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Finitely generated subgroups of algebraic elements of plane Cremona groups are bounded

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FINITELY GENERATED SUBGROUPS OF ALGEBRAIC ELEMENTS OF PLANE CREMONA GROUPS ARE BOUNDED

by Anne Lonjou, Piotr Przytycki & Christian Urech

Abstract. — We prove that any finitely generated subgroup of the plane Cremona group consisting only of algebraic elements is of bounded degree. This follows from a more general result on 'decent' actions on infinite restricted products. We apply our results to describe the degree growth of finitely generated subgroups of the plane Cremona group.

Résumé (Les sous-groupes de type fini d'éléments algébriques du groupe de Cremona planaire sont bornés)

Nous montrons que tout sous-groupe de type fini du groupe de Cremona planaire contenant seulement des éléments algébriques est de degré borné. Cela découle d'un résultat plus général sur les actions « décentes » sur les produits infinis restreints. Nous appliquons nos résultats pour décrire la croissance des degrés des sous-groupes de type fini du groupe de Cremona planaire.

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1. Introduction

The Cremona group $\mathrm{Cr}_2(k)$ over a field k is the group of birational transformations of the projective plane $\mathbb{P}^2$ over k. Cremona groups have been the delight of algebraic geometers and group theorists in both, classical and modern times. As of today, many aspects of $\mathrm{Cr}_2(k)$ are well understood and there are many tools at hand to study those groups. For instance, $\mathrm{Cr}_2(k)$ acts by isometries on an infinite dimensional hyperbolic

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space $\mathbb{H}^\infty$ [Can11] and on various CAT(0) cube complexes [LU21]. Nevertheless, some questions have remained open. The goal of this article is to positively answer a question asked more than a decade ago by Favre in [Fav10, Quest. 1].

Let us fix projective coordinates $[x : y : z]$ of $\mathbb{P}^2$. An element $f \in \mathrm{Cr}_2(k)$ is given by

$$[x : y : z] \dashmapsto [f_0 : f_1 : f_2],$$

where the $f_i \in k[x, y, z]$ are homogeneous of the same degree and without non-constant common factor. The *degree* $\deg(f)$ of f is defined as the degree of the polynomials f_i. An element $f \in \mathrm{Cr}_2(k)$ is *algebraic*, if $\deg(f^n)$ is uniformly bounded for all $n \in \mathbb{Z}$. A subgroup $G < \mathrm{Cr}_2(k)$ is *bounded* if the degree of all elements in G is uniformly bounded. Clearly, a bounded subgroup consists of algebraic elements. However, the converse is not true. For instance, consider the subgroup defined in the affine coordinates (x, y) of $\mathbb{P}^2$ by

$$G = \{(x, y) \dashmapsto (x + R(y), y) \mid R \in k(y)\}.$$

Every element in G is algebraic, but G is not bounded. In this paper, we show the following theorem, which solves Favre's question:

Theorem 1.1. — *Let k be a field and let $G < \mathrm{Cr}_2(k)$ be a finitely generated subgroup such that every element of G is algebraic. Then G is bounded.*

Remark 1.2. — Note that the properties of being algebraic and being bounded are invariant under field extensions, so it is enough to show Theorem 1.1 for algebraically closed fields.

Remark 1.3. — The first step towards the proof of Theorem 1.1 is due to Cantat, who showed in [Can11] that a finitely generated subgroup $G < \mathrm{Cr}_2(k)$ consisting of algebraic elements is bounded or preserves a rational fibration (see [Lam24] for a proof of this result for fields of arbitrary characteristic). It is therefore enough to show Theorem 1.1 for finitely generated groups that preserve a rational fibration.

In order to show Theorem 1.1 for finitely generated groups that preserve a rational fibration, we introduce the *Jonquières complex* — a CAT(0) cube complex with an isometric action of the group of birational transformations preserving a rational fibration, whose vertex stabilizers are bounded subgroups. Using a suitable description of this complex and the dynamics of $\mathrm{PGL}_2(k)$ on $\mathbb{P}^1$, we show that the action of our group on this complex is decent, as defined below.

Definition 1.4. — Let X_0 be a set. We say that a group G_0 acts on X_0 *purely elliptically*, if each element of G_0 fixes a point of X_0. We say that a group G_0 acts on X_0 *decently* if

– each subgroup of G_0 with a finite orbit fixes a point of X_0, and
– each finitely generated subgroup of G_0 acting purely elliptically fixes a point of X_0.

It is an easy exercise that if G_0 is the isometry group of a simplicial tree X_0, then G_0 acts on X_0 decently (see [Ser80, Cor. 3 in §6.5 & Ex. 6.3.4]). More generally, if G_0 is the isometry group of a CAT(0) cube complex X_0 with no infinite cubes, then G_0 acts on X_0 decently [GLU24]. Similarly, if G_0 is the isometry group of a CAT(0) 2-complex X_0 with rational angles, then G_0 acts on X_0 decently [NOP22]. For further examples, see [HO21].

1.1. Applications. — The Cremona group $\mathrm{Cr}_2(k)$ can be equipped with the *Zariski topology* (see for instance [Ser10]). An *algebraic subgroup* of $\mathrm{Cr}_2(k)$ is a Zariski closed bounded subgroup. This explains the terminology: an element in $\mathrm{Cr}_2(k)$ is algebraic if and only if it is contained in an algebraic subgroup. Note that for any $d \geqslant 1$, the subset of $\mathrm{Cr}_2(k)$ consisting of elements of degree at most d is closed (see [BF13]), hence a bounded subgroup is contained in an algebraic subgroup. An algebraic subgroup G is always *projectively regularizable*, i.e., there exists a birational map $\varphi\colon \mathbb{P}^2 \dashrightarrow S$ such that $\varphi G \varphi^{-1} < \mathrm{Aut}(S)$ for some regular projective surface S. This follows from the theorems of Weil and Sumihiro or from the fact that the number of base-points of elements in an algebraic subgroup is uniformly bounded (we refer to [Lam24] for references and a proof of this fact, or to [LU21]). Theorem 1.1 has therefore the following direct consequence:

Corollary 1.5. — *Let k be a field and let $G < \mathrm{Cr}_2(k)$ be a finitely generated subgroup such that every element of G is algebraic. Then G is projectively regularizable.*

From another point of view, algebraic elements correspond exactly to elements in $\mathrm{Cr}_2(k)$ inducing an elliptic isometry on the infinite dimensional hyperbolic space $\mathbb{H}^\infty$, on which $\mathrm{Cr}_2(k)$ acts [Can11]. Theorem 1.1 therefore states that the action of $\mathrm{Cr}_2(k)$ on $\mathbb{H}^\infty$ is decent.

While all algebraic elements are projectively regularizable, there exist also non-algebraic elements that are projectively regularizable. It is still unknown whether a finitely generated subgroup of the Cremona group containing only projectively regularizable elements is projectively regularizable or not. This question is equivalent to the question whether $\mathrm{Cr}_2(k)$ acts decently on the *blow-up complex* — a CAT(0) cube complex constructed in [LU21]. In [GLU24], the first and third authors together with Genevois positively answer this question when the base-field k is finite.

Since elements in $\mathrm{Cr}_2(k)$ preserving a rational fibration are projectively regularizable if and only if they are algebraic (see for instance [LU21, Table 1]), Corollary 1.5 directly implies the following. Let $G < \mathrm{Cr}_2(k)$ be a finitely generated subgroup preserving a rational fibration such that every element in G is projectively regularizable. Then G is projectively regularizable.

The notions of algebraic elements and bounded subgroups generalize to birational transformations of arbitrary regular projective surfaces and Theorem 1.1 generalizes to this setting:

Theorem 1.6. — *Let S be a regular projective surface over a field k and let $G < \mathrm{Bir}(S)$ be a finitely generated subgroup such that every element of G is algebraic. Then G is bounded.*

However, the most interesting and difficult case is the one of rational surfaces. In order to keep the notation more accessible, we discuss and prove the general case in the separate Section 5.1.

In Section 5.2 we apply Theorem 1.1 to give a first description of the asymptotic degree growth of finitely generated subgroups of $\mathrm{Cr}_2(k)$. This opens up new interesting questions about the dynamical behaviour of finitely generated subgroups of $\mathrm{Cr}_2(k)$. Let $G < \mathrm{Cr}_2(k)$ be a finitely generated subgroup with a finite generating set T. Denote by $B_T(n)$ the set of all elements in G of word length ℓ_T at most n with respect to the generating set T, and define

$$D_T(n) := \max_{f \in B_T(n)} \{\deg(f)\}.$$

It has been shown in [Ure18] that there are only countably many integer sequences that can appear in this way. However, still very little is known about them. In the case where T consists of a single element, the growth of the function D_T has been extensively studied (see Theorem 5.1). Theorem 1.1 is the main ingredient for the following result, which we prove in Section 5.2. For two functions, f and g on $\mathbb{Z}_{\geqslant 0}$, we write $f \asymp g$ if $f(x) \leqslant ag(bx)$ and $g(x) \leqslant cf(dx)$ for some $a, b, c, d > 0$.

Corollary 1.7. — *Let k be an algebraically closed field and let $G < \mathrm{Cr}_2(k)$ be a finitely generated subgroup with generating set T. Then one of the following is true:*

(1) *All elements in G are algebraic, G fixes a point in $\mathbb{H}^\infty$, and $D_T(n)$ is bounded.*

(2a) *The group G contains an element that induces a parabolic isometry of $\mathbb{H}^\infty$, $D_T(n) \asymp n$, and G preserves a rational fibration.*

(2b) *The group G contains an element that induces a parabolic isometry of $\mathbb{H}^\infty$, $D_T(n) \asymp n^2$, and G preserves an elliptic fibration.*

(3) *The group G contains an element that induces a loxodromic isometry of $\mathbb{H}^\infty$ and $D_T(n) \asymp \lambda^n$ for $\lambda > 0$.*

It would be interesting to study the dynamical behaviour of the degrees of finitely generated subgroups in more detail. In Corollary 1.7, the asymptotic behaviour does not depend on the choice of the finite generating set T, since for any other generating set T' there is a constant $d > 0$ such that for all n we have $B_{T'}(n) \subseteq B_T(dn)$, and so $D_{T'}(n) \leqslant D_T(dn)$. For instance, one could ask, after fixing T, what is the precise asymptotic growth of $D_T(n)$, if the group G contains a loxodromic element:

Question 1.8. — Let $G < \mathrm{Cr}_2(k)$ be a finitely generated subgroup containing an element that induces a loxodromic isometry of $\mathbb{H}^\infty$, with finite generating set T. What is the asymptotic growth of $D_T(n)$? Do we always have $D_T(n) \sim c\lambda^n$ for some constants $c, \lambda > 0$? Here, we write $f \sim g$ if f and g are functions such that $\lim_{n\to\infty} f(n)/g(n) = 1$.

In another direction, it has been shown that the *dynamical degree* of $f \in \mathrm{Cr}_2(k)$, i.e., the number $\lambda(f) := \lim_{n\to\infty} \deg(f)^{1/n}$ is always an algebraic integer (more precisely, it is always a Pisot or Salem number, see [BC16]). This leads to the following natural question, which is due to Cantat:

Question 1.9. — Let $G < \mathrm{Cr}_2(k)$ be a finitely generated subgroup with finite generating set T. Which real numbers can be realized as $\lim_{n\to\infty} D_T(G)^{1/n}$?

For instance, it could be interesting to construct examples of such subgroups and generating sets such that $\lim_{n\to\infty} D_T(G)^{1/n}$ is transcendental. If S is a regular projective surface and $f \in \mathrm{Aut}(S)$, then $\lambda(f)$ is the spectral radius of the induced transformation f_* of f on the Neron–Severi lattice of S. For this reason, the limit $\lim_{n\to\infty} D_T(G)^{1/n}$ should be seen as an analogue to the *joint spectral radius*, which has been introduced in [RS60] and has been studied by many authors (see for instance [Jun09] or [BF21]).

1.2. Decent actions. — In Section 3 we will show that Theorem 1.1 is a special case of a following more general result on groups acting decently on an infinite restricted product.

A *pointed set* (X_0, x_0) is a set X_0 and a point $x_0 \in X_0$. The *restricted product* $\bigoplus_{p\in P}(X_p, x_p)$ of a family $\{(X_p, x_p)\}_{p\in P}$ of pointed sets is the set of sections $\{y_p\}_{p\in P}$ with $y_p \in X_p$ such that all but finitely many y_p are equal to x_p.

Note that for finite P we have $\bigoplus_{p\in P}(X_p, x_p) = \Pi_{p\in P} X_p$. For infinite P, we have that $\bigoplus_{p\in P}(X_p, x_p)$ is a proper subset of $\Pi_{p\in P} X_p$.

If each X_p is a simplicial tree and each x_p is a vertex (which will be the case in our application towards Theorem 1.1), then $\bigoplus_{p\in P}(X_p, x_p)$ has a structure of a cube complex whose cubes have the form $\Pi_{p\in P} I_p$, where all but finitely many I_p are equal to x_p and the remaining I_p are edges of X_p. This cube complex is CAT(0) though this will not be exploited explicitly in the current article.

Let G_0 be a group acting on X_0 and let H be a group acting on P. The (unrestricted) *wreath product* $G_0 \wr_P H$ of G_0 and H over P is the semi-direct product of $\Pi_{p\in P} G_p$, where $G_p = G_0$, with the group H acting on $\Pi_{p\in P} G_p$ by $h \cdot \{g_p\}_{p\in P} = \{g_{h^{-1}(p)}\}_{p\in P}$.

Let $x_0 \in X_0$. We will be considering the subgroup $G^{\oplus}$ of $G_0 \wr_P H$ preserving $\bigoplus_{p\in P}(X_p, x_p)$, where $X_p = X_0$ and $x_p = x_0$, and where the action is defined as follows. For $g = \{g_p\}_{p\in P} \in \Pi_{p\in P} G_p$, we have $g \cdot \{y_p\}_{p\in P} = \{g_p(y_p)\}_{p\in P}$. For $h \in H$, we have $h \cdot \{y_p\}_{p\in P} = \{y_{h^{-1}(p)}\}_{p\in P}$.

It is not hard to verify that if X_0 is a simplicial tree, and f is a combinatorial isometry of the cube complex $\bigoplus_{p\in P}(X_p, x_p)$, then f induces a bijection of P. Furthermore, if G_0 is the group of all simplicial isometries of X_0, then we have $f \in G^{\oplus}$ if and only if H contains that bijection. For example, if $P = \{p, q\}$ with X_0 the real line tiled by unit intervals, then $\bigoplus_{p\in P}(X_p, x_p)$ is the square tiling of the Euclidean plane, and a 90° rotation of the plane induces the bijection interchanging p and q.

Theorem 1.10. — *Let G_0 be a group acting decently on X_0. Let P be the set of the points of the projective line $\mathbb{P}^1(k)$ over an algebraically closed field k, and let $H = \mathrm{Aut}(\mathbb{P}^1)$. Then $G^{\oplus}$ acts decently on $\bigoplus_{p\in P}(X_p, x_p)$.*

We will prove Theorem 1.10 in Section 4. We will deduce Theorem 1.1 from Theorem 1.10 at the end of Section 3. Note that in Theorem 1.10, we have to make some assumptions on P and H. For example, assume that the set X_0 is a finite simplicial tree and G_0 is a nontrivial subgroup of the group of the simplicial isometries of X_0. Suppose that H is a finitely generated infinite torsion group acting by left multiplication on $P = H$. Then the entire $G^{\oplus}$ is finitely generated and acts purely elliptically on $\bigoplus_{p\in P}(X_p, x_p)$, but does not have a fixed-point.

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2. Preliminaries

In this section, we briefly recall some well-known facts about blow-ups and conic bundles. By Remark 1.2, it is enough to work over an algebraically closed field for our problem, hence we assume our base-field k to be algebraically closed. Unless mentioned otherwise, surfaces are assumed to be projective and smooth.

2.1. Subgroups of algebraic elements of automorphism groups. — Let S be a surface over k. An ample divisor H on S defines a degree function $\deg_H$ on $\mathrm{Bir}(S)$ by $\deg_H(f) = f_*H \cdot H$. An element $g \in \mathrm{Bir}(S)$ is then *algebraic* if the degree sequence $\{\deg_H(f^n)\}_n$ is bounded. A subgroup $G < \mathrm{Bir}(S)$ is *bounded* if $\{\deg_H(f) \mid f \in G\}$ is bounded. The properties of being algebraic and being bounded do not depend on the choice of an ample divisor (see for example [Dan20]).

Let us observe that Theorem 1.1 holds if we work with finitely generated groups of automorphism groups:

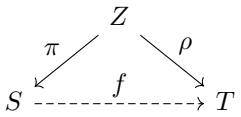
Lemma 2.1. — *Let S be a surface and let $G < \mathrm{Aut}(S)$ be a finitely generated subgroup such that every element in G is algebraic. Then G is bounded.*

Proof. — The linear action of $\mathrm{Aut}(S)$ on the Néron–Severi lattice $N^1(S)$ of S yields a homomorphism $\mathrm{Aut}(S) \to \mathrm{GL}(N^1(S)) \simeq \mathrm{GL}_n(\mathbb{Z})$, whose kernel is an algebraic group and therefore bounded. The image of an element $g \in G$ in $\mathrm{GL}(N^1(S))$ is of finite order, since g is algebraic (see for example [Can18, Th. 4.6]). Using that G is finitely generated, we obtain that the image of G in $\mathrm{GL}(N^1(S))$ is finite. Therefore, G is a finite extension of a bounded group and therefore bounded itself. □

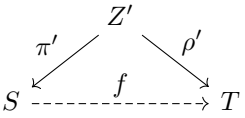
2.2. BUBBLE SPACE, STRONG FACTORIZATION, AND BASE-POINTS. — Let S be a surface and let $s \in S$. Then there exists a surface $\widetilde{S}$ and a morphism $\pi\colon \widetilde{S} \to S$ such that the fiber E over s is isomorphic to $\mathbb{P}^1$ and π induces an isomorphism between $\widetilde{S} \smallsetminus E$ and $S \smallsetminus \{s\}$. The morphism $\pi\colon \widetilde{S} \to S$ is called the *blow-up* of S in s, and it is unique up to isomorphism. Consider two distinct points s, s' on a surface S and their respective blow-ups π_s and $\pi_{s'}$. Blowing-up first s and then s' gives the same transformation up to isomorphism as blowing-up first s' and then s. By abuse of language, we will say that π_s and $\pi_{s'}$ *commute* and we denote the successive blow-up of s and then s' by $\pi_{s'}\pi_s = \pi_s\pi_{s'}$.

The *bubble space* of S is the set $\mathcal{B}(S)$ of triples (t, T, π), where $\pi\colon T \to S$ is a birational morphism from a surface T and t is a point on T; two triples (t, T, π) and (t', T', π') are identified if $\pi^{-1}\pi'$ is a local isomorphism around t' mapping t' to t. The bubble space can be thought of as the set of all points on S and on surfaces obtained from S by successively blowing up points. The points in $\mathcal{B}(S)$ contained in S are called *proper* points.

Zariski's strong factorization theorem states that every birational transformation $f\colon S \dashrightarrow T$ between surfaces can be factored into blow-ups of points. More precisely, there exists a surface Z and a factorization



where π and ρ are compositions of blow-ups of points. Note that the points blown up by π and ρ can be seen as elements of the bubble space $\mathcal{B}(S)$ and $\mathcal{B}(T)$ respectively. Moreover, Z can be chosen in such a way, and will be denoted by Z_f, that for any other such factorization



there exists a surjective morphism $\eta\colon Z' \to Z_f$ such that $\pi' = \pi\eta$ and $\rho' = \rho\eta$. If we require Z_f to be minimal in this sense, this factorization is unique up to isomorphism and up to possibly changing the order of blowing up the points. The morphism $\pi\colon Z_f \to S$ is called the *minimal resolution* of f, and the points (in $\mathcal{B}(S)$) blown up by π are called the *base-points* of f.

A base-point s of f is called *persistent* if there exists $\ell \geqslant 1$ such that, for all $n \geqslant \ell$, the point s is a base-point of f^n but s is not a base-point of f^{-n}.

2.3. CONIC BUNDLES. — A *rational fibration* on a surface S is a morphism $\pi\colon S \to C$, where C is a curve, such that all the fibers are isomorphic to $\mathbb{P}^1$. Note that this is a more restrictive definition than the more usual one, which only asks that the fibers are rational curves.

Let $\pi\colon S \to C$ and $\pi'\colon S' \to C$ be rational fibrations. We say that a birational transformation $f\colon S \dashrightarrow S'$ *preserves* the fibrations if there exists an automorphism $\hbar(f)\colon C \to C$ such that the following diagram commutes

$$\begin{array}{ccc} S & \xdashrightarrow{f} & S' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi'} \\ C & \xrightarrow{\hbar(f)} & C. \end{array}$$

Let $\pi\colon S \to C$ be a rational fibration. An *elementary transformation* $f\colon S \to S'$ is the composition of blowing up a point $s \in S$ followed by blowing down the strict transform of the fiber containing s. The surface S' comes equipped with the rational fibration $\pi'\colon S' \to C$ defined by $\pi' = \pi f^{-1}$. Note that f preserves these fibrations for $\hbar(f) = \mathrm{id}$.

The following fact is well-known (see for instance [Sch22, Cor. 3.2]):

Proposition 2.2. — *Let $\pi\colon S \to C$ and $\pi'\colon S' \to C$ be rational fibrations. Every birational transformation $f\colon S \dashrightarrow S'$ preserving the fibrations can be factored into a sequence of elementary transformations and a (fibration preserving) isomorphism.*

A *conic bundle* is a composition with a rational fibration $\pi\colon S \to C$ of a sequence of blow-ups $\widetilde{S} \to S$, such that we blow up in total at most one point $s \in S$ in each fiber. In other words, $\widetilde{S} \to C$ has only finitely many singular fibers, each of them consisting of two rational curves of self-intersection -1, called also -1-*curves* (the preimage of s under the blow-up and the strict transform of the fiber of s).

3. The Jonquières complex

In this section we reduce Theorem 1.1 to Theorem 1.10. We always assume our base-field k to be algebraically closed, which is justified by Remark 1.2, and we assume the surfaces to be projective and smooth.

3.1. The blow-up complex. — Let T be a surface. In [LU21], the authors constructed the *blow-up complex* $\mathcal{C}(T)$ — a CAT(0) cube complex with an isometric action of $\mathrm{Bir}(T)$. Let us briefly recall the construction of $\mathcal{C}(T)$. The vertices of $\mathcal{C}(T)$ are equivalence classes of *marked surfaces*, i.e., pairs (S, φ), where S is a surface and $\varphi\colon S \dashrightarrow T$ a birational transformation. Marked surfaces (S, φ) and (S', φ') are *equivalent* if $\varphi^{-1}\varphi'\colon S' \to S$ is an isomorphism. Vertices (represented by) (S, φ) and $(\widetilde{S}, \widetilde{\varphi})$, are connected by an edge if $\varphi^{-1}\widetilde{\varphi}\colon \widetilde{S} \to S$ is the blow-up of a point (or its inverse). More generally, 2^n different vertices form an n-cube if there is a marked surface (S, φ) and distinct points $s_1, \dots, s_n \in S$ such that these vertices are obtained by blowing up subsets of $\{s_i\}_{\{1 \leqslant i \leqslant n\}}$.

3.2. The Jonquières complex. — Let $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ with the rational fibration onto the first factor $\pi_0\colon \mathbb{F}_0 \to \mathbb{P}^1$. We define the *Jonquières complex* $\mathscr{J}$ as the subcomplex of the blow-up complex $\mathcal{C}(\mathbb{F}_0)$ induced on the set of vertices represented by marked surfaces (S, φ) such that, for $\pi = \pi_0\varphi$, the rational map $\pi\colon S \dashrightarrow \mathbb{P}^1$ is a conic bundle.

Remark 3.1. — Consider 2^n vertices of $\mathscr{J}$ spanning a cube. This means that there exists a surface (S,φ) and distinct points $s_1,\dots,s_n \in S$ such that these vertices are obtained by blowing up subsets of $\{s_i\}_{\{1\leqslant i\leqslant n\}}$. Since these vertices belong to the Jonquières complex, we have that the points s_i belong to distinct fibers of π. In other words, we have $\pi(s_i)\neq\pi(s_j)$ for all $i\neq j$.

Remark 3.2. — Note that the Jonquières complex $\mathscr{J}$ is not a convex subcomplex of the blow-up complex $\mathcal{C}(\mathbb{F}_0)$. Indeed, consider the birational transformation $\varphi:(x,y)\mapsto(x,x^2+y)$. It has three base-points, but only one of them is proper. Let $\rho\colon S\to\mathbb{F}_0$ be the minimal resolution of φ. Then the vertex represented by (S,ρ) is not a vertex of the Jonquières complex because more than one point has been blown-up in the same fiber. However, the vertex represented by (S,ρ) lies on a geodesic edge-path between the vertices $(\mathbb{F}_0,\mathrm{id})$ and $(\mathbb{F}_0,\varphi^{-1})$.

Nevertheless, by Proposition 2.2, the 1-skeleton of $\mathscr{J}$ it is isometrically embedded in the 1-skeleton of $\mathcal{C}(\mathbb{F}_0)$.

For any $p\in\mathbb{P}^1$, consider the subcomplex X_p of $\mathscr{J}$ induced on the vertices represented by the marked surfaces (S,φ), where $\varphi\colon S\dashrightarrow\mathbb{F}_0$ induces an isomorphism between $S\smallsetminus\pi^{-1}(p)$ and $\mathbb{F}_0\smallsetminus\pi_0^{-1}(p)$.

Lemma 3.3. — *For any $p\in\mathbb{P}^1$, the subcomplex X_p is a tree.*

Proof. — First note that X_p is connected as a consequence of Proposition 2.2 and the fact that two elementary transformations performed in distinct fibers commute.

Second, consider the vertex $v_0=(\mathbb{F}_0,\mathrm{id})$ in X_p, and let $\mathbb{P}^1_p\subset\mathbb{F}_0$ be the fiber over p. Let $v_0,v_1,v_2,v_3,\dots$ be the consecutive vertices on an edge-path without backtracks in X_p. The surface v_1 is obtained from v_0 by blowing up a point s in $\mathbb{P}^1_p$ to a -1-curve. The surface $v_2=(S_2,\varphi_2)$ is obtained from v_1 by blowing down the other -1-curve in the fiber over p to a point s_2. In particular, the birational transformation φ_2^{-1} sends the entire $\mathbb{P}^1_p\smallsetminus\{s\}$ to s_2.

Continuing, for $i\geqslant 2$ the surface v_{2i-1} is obtained from v_{2i-2} by blowing up a point in the fiber over p distinct from s_{2i-2} to a -1-curve, and the surface v_{2i} is obtained from v_{2i-1} by blowing down the other -1-curve in the fiber over p to a point s_{2i}. Thus the birational transformation φ_{2i}^{-1} sends the entire $\mathbb{P}^1_p\smallsetminus\{s\}$ to s_{2i}. In particular, φ_{2i} is not an isomorphism, and so $v_{2i}\neq v_0$. This proves that there is no cycle in X_p, and so X_p is a tree. □

We choose a preferred vertex $x_p=(\mathbb{F}_0,\mathrm{id})$ in each X_p, and we consider the family of pointed sets $\{(X_p,x_p)\}_{p\in\mathbb{P}^1}$.

Lemma 3.4. — *The cube complex $\mathscr{J}$ is isomorphic to $\bigoplus_{p\in\mathbb{P}^1}(X_p,x_p)$.*

Proof of Lemma 3.4. — Let (S,φ) be a vertex of $\mathscr{J}$. The birational transformation $\varphi\colon S\dashrightarrow\mathbb{F}_0$ is decomposed as $\varphi=\sigma_n\cdots\sigma_1$, where each σ_i is the blow-up of a point or the blow-down of a -1-curve in the fiber over a point $p_i\in\mathbb{P}^1$. For $p_i\neq p_j$, we have that σ_i and σ_j commute. Thus there is a finite subset $Q\subset\mathbb{P}^1$ such that $\varphi=\Pi_{p\in Q}\varphi_p$,

where each (of the commuting) φ_p is a composition of σ_i with $p_i = p$. Consider the marked surfaces $\varphi_p \colon S_p \dashrightarrow \mathbb{F}_0$ for $p \in Q$, and $\varphi_p = \mathrm{id}, S_p = \mathbb{F}_0$ for $p \in \mathbb{P}^1 \smallsetminus Q$. They represent vertices of X_p. Then $\{(S_p, \varphi_p)\}_{p\in\mathbb{P}^1}$ defines a vertex of $\bigoplus_{p\in\mathbb{P}^1}(X_p, x_p)$.

This is a bijective correspondence between the vertices of $\mathscr{J}$ and $\bigoplus_{p\in\mathbb{P}^1}(X_p, x_p)$, and it extends to an isomorphism on the entire complexes. □

The coordinate y_p of a point $y \in \bigoplus_{p\in\mathbb{P}^1}(X_p, x_p) = \mathscr{J}$ should be thought of as the "marked fiber" in the surface corresponding to y over the point p. Thus modifying the coordinate y_p of y corresponds to performing an alternating sequence of blow-ups of points and blow-downs of -1 curves in the fiber over p of the surface corresponding to y.

The *Jonquières group* is the subgroup of $\mathrm{Bir}(\mathbb{F}_0)$ consisting of the *Jonquières transformations* f that preserve the rational fibration $\pi_0 \colon \mathbb{F}_0 \to \mathbb{P}^1$. The Jonquières group acts on the vertex set of $\mathscr{J}$ by $f \cdot (S, \varphi) = (S, f\varphi)$, and this action extends to an action by isometries on the entire $\mathscr{J}$.

Remark 3.5. — A subgroup G of the Jonquières group is a subgroup of $\mathrm{Aut}(S)$ for some conic bundle $\pi \colon S \to \mathbb{P}^1$ if and only if G fixes a point in $\mathscr{J}$. This is because if G fixes an interior point of a cube in $\mathscr{J}$ described via a surface S in Remark 3.1, then G fixes this cube and in particular G fixes the vertex corresponding to S. By the definition of a marked surface this is equivalent to $G < \mathrm{Aut}(S)$.

We conclude with the following.

Proof of Theorem 1.1. — By Remark 1.2, we can assume the base-field k to be algebraically closed. By Remark 1.3, we can assume G to be a subgroup of the Jonquières group.

As a consequence of Lemma 3.4, the complex $\bigoplus_{p\in\mathbb{P}^1}(X_p, x_p)$ inherits the action of the Jonquières group from $\mathscr{J}$. Because X_p are trees (see Lemma 3.3), their isometry groups are decent. All the elements of the Jonquières group induce elements of $\mathrm{Aut}(\mathbb{P}^1)$ on $\mathbb{P}^1$. Moreover, by Remark 3.5 and by Lemma 2.1, to prove Theorem 1.1 for a subgroup G of the Jonquières group, we need to find a fixed-point for G in $\mathscr{J}$, which is guaranteed by Theorem 1.10. □

4. Proof of the main theorem

In this section we prove Theorem 1.10. We keep the notation $\mathscr{J}$ for the general $\bigoplus_{p\in\mathbb{P}^1}(X_p, x_p)$ (and not just for the Jonquières complex). We denote by $\hbar$ the quotient map $G_0 \wr_P H \to H$. Abusing the notation, for $f \in G_0 \wr_P H$ and $p \in P$, we denote by $f(p) \in P$ the point $\hbar(f)(p)$.

Convention 4.1. — Note that for $f = hg \in G^{\oplus}$, with $h \in H$ and $g \in \Pi_{p\in P} G_p$, and for $y \in \mathscr{J}$, the value $f(y)_p$ equals $g_{h^{-1}(p)}\big(y_{h^{-1}(p)}\big)$ and so it depends only on $h, g_{h^{-1}(p)}$ and $y_{h^{-1}(p)}$. Henceforth, slightly abusing the notation, we will refer to $f(y)_p$ as $f(y_{f^{-1}(p)})$.

This conveys the fact that for a Jonquières transformation f, the "marked fiber" in $f(S)$ over p depends only on f and the "marked fiber" in S over $f^{-1}(p)$.

Lemma 4.2. — *Let G_0 be a group acting on X_0 so that each subgroup of G_0 with a finite orbit fixes a point of X_0 (which is the first item of Definition* 1.4 *for G_0). Then each subgroup of $G < G^{\oplus}$ with a finite orbit Y in $\mathscr{J}$ fixes a point of $\mathscr{J}$ (which is the first item of Definition* 1.4 *for G).*

Proof. — For $p \in P$, let $Y_p = \{y_p \colon y \in Y\}$ denote the finite set of the coordinates of Y in the factor X_p.

First, we consider each finite orbit $O \subset P$ of $\hbar(G) < H$. If for all $p \in O$ we have $Y_p = \{x_p\}$, then we define $z_p = x_p$ for all $p \in O$. Since $Y \subset \mathscr{J}$, there are only finitely many finite orbits O of $\hbar(G) < H$ with $Y_p \neq \{x_p\}$ for some $p \in O$. For each such O, we choose $o \in O$. Let G'_o be the projection to G_o of the stabilizer of o in G, that is, the group of all g_o over $g \in G$ fixing o. Note that Y_o contains an orbit of G'_o. Thus G'_o has a finite orbit and hence a fixed-point $z_o \in X_o$ by the hypothesis of the lemma. We set $z_p = f(z_o)$ (see Convention 4.1) for any $f \in G$ with $f(o) = p$, which only depends on p and not on f since z_o was fixed by G'_o.

Now, let $O \subset P$ be an infinite orbit of $\hbar(G) < H$. Since $Y \subset \mathscr{J}$, there is $o \in O$ with $Y_o = \{x_o\}$. Consequently, for any $p \in O$, we have that Y_p consists only of a single element, which we call z_p. Since $Y \subset \mathscr{J}$, this z_p equals x_p for all but finitely many p.

Consequently, we have $z = \{z_p\}_p \in \mathscr{J}$. By construction, z is a fixed-point of G. □

4.1. Biregularity. — The following encapsulates the idea of a Jonquières transformation f having a persistent base-point in the fiber over a point $p \in P$.

Definition 4.3. — Let $z = \{z_r\}_r \in \mathscr{J}$ be a distinguished vertex. Let $p \in P$ and $f \in G^{\oplus}$. We say that f is *biregular over p* (with respect to z) if $f(z)_{f(p)} = z_{f(p)}$ (or, in our notation from Convention 4.1, $f(z_p) = z_{f(p)}$). Equivalently, for $f = hg$ with $h \in H, g \in \Pi_p G_0$, we have $g_p(z_p) = z_{f(p)}$ (which equals $z_{h(p)}$). Otherwise, we say that f is *singular* (with respect to z) *over p*.

An element $f \in G^{\oplus}$ *has persistent fiber over $p \in P$* if there exists $\ell \geqslant 1$ such that, for all $n \geqslant \ell$, we have that f^n is singular over p and f^{-n} is biregular over p.

Remark 4.4. — Note that f is biregular over p if and only if f^{-1} is biregular over $f(p)$. Furthermore, if f is biregular over p and f' is biregular over $f(p)$, then $f'f$ is biregular over p.

Remark 4.5. — The point z is a fixed-point for f if and only if f is biregular over all $p \in P$.

Remark 4.6. — If f has persistent fiber over p, then it does not have a fixed-point $y \in \mathscr{J}$. Indeed, the orbit of p under $\langle f \rangle$ is infinite, since $f^m(p) = p$ with $m > 0$ would imply, by Remark 4.4, that $f^{\ell m}$ is simultaneously biregular and singular over p. Furthermore, for all $n \geqslant \ell$, we have $z_{f^n(p)} \neq f^n(z_p) = f^n\big(f^n(z_{f^{-n}(p)})\big)$. For large n, we have $y_{f^{\pm n}(p)} = z_{f^{\pm n}(p)}$, contradicting $y_{f^n(p)} = f^{2n}(y_{f^{-n}(p)})$.

4.2. Abelian case. — The following proves Theorem 1.10 in the case where $\hbar(G)$ is abelian. Note that here, as well as in Lemma 4.9, we do not need to assume that P is the projective line $\mathbb{P}^1$.

Lemma 4.7. — *Let G_0 be a group acting decently on X_0. Let H be a group acting on P, and let $G < G^{\oplus}$ be such that either $\hbar(G) < H$ is trivial or it contains an element that has only finitely many finite orbits on P.*(1) *If G is finitely generated, acts purely elliptically on $\mathscr{J}$, and $\hbar(G)$ is abelian, then G fixes a point of $\mathscr{J}$.*

Proof. — If $\hbar(G)$ is trivial, then $G < \Pi_{p\in P} G_p$. Let $Q \subset P$ be a finite set such that each generator $\{g_p\}_{p\in P}$ of G satisfies $g_p(x_p) = x_p$ for $p \notin Q$. For each $q \in Q$, the projection of G to G_q is a purely elliptic subgroup, so it fixes a point $y_q \in X_q$. Setting $y_p = x_p$ for $p \notin Q$, we obtain a fixed-point $\{y_p\}_p$ for G in $\mathscr{J}$.

Thus from now on we can assume that there is $t \in G$ with $\hbar(t)$ having only finitely many finite orbits on P. Let $z \in \mathscr{J}$ be a fixed-point of t. Then t is biregular with respect to z over each $p \in P$ by Remark 4.5. Let $Q \subset P$ be the union of the finite orbits of $\langle t\rangle$. Note that Q is preserved by $\hbar(G)$, since $\hbar(G)$ is abelian.

Let $p \in P \smallsetminus Q$. We claim that any $f \in G$ is biregular with respect to z over p. To justify the claim, let B be the finite set of $b \in P \smallsetminus Q$ over which f or f^{-1} is singular. We will show that for some $n > 0$, setting $f_n = t^n f$, we have $f_n^i(p) \notin B$, for all $i \in \mathbb{Z}\smallsetminus\{0\}$. To find such n, suppose first that $f^\ell(p) = t^k(p)$ for some $k, \ell \in \mathbb{Z}$ with $\ell > 0$. Then choose $m_0 > 0$ such that, for all $m \geqslant m_0$, and all $0 \leqslant j < \ell$, we have $t^{\pm m} f^j(p) \notin B$. It suffices then to take $n = m_0 + |k|$, since for any $i \neq 0$ we have $f_n^i(p) = t^{ni} f^i(p) = t^{\pm m} f^j(p)$ for some $0 \leqslant j < \ell$ and $m \geqslant m_0$. Otherwise, if there is no $\ell \neq 0$ with $f^\ell(p) \in \langle t\rangle(p)$, then there are finitely many $k \in \mathbb{Z}$ with $t^k(p) \in \langle f\rangle B$, since B is finite. It suffices then to take n larger than the maximum of their $|k|$.

Thus we have $f_n^i(p) \notin B$, for all $i \in \mathbb{Z}\smallsetminus\{0\}$. If f was singular with respect to z over p, then by Remark 4.4 $f_n \in G$ would have persistent fiber over p (with $\ell = 1$), contradicting Remark 4.6 and justifying the claim.

For each orbit O of $\hbar(G)$ in Q, choose $o \in O$ and $y_o \in X_o$ that is fixed by the projection to G_o of the stabilizer of o in G, which is of finite index in G, hence finitely generated. For each $f \in G$, choose $y_{f(o)} = f(y_o)$ (notation from Convention 4.1), which only depends on $f(o)$ and not on f, since y_o was fixed by the projection to G_o of the stabilizer of o in G. Setting $y_p = z_p$ for $p \notin Q$ gives us a fixed-point y for G. □

4.3. Semisimple case. — In order to treat the case of non-abelian $\hbar(G)$, we use the dynamics of elements in $\mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(k)$. An element $a \in \mathrm{PGL}_2(k)$ is *semisimple*, if it is conjugate to a diagonal element. With respect to suitable local coordinates, the automorphism of $\mathbb{P}^1$ induced by a is given by $z \mapsto \lambda z$ for some $\lambda \in k^*$. Let us observe that a is of infinite order if and only if λ is not a root of unity. In this case, a fixes exactly two points of $\mathbb{P}^1$ and it does not have any other finite orbit. If k admits

(1) Chris Karpinski proved that this assumption can be removed [Kar24].

a norm $|\cdot|$ with $|\lambda| \neq 1$, then a has north-south dynamics, defined below, in the topology of $\mathbb{P}^1$ induced by $|\cdot|$.

Definition 4.8. — Let a be a homeomorphism of a Hausdorff topological space P, fixing $p, q \in P$, and having the following property. For any disjoint open sets $U \ni p$, $V \ni q$, there is n such that $a^n(P \smallsetminus V) \subset U$ and $a^{-n}(P \smallsetminus U) \subset V$. We then say that a has *north-south dynamics*.

Lemma 4.9. — *Let G_0 be a group acting decently on X_0. Let H be a group acting on P. Let $a \in H$ have north-south dynamics with p, q as in Definition* 4.8 *for some Hausdorff topology on P in which H acts by homeomorphisms. Suppose also that* $\mathrm{Stab}(p) \cap \mathrm{Stab}(q) < H$ *is abelian. If $G < G^{\oplus}$ is finitely generated, acts purely elliptically on $\mathscr{J}$, and $\hbar(G)$ contains a, then G has a fixed-point in $\mathscr{J}$.*

Proof. — Choose $t \in G$ with $\hbar(t) = a$. Let $z \in \mathscr{J}$ be a fixed-point of t. We claim that for any $f \in G$ and any $r \in P \smallsetminus \{p, q\}$ with $f(r) \neq p, q$, we have that f is biregular with respect to z over r.

To justify the claim, first consider the case where $\hbar(f)$ interchanges p and q. Then the group $\langle a, \hbar(f)\rangle$ is virtually abelian, since it has an index 2 subgroup contained in $\mathrm{Stab}(p) \cap \mathrm{Stab}(q)$. By Lemma 4.7, the group $\langle t, f\rangle$ has a finite orbit, hence a fixed-point y in $\mathscr{J}$ by Lemma 4.2. Since both y and z belong to $\mathscr{J}$, all but finitely many coordinates of y have to be that of z. Thus $t(y) = y$ implies $y_r = z_r$ for all r in all infinite orbits of $\langle t\rangle$, so for all $r \neq p, q$. Since f fixes y, we have that f is biregular with respect to z over r, as desired.

Second, consider the case where $\hbar(f)$ does not interchange p and q. Then after possibly replacing t with t^{-1} and interchanging p with q, we can assume $f(p) \neq q$. Let $B \subset P$ be the finite set of points over which f or f^{-1} is singular with respect to z. Let $U \ni p$ (respectively, $V \ni q$) be an open neighbourhood intersecting $B \cup \{f^{-1}(p), f(q)\}$ only possibly at p (respectively, q), and such that $f(U)$ is disjoint from V, which is possible since $f(p) \neq q$. Assume also that U is disjoint from V. Let $n > 0$ be as in Definition 4.8.

If f is singular over r, then we have $r, f(r) \in P \smallsetminus V$. Thus $t^n f(r) \subset U$, but $t^n f(r) \neq p$ since $f(r) \neq p$. Since $f(U)$ is disjoint from V, and U does not contain $f^{-1}(p)$, unless $f(p) = p$, we obtain inductively, for all $m > 0$, that $(t^n f)^m(r) \in U \smallsetminus p$. We also have $t^{-n}(r) \in V \smallsetminus q$ and analogously we obtain $t^{-n}(t^n f)^m(r) \in V \smallsetminus q$ for all $m < 0$. Consequently, $t^n f$ has persistent fiber over r (with $\ell = 1$), which contradicts Remark 4.6 and finishes the proof of the claim.

Note that if the entire $\hbar(G)$ stabilizes $\{p, q\}$, then we are done by Lemma 4.7. Suppose also for the moment that $\hbar(G)$ does not fix p or q. Then there is $f \in G$ and $r \neq p, q$ with $f(r) = p$. Consider another $f' \in G$ with $r' \neq p, q$ and $f'(r') = p$. Then $f^{-1}f'(r') = r$. Applying the claim above to $f^{-1}f'$, we have $f^{-1}f'(z)_{r'} = z_r$. Consequently, $f'(z)_p = f(z)_p$. We now replace the coordinate z_p of z by $f(z)_p$, which, as we have seen, does not depend on f. Note that the new z is still fixed by t, which

can be verified by substituting above $f' = tf$. Furthermore, now f is biregular over r and f^{-1} is biregular over p. We analogously change the z_q coordinate of z.

We will verify that the new z is a fixed-point for G. It remains to verify the biregularity of $f \in G$ over $u \in \{p, q\}$ in the case where $f(u) \in \{p, q\}$. Choose any $r \neq p, q$ and $f' \in G$ with $f'(r) = u$. Introduce $f'' = ff'$, which satisfies $f''(r) = f(u)$. From the previous paragraph it follows that both f', f'' are biregular over r. This implies that f is biregular over u, as desired.

In the case where $\hbar(G)$ fixes, say, p, we redefine z_p to be the fixed-point of the projection of G to G_p. □

4.4. Conclusion. — Recall that an element $h \in \mathrm{PGL}_2(k)$ is *unipotent*, if it is conjugate to an element of the form $\left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$ for some $c \in k$. By considering the Jordan decomposition, we observe that, for k algebraically closed, every element of $\mathrm{PGL}_2(k)$ is either unipotent or semisimple.

Proof of Theorem 1.10. — Suppose first that $\hbar(G)$ contains a semisimple element a of infinite order with eigenvalues λ, λ^{-1} and fixed-points $p, q \in \mathbb{P}^1$. Since a is of infinite order, λ is not a root of unity. Let $z \in \mathscr{J}$. Let $\widetilde{k} \subset k$ be the smallest field such that the points p and q, the scalar λ, the elements of $\hbar(G)$, and all the points over which the elements of G are singular with respect to z, are defined over $\widetilde{k}$. Since G is finitely generated, the field $\widetilde{k}$ is a finitely generated field extension over the prime field of k.

Let $\mathscr{J}_{\widetilde{k}} = \bigoplus_{p \in \mathbb{P}^1(\widetilde{k})} (X_p, x_p)$, where $\mathbb{P}^1(\widetilde{k})$ is the set of the $\widetilde{k}$-rational points of $\mathbb{P}^1$. Note that the action of G on $\mathscr{J}$ projects to an action on $\mathscr{J}_{\widetilde{k}}$.

By [DK18, Th. 2.65], we can embed $\widetilde{k}$ as a subfield into some local field K with norm $|\cdot|$ such that $|\lambda| \neq 1$. Then a has north-south dynamics on $\mathbb{P}^1(\widetilde{k})$ with respect to the Hausdorff topology on $\mathbb{P}^1(\widetilde{k})$ induced by the topology of $\mathbb{P}^1(K)$. By Lemma 4.9 applied with $P = \mathbb{P}^1(\widetilde{k})$, we have that G fixes a point $\{y_p\}_{p \in \mathbb{P}^1(\widetilde{k})} \in \mathscr{J}_{\widetilde{k}}$. Since all the elements of G are biregular with respect to z over all the points outside $\mathbb{P}^1(\widetilde{k})$, we obtain that G fixes the point $\{y_p\}_{p \in \mathbb{P}^1} \in \mathscr{J}$, where $y_p = z_p$ for $p \notin \mathbb{P}^1(\widetilde{k})$.

Otherwise, if $\hbar(G)$ does not contain a semisimple element of infinite order, then all the elements of $\hbar(G)$ are unipotent or of finite order. By [DK18, Prop. 14.46], there is a finite index subgroup $G' < G$ with $\hbar(G')$ conjugate into the abelian subgroup of the elements of the form $\left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$. Lemma 4.7 implies that also in this case G' (and hence G) fixes a point of $\mathscr{J}$. □

5. Proofs of Theorem 1.6 and Corollary 1.7

5.1. Non-rational surfaces. — Here, we give a proof of Theorem 1.6.

Proof of Theorem 1.6. — Again, by Remark 1.2, we can assume that k is algebraically closed. If S is rational, then the result follows from Theorem 1.1.

If the Kodaira dimension of S is non-negative, then there exists a smooth projective surface T birationally equivalent to S such that $\mathrm{Bir}(T) = \mathrm{Aut}(T)$ and the result follows from Lemma 2.1.

Finally, if the Kodaira dimension of S is $-\infty$, but S is not rational, then S is birationally equivalent to $\mathbb{P}^1 \times C$ for some non-rational smooth curve C. In this case, all the elements $f \in \mathrm{Bir}(S)$ preserve the rational fibration given by the projection to C. Indeed, if F is a general fiber of the second projection $\pi_2 \colon S \to C$, then the restriction of $\pi_2 \circ f$ to F induces a rational map to C. Since $F \cong \mathbb{P}^1$, this rational map cannot be dominant, hence its image is a point, which implies that $f(F)$ is another fiber. If the genus of C is > 1, then $\mathrm{Bir}(C) = \mathrm{Aut}(C)$ is finite. If the genus of C is 1, then C is an elliptic curve and $\mathrm{Bir}(C) = \mathrm{Aut}(C)$ is virtually abelian and virtually consists of translations given by the points $C(k)$. We can thus apply Lemma 4.7. □

5.2. Degree growth of finitely generated groups. — There is a well-known and important correspondence between the dynamical behaviour of birational transformations in $\mathrm{Cr}_2(k)$ and the type of isometry they induce on the infinite dimensional hyperbolic space $\mathbb{H}^\infty$. The following theorem gives in particular a precise description of the degree growth. It is due to several people. We refer to [Can18] for details and references.

Theorem 5.1 (Gizatullin; Diller and Favre; Cantat). — *Let k be an algebraically closed field and $f \in \mathrm{Cr}_2(k)$. Then one of the following is true:*

(1) *The transformation f is algebraic, the isometry of $\mathbb{H}^\infty$ induced by f is elliptic, and the degree sequence $\{\deg(f^n)\}$ is bounded.*

(2a) *The isometry of $\mathbb{H}^\infty$ induced by f is parabolic, $\deg(f^n) \sim cn$ for some $c > 0$, and f preserves a rational fibration.*

(2b) *The isometry of $\mathbb{H}^\infty$ induced by f is parabolic, $\deg(f^n) \sim cn^2$ for some $c > 0$, and f preserves a fibration of genus 1 curves.*

(3) *The isometry of $\mathbb{H}^\infty$ induced by f is loxodromic, $\deg(f^n) = c\lambda^n + \mathcal{O}(1)$ for some $c > 0$ and $\lambda > 1$.*

We prove now Corollary 1.7:

Proof of Corollary 1.7. — If all elements in G induce elliptic isometries on $\mathbb{H}^\infty$, i.e., they are algebraic, then G and hence $D_T(n)$ are bounded by Theorem 1.1. This implies that the orbit of G on $\mathbb{H}^\infty$ is bounded and hence that G fixes a point in $\mathbb{H}^\infty$.

Next, consider the case, where no element in G induces a loxodromic isometry of $\mathbb{H}^\infty$, but there is an element $f \in G$ inducing a parabolic isometry.

First, assume that f preserves a rational fibration. Then all elements in G preserve this same rational fibration (see for instance [Ure17, Lem. 5.3.4]) and after conjugation we may assume that G is a subgroup of the Jonquières group (note that the asymptotic growth of $D_T(n)$ is invariant under conjugation). For an element g in the Jonquières group we have $\deg(g) = (\#\mathfrak{b}(g) + 1)/2$ (see for instance [Lam24]), where $\#\mathfrak{b}(g)$ denotes the number of base-points of g. Since $\#\mathfrak{b}(gh) \leqslant \#\mathfrak{b}(g) + \#\mathfrak{b}(h)$ we obtain that that $D_T(n) \leqslant Kn$, where $K = \max_{g \in T}\{\#\mathfrak{b}(g)\}$. At the same time, since by assumption G contains an element whose degree growth is linear, we have $kn \leqslant D_T(n)$, for some $k > 0$. Hence, $D_T(n) \asymp n$.

Now, assume that f preserves a fibration of curves of genus 1. Again, this implies that all of G preserves the same fibration of curves of genus 1 and after conjugation we may assume that G is a subgroup of $\mathrm{Aut}(S)$, where S is a Halphen surface [Ure17, Lem. 5.3.4]. In this case, the statement follows from Lemma 5.2 below.

Finally, we consider the case, where G contains an element f inducing a loxodromic isometry on $\mathbb{H}^\infty$. By Theorem 5.1, there exist c and λ such that $\deg(f^n) = c\lambda^n + \mathcal{O}(1)$. On the other hand, for $\lambda_2 = \max_{g\in T}\{\deg(g)\}$ we have $D_T(n) \leqslant \lambda_2^n$. This shows that $f \asymp \lambda^n$. □

A *Halphen surface* is a rational smooth projective surface S such that $|-mK_S|$ is a pencil of genus 1 curves with empty base locus for some $m > 0$. The ideas used in the following lemma have been described in [Giz80] (see also [Gri16] and [CD12]). We follow the account described in [Lam24].

Lemma 5.2. — *Let S be a Halphen surface and let $G < \mathrm{Aut}(S)$ be a finitely generated subgroup containing a non-algebraic element f. Then $D_T(n) \asymp cn^2$ for some $c > 0$.*

Proof. — Let us first recall that after possibly passing to a finite index subgroup (which does not change the asymptotic growth of $D_T(n)$), we may assume that G is abelian. Moreover, all algebraic elements in $\mathrm{Aut}(S)$ are of finite order and $\mathrm{Aut}(S)$ does not contain any element inducing a loxodromic isometry on $\mathbb{H}^\infty$ (see for instance [Can18] for these facts). Hence, again up to passing to a finite index subgroup, we may assume that all elements in G induce a parabolic isometry on $\mathbb{H}^\infty$.

Let $N_\mathbb{R}(S)$ be the Néron–Severi space of S. Recall that $N_\mathbb{R}(S)$ comes with an intersection form of signature $(1, \dim(N_\mathbb{R}(S)) - 1)$, which is preserved by the action of $\mathrm{Aut}(S)$ by push-forwards. There exists a nef divisor class $D_0 \in N_\mathbb{R}(S)$ such that $D_0 \cdot D_0 = 0$ and such that $g_* D_0 = D_0$ for all $g \in G$ (in fact, we can take $D_0 = mK_S$). The assumption that all elements in G induce parabolic isometries on $\mathbb{H}^\infty$ implies that for all $f \in G$, the only eigenvectors of f are multiples of D_0. For all $g \in G$, the restriction of g_* to $D_0^\perp/D_0$ has finite order, since g_* preserves an integral lattice and the induced intersection form on $D_0^\perp/D_0$ is negative definite. Hence, up to passing to a finite index subgroup of G, we may assume that the restriction of G to $D_0^\perp/D_0$ is the identity. Let $f_1, \dots, f_k$ be generators of G.

Let $A \in N_\mathbb{R}(S)$ be an ample divisor. Note that we have $f_* A \neq A$ for all $f \in G$ and $A \notin D_0^\perp$. Write $(f_i)_* A = A + R_i$, where $R_i \in D_0^\perp$. Since the restriction of $(f_i)_*$ to $D_0^\perp/D_0$ is the identity, we can write $(f_i)_* R_j = R_j + t_{ij} D_0$ for some $t_{ij} \in \mathbb{R}$. Let us observe that $(f_i)_*(f_j)_* A = (f_j)_*(f_i)_* A$ for all i and j implies that $t_{ij} = t_{ji}$ for all i and j.

By induction, we obtain

$$(f_i)_*^n A = A + nR_i + \frac{n(n-1)t_{ii}}{2} D_0,$$

and, as a consequence,

$$(f_1)_*^{n_1} \cdots (f_k)_*^{n_k} A = A + \sum_i n_i R_i + \left(\sum_i \frac{n_i(n_i-1)}{2} t_{ii} + \sum_{i<j} n_i n_j t_{ij} \right) D_0.$$

Since, by assumption, all the f_i are non-algebraic and preserving a fibration of genus 1 curves, the sequence $\deg(f^n)$ grows quadratically in n, by Theorem 5.1, and hence the t_{ii} are positive. We have that

$$\deg_A(f_1^{n_1} \cdots f_k^{n_k}) = ((f_1)_*^{n_1} \cdots (f_k)_*^{n_k} A) \cdot A.$$

Since A is ample, we have that $A \cdot D_0 > 0$. Hence, $D_T(n)$ has quadratic growth. □

References

[BC16] J. Blanc & S. Cantat – "Dynamical degrees of birational transformations of projective surfaces", *J. Amer. Math. Soc.* **29** (2016), no. 2, p. 415–471.

[BF13] J. Blanc & J.-P. Furter – "Topologies and structures of the Cremona groups", *Ann. of Math. (2)* **178** (2013), no. 3, p. 1173–1198.

[BF21] E. Breuillard & K. Fujiwara – "On the joint spectral radius for isometries of non-positively curved spaces and uniform growth", *Ann. Inst. Fourier (Grenoble)* **71** (2021), no. 1, p. 317–391.

[Can11] S. Cantat – "Sur les groupes de transformations birationnelles des surfaces", *Ann. of Math. (2)* **174** (2011), no. 1, p. 299–340.

[Can18] ———, "The Cremona group", in *Algebraic geometry (Salt Lake City, 2015)*, Proc. Sympos. Pure Math., vol. 97.1, American Mathematical Society, Providence, RI, 2018, p. 101–142.

[CD12] S. Cantat & I. Dolgachev – "Rational surfaces with a large group of automorphisms", *J. Amer. Math. Soc.* **25** (2012), no. 3, p. 863–905.

[Dan20] N.-B. Dang – "Degrees of iterates of rational maps on normal projective varieties", *Proc. London Math. Soc. (3)* **121** (2020), no. 5, p. 1268–1310.

[DK18] C. Druţu & M. Kapovich – *Geometric group theory*, Amer. Math. Soc. Colloquium Publ., vol. 63, American Mathematical Society, Providence, RI, 2018.

[Fav10] C. Favre – "Le groupe de Cremona et ses sous-groupes de type fini", in *Séminaire Bourbaki. Volume 2008/2009*, Astérisque, vol. 332, Société Mathématique de France, Paris, 2010, Exp. No. 998, p. 11–43.

[GLU24] A. Genevois, A. Lonjou & C. Urech – "Cremona groups over finite fields, Neretin groups, and non-positively curved cube complexes", *Internat. Math. Res. Notices* (2024), no. 1, p. 554–596.

[Giz80] M. K. Gizatullin – "Rational G-surfaces", *Izv. Akad. Nauk SSSR Ser. Mat.* **44** (1980), no. 1, p. 110–144.

[Gri16] J. Grivaux – "Parabolic automorphisms of projective surfaces (after M. H. Gizatullin)", *Moscow Math. J.* **16** (2016), no. 2, p. 275–298.

[HO21] T. Haettel & D. Osajda – "Locally elliptic actions, torsion groups, and nonpositively curved spaces", 2021, `arXiv:2110.12431`.

[Jun09] R. Jungers – *The joint spectral radius: theory and applications*, Lect. Notes in Control and Information Sci., vol. 385, Springer, Berlin, Heidelberg, 2009.

[Kar24] C. Karpinski – "Decency of group actions on restricted products", in preparation, 2024.

[Lam24] S. Lamy – "The Cremona group", in preparation, available at `https://www.math.univ-toulouse.fr/~slamy/blog/cremona.html`, 2024.

[LU21] A. Lonjou & C. Urech – "Actions of Cremona groups on CAT(0) cube complexes", *Duke Math. J.* **170** (2021), no. 17, p. 3703–3743.

[NOP22] S. Norin, D. Osajda & P. Przytycki – "Torsion groups do not act on 2-dimensional CAT(0) complexes", *Duke Math. J.* **171** (2022), no. 6, p. 1379–1415.

[RS60] G.-C. Rota & G. Strang – "A note on the joint spectral radius", *Indag. Math.* **22** (1960), no. 4, p. 379–381.

[Sch22] J. Schneider – "Relations in the Cremona group over a perfect field", *Ann. Inst. Fourier (Grenoble)* **72** (2022), no. 1, p. 1–42.

[Ser80] J.-P. Serre – *Trees*, Springer-Verlag, Berlin-New York, 1980.

[Ser10] ———, "Le groupe de Cremona et ses sous-groupes finis", in *Séminaire Bourbaki. Volume 2008/2009*, Astérisque, vol. 332, Société Mathématique de France, Paris, 2010, Exp. No. 1000, p. 75–100.
[Ure17] C. Urech – "Subgroups of Cremona groups", PhD Thesis, University of Basel, 2017.
[Ure18] ———, "Remarks on the degree growth of birational transformations", *Math. Res. Lett.* **25** (2018), no. 1, p. 291–308.

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Anne Lonjou, Department of Mathematics, University of the Basque Country UPV/EHU,
Sarriena s/n, 48940 Leioa, Bizkaia, Spain
& IKERBASQUE, Basque Foundation for Science,
Bilbao, Spain
E-mail : anne.lonjou@ehu.eus

Piotr Przytycki, Department of Mathematics and Statistics, McGill University,
Burnside Hall, 805 Sherbrooke Street West, Montreal, QC, H3A 0B9, Canada
E-mail : piotr.przytycki@mcgill.ca

Christian Urech, Department of Mathematics, ETH Zürich,
Rämistrasse 101, 8092 Zürich, Switzerland
E-mail : christian.urech@math.ethz.ch