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ERRATUM TO "TOPOLOGICAL PROPERTIES OF WAŻEWSKI DENDRITE GROUPS"

BY Bruno Duchesne

Abstract. — In [Duc20], the universal minimal flow of the group of homeomorphisms of the universal Ważewski dendrite was identified as the completion of some coset space. A more concrete description was erroneously given in that paper as Basso and Tsankov explained in [BT23]. The aim of this note is to give a corrected description of this universal minimal flow.

More Precisely, Theorem 1.12 in [Duc20] (that is Theorem 7.16 with a more precise statement in the body of the article) is wrong and should be replaced by Theorem 2 and Theorem 4 below. The mistake is located in the proof of Theorem 7.16 and the other statements are not affected by the mistake.

Résumé (Erratum à « Propriétés topologiques des groupes d'homéomorphismes des dendrites de Ważewski »)

Dans [Duc20], le flot minimal universel du groupe des homéomorphismes de la dendrite universelle de Ważewski a été identifié comme la complétion d'un certain espace de classes d'équivalence. Une description plus concrète a été donnée par erreur dans cet article, comme Basso et Tsankov l'ont expliqué dans [BT23]. Le but de cette note est de donner une description corrigée de ce flot minimal universel.

Plus précisément, le théorème 1.12 dans [Duc20] (qui est le théorème 7.16 avec un énoncé plus précis dans le corps de l'article) est erroné et devrait être remplacé par le théorème 2 et le théorème 4 ci-dessous. L'erreur se situe dans la preuve du théorème 7.16 et les autres énoncés ne sont pas affectés par l'erreur.

Contents

1. Corrected statement

We use the same notations as in [Duc20] and refer to it for the context. So D_∞ is the universal Ważewski dendrite, that is the unique dendrite with dense branch points set and such that any branch point has infinite order. Let G be its homeomorphism

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group. We denote by $\mathrm{CCLO}(D_\infty)$ the set of convex and converging linear orders on the set of branch points $\mathrm{Br}(D_\infty)$ [Duc20, Def. 7.3]. Basso and Tsankov gave a simpler definition [BT23, §9] of its elements. Let us recall it:

Definition 1. — Let $\prec$ be a linear order on the set $\mathrm{Br}(D_\infty)$.

– The order $\prec$ is *converging* if for any three distinct branch points x_1, x_2, x_3 such that $x_1 \prec x_3 \prec x_2$ then x_2 is not between x_1 and x_3.

– A converging order is, moreover, *convex* if for any points x_1, x_2, x_3, x_4 such that $x_2 \in (x_1, x_3)$ and $x_3 \in (x_2, x_4)$ then it is not the case that $x_2 \prec x_3 \prec x_1 \prec x_4$.

Let us recall that for a converging linear order on a dendrite, any minimizing sequence has a limit and this limit does not depend on the minimizing sequence [Duc20, Lem. 7.2]. It is called the root of the linear order.

The space $\mathrm{CCLO}(D_\infty)$ of convex converging linear orders is a closed G-invariant subspace of the space of all linear orders on all branch points and thus a G-flow. In [Duc20, Th. 1.12], it is stated that $\mathrm{CCLO}(D_\infty)$ is the universal minimal flow of G. This is false as Basso and Tsankov proved in [BT23, Th. 9.3]. Among convex converging linear orders, there is a generic G-orbit. The elements of the generic orbit can be described in the following way: the root is some end point ξ and for each branch point b, the linear order induced on branches around b not containing ξ is isomorphic to the standard order on rational numbers. Let us fix one of these convex converging linear orders and denote it $\prec_0$. Its stabilizer is denoted $G_{\prec_0}$.

Let K be the blown-up dendrite $\overline{\mathrm{Ends}(D_\infty)}$, that is the completion of the space of ends of D_∞ with respect to right uniformity when $\mathrm{Ends}(D_\infty)$ is identified with G/G_ξ where G_ξ is the stabilizer of an end point $\xi \in D_\infty$ in G. This appears to be the universal Furstenberg boundary of G [Duc20, §8.2].

An element C of K can be identified with a collection $(C_b)_{b\in\mathrm{Br}(D_\infty)}$ where C_b is an element of $\overline{b} = \widehat{b} \cup \{\infty\}$, the one-point compactification of the discrete space $\widehat{b}$ of branches around $b \in \mathrm{Br}(D_\infty)$.

We introduce the following subset of $\mathrm{CCLO}(D_\infty) \times K$:

$$M = \big\{(\prec, C) \in \mathrm{CCLO}(D_\infty) \times K,\ \forall b, b' \in \mathrm{Br}(D_\infty),\ b \prec b' \implies b \in C'_b\big\}.$$

Since $\mathrm{CCLO}(D_\infty)$ and K are G-flows, it follows that M is a G-flow as well since it is invariant and closed.

For a uniform space X, we denote by $\widehat{X}$ its completion. For example, $K = \widehat{G/G_\xi}$. We can now give the correct concrete identification of the universal minimal flow of G.

Theorem 2. — *The universal minimal flow of G is $M \simeq \widehat{G/G_{\prec_0}}$.*

Remark 3. — This universal minimal flow can also be recovered using highly proximal extensions. Since $G \curvearrowright \mathrm{CCLO}(D_\infty)$ is minimal and highly proximal extensions preserve minimality [AG77], $G \curvearrowright S_G(\mathrm{CCLO}(D_\infty))$ (the universal highly proximal extension of $\mathrm{CCLO}(D_\infty)$) is minimal. By [BT23, Prop. 2.7], $S_G(\mathrm{CCLO}(D_\infty)) \simeq \mathrm{Sa}(G/G_{\prec_0})$

(the Samuel compactification of $G/G_{\prec_0}$), so by [Zuc21, Prop. 6.6], $G_{\prec_0}$ is pre-syndetic. Since $G_{\prec_0}$ is also extremely amenable [Duc20, Prop. 7.9], by [Zuc21, Th. 7.5],

$$M(G) \simeq \mathrm{Sa}(G/G_{\prec_0}) \simeq S_G(\mathrm{CCLO}(D_\infty)).$$

The mistake in the proof of [Duc20, Th. 7.16] is located in the statement that the embedding $G/G_\xi \simeq \mathrm{CCLO}(D_\infty)$ is bi-uniformly continuous. It is uniformly continuous but its inverse is not. Let us conclude with points in [Duc20] that depend on this theorem and see how they are affected.

Corollary 7.17 in [Duc20] still holds since M is metrizable (and thus has a comeager orbit). The universal minimal flow of G_ξ is as depicted in [Duc20]. Let us denote by $M_\xi = \{m \in M,\ \pi_1(m) = \xi\}$ where $\pi_1(\prec, C) \in D_\infty$ is the root of $\prec$.

Theorem 4. — *The universal minimal flow of G_ξ is indeed $M_\xi \simeq \mathrm{CCLO}(D_\infty)_\xi$.*

This result is used in the proof of the amenability of G_ξ [Duc20, Th. 8.5] and this amenability result is unaffected.

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2. Proofs

Proof of Theorem 2

Identification between M and $\widehat{G/G_{\prec_0}}$. — For $\xi \in \mathrm{Ends}(D_\infty)$, we denote by C^ξ the element of K such that for any $b \in \mathrm{Br}(D_\infty)$, C^ξ_b is the branch around b containing ξ (denoted $C_b(\xi)$ in the original paper). Let ξ_0 be the root of $\prec_0$ and $C^0 = C^{\xi_0}$. Let us consider the orbit map $g \mapsto (g \prec_0, gC^0)$ from G to M. On the image in M of this orbit, a basis of entourages for the uniform structure is given by entourages

$$V_F = \{((\prec_1, C^{\xi_1}), (\prec_2, C^{\xi_2}) \colon\ \forall x \neq y \in F,\ x \prec_1 y \Leftrightarrow x \prec_2 y,\ \exists \gamma \in U_F,\ \gamma\xi_1 = \xi_2\},$$

where F is some finite subset of $\mathrm{Br}(D_\infty)$ and U_F is its pointwise stabilizer. Moreover, it suffices to consider finite c-closed subsets F since the family given by these subsets is cofinal for inclusion. The description of the uniform structure comes from the identification proved in [Duc20, Prop. 8.13]. Since $G_{\prec_0}$ fixes ξ_0 and thus C^0, this orbit map induces a map $G/G_{\prec_0} \to M$ that is uniformly continuous and thus extends to an equivariant continuous map $\widehat{G/G_{\prec_0}} \to M$. Let us see that the inverse map from the G-orbit of $(\prec_0, C^0)$ to $G/G_{\prec_0}$ is uniformly continuous as well.

The left uniform structure on the quotient space $G/G_{\prec_0}$ is generated by the system of basic entourages $E_F = \{(gG_{\prec_0}, hG_{\prec_0}), g \in U_F hG_{\prec_0}\}$ where F is a finite subset of $\mathrm{Br}(D_\infty)$ as above.

So let us fix some finite subset $F \subset \mathrm{Br}(D_\infty)$ and let us prove that for two elements $g, h \in G$, if $((g \prec_0, g\xi_0), (h \prec_0, h\xi_0)) \in V_F$ then $(gG_{\prec_0}, hG_{\prec_0}) \in E_F$. This will prove the uniform continuity of the inverse of the orbit map $G/G_{\prec_0} \to M$ on its image.

In that case by the description of V_F, there is $u \in U_F$ such that $g\xi_0 = uh\xi_0$. Then, applying Lemma 5 (see below) to $\prec\ = uh \prec_0$ and $\prec'\ = g \prec_0$, there is $v \in G$ fixing pointwise F and $\xi = g\xi_0$ such that $g \prec_0\ = vuh \prec_0$ and thus for $u' = vu \in U_F$ one has $g \in u'hG_{\prec_0}$ as required.

Minimality. — Let $(\prec_1, C^1), (\prec_2, C^2) \in M$. We aim to show that the closure of the G-orbit of $(\prec_1, C^1)$ contains $(\prec_2, C^2)$. Fix $F \subset \mathrm{Br}(D_\infty)$ finite and for each $b \in F$, choose a neighborhood N_b of C^2_b. We may assume that N_b is a singleton if $C^2_b \neq \infty$. To prove that the closure of the G-orbit of $(\prec_1, C^1)$ contains $(\prec_2, C^2)$, it suffices to prove there is $g \in G$ such that $g \prec_1$ coincides with $\prec_2$ on F and $gC^1_b \in N_b$ for all $b \in F$.

Let r_1 be the root of $\prec_1$. Let $g \in G$ such that $gr_1 \in \bigcap_{b\in F} N_b$ (this intersection is non-empty since eventually it contains a minimizing sequence for $\prec_2$). Let b_0 be the projection (i.e., the image by the first-point map) of gr_1 on $[F]$. Let $g_1 \in G$ fixing pointwise the branch around b_0 containing gr_1 and permuting the branches around b_0 containing elements of F in order that the orders induced on these branches coincide for $\prec_2$ and $g_1 g \prec_1$. The closure $B_1, \dots, B_n$ of each of these branches is homeomorphic to D_∞ itself and b_0 is an end point for these subdendrites. One can now use [Duc20, Lem. 7.14] for each b_i and find an element $h_i \in \mathrm{Homeo}(B_i)$ fixing b_0 such that $h_i g_1 g \prec_1$ and $\prec_2$ coincide on $F \cap B_i$. Patching the h_i's (extended by the identity on the other branches around b_0), one gets an element g_2 such that $g_2 g_1 g \prec_1$ and $\prec_2$ coincides on F. Now, it suffices to observe that for each $b \in F$, $g_2 g_1 g C^1_b$ is the branch around $g_2 g_1 g b$ containing the root of $g_2 g_1 g \prec_1$ that is C^2_b or lies in N_b if b is the root of $\prec_2$.

Universality. — Since $G_{\prec_0}$ is extremely amenable, for any G-flow X, there is a $G_{\prec_0}$-fixed point x. The orbit map $g \mapsto gx$ is uniformly continuous and thus extends to a G-map $\widehat{G/G_{\prec_0}} \to X$. If X is moreover minimal, the image is X itself. ☐

Lemma 5. — *Let $\prec, \prec'$ be two convex converging linear orders with root $\xi \in \mathrm{Ends}(D_\infty)$ in the G-orbit of $\prec_0$ and F be a finite c-closed subset of $\mathrm{Br}(D_\infty)$ such that for any $x, y \in F$, $x \prec y \Leftrightarrow x \prec' y$. Then there exists $v \in G_\xi$ fixing pointwise F such that $v \prec\ =\ \prec'$.*

Proof. — First, we may assume that F contains the projection p of ξ to the subdendrite generated by F. Actually if this point p does not belong to F, we observe that since these orders are converging and they have the same root, p has to be smaller than any other element of F for $\prec$ and $\prec'$.

We use a back and forth argument to construct v. Actually, we construct v as a bijection of $\mathrm{Br}(D_\infty)$ preserving the betweenness relation and use the identification of G with the group of betweenness preserving bijections of $\mathrm{Br}(D_\infty)$.

Let $\{x_n\}_{n\geqslant 1}$ be an enumeration of $\mathrm{Br}(D_\infty) \smallsetminus F$. We construct increasing sequences X_n, Y_n of finite subsets of $\mathrm{Br}(D_\infty)$ containing F and bijections $v_n \colon X_n \to Y_n$ such that for any $n \in \mathbb{N}$:

(1) X_n, Y_n are c-closed and contain the projection of ξ on the subdendrite they generate.

(2) $x_n \in X_n \cap Y_n$,

(3) v_n preserves the betweenness relation and for any $x, y \in X_n$,

$$x \prec y \iff v_n(x) \prec' v_n(y),$$

(4) if $p_n \in X_n$ is the projection of ξ on the subdendrite generated by X_n then $v_n(p_n)$ is the projection of ξ on the subdendrite generated by Y_n,

(5) $v_{n+1}|_{X_n} = v_n$.

We set $X_0 = Y_0 = F$ and v_0 to be the identity on F. We proceed by induction. Assume for some $n \in \mathbb{N}$, X_n, Y_n and v_n have been defined and satisfy the above properties. Let c_n be the projection of x_{n+1} on the subdendrite generated by $X_n \cup \{\xi\}$. Let $[a_n, b_n]$ be the minimal arc (maybe reduced to c_n if $c_n \in X_n$) such that $a_n, b_n \in X_n \cup \{\xi\}$ containing c_n. Let c'_n be a branch point in $(v_n(a_n), v_n(b_n))$ (with the convention that $v_n(\xi) = \xi$) or $c'_n = v_n(a_n) = v_n(b_n)$ if $c_n \in X_n$.

Let us observe that for any $f \in X_n$, $c_n \prec f \Leftrightarrow c'_n \prec' v_n(f)$. Actually, let $d_n = c(\xi, c_n, f) \in X_n$, if $d_n = c_n$ then $c_n \prec d_n$, if $d_n = f$ then $f \prec c_n$ (by the converging property) and otherwise

$$f \prec c_n \iff f \prec a_n \iff f \prec b_n$$

(see [Duc20, Lem. 7.5]). These properties are true because $c_n \in [a_n, b_n]$ and thus, the same holds for any $c'_n \in [v_n(a_n), v_n(b_n)]$.

If $x_n = c_n$ then let x'_n be c'_n and otherwise choose a point x'_n in some component not containing $\xi, v_n(a_n), v_n(b_n)$ around c'_n such that for any $x \in X_n$,

$$x_{n+1} \prec x \iff x'_n \prec' v_n(x).$$

This is possible because $\prec'$ is in the G-orbit of $\prec_0$ and thus the linear order induced by $\prec'$ on branches around c'_n not containing ξ is isomorphic to the standard order on rational numbers. Let $Y'_n = Y_n \cup \{x'_n, c'_n\}$ and $X'_n = X_n \cup \{x_{n+1}, c_n\}$ and let us extend v_n on X'_n with $v_n(x_{n+1}) = x'_n$ and $v_n(c_n) = c'_n$.

Similarly let d'_n be the projection of x_{n+1} on the subdendrite generated by $Y'_n \cup \{\xi\}$. If $d'_n \notin Y'_n$, let $[a'_n, b'_n]$ be the minimal arc such that $a'_n, b'_n \in Y'_n \cup \{\xi\}$ and containing d'_n. Let d_n be a branch point in $\big(v_n^{-1}(a'_n), v_n^{-1}(b'_n)\big)$ and choose a point y_n in some component around d_n not intersecting $X'_n \cup \{\xi\}$ and such that for any $x \in X_n$,

$$y_n \prec x \iff x_{n+1} \prec' v_n(x).$$

Now let $X_{n+1} = X'_n \cup \{d_n, y_n\}$, $Y_{n+1} = Y'_n \cup \{d'_n, x_{n+1}\}$ and $v_{n+1}(x) = v_n(x)$ for all $x \in X'_n$, $v_{n+1}(d_n) = d'_n$ and $v_{n+1}(y_n) = x_{n+1}$ (if these new points do not already belong to X'_n).

By construction, X_n, Y_n and v_n have the announced properties and finally, for $x \in \mathrm{Br}(D_\infty)$ we define $v(x) = v_n(x)$ for any n large enough such that $x \in X_n$. The map v is a bijection that preserves the betweenness relation and maps $\prec$ to $\prec'$. Because of the fourth property, v fixes ξ. Thus v fixes pointwise $F \cup \{\xi\}$. □

Proof of Theorem 4. — Since G acts transitively on $\mathrm{Ends}(D_\infty)$, we continue with $\xi = \xi_0$ the root of $\prec_0$. The space M_{ξ_0} is closed and thus compact since π_1 is continuous. Let π be the projection $M \to \mathrm{CCLO}(D_\infty)$. This map is continuous on M and injective on M_{ξ_0} (in that case, the second component is constant equal to C^0) with image $\mathrm{CCLO}(D_\infty)_{\xi_0}$. Since $\mathrm{CCLO}(D_\infty)_{\xi_0}$ and M_{ξ_0} are both compact, these two spaces are homeomorphic as G_{ξ_0}-flows. The first part of the proof of Theorem 2 also proves that $G_{\xi_0}/G_{\prec_0}$ is bi-uniformly embedded in M_{ξ_0} and since it is dense $\widehat{G_{\xi_0}/G_{\prec_0}} \simeq M_{\xi_0}$. Minimality and universality go as in the proof of Theorem 2. □

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