 0$. We consider the period lattice of ϕ:

$$\Lambda(\phi) = \{x \in \mathbb{C}_\infty, \exp_\phi(x) = 0\}.$$

Then $\Lambda(\phi)$ is a finitely generated A-module of rank one and we have an exact sequence of A-modules induced by $\exp_\phi$:

$$0 \to \Lambda(\phi) \to \mathbb{C}_\infty \to \phi(\mathbb{C}_\infty) \to 0,$$

where $\phi(\mathbb{C}_\infty)$ is the $\mathbb{F}_q$-vector space $\mathbb{C}_\infty$ viewed as an A-module via ϕ.

Lemma 3.4. *We have:*

$$\Lambda(\phi) \subset \sqrt[q^{d_\infty}-1]{-\pi}^{\,-q^{n(\phi)}} K_\infty,$$

and for all $I \in \mathcal{I}(A)$:

$$\Lambda(\phi^{\sigma_I}) = \psi_\phi(I) I^{-1} \Lambda(\phi).$$

Proof. Observe that $\Lambda(\phi)K_\infty$ is a K_∞-vector space of dimension one. Let J be a non-zero ideal of A, and let $\lambda_J \neq 0$ be a generator of the A-module of J-torsion points of ϕ. By the proof of [16, Prop. 7.5.16], we have:

$$\lambda_J \in \Lambda(\phi)K_\infty.$$

By class field theory (see [16, §7.5]), we have:

$$E := H(\lambda_J) \subset K_\infty\Big(\sqrt[q^{d_\infty}-1]{-\pi}\Big).$$

Furthermore, by [16, Rem. 7.5.17],

$$\lambda_J^{q^{d_\infty}-1} \in K_\infty^\times.$$

By local class field theory, for $x \in K_\infty^\times$, we have:

$$\Big(x, K_\infty\Big(\sqrt[q^{d_\infty}-1]{-\pi}\Big)/K_\infty\Big)\Big(\sqrt[q^{d_\infty}-1]{-\pi}\Big) = \frac{\sqrt[q^{d_\infty}-1]{-\pi}}{\operatorname{sgn}(x)}.$$

By [16, Cor. 7.5.7], for all $a \in K^\times, a \equiv 1 \pmod J$, we get:

$$(aA, E/K)(\lambda_J) = \operatorname{sgn}(a)^{-q^{n(\phi)}} \lambda_J.$$

Thus, for all $a \in K^\times, a \equiv 1 \pmod J$):

$$\Big(a, K_\infty\Big(\sqrt[q^{d_\infty}-1]{-\pi}\Big)/K_\infty\Big)(\lambda_J) = \operatorname{sgn}(a)^{q^{n(\phi)}} \lambda_J.$$

Therefore, by the approximation Theorem, we get:

$$\forall\, x \in K_\infty^\times, \quad \Big(x, K_\infty\Big(\sqrt[q^{d_\infty}-1]{-\pi}\Big)/K_\infty\Big)(\lambda_J) = \operatorname{sgn}(x)^{q^{n(\phi)}} \lambda_J.$$

It implies:

$$\lambda_J \in \sqrt[q^{d_\infty}-1]{-\pi}^{\,-q^{n(\phi)}} K_\infty.$$

Hence,

$$\Lambda(\phi) \subset \sqrt[q^{d_\infty}-1]{-\pi}^{\,-q^{n(\phi)}} K_\infty .$$

The second assertion comes from the fact that we have the following equality in $H\{\{\tau\}\}$:

$$\phi_I \exp_\phi = \exp_{\phi^{\sigma_I}} \psi_\phi(I) . \qquad \square$$

Set:

$$L = \rho(K)(\mathbb{F}_\infty)\Big(\Big(\sqrt[q^{d_\infty}-1]{-\pi}\Big)\Big) .$$

Then, by the above Lemma, $H \subset \mathbb{F}_\infty((\sqrt[q^{d_\infty}-1]{-\pi})) \subset L$. Let $v_\infty : L \to \mathbb{Q} \cup \{+\infty\}$ be the valuation on L which is trivial on $\rho(K)(\mathbb{F}_\infty)$ and such that $v_\infty(\sqrt[q^{d_\infty}-1]{-\pi}) = \frac{1}{q^{d_\infty}-1}$. Let $\tau : L \to L$ be the continuous homomorphism of $\rho(K)$-algebras such that:

$$\forall\, x \in \mathbb{F}_\infty\Big(\Big(\sqrt[q^{d_\infty}-1]{-\pi}\Big)\Big), \quad \tau(x) = x^q .$$

Observe that:

$$\forall\, x \in L, \quad v_\infty(\tau(x)) = q v_\infty(x) .$$

Lemma 3.5. *We have:*

$$\mathrm{Ker} \exp_\phi |_L = \Lambda(\phi)\rho(K) ,$$

where $\Lambda(\phi)\rho(K)$ is the $\rho(K)$-vector space generated by $\Lambda(\phi)$.

Proof. The proof is standard in non-archimedean functional analysis, we give a sketch of the proof for the convenience of the reader. We have:

$$\Lambda(\phi)\rho(K) \subset \mathrm{Ker} \exp_\phi |_L .$$

Let:

$$\mathfrak{M} = \sqrt[q^{d_\infty}-1]{-\pi}\,\rho(K)(\mathbb{F}_\infty)[\![\sqrt[q^{d_\infty}-1]{-\pi}]\!] .$$

Let $\log_\phi \in H\{\{\tau\}\}$ such that $\log_\phi \exp_\phi = \exp_\phi \log_\phi = 1$. If we write: $\log_\phi = \sum_{i\geq 0} l_i(\phi)\tau^i$, then there exists $C \in \mathbb{R}$ such that, for all $i \geq 0$, $v_\infty(l_i(\phi)) \geq Cq^i$. It implies that there exists an integer $N \geq 0$ such that $\exp_\phi$ is an isometry on $\mathfrak{M}^N$.

Now, select $\theta \in A \setminus \mathbb{F}_q$. Then:

$$\mathrm{Ker} \exp_\phi |_{\mathbb{F}_\infty[\rho(\theta)]((\sqrt[q^{d_\infty}-1]{-\pi}))} = \Lambda(\phi)\mathbb{F}_q[\rho(\theta)] .$$

Since $\rho(A)$ is finitely generated and free as an $\mathbb{F}_q[\rho(\theta)]$-module, it implies:

$$\mathrm{Ker} \exp_\phi |_{\rho(A)[\mathbb{F}_\infty]((\sqrt[q^{d_\infty}-1]{-\pi}))} = \Lambda(\phi)\rho(A) .$$

Let V be the $\rho(K)$-vector space generated by $\rho(A)[\mathbb{F}_\infty]((\sqrt[q^{d_\infty}-1]{-\pi}))$. Then:

$$\mathrm{Ker} \exp_\phi |_V = \Lambda(\phi)\rho(K) .$$

Let $x \in \operatorname{Ker} \exp_\phi |_L$, then there exists $y \in V$ such that:

$$x - y \in \mathfrak{M}^N .$$

Thus,

$$\exp_\phi(y - x) = \exp_\phi(y) \in \mathfrak{M}^N \cap V = \exp_\phi(\mathfrak{M}^N \cap V) .$$

Therefore, $y = z + v$, for some $z \in \mathfrak{M}^N \cap V$, and some $v \in \Lambda(\phi)\rho(K)$. It implies that $x - v \in \mathfrak{M}^N$, and hence:

$$x = v \in \Lambda(\phi)\rho(K) . \qquad \square$$

Lemma 3.6. *We consider the following $\rho(K)$-vector space:*

$$V = \bigcap_{a \in A \setminus \mathbb{F}_q} \exp_\phi \left(\frac{1}{a - \rho(a)} \Lambda(\phi)\rho(K) \right).$$

Then, we have:

$$\dim_{\rho(K)} V = 1 .$$

Proof. For any $a \in A$, we set:

$$V_a = \{x \in L, \phi_a(x) = \rho(a)x\} .$$

Then, if $a \notin \mathbb{F}_q$, by Lemma 3.5, we have:

$$V_a = \exp_\phi \left(\frac{1}{a - \rho(a)} \Lambda(\phi)\rho(K) \right),$$

and:

$$\dim_{\rho(K)} V_a = \deg a = [K : \mathbb{F}_q(a)] .$$

Select $\theta \in A \setminus \mathbb{F}_q$ such that $K/\mathbb{F}_q(\theta)$ is a finite separable extension. Let $b \in A \setminus \mathbb{F}_q$ and let $P_b(X) \in \mathbb{F}_q[\theta][X]$ be the minimal polynomial of b over $\mathbb{F}_q(\theta)$. Since V_θ is an A-module via ϕ and ϕ_b induces a $\rho(K)$-linear endomorphism of V_θ, it follows that:

$$\rho(P_b)(\phi_b) = 0 .$$

This implies that the minimal polynomial of ϕ_b viewed as an $\mathbb{F}_q(\rho(\theta))$-linear endomorphism of V_θ is $\rho(P_b(X))$. Observe that V_θ is the $\rho(K)$-vector space generated by:

$$\exp_\phi \left(\frac{1}{\theta - \rho(\theta)} \Lambda(\phi)\mathbb{F}_q(\rho(\theta)) \right),$$

and:

$$\dim_{\mathbb{F}_q(\rho(\theta))} \exp_\phi \left(\frac{1}{\theta - \rho(\theta)} \Lambda(\phi)\mathbb{F}_q(\rho(\theta)) \right) = \deg \theta .$$

Therefore, $\rho(P_b(X))$ is the minimal polynomial of ϕ_b viewed as a $\rho(K)$-linear endomorphism of V_θ.

Select $\theta' \in A \setminus \mathbb{F}_q$ such that $K = \mathbb{F}_q(\theta, \theta')$. Then the characteristic polynomial of $\phi_{\theta'}$ on the $\rho(K)$-vector space V_θ is $\rho(P_{\theta'}(X))$. Since $P_{\theta'}(X)$ has simple roots, if $V' = V_\theta \cap V_{\theta'}$, we get:

$$\dim_{\rho(K)} V' = 1\,.$$

Now, let $b \in A$, there exist $x, y \in A[\theta, \theta']$, such that $b = \frac{x}{y}$. Let $\lambda_b \in \rho(K)$ such $\phi_b|V'$ is the multiplication by λ_b, then for any $v \in V' \setminus 0$, we have:

$$\rho(y)\lambda_b v = \phi_{yb} v = \rho(x) v\,.$$

It follows that:

$$\lambda_b = \rho(b)\,. \qquad \square$$

Let $\mathrm{sgn} : \rho(K)(\mathbb{F}_\infty)((\pi))^\times \to \rho(K)(\mathbb{F}_\infty)^\times$ be the group homomorphism such that $\mathrm{Ker}\,\mathrm{sgn} = \pi^{\mathbb{Z}} \times (1 + \pi\rho(K)(\mathbb{F}_\infty)[\![\pi]\!])$, and $\mathrm{sgn}\,|_{\rho(K)(\mathbb{F}_\infty)^\times} = \mathrm{Id}\,|_{\rho(K)(\mathbb{F}_\infty)^\times}$. Let $\pi_* = (\sqrt[q^{d_\infty}-1]{-\pi})^{(q-1)q^{n(\phi)}}$.

Lemma 3.7. *We have:*

$$f\pi_* \in \rho(K)(\mathbb{F}_\infty)((\pi))\,,$$

$$v_\infty(f) \equiv -\frac{(q-1)q^{n(\phi)}}{q^{d_\infty}-1} \pmod{(q-1)\mathbb{Z}}\,,$$

and:

$$N_{\rho(K)(\mathbb{F}_\infty)/\rho(K)}(\mathrm{sgn}(f\pi_*)) = 1\,.$$

Proof. Recall that:

$$V = \bigcap_{a \in A \setminus \mathbb{F}_q} \exp_\phi\left(\frac{1}{a - \rho(a)}\Lambda(\phi)\rho(K)\right).$$

By Lemma 3.4, we have:

$$V \subset \left(\sqrt[q^{d_\infty}-1]{-\pi}\right)^{-q^{n(\phi)}} \rho(K)(\mathbb{F}_\infty)((\pi)).$$

Thus, by Lemma 3.6, there exists $U \in (\sqrt[q^{d_\infty}-1]{-\pi})^{-q^{n(\phi)}}\rho(K)(\mathbb{F}_\infty)((\pi)) \setminus \{0\}$, such that:

$$\forall\ a \in A, \quad \phi_a(U) = \rho(a)U\,.$$

Write $f = \frac{\sum_i \rho(a_i)b_i}{\sum_k \rho(a'_k)b'_k}$, $a_i, a'_k \in A$, $b_i, b'_k \in B$. Then, by the proof of Lemma 3.1, we have:

$$\sum_i b_i\phi_{a_i} = \sum_k b'_k\tau\phi_{a'k}\,.$$

Thus,

$$\left(\sum_i \rho(a_i)b_i\right)U = \left(\sum_k \rho(a'_k)b'_k\right)\tau(U)\,.$$

Therefore:

$$\tau(U) = fU\,.$$

In particular,

$$\{x \in L, \tau(x) = fx\} = \rho(K)U\,.$$

We also get:

$$f \in \pi_*^{-1}\rho(K)(\mathbb{F}_\infty)((\pi))\,.$$

Let $F = f\pi_* \in \rho(K)(\mathbb{F}_\infty)((\pi))$. Set

$$R = U\left(\sqrt[q^{d_\infty}-1]{-\pi}\right)^{q^{n(\phi)}} \in \rho(K)(\mathbb{F}_\infty)((\pi))\,.$$

We have:

$$\tau(R) = FR.$$

Let $i_0 = v_\infty(F) \in \mathbb{Z}$, and write:

$$F = \sum_{i \geq i_0} F_i(-\pi)^i, F_i \in \rho(K)(\mathbb{F}_\infty)\,.$$

Let $\lambda = F_{i_0}$. Set:

$$\alpha = \sqrt[q-1]{-\pi}^{\,i_0}\left(\prod_{i\geq 0}\frac{F^{(i)}}{\lambda^{(i)}(-\pi)^{q^i i_0}}\right)^{-1} \in L^\times,$$

where $\sqrt[q-1]{-\pi} = \left(\sqrt[q^{d_\infty}-1]{-\pi}\right)^{\frac{q^{d_\infty}-1}{q-1}}$. Then clearly:

$$\tau(\alpha) = \frac{F}{\lambda}\alpha\,.$$

Thus:

$$\tau\left(\frac{R}{\alpha}\right) = \lambda\frac{R}{\alpha}\,.$$

This implies:

$$R = \mu\alpha, \quad \mu \in \rho(K)(\mathbb{F}_\infty)^\times\,.$$

In particular, $i_0 \equiv 0 \pmod{q-1}$, i.e. $v_\infty(f) \equiv -\frac{(q-1)q^{n(\phi)}}{q^{d_\infty}-1} \pmod{q-1}$. Also:

$$\operatorname{sgn}(R) = \mu\operatorname{sgn}(\alpha)\,.$$

Since $\operatorname{sgn}(\alpha) = (-1)^{\frac{i_0}{q-1}}$, we get:

$$\frac{\tau(\mu)}{\mu} = \lambda\,. \qquad \square$$

We set:

$$\mathbb{T} := \rho(A)[\mathbb{F}_\infty]\left(\left(\sqrt[q^{d_\infty}-1]{-\pi}\right)\right) \subset L\,.$$

Then $\mathbb{T}$ is complete with respect to the valuation v_∞, and:

$$\{x \in \mathbb{T}, \tau(x) = x\} = \rho(A)\,.$$

Furthermore, we have (see the proof of Lemma 3.5):

$$\operatorname{Ker}\exp_\phi|_{\mathbb{T}} = \Lambda(\phi)\rho(A)\,.$$

Let $\mathrm{ev} : \rho(A)[\mathbb{F}_\infty] \to \overline{\mathbb{F}}_q \subset \mathbb{C}_\infty$ be a homomorphism of $\mathbb{F}_\infty$-algebras. Such a homomorphism induces a continuous homomorphism $\mathbb{F}_\infty((\sqrt[q^{d_\infty}-1]{-\pi}))$-algebras:

$$\mathrm{ev} : \mathbb{T} \to \mathbb{C}_\infty\,.$$

We denote by $\mathcal{E}$ the set of such continuous homomorphisms from $\mathbb{T}$ to $\mathbb{C}_\infty$.

Proposition 3.8. *We have:*

$$f \in \mathbb{T}^\times,$$

$$\operatorname{sgn}(f\pi_*) \in \rho(A)[\mathbb{F}_\infty]^\times.$$

Furthermore there exists $U \in \mathbb{T}\setminus\{0\}$ such that:

$$\{x \in L, \tau(x) = fx\} = U\rho(K)\,.$$

If $d_\infty = 1$, then $\operatorname{sgn}(f\pi_*) = 1$, and we can take:

$$U = \sqrt[q^{d_\infty}-1]{-\pi}^{\,-1}\ \sqrt[q-1]{-\pi}^{\,i_0}\left(\prod_{i\geq 0}\frac{(f\pi_*)^{(i)}}{(-\pi)^{q^i i_0}}\right)^{-1} \in \mathbb{T}^\times,$$

where $i_0 := v_\infty(f\pi_*)$.

Proof. Recall that $f \in \mathbb{H} \subset L$. Le P be a point in $\bar{X}(\overline{\mathbb{F}}_q)$ above a maximal ideal of $\rho(A)$. Then P is above a maximal ideal of $\rho(A)[\mathbb{F}_\infty]$ which can be viewed as the kernel of some homomorphism of $\mathbb{F}_\infty$-algebras $\mathrm{ev} : \rho(A)[\mathbb{F}_\infty] \to \overline{\mathbb{F}}_q$. Since the field of constants of H is $\mathbb{F}_\infty$, we deduce that ev can be uniquely extended to a homomorphism of H-algebras:

$$\mathrm{ev} : \rho(A)[H] \to \mathbb{C}_\infty\,.$$

Furthermore, the kernel of the above homomorphism corresponds to $P \cap \mathbb{H}$ (recall that $\mathbb{H} = \operatorname{Frac}(\rho(A)[H])$). Then ev extends to a continuous homomorphism of $\mathbb{F}_\infty((\sqrt[q^{d_\infty}-1]{-\pi}))$-algebras:

$$\mathrm{ev} : \mathbb{T} \to \mathbb{C}_\infty\,.$$

We deduce that, by [23, Lem. 1.1], for any $\mathrm{ev} \in \mathcal{E}$, $\mathrm{ev}(f)$ is well-defined. Thus $f \in \mathbb{T}$. Therefore, by Lemma 3.7, we have:

$$f \in \pi_*^{\mathbb{Z}} \times (\operatorname{sgn}(f\pi_*) + \pi\rho(A)[\mathbb{F}_\infty][\![\pi]\!])\,,$$

where $\operatorname{sgn}(f\pi_*) \in \rho(A)[\mathbb{F}_\infty]$ is such that:

$$N_{\rho(K)(\mathbb{F}_\infty)/\rho(K)}(\operatorname{sgn}(f\pi_*)) = 1\,.$$

Thus:

$$\operatorname{sgn}(f\pi_*) \in \rho(A)[\mathbb{F}_\infty]^\times,$$

and there exists $\mu \in \rho(A)[\mathbb{F}_\infty] \setminus \{0\}$ such that:

$$\operatorname{sgn}(f\pi_*) = \frac{\tau(\mu)}{\mu}.$$

In particular, $f \in \mathbb{T}^\times$. Furthermore, there exists a non-zero ideal I of A such that:

$$\mu\rho(A)[\mathbb{F}_\infty] = \rho(I)\rho(A)[\mathbb{F}_\infty].$$

Now, we use the proof of Lemma 3.7. We put $i_0 = v_\infty(f\pi_*)$ (observe that $i_0 \equiv 0 \pmod{q-1}$) and set:

$$U = \mu\alpha \sqrt[q^{d_\infty}-1]{-\pi}^{-q^{n(\phi)}},$$

where :

$$\alpha = \sqrt[q-1]{-\pi}^{i_0} \left(\prod_{i \geq 0} \frac{(f\pi_*)^{(i)}}{\operatorname{sgn}(f\pi_*)^{(i)}(-\pi)^{q^i i_0}} \right)^{-1} \in \mathbb{T}^\times.$$

Then:

$$\tau(U) = fU,$$
$$U \in \mathbb{T}.$$

Note that U is well-defined modulo $\rho(K)^\times$ and if $d_\infty = 1$, then $U \in \mathbb{T}^\times$. $\square$

Definition 3.9. A non-zero element in $\{x \in L, \tau(x) = fx\}$ will be called a *special function* attached to the shtuka function f.

Remark 3.10. Let $M = \{x \in \mathbb{T}, \tau(x) = fx\}$. Then, by the above Proposition, there exists $U \in \mathbb{T} \setminus \{0\}$ such that:

$$U\rho(A) \subset M \subset U\rho(K).$$

Furthermore (see the proof of Lemma 3.7):

$$M = \bigcap_{a \in A \setminus \mathbb{F}_q} \exp_\phi \left(\frac{1}{a - \rho(a)} \Lambda(\phi)\rho(A) \right).$$

Thus M is a finitely generated $\rho(A)$-module of rank one. When $d_\infty = 1$, the above Proposition tells us that M is a free $\rho(A)$-module. In general, we have:

$$M = U'\rho(\mathcal{B}),$$

where $\mathcal{B} \in \mathcal{I}(A), U' \in L^\times$, and $M = U''\rho(\mathcal{B}')$ if and only if $U' = xU''$ where $x \in \rho(K)^\times$ is such that $x\mathcal{B} = \mathcal{B}'$.

Let I be a non-zero ideal of A, and let $\sigma = \sigma_I \in G$. Recall that, by Lemma 3.1, we have:

$$\sigma(f) = f\frac{\tau(u_I)}{u_I}.$$

Now observe that $u_I \in \mathbb{T}$, $\frac{\tau(u_I)}{u_I} \in \mathbb{T}^\times$, but in general we don't have $u_I \in \mathbb{T}^\times$. By Lemma 3.1, we have:

$$\frac{u_I}{\rho(x_I)} \in \mathbb{T}^\times,$$

where $I^n = x_I A$, n being the order of I in $\mathrm{Pic}(A)$. Thus:

$$M_\sigma := \{x \in \mathbb{T}, \tau(x) = \sigma(f)x\} = \frac{\rho(x_I)}{u_I} M\,.$$

We leave open the following question: is M a free $\rho(A)$-module? We will show in section 4 that the answer is positive if $g = 0$.

3.3. The period $\widetilde{\pi}$. By Lemma 2.2, and Lemma 3.4, let f be the unique shtuka function in $\mathfrak{F}$ such that, if ϕ is the Drinfeld module associated to f, we have:

$$\mathrm{Ker}\exp_\phi|_L = \widetilde{\pi} A[\rho(A)]\,,$$

where $\widetilde{\pi} \in \sqrt[q^{d_\infty}-1]{-\pi}^{\,-q^{n(\phi)}} K_\infty$, $\mathrm{sgn}(\widetilde{\pi}\,(\sqrt[q^{d_\infty}-1]{-\pi})^{q^{n(\phi)}}) = 1$.

Proposition 3.11. *There exist $\theta \in A \setminus \mathbb{F}_q$, $a \in A[\rho(A)]$, and a special function $U \in \mathbb{T}$, such that for all $i \geq 0$:*

$$\frac{\rho(\theta) - \theta^{q^i}}{a^{(i)}} U|_{\xi^{(i)}} = e_i(\phi)\widetilde{\pi}^{q^i}.$$

In particular, for any special function U' associated to f, we have :

$$\forall\ i \geq 0, \quad f^{(i)} U'|_{\xi^{(i)}} \in \widetilde{\pi}^{q^i} H\,.$$

Proof. Let $\mathbb{A} = A[\rho(K)]$. We still denote by ρ the obvious $\rho(K)$-linear map $\mathbb{A} \to \rho(K)$. We observe that:

$$\mathrm{Ker}\,\rho = \sum_{a \in A} (a - \rho(a))\mathbb{A}\,.$$

We also observe that there exists $\theta \in A \setminus \mathbb{F}_q$ such that $\rho(\theta) - \theta \in \mathrm{Ker}\,\rho \setminus (\mathrm{Ker}\,\rho)^2$. Set $z = \rho(\theta)$. Then $z - \theta$ has a zero of order one at ξ (observe that $z - \theta^{q^i}$ has a zero of order one at $\xi^{(i)}$). Note that $K/\mathbb{F}_q(\theta)$ is a finite separable extension, therefore there exists $y \in A$ such that $K = \mathbb{F}_q(\theta, y)$. Let $P(X) \in \mathbb{F}_q[\theta][X]$ be the minimal polynomial of y over $\mathbb{F}_q(\theta)$ and set:

$$a = \frac{P(X)}{X - y}|_{X = \rho(y)} \in A[\rho(A)] \subset \mathbb{A}\,.$$

Since $P(X)$ has a zero of order one at y, we have:

$$a \notin \mathrm{Ker}\,\rho\,.$$

Let's set:

$$U = \exp_\phi\left(\frac{a}{z - \theta}\widetilde{\pi}\right) \in \mathbb{T}\,.$$

Since $\frac{a}{z-\theta} \notin \mathbb{A}$, we have:

$$U \neq 0\,.$$

Furthermore, observe that $\mathbb{F}_q[\theta, y] \subset A \subset \mathrm{Frac}(\mathbb{F}_q[\theta, y])$. Thus:

$$\forall\ b \in A, \quad \phi_b(U) = \rho(b)U\,.$$

We conclude that:

$$U \in (\{x \in L, \tau(x) = fx\} \setminus \{0\}) \cap \mathbb{T}\,.$$

Let's set:

$$\delta = \frac{a}{z-\theta}\,.$$

We have:

$$U = \sum_{i \geq 0} \delta^{(i)} e_i(\phi) \widetilde{\pi}^{q^i}\,.$$

We therefore get:

$$\forall\ i \geq 0, \quad (\delta^{-1})^{(i)} U|_{\xi^{(i)}} = e_i(\phi)\widetilde{\pi}^{q^i}\,.$$

The last assertion comes from the fact that $f^{(i)}$ has a zero of order at least one at $\xi^{(i)}$. $\square$

We refer the reader to [1] for the explicit construction of f in the case $d_\infty = 1$, and to [17] for the explicit construction of the special functions attached to f in the case $g = 1$ and $d_\infty = 1$.

4. A basic example: the case $g = 0$

In this section, we assume that the genus of K is zero. Let's select $x \in K$ such that $K = \mathbb{F}_q(x)$ and $v_\infty(x) = 0$. Let $P_\infty(x) \in \mathbb{F}_q[x]$ be the monic irreducible polynomial corresponding to ∞, then $\deg_x P_\infty(x) = d_\infty$. Let $\mathrm{sgn} : K_\infty^\times \to \mathbb{F}_\infty^\times$ be the sign function such that $\mathrm{sgn}(P_\infty(x)) = 1$. Then $A = \{\frac{f(x)}{P_\infty(x)^k}, k \in \mathbb{N}, f(x) \in \mathbb{F}_q[x], f(x) \not\equiv 0 \pmod{P_\infty(x)}, \deg_x(f(x)) \leq kd_\infty\}$. Observe that:

$$\mathrm{Pic}(A) \simeq \frac{\mathbb{Z}}{d_\infty \mathbb{Z}}\,.$$

Let P be the maximal ideal of A which corresponds to the pole of x, i.e. $P = \{\frac{f(x)}{P_\infty(x)^k}, k \in \mathbb{N}, f(x) \in \mathbb{F}_q[x], f(x) \not\equiv 0 \pmod{P_\infty(x)}, \deg_x(f(x)) < kd_\infty\}$, the order of P in $\mathrm{Pic}(A)$ is exactly d_∞, and $P^{d_\infty} = \frac{1}{P_\infty(x)} A$. We also observe that the Hilbert class field of A is $K(\mathbb{F}_\infty)$. Let $\zeta = \mathrm{sgn}(x) \in \mathbb{F}_\infty^\times$. Then $P_\infty(\zeta) = 0$. Note that:

$$v_\infty(x - \zeta) = 1\,,$$
$$\mathrm{sgn}(x - \zeta) = P_\infty'(\zeta)^{-1}\,.$$

The integral closure of A in $K(\mathbb{F}_\infty)$ is $A[\mathbb{F}_\infty]$. The abelian group $A[\mathbb{F}_\infty]^\times$ is equal to:

$$\mathbb{F}_\infty^\times \prod_{k=1}^{d_\infty - 1} \left(\frac{x-\zeta}{x-\zeta^{q^k}} \right)^{\mathbb{Z}}.$$

We know that $A[\mathbb{F}_\infty]$ is a principal ideal domain and we have:

$$PA[\mathbb{F}_\infty] = \frac{1}{x-\zeta} A[\mathbb{F}_\infty].$$

Furthermore $B = A[\mathbb{F}_\infty][u]$, where $u \in B^\times$ is such that:

$$u^{\frac{q^{d_\infty}-1}{q-1}} = \prod_{k=0}^{d_\infty - 1} \frac{\zeta - x^{q^k}}{\zeta^{q^k} - x^{q^k}}.$$

Indeed, using Thakur Gauss sums ([22]), there exists $g \in \overline{K}$ such that $K(\mathbb{F}_\infty, g)/K$ is a finite abelian extension and:

$$g^{q^{d_\infty}-1} = \prod_{k=0}^{d_\infty - 1} (\zeta - x^{q^k}).$$

Furthermore $K(\mathbb{F}_\infty, g)/K$ is unramified outside ∞ and the pole of x, and $P_\infty(x)$ is a local norm for every place of $K(\mathbb{F}_\infty, g)$ above ∞.

Let $z = \rho(x) \in \rho(K)^\times$. Then:

$$\mathbb{H} = H(z).$$

Let $Q \in \bar{X}(\mathbb{F}_q)$ be the unique point which is a pole of z, then:

$$(z - x) = (\xi) - (Q).$$

We choose $\bar{\infty}$ to be the point of $\bar{X}(\mathbb{F}_\infty)$ which is the zero of $z - \zeta$. Then:

$$\left(\frac{z-x}{z-\zeta} \right) = (\xi) - (\bar{\infty}).$$

We easily deduce that if f is a shtuka function relative to $\bar{\infty}$ (note that f is well-defined modulo $\{x \in \mathbb{F}_\infty^\times, x^{\frac{q^{d_\infty}-1}{q-1}} = 1\}$), then f is of the form:

$$\frac{z-x}{z-\zeta} v, \quad v \in H^\times.$$

Let $\theta = \frac{1}{P_\infty(x)} \in A$. Then:

$$\operatorname{sgn}(\theta) = 1,$$
$$\deg \theta = d_\infty.$$

Let ϕ be the Drinfeld module attached to f, then:

$$\phi_\theta = \theta + \cdots + \tau^{d_\infty}.$$

We have:

$$f\dots f^{(d_\infty-1)} = \frac{\prod_{k=0}^{d_\infty-1}(z-x^{q^k})}{P_\infty(z)} v^{\frac{q^{d_\infty}-1}{q-1}}.$$

We get:

$$1 = \prod_{k=0}^{d_\infty-1}(\zeta - x^{q^k}) v^{\frac{q^{d_\infty}-1}{q-1}}.$$

Thus:

$$(vg^{q-1})^{\frac{q^{d_\infty}-1}{q-1}} = 1,$$

So that,

$$f = \frac{z-x}{z-\zeta} g^{1-q}\zeta',$$

where $\zeta' \in \mathbb{F}_\infty^\times$ is such that:

$$(\zeta')^{\frac{q^{d_\infty}-1}{q-1}} = 1.$$

Furthermore, if we write $\exp_\phi = \sum_{i\geq 0} e_i(\phi)\tau^i, e_i(\phi) \in H$, then:

$$e_i(\phi) = g^{q^i-1}(\zeta')^{-\frac{q^i-1}{q-1}} \prod_{k=0}^{i-1} \frac{x^{q^i}-\zeta^{q^k}}{x^{q^i}-x^{q^k}}.$$

We also deduce that:

$$\forall\ a \in A, \phi_a = a + \cdots + \mathrm{sgn}(a)\tau^{\deg a}.$$

Recall that $H \subset \mathbb{C}_\infty$, and $v_\infty(x-\zeta) = 1$. We now work in

$$L = \mathbb{F}_\infty(z)\left(\left(\sqrt[q^{d_\infty}-1]{-P_\infty(x)}\right)\right).$$

Recall that g is the Thakur–Gauss sum associated to sgn, i.e. let $C : \mathbb{F}_q[x] \to \mathbb{F}_q[x]\{\tau\}$ be the homomorphism of $\mathbb{F}_q$-algebras such that $C_x = x + \tau$, we have chosen $\lambda \in H \setminus \{0\}$ such that $C_{P_\infty(x)}(\lambda) = 0$, and:

$$g = -\sum_{\substack{y\in\mathbb{F}_q[x]\setminus\{0\}\\ \deg_x y<d_\infty}} \mathrm{sgn}(y)^{-1} C_y(\lambda).$$

Furthermore, λ is chosen is such a way that:

$$\lambda \in \sqrt[q^{d_\infty}-1]{-P_\infty(x)}K_\infty,$$

$$\mathrm{sgn}\left(\frac{\lambda}{\sqrt[q^{d_\infty}-1]{-P_\infty(x)}}\right) = 1.$$

Thus:

$$\mathrm{sgn}\left(\frac{g}{\sqrt[q^{d_\infty}-1]{-P_\infty(x)}}\right) = 1.$$

Recall also that:

$$\mathbb{T} = \rho(A)[\mathbb{F}_\infty]\left(\left(\sqrt[q^{d_\infty}-1]{-P_\infty(x)}\right)\right).$$

We can choose f such that $\zeta' = 1$, i.e. $f = \frac{z-x}{z-\zeta}g^{1-q}$. Now, recall that:

$$f, \frac{z-x}{z-\zeta} \in \mathbb{T}^\times.$$

Set:

$$U = \prod_{i\geq 0}\left(1 + \frac{(\zeta - x)^{q^i}}{z - \zeta^{q^i}}\right)^{-1} \in L^\times.$$

Then:

$$U \in \mathbb{T}^\times.$$

Furthermore:

$$\tau(U) = \frac{z-x}{z-\zeta}U.$$

Let's set:

$$\omega = g^{-1}U,$$

Then:

$$\tau(\omega) = f\omega,$$
$$\operatorname{sgn}\left(\omega \sqrt[q^{d_\infty}-1]{-P_\infty(x)}\right) = 1,$$
$$\omega \in \mathbb{T}^\times,$$
$$\{x \in \mathbb{T}, \tau(x) = fx\} = \omega\rho(A).$$

Finally observe that:

$$(z-x)\omega|_\xi = g^{-1}(x-\zeta)\prod_{i\geq 1}\left(1 + \frac{(\zeta - x)^{q^i}}{x - \zeta^{q^i}}\right)^{-1}.$$

Thus, there exist $b \in K^\times$, $\operatorname{sgn}(b) = 1$, ζ' a root of $P_\infty(x)$, such that:

$$\widetilde{\pi} = bg'^{-1}(x-\zeta')\prod_{i\geq 1}\left(1 + \frac{(\zeta' - x)^{q^i}}{x - (\zeta')^{q^i}}\right)^{-1},$$

for some well-chosen Thakur–Gauss sum g' relative to a twist of sgn.

Let's treat the elementary (and well-known, see [3], and especially the proof of Lemma 2.5.4) case $d_\infty = 1$. Then $A = \mathbb{F}_q[\theta]$ for some $\theta \in K$, $\operatorname{sgn}(\theta) = 1$. Let's take $x = \frac{\theta+1}{\theta}$. Then $P_\infty(x) = x - 1$, and $\zeta = 1$. In that case:

$$g = \sqrt[q-1]{-P_\infty(x)} = \sqrt[q-1]{-\frac{1}{\theta}}.$$

We get:

$$f=\frac{z-x}{z-1}g^{1-q}=t-\theta\,,$$

where $t=\rho(\theta)$. We have:

$$\phi_{\frac{1}{P_\infty(x)}}=\phi_\theta=\theta+\tau\,.$$

We get:

$$\omega=\sqrt[q-1]{-\theta}\prod_{i\geq 0}\left(1-\frac{t}{\theta^{q^i}}\right)^{-1}\in\mathbb{T}=\mathbb{F}_q[t]\left(\left(\sqrt[q-1]{\frac{-1}{\theta}}\right)\right).$$

In this case ϕ is standard, thus we have:

$$\operatorname{Ker}\exp_\phi=\widetilde{\pi}A\,,$$

for $\widetilde{\pi}\in\sqrt[q-1]{-\theta}K_\infty, \operatorname{sgn}(\widetilde{\pi}\sqrt[q-1]{\frac{-1}{\theta}})=1$. Let's set:

$$\omega'=\exp_\phi\left(\frac{\widetilde{\pi}}{f}\right)\in\mathbb{T}\setminus\{0\}\,.$$

Then, one has:

$$\phi_\theta(\omega')=\exp_\phi\left(\theta\frac{\widetilde{\pi}}{t-\theta}\right)=t\omega'\,.$$

Thus:

$$\forall\ a\in A,\quad \phi_a(\omega')=\rho(a)\omega'\,.$$

Therefore there exists $a\in A\setminus\{0\}$ such that:

$$\omega'=\omega\rho(a)\,.$$

But, since $\forall\ i\geq 0$, $v_\infty(e_i(\phi))=iq^i$, by examining the Newton polygon of $\sum_{i\geq 0}e_i(\phi)\tau^i$, we get:

$$v_\infty(\widetilde{\pi})=\frac{-q}{q-1}\,.$$

This implies:

$$v_\infty\left(\omega'-\frac{\widetilde{\pi}}{f}\right)\geq q-\frac{q}{q-1}\,.$$

Therefore:

$$\operatorname{sgn}\left(\omega'\sqrt[q-1]{\frac{-1}{\theta}}\right)=\operatorname{sgn}\left(\frac{\widetilde{\pi}}{f}\sqrt[q-1]{\frac{-1}{\theta}}\right)=-1\,.$$

Thus:

$$\omega'=-\omega\,.$$

We get:

$$\frac{-\widetilde{\pi}}{\theta^2}=(z-x)\omega'|_\xi=-(z-x)\omega|_\xi\,.$$

Thus:

$$(z-x)\omega|_\xi=\frac{\widetilde{\pi}}{\theta^2}\,,$$

and therefore:

$$\widetilde{\pi} = \theta^2(z-x)\omega|_\xi = \sqrt[q-1]{-\theta}\,\theta \prod_{i\geq 1}\left(1-\theta^{1-q^i}\right)^{-1}.$$

5. A rationality result for twisted L-series

Let s be an integer, $s \geq 1$. We introduce:

$$\mathcal{A}_s = A \otimes_{\mathbb{F}_q} \cdots \otimes_{\mathbb{F}_q} A = A^{\otimes s},$$

and set:

$$k_s = \mathrm{Frac}(\mathcal{A}_s).$$

For $i = 1, \ldots, s$, let $\rho_i : K \to k_s$ be the homomorphism of $\mathbb{F}_q$-algebras such that $\forall\ a \in A, \rho_i(a) = 1 \otimes \ldots 1 \otimes a \otimes 1 \cdots \otimes 1$, where a appears at the ith position. We set:

$$\mathbb{A}_s = A \otimes_{\mathbb{F}_q} k_s\,,$$
$$\mathbb{K}_s = \mathrm{Frac}(\mathbb{A}_s)\,,$$
$$\mathbb{H}_s = \mathrm{Frac}(B \otimes_{\mathbb{F}_q} k_s)\,.$$

We identify H with its image $H \otimes 1$ in $\mathbb{H}_s$, and k_s with its image $1 \otimes k_s$. Thus:

$$\mathbb{A}_s = A[k_s]\,.$$

We also identify G with the Galois group of $\mathbb{H}_s/\mathbb{K}_s$. For $i = 1, \ldots, s$, ρ_i induces a homomorphism of H-algebras:

$$\rho_i : \mathbb{H} \to \mathbb{H}_s\,.$$

Let $\mathbb{K}_{s,\infty}$ be the ∞-adic completion of $\mathbb{K}_s$, i.e.:

$$\mathbb{K}_{s,\infty} = k_s[\mathbb{F}_\infty]((\pi))\,.$$

We set:

$$\mathbb{H}_{s,\infty} = \mathbb{H}_s \otimes_{\mathbb{K}_s} \mathbb{K}_{s,\infty}\,.$$

Then we have an isomorphism of $\mathbb{K}_{s,\infty}$-algebras:

$$\kappa : \mathbb{H}_{s,\infty} \simeq k_s[\mathbb{F}_\infty]((\pi_*))^{|\,\mathrm{Pic}(A)|}\,,$$

where we set $\pi_* := \sqrt[\frac{q^{d_\infty}-1}{q-1}]{-\pi}$.

Let V be a finite dimensional $\mathbb{K}_{s,\infty}$-vector space. An $\mathbb{A}_s$-module M, $M \subset V$, will be called an $\mathbb{A}_s$-lattice in V, if M is a finitely generated $\mathbb{A}_s$-module which is discrete in V and such that M contains a $\mathbb{K}_{s,\infty}$-basis of V. For example, $\mathbb{B}_s := B[k_s]$ is an $\mathbb{A}_s$-lattice in $\mathbb{H}_{s,\infty}$.

Let $\phi \in$ Drin and let f be its associated shtuka function. For $i = 1, \ldots, s$ we set:

$$f_i = \rho_i(f)\,.$$

Let $\tau : \mathbb{H}_{s,\infty} \to \mathbb{H}_{s,\infty}$ be the continuous homomorphism of k_s-algebras such that:

$$\forall\ x \in H \otimes_K K_\infty\,, \quad \tau(x) = x^q\,.$$

Let $\varphi_s : \mathbb{A}_s \to \mathbb{H}_s\{\tau\}$ be the homomorphism of k_s-algebras such that:

$$\forall\ a \in A, \quad \varphi_{s,a} = \sum_{k=0}^{\deg a} \phi_{a,k} \prod_{i=1}^{s} \prod_{j=0}^{k-1} f_i^{(j)}\, \tau^k.$$

We consider:

$$\exp_{\varphi_s} = \sum_{k\geq 0} e_k(\phi) \prod_{i=1}^{s} \prod_{j=0}^{k-1} f_i^{(j)}\, \tau^k \in \mathbb{H}_s\{\{\tau\}\}\,.$$

Then:

$$\forall\ a \in \mathbb{A}_s\,, \quad \exp_{\varphi_s} a = \varphi_{s,a} \exp_{\varphi_s}\,.$$

Furthermore $\exp_{\varphi_s}$ converges on $\mathbb{H}_{s,\infty}$.

Proposition 5.1. *Assume that $s \equiv 1 \pmod{q-1}$. The $\mathbb{A}_s$-module $\mathrm{Ker}(\exp_{\varphi_s} : \mathbb{H}_{s,\infty} \to \mathbb{H}_{s,\infty})$ is a finitely generated $\mathbb{A}_s$-module, discrete in $\mathbb{H}_{s,\infty}$ and of rank $|\operatorname{Pic}(A)|$. In particular, $\mathrm{Ker}\exp_{\varphi_s}$ is an $\mathbb{A}_s$-lattice in $\{x \in \mathbb{H}_{s,\infty}, \forall\ a \in A \setminus \{0\}, \sigma_{aA}(x) = \operatorname{sgn}(a)^{q^{n(\phi)}(s-1)}x\}$. Furthermore, if $s \not\equiv 1 \pmod{q-1}$, then:*

$$\mathrm{Ker}\exp_{\varphi_s} = \{0\}\,.$$

Proof. One can show that, for any s, $\mathrm{Ker}\exp_{\varphi_s}$ is a finitely generated $\mathbb{A}_s$-module and is discrete in $\mathbb{H}_{s,\infty}$.

We view $\mathbb{H}_s$ as a subfield of $k_s[\mathbb{F}_\infty]((\pi_*))$. There exists $\mathcal{G} \subset G$ a system of representatives of $\frac{G}{\mathrm{Gal}(H/H_A)}$, such that:

$$\forall\ x \in \mathbb{H}_s, \quad \kappa(x) = (\sigma(x))_{\sigma\in\mathcal{G}}\,.$$

By Proposition 3.8, for $i = 1, \dots, s$, $\sigma \in \mathcal{G}$, we can select a non-zero element $U_{i,\sigma} \in L_s = k_s[\mathbb{F}_\infty]((\sqrt[q^{d_\infty}-1]{-\pi}))$ such that:

$$\tau(U_{i,\sigma}) = \sigma(f_i) U_{i,\sigma}\,.$$

Thus, by similar arguments to those of the proof of Lemma 3.5, we get:

$$\mathrm{Ker}\exp_{\sigma(\varphi_s)}|_{L_s} = \frac{\Lambda(\phi^\sigma)k_s}{\prod_{i=1}^{s} U_{i,\sigma}}\,.$$

Recall that (by Proposition 3.8):

$$U_{i,\sigma} \in \Lambda(\phi^\sigma)k_s \subset (\sqrt[q^{d_\infty}-1]{-\pi})^{-q^{n(\phi)}}\mathbb{K}_{s,\infty}\,,$$

and (by Lemma 3.4):

$$\Lambda(\phi^\sigma)k_s \subset (\sqrt[q^{d_\infty}-1]{-\pi})^{-q^{n(\phi)}}\mathbb{K}_{s,\infty}\,.$$

Thus:

$$\operatorname{Ker}\exp_{\sigma(\varphi_s)}|_{L_s} \subset (\sqrt[q^{d_\infty}-1]{-\pi})^{q^{n(\phi)}(s-1)}\mathbb{K}_{s,\infty}\,.$$

Thus, if $s \equiv 1 \pmod{q-1}$, we get:

$$\operatorname{Ker}\exp_{\sigma(\varphi_s)}|_{k_s[\mathbb{F}_\infty]((\pi_*))} = \frac{\Lambda(\phi^\sigma)k_s}{\prod_{i=1}^s U_{i,\sigma}}\,,$$

and if $s \not\equiv 1 \pmod{q-1}$:

$$\operatorname{Ker}\exp_{\sigma(\varphi_s)}|_{k_s[\mathbb{F}_\infty]((\pi_*))} = \{0\}\,. \qquad \square$$

Remark 5.2. Let $\mathbb{H}'_s = \operatorname{Frac}(H_A \otimes_{\mathbb{F}_q} k_s)$. Let $I = aA, a \in A \setminus \{0\}$, and $\sigma = \sigma_I \in \operatorname{Gal}(H/H_A)$. We have already noticed that:

$$\sigma(f) = \operatorname{sgn}(a)^{q^{n(\phi)}-q^{n(\phi)+1}} f\,.$$

We verify that:

$$\forall\ \sigma \in \operatorname{Gal}(H/H_A), \quad \varphi_s^\sigma = \varphi_s \Leftrightarrow s \equiv 1 \quad \left(\operatorname{mod}\ \frac{q^{d_\infty}-1}{q-1}\right).$$

In particular, when $s \equiv 1 \pmod{q^{d_\infty}-1}$, φ_s is defined over $\mathbb{H}'_s$, $\exp_{\varphi_s} : \mathbb{H}_s \to \mathbb{H}_s$ is $\operatorname{Gal}(H/H_A)$-equivariant, and $\operatorname{Ker}\exp_{\varphi_s}$ is an $\mathbb{A}_s$-lattice in $\mathbb{H}'_{s,\infty} := \mathbb{H}'_s \otimes_{\mathbb{K}_s} \mathbb{K}_{s,\infty}$.

We introduce (see [6]):

$$\mathcal{L}_s = \sum_{I \in \mathcal{I}(A), I \subset A} \frac{\prod_{k=1}^s \rho_k(u_I)}{\psi_\phi(I)}\sigma_I \in \mathbb{H}_{s,\infty}[G]^\times.$$

Theorem 5.3. *Let* $s \equiv 1 \pmod{\frac{q^{d_\infty}-1}{q-1}}$. *Set:*

$$W'_s = (\oplus_{i_1,\ldots,i_s \geq 0} B \prod_{k=1}^s f_k \ldots f_k^{(i_k-1)})^{\operatorname{Gal}(H/H_A)}\,.$$

Then:

$$\exp_{\varphi_s}(\mathcal{L}_s W'_s) \subset W'_s\,.$$

Proof. By our assumption on s, and by Lemma 3.1, we get:

$$\mathcal{L}_s \in \mathbb{H}'_{s,\infty}[G]^\times.$$

The result is then a consequence of the above remark and [6, Cor. 4.10]. $\square$

Remark 5.4. Set

$$W'_s = \left(\oplus_{i_1,\ldots,i_s \geq 0} B \prod_{k=1}^s f_k \ldots f_k^{(i_k-1)}\right)^{\operatorname{Gal}(H/H_A)}.$$

By Lemma 3.3, there exists $u \in B^\times$ such that:

$$\frac{f}{u} \in \operatorname{Frac}(H_A \otimes_{\mathbb{F}_q} A)\,.$$

In particular:

$$B = B'[u]\,,$$

where we recall that B' is the integral closure of A in H_A. Thus:

$$W'_s = \oplus_{i_1,\ldots,i_s\geq 0} B' u^{-\sum_{k=1}^{s} \frac{q^{i_k}-1}{q-1}} \prod_{k=1}^{s} f_k \ldots f_k^{(i_k-1)}\,.$$

Let $\mathbb{W}'_s$ be the k_s-vector space generated by W'_s. Then, by the proof of [6, Lem. 4.4], $\mathbb{W}'_s$ is a fractional ideal of $\mathbb{B}'_s := B'[k_s]$, and therefore $\mathbb{W}'_s$ is an $\mathbb{A}_s$-lattice in $\mathbb{H}'_{s,\infty}$.

Proposition 5.5. *Let $s \equiv 1 \pmod{\frac{q^{d_\infty}-1}{q-1}}$. We set:*

$$\mathbb{U}_s = \{x \in \mathbb{H}'_{s,\infty}, \exp_{\varphi_s}(x) \in \mathbb{W}'_s\}.$$

Then $\mathbb{U}_s$ is an $\mathbb{A}_s$-lattice in $\mathbb{H}'_{s,\infty}$ and:

$$\mathcal{L}_s \mathbb{W}'_s \subset \mathbb{U}_s\,.$$

If furthermore $s \equiv 1 \pmod{q^{d_\infty}-1}$, then $\frac{\mathbb{U}_s}{\operatorname{Ker}\exp_{\varphi_s}}$ is a finite dimensional k_s-vector space. In particular, there exists $a \in \mathbb{A}_s \setminus \{0\}$ such that:

$$a\mathcal{L}_s \mathbb{W}'_s \subset \operatorname{Ker}\exp_{\varphi_s}\,.$$

Proof. Since $\mathbb{W}'_s$ is an $\mathbb{A}_s$-lattice in $\mathbb{H}'_{s,\infty}$, we deduce that $\mathbb{U}_s$ is discrete in $\mathbb{H}'_{s,\infty}$ and is a finitely generated $\mathbb{A}_s$-module. By Theorem 5.3, we have:

$$\mathcal{L}_s \mathbb{W}'_s \subset \mathbb{U}_s.$$

Let $G' = \operatorname{Gal}(H_A/K)$, and let $\operatorname{res} : \mathbb{H}'_{s,\infty}[G] \to \mathbb{H}'_{s,\infty}[G']$ be the usual restriction map, then:

$$\operatorname{res}(\mathcal{L}_s) \in \mathbb{H}'_{s,\infty}[G']^\times.$$

Therefore $\mathcal{L}_s \mathbb{W}'_s$ is an $\mathbb{A}_s$-lattice in $\mathbb{H}'_{s,\infty}$. We conclude that $\mathbb{U}_s$ is an $\mathbb{A}_s$-lattice in $\mathbb{H}'_{s,\infty}$.

If $s \equiv 1 \pmod{q^{d_\infty}-1}$, then $\operatorname{Ker}\exp_{\varphi_s}$ is an $\mathbb{A}_s$-lattice in $\mathbb{H}'_{s,\infty}$ by Proposition 5.1. The proposition follows. □

Theorem 5.6. *Let $s \equiv 1 \pmod{q^{d_\infty}-1}$. We work in*

$$L_s := k_s[\mathbb{F}_\infty]\left(\left(\sqrt[q^{d_\infty}-1]{-\pi}\right)\right).$$

There exist non-zero elements $\omega_1, \ldots, \omega_s \in \mathbb{T}_s := \mathcal{A}_s[\mathbb{F}_\infty]((\sqrt[q^{d_\infty}-1]{-\pi}))$ such that:

$$\tau(\omega_i) = f_i \omega_i\,.$$

There also exists $h \in B \setminus \{0\}$ such that:

$$\forall\ x \in \mathbb{W}'_s, \quad \frac{\mathcal{L}_s(x)\prod_{k=1}^{s}\omega_i}{\widetilde{\pi}} \in h\mathbb{K}_s\,.$$

Furthermore, if ϕ is standard, then $h \in \mathbb{F}_\infty^\times$.

Proof. By Proposition 3.8, we have:

$$f_1, \dots, f_s \in \mathbb{T}_s^\times .$$

By the same proposition, there exist $\omega_1, \dots, \omega_s \in \mathbb{T}_s \setminus \{0\}$ such that:

$$\tau(\omega_i) = f_i \omega_i .$$

We deduce, by Lemma 3.4 and Lemma 3.5, that:

$$\operatorname{Ker} \exp_{\varphi_s} |_L = \frac{h \widetilde{\pi} I \mathbb{A}_s}{\prod_{k=1}^s \omega_i} ,$$

where I is some fractional ideal of A, $h \in H^\times$. Let $x \in \mathbb{W}_s'$, by Proposition 5.5, we get:

$$\frac{\mathcal{L}_s(x) \prod_{k=1}^s \omega_i}{\widetilde{\pi}} \in h\mathbb{K}_s . \qquad \square$$

We end this section with an application of the above Theorem. Let $\phi \in$ Drin such that ϕ is standard, i.e.

$$\operatorname{Ker} \exp_\phi = \widetilde{\pi} A .$$

Let $f \in \mathfrak{F}$ be the shtuka function associated to ϕ.

Theorem 5.7. *Let $n \geq 1$, $n \equiv 0 \pmod{q^{d_\infty} - 1}$. Then, there exists $b \in B' \setminus \{0\}$ such that we have the following property in $\mathbb{C}_\infty$:*

$$\frac{\sum_I \frac{\sigma_I(b)}{\psi_\phi(I)^n}}{\widetilde{\pi}^n} \in H_A^\times .$$

Proof. Write $n = q^k - s$, $k \equiv 0 \pmod{d_\infty}, s \equiv 1 \pmod{q^{d_\infty} - 1}$.

Observe that the map $u_.$ extends naturally into a map $u_. : \mathcal{I}(A) \to \mathbb{H}^\times$, such that:

$$\forall\ x \in K^\times, \quad u_{xA} = \frac{\rho(x)}{\operatorname{sgn}(x)} ,$$
$$\forall\ I, J \in \mathcal{I}(A), \quad u_{IJ} = \sigma_I(u_J) u_I .$$

By Lemma 3.1, we deduce that for all $l \geq 0$, $\frac{\tau^l(u_I)}{u_I}$ has no zero and no pole at ξ. For $m \geq 1$, $m \equiv 0 \pmod{d_\infty}$, let $\chi_m : \mathcal{I}_A \to H_A^\times$, such that:

$$\forall\ I \in \mathcal{I}(A), \quad \chi_m(I) = \frac{\tau^m(u_I)}{u_I}|_\xi .$$

We observe that:

$$\forall\ x \in K^\times, \quad \chi_m(xA) = 1 ,$$
$$\forall\ I, J \in \mathcal{I}(A), \quad \chi_m(IJ) = \sigma_I(\chi_m(J)) \chi_m(I) .$$

In particular, there exists $b_m \in B' \setminus \{0\}$ such that:

$$\forall\; I \in \mathcal{I}(A)\,, \quad \chi_m(I) = \frac{\sigma_I(b_m)}{b_m}\,.$$

By Theorem 5.3, we have:

$$\frac{\mathcal{L}_s(1)\prod_{j=1}^{s}\omega_j}{\widetilde{\pi}} \in \mathbb{K}_s\,.$$

We now apply τ^k to the above rationality result. We get:

$$\frac{\prod_{j=1}^{s}(f_j \dots f_j^{(k-1)}\omega_j)\,\tau^k(\mathcal{L}_s(1))}{\widetilde{\pi}^{q^k}} \in \mathbb{K}_s\,.$$

Let $j \in \{1,\dots,s\}$. Let $\mathbb{H}_{s,j} = H(\rho_k(K), k = 1,\dots,s, k \neq j)$. Let ξ_j be the place of $\mathbb{H}_s/\mathbb{H}_{s,j}$ which corresponds to the kernel of the homomorphism of $\mathbb{H}_{s,j}$-algebras: $\rho_j(A)[\mathbb{H}_{s,j}] \to \mathbb{H}_{s,j}, \rho_j(a) \mapsto a$. By Proposition 3.11, there exists $x_j \in K(\rho_j(K))^\times$ such that we have :

$$x_j f_j \dots f_j^{(k-1)}\omega_j|_{\xi_j} \in \widetilde{\pi} H_A^\times\,.$$

Now:

$$\tau^k(\mathcal{L}_s(1)) = \sum_I \frac{\prod_{j=1}^{s}\rho_j(u_I)}{\psi_\phi(I)^{q^k}} \prod_{j=1}^{s} \frac{\tau^k(\rho_j(u_I))}{\rho_j(u_I)}\,.$$

Therefore, there exists $b \in B' \setminus \{0\}$ such that:

$$\tau^k(\mathcal{L}_s(1))|_{\xi_1,\dots,\xi_s} = \frac{1}{b}\prod_P \left(1 - \frac{1}{\psi_\phi(P)^{q^k-s}}(P, H/K)\right)^{-1}(b) \in K_\infty^\times\,.$$

The Theorem follows. □

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Bruno Anglès
Normandie Université
Université de Caen Normandie
Laboratoire de Mathématiques Nicolas Oresme, CNRS UMR 6139
Campus II, Boulevard Maréchal Juin
B.P. 5186, 14032 Caen Cedex, France
E-mail: `bruno.angles@unicaen.fr`

Tuan Ngo Dac
CNRS - Normandie Université
Université de Caen Normandie
Laboratoire de Mathématiques Nicolas Oresme, CNRS UMR 6139
Campus II, Boulevard Maréchal Juin
B.P. 5186, 14032 Caen Cedex, France
E-mail: `tuan.ngodac@unicaen.fr`

Floric Tavares Ribeiro
Normandie Université
Université de Caen Normandie
Laboratoire de Mathématiques Nicolas Oresme, CNRS UMR 6139
Campus II, Boulevard Maréchal Juin
B.P. 5186, 14032 Caen Cedex, France
E-mail: `floric.tavares-ribeiro@unicaen.fr`