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Indranil Biswas & Oscar García-Prada

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ANTI-HOLOMORPHIC INVOLUTIONS OF THE MODULI SPACES OF HIGGS BUNDLES

by Indranil Biswas & Oscar García-Prada

Abstract. — We study anti-holomorphic involutions of the moduli space of G-Higgs bundles over a compact Riemann surface X, where G is a complex semisimple Lie group. These involutions are defined by fixing anti-holomorphic involutions on both X and G. We analyze the fixed point locus in the moduli space and their relation with representations of the orbifold fundamental group of X equipped with the anti-holomorphic involution. We also study the relation with branes. This generalizes work by Biswas–García-Prada–Hurtubise and Baraglia–Schaposnik.

Résumé (Involutions anti-holomorphes des espaces de modules de fibrés de Higgs)
Nous étudions les involutions anti-holomorphes des espaces de modules de G-fibrés de Higgs sur une surface de Riemann compacte X, où G est un groupe de Lie semi-simple complexe. Ces involutions sont définies en fixant des involutions anti-holomorphes à la fois sur X et G. Nous en analysons le lieu des points fixes dans l'espace de modules et leur relation avec les représentation du groupe fondamental orbifold de X muni de l'involution anti-holomorphe. Nous étudions aussi la relation avec les « branes ». Ceci généralise les travaux de Biswas–García-Prada–Hurtubise et Baraglia–Schaposnik.

Contents

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1. Introduction

Let G be a complex semisimple affine algebraic group with Lie algebra $\mathfrak{g}$. Let X be a compact connected Riemann surface. A G–Higgs bundle over X is a pair (E, φ), where E is a holomorphic principal G-bundle over X and φ is a holomorphic section of $E(\mathfrak{g}) \otimes K$ with $E(\mathfrak{g})$ being the vector bundle associated to E for the adjoint action of G on $\mathfrak{g}$ and K being the canonical line bundle on X. We consider the moduli space of polystable G-Higgs bundles $\mathcal{M}(G)$. This has the structure of a hyper-Kähler manifold outside the singular locus.

Let $\alpha : X \to X$ and $\sigma : G \to G$ be anti-holomorphic involutions. We define the two involutions (see Section 4.1 for details)

$$\begin{aligned} \iota(\alpha,\sigma)^{\pm} : \mathcal{M}(G) &\longrightarrow \mathcal{M}(G) \\ (E,\varphi) &\longmapsto (\alpha^*\sigma(E), \pm\alpha^*\sigma(\varphi)). \end{aligned} \tag{1.1}$$

The goal of this paper is to describe the fixed points of these involutions. The fixed points are given by the image of moduli spaces of G-Higgs bundles satisfying a reality condition determined by α and σ, and an element $c \in Z_2^\sigma$, where Z is the center of G and Z_2^σ is the group of elements of order two in Z fixed by σ. For the involution $\iota(\alpha,\sigma)^+$, these are the moduli space of pseudo-real Higgs bundles considered in [5], to which we refer here as $(\alpha, \sigma, c, +)$-pseudo-real G-Higgs bundles. For $\iota(\alpha,\sigma)^-$, the reality condition on the bundle E is the same as that for $\iota(\alpha,\sigma)^+$, but the different sign on φ gives a different reality condition on the moduli space of Higgs bundles, defining objects that we call $(\alpha, \sigma, c, -)$-pseudo-real G-Higgs bundles. When the element $c \in Z_2^\sigma$ is trivial we call these objects real G-Higgs bundles.

The involution $\iota(\alpha,\sigma)^-$ is studied by Baraglia-Schaposnik [3] when σ is the anti-holomorphic involution τ corresponding to a compact real form of G (see also [19]). In [4], they consider the involutions $\iota(\alpha,\sigma)^+$ obtained as a result of composing $\iota(\alpha,\tau)^-$ with the holomorphic involution $\iota^-(\theta)$ of $\mathcal{M}(G)$ defined by $\iota^-(E,\varphi) = (\theta(E), -\theta(\varphi))$, where θ is the holomorphic involution of G given by $\theta = \sigma\tau$ (here one takes a compact conjugation τ commuting with σ). In fact, if we consider the involutions

$$\begin{aligned} \iota(\theta)^{\pm} : \mathcal{M}(G) &\longrightarrow \mathcal{M}(G) \\ (E,\varphi) &\longmapsto (\theta(E), \pm\theta(\varphi)), \end{aligned} \tag{1.2}$$

one has

$$\iota(\alpha,\sigma)^{\pm} = \iota^{\mp}(\theta) \circ \iota^-(\alpha,\tau).$$

The involutions (1.2) have been studied in [12, 13, 15].

In the language of branes [20], the fixed points of $\iota(\alpha,\sigma)^+$ are (A, A, B)–branes, while the fixed points of $\iota(\alpha,\sigma)^-$ are (A, B, A)–branes. What these mean is that the fixed points of $\iota(\alpha,\sigma)^-$ are complex Lagrangian submanifolds with respect to the complex structure J_2 defined on $\mathcal{M}(G)$ by the complex structure of G, while the fixed points of $\iota(\alpha,\sigma)^+$ are complex Lagrangian submanifolds with respect to the complex structure $J_3 = J_1 J_2$ obtained by combining J_2 with the natural complex structure J_1

defined on the moduli space of Higgs bundles for the Riemann surface X. The study of these branes is of great interest in connection with mirror symmetry and the Langlands correspondence in the theory of Higgs bundles (see [20, 18, 3, 2]).

We then identify these involutions in the moduli space of representations of the fundamental group of X in G, and describe the fixed points corresponding to the $(\alpha, \sigma, c, \pm)$-pseudo-real G-Higgs bundles in terms of representations of the orbifold fundamental group of (X, α) in a group whose underlying set is $G \times \mathbb{Z}/2\mathbb{Z}$. The group structure on $G \times \mathbb{Z}/2\mathbb{Z}$ is constructed using the element $c \in Z_2^\sigma$ and an action of $\mathbb{Z}/2\mathbb{Z}$ on G which depends on the sign of the pseudo-reality condition; more precisely, this action is given by the conjugation σ in the "+" case, and the action of $\theta = \sigma\tau$ in the "$-$" case, where τ is a compact conjugation commuting with σ. When c is trivial we obtain the semi-direct products of G with $\mathbb{Z}/2\mathbb{Z}$ for the action σ.

The results of this paper have a straightforward generalization to the case in which G is reductive. In this situation the fundamental group of X is replaced by its universal central extension.

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2. G-Higgs bundles and representations of the fundamental group

2.1. Moduli space of G-Higgs bundles. — Let G be a complex semisimple affine algebraic group. Its Lie algebra will be denoted by $\mathfrak{g}$. Let X be an irreducible smooth projective curve defined over $\mathbb{C}$, equivalently, it is a compact connected Riemann surface. Let g_X be the genus of X; throughout we assume that $g_X \geqslant 2$. The canonical line bundle of X will be denoted by K. For a principal G-bundle E, let $E(\mathfrak{g}) := E \times^G \mathfrak{g}$ be the adjoint vector bundle for E.

A *G-Higgs bundle* over X is a pair (E, φ), where E is a holomorphic principal G-bundle E over X and φ is a holomorphic section of $E(\mathfrak{g}) \otimes K$. Two G-Higgs bundles (E, φ) and (F, ψ) are isomorphic if there is a holomorphic isomorphism of principal G-bundles $f : E \to F$ such that the induced isomorphism

$$\mathrm{Ad}(f) \otimes \mathrm{Id}_K : E(\mathfrak{g}) \otimes K \longrightarrow F(\mathfrak{g}) \otimes K$$

sends φ to ψ.

There are notions of (semi)stability and polystability for G-Higgs bundles (see [8, 14, 7] for example). A G-Higgs bundle (E, φ) is said to be *stable* (respectively, *semistable*) if for every parabolic subgroup $P \subset G$, every holomorphic reduction $\sigma : E_P \to E$ of E to P such that

$$\varphi \in H^0(X, E_P(\mathfrak{p}) \otimes K) \subset H^0(X, E(\mathfrak{g}) \otimes K)$$

and every strictly antidominant character χ of P, we have that $\deg E_P(\chi) > 0$ (respectively, $\deg E_P(\chi) \geqslant 0$). A Higgs bundle (E, φ) is *polystable* if it is semistable and for every P, every reduction and every χ as above such that $\deg E_P(\chi) = 0$, there is a holomorphic reduction $E_L \subset E$ to a Levi subgroup $L \subset P$ such that $\varphi \in H^0(X, E_L(\mathfrak{l}) \otimes K)$.

Let $\mathcal{M}(G)$ denote the *moduli space of semistable G-Higgs bundles* of fixed topological type. This moduli space has the structure of a complex normal quasiprojective variety of dimension $\dim G(g_X - 1)$.

2.2. G-Higgs bundles and Hitchin equations. — As above, let G be a complex semisimple affine algebraic group. Let $H \subset G$ be a maximal compact subgroup. Let (E, φ) be a G-Higgs bundle over a compact Riemann surface X. By a slight abuse of notation, we shall denote the C^∞-objects underlying E and φ by the same symbols. In particular, the Higgs field can be viewed as a $(1,0)$-form $\varphi \in \Omega^{1,0}(E(\mathfrak{g}))$ with values in $E(\mathfrak{g})$. Let

$$\tau \colon \Omega^{1,0}(E(\mathfrak{g})) \longrightarrow \Omega^{0,1}(E(\mathfrak{g}))$$

be the isomorphism induced by the compact conjugation of $\mathfrak{g}$ (with respect to H) combined with the complex conjugation on complex 1-forms. Given a C^∞ reduction of structure group h of the principal G-bundle E to H, we denote by F_h the curvature of the unique connection compatible with h and the holomorphic structure on E; see [1, pp. 191–192, Prop. 5] for the connection.

Theorem 2.1. — *There is a reduction h of structure group of E from G to H that satisfies the Hitchin equation*

$$F_h - [\varphi, \tau(\varphi)] = 0$$

if and only if (E, φ) is polystable.

Theorem 2.1 was proved by Hitchin [17] for $G = \mathrm{SL}(2, \mathbb{C})$, and in [23, 24, 7] for the general case.

Remark 2.2. — When G is reductive the equation in Theorem 2.1 is replaced by the equation

$$F_h - [\varphi, \tau(\varphi)] = c\,\omega,$$

where ω is a Kähler form on X and c is an element in the center of the Lie algebra of G, which is determined by the topology of E.

From the point of view of moduli spaces it is convenient to fix a C^∞ principal H–bundle $\boldsymbol{E}_H$ and study the moduli space of solutions to *Hitchin's equations* for a pair (A, φ) consisting of a H–connection A on $\boldsymbol{E}_H$ and a section $\varphi \in \Omega^{1,0}(X, \boldsymbol{E}_H(\mathfrak{g}))$:

$$\begin{aligned} &F_A - [\varphi, \tau(\varphi)] = 0 \\ &\overline{\partial}_A \varphi = 0. \end{aligned} \tag{2.1}$$

Here d_A is the covariant derivative associated to A, and $\overline{\partial}_A$ is the $(0,1)$ part of d_A. The $(0,1)$ part of d_A defines a holomorphic structure on $\boldsymbol{E}_H$. The gauge group $\mathscr{H}$

of $\boldsymbol{E}_H$ acts on the space of solutions and the moduli space of solutions is

$$\mathcal{M}^{\text{gauge}}(G) := \{(A, \varphi) \text{ satisfying } (2.1)\}/\mathscr{H}.$$

Now, Theorem 2.1 can be reformulated as follows.

Theorem 2.3. — *There is a homeomorphism*

$$\mathcal{M}(G) \cong \mathcal{M}^{\text{gauge}}(G).$$

To explain this correspondence we interpret the moduli space of G-Higgs bundles in terms of pairs $(\overline{\partial}_E, \varphi)$ consisting of a $\overline{\partial}$-operator (holomorphic structure) $\overline{\partial}_E$ on the C^∞ principal G-bundle $\boldsymbol{E}_G$ obtained from $\boldsymbol{E}_H$ by the extension of structure group $H \hookrightarrow G$, and $\varphi \in \Omega^{1,0}(X, \boldsymbol{E}_G(\mathfrak{g}))$ satisfying $\overline{\partial}_E \varphi = 0$. Such pairs are in one-to-one correspondence with G-Higgs bundles (E, φ), where E is the holomorphic G-bundle defined by the operator $\overline{\partial}_E$ on $\boldsymbol{E}_G$. The equation $\overline{\partial}_E \varphi = 0$ is equivalent to the condition that $\varphi \in H^0(X, E(\mathfrak{g}) \otimes K)$. The moduli space of polystable G-Higgs bundles $\mathcal{M}_d(G)$ can now be identified with the orbit space

$$\{(\overline{\partial}_E, \varphi) \mid \overline{\partial}_E \varphi = 0 \text{ and the } G\text{-Higgs bundle is polystable}\}/\mathscr{G},$$

where $\mathscr{G}$ is the gauge group of $\boldsymbol{E}_G$, which is in fact the complexification of $\mathscr{H}$. Since there is a one-to-one correspondence between H-connections on $\boldsymbol{E}_H$ and $\overline{\partial}$-operators on $\boldsymbol{E}_G$, the correspondence given in Theorem 2.3 can be reformulated by saying that in the $\mathscr{G}$–orbit of a polystable G-Higgs bundle $(\overline{\partial}_{E_0}, \varphi_0)$ we can find another Higgs bundle $(\overline{\partial}_E, \varphi)$ whose corresponding pair (d_A, φ) satisfies the Hitchin equation $F_A - [\varphi, \tau(\varphi)] = 0$ with this pair (d_A, φ) being unique up to H-gauge transformations.

2.3. Higgs bundles and representations. — Fix a base point $x_0 \in X$. By a *representation* of $\pi_1(X, x_0)$ in G we mean a homomorphism $\pi_1(X, x_0) \to G$. After fixing a presentation of $\pi_1(X, x_0)$, the set of all such homomorphisms, $\operatorname{Hom}(\pi_1(X, x_0), G)$, can be identified with the subset of G^{2g_X} consisting of $2g_X$-tuples $(A_1, B_1, \ldots, A_{g_X}, B_{g_X})$ satisfying the algebraic equation $\prod_{i=1}^{g_X}[A_i, B_i] = 1$. This shows that $\operatorname{Hom}(\pi_1(X, x_0), G)$ is a complex algebraic variety.

The group G acts on $\operatorname{Hom}(\pi_1(X, x_0), G)$ by conjugation:

$$(g \cdot \rho)(\gamma) = g\rho(\gamma)g^{-1},$$

where $g \in G$, $\rho \in \operatorname{Hom}(\pi_1(X, x_0), G)$ and $\gamma \in \pi_1(X, x_0)$. If we restrict the action to the subspace $\operatorname{Hom}^+(\pi_1(X, x_0), G)$ consisting of reductive representations, the orbit space is Hausdorff. We recall that a *reductive representation* is one whose composition with the adjoint representation in $\mathfrak{g}$ decomposes as a direct sum of irreducible representations. This is equivalent to the condition that the Zariski closure of the image of $\pi_1(X, x_0)$ in G is a reductive group. Define the *moduli space of representations* of $\pi_1(X, x_0)$ in G to be the orbit space

$$\mathcal{R}(G) = \operatorname{Hom}^+(\pi_1(X, x_0), G)/G.$$

For another point $x' \in X$, the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x')$ are identified by an isomorphism unique up to an inner automorphism. Consequently, $\mathcal{R}(G)$ is independent of the choice of the base point x_0.

One has the following (see e.g. [16], [25]).

Theorem 2.4. — *The moduli space $\mathcal{R}(G)$ has the structure of a normal complex variety. Its smooth locus is equipped with a holomorphic symplectic form.*

Given a representation $\rho\colon \pi_1(X, x_0) \to G$, there is an associated flat principal G-bundle on X, defined as

$$E_\rho = \widetilde{X} \times^\rho G,$$

where $\widetilde{X} \to X$ is the universal cover associated to x_0 and $\pi_1(X, x_0)$ acts on G via ρ. This gives in fact an identification between the set of equivalence classes of representations $\mathrm{Hom}(\pi_1(X), G)/G$ and the set of equivalence classes of flat principal G-bundles, which in turn is parametrized by the (nonabelian) cohomology set $H^1(X, G)$.

We have the following:

Theorem 2.5. — *There is a homeomorphism $\mathcal{R}(G) \cong \mathcal{M}(G)$.*

The moduli spaces $\mathcal{M}(G)$ and $\mathcal{R}(G)$ are sometimes referred as the *Dolbeault* and *Betti* moduli spaces, respectively.

The proof of Theorem 2.5 is the combination of two existence theorems for gauge-theoretic equations. To explain this, let $\boldsymbol{E}_G$ be, as above, a C^∞ principal G-bundle over X and $\boldsymbol{E}_H$ a C^∞ reduction of structure group of it to H. Every G–connection D on $\boldsymbol{E}_G$ decomposes uniquely as

$$D = d_A + \psi,$$

where d_A is an H-connection on $\boldsymbol{E}_H$ and $\psi \in \Omega^1(X, \boldsymbol{E}_H(\sqrt{-1}\,\mathfrak{h}))$. Let F_A be the curvature of d_A. We consider the following set of equations for the pair (d_A, ψ):

$$\begin{aligned} &F_A + \tfrac{1}{2}[\psi, \psi] = 0\\ &d_A\psi = 0\\ &d_A^*\psi = 0. \end{aligned} \tag{2.2}$$

These equations are invariant under the action of $\mathscr{H}$, the gauge group of $\boldsymbol{E}_H$. A theorem of Corlette [10], and Donaldson [11] for $G = \mathrm{SL}(2, \mathbb{C})$, says the following.

Theorem 2.6. — *There is a homeomorphism between*

$$\{\textit{Reductive } G\textit{-connections } D \mid F_D = 0\}/\mathscr{G}$$

and

$$\{(d_A, \psi) \textit{ satisfying } (2.2)\}/\mathscr{H}.$$

The first two equations in (2.2) are equivalent to the flatness of $D = d_A + \psi$, and Theorem 2.6 simply says that in the $\mathscr{G}$-orbit of a reductive flat G-connection D_0 we can find a flat G-connection $D = \widetilde{g}(D_0)$ such that if we write $D = d_A + \psi$, the additional condition $d_A^*\psi = 0$ is satisfied. This can be interpreted more geometrically in terms of the reduction $h = \widetilde{g}(h_0)$ of $\boldsymbol{E}_G$ to a principal H-bundle obtained by the action of $\widetilde{g} \in \mathscr{G}$ on h_0. The equation $d_A^*\psi = 0$ is equivalent to the harmonicity of the $\pi_1(X)$-equivariant map $\widetilde{X} \to G/H$ corresponding to the new reduction of structure group h.

To complete the argument, leading to Theorem 2.5, we just need Theorem 2.1 and the following simple result.

Proposition 2.7. — *The correspondence* $(d_A, \varphi) \mapsto (d_A, \psi := \varphi - \tau(\varphi))$ *defines a homeomorphism*

$$\{(d_A, \varphi) \text{ satisfying } (2.1)\}/\mathscr{H} \cong \{(d_A, \psi) \text{ satisfying } (2.2)\}/\mathscr{H}.$$

2.4. The moduli space as a hyper-Kähler quotient. — We will see now that the moduli space $\mathcal{M}(G)$ has a hyper-Kähler structure. For this, recall first that a hyper-Kähler manifold is a differentiable manifold M equipped with a Riemannian metric g and complex structures J_i, $i = 1, 2, 3$ satisfying the quaternion relations $J_i^2 = -I$, $J_3 = J_1J_2 = -J_2J_1$, $J_2 = -J_1J_3 = J_3J_1$ and $J_1 = J_2J_3 = -J_3J_2$ such that if we define $\omega_i(\cdot\,,\cdot) = g(J_i\cdot\,,\cdot)$, then (g, J_i, ω_i) is a Kähler structure on M. Let Ω_i denote the holomorphic symplectic structure on $\mathcal{M}(G)$ with respect to the complex structure J_i. In fact, $\Omega_1 = \omega_2 + \sqrt{-1}\omega_3$, $\Omega_2 = \omega_3 + \sqrt{-1}\omega_1$ and $\Omega_3 = \omega_1 + \sqrt{-1}\omega_2$.

One way to understand the non-abelian Hodge theory correspondence mentioned above is through the analysis of the hyper-Kähler structure of the moduli spaces involved. We explain how these can be obtained as hyper-Kähler quotients. For this, let $\boldsymbol{E}_G$ be a smooth principal G-bundle over X, and let $\boldsymbol{E}_H$ be a fixed reduction of $\boldsymbol{E}_G$ to the maximal compact subgroup H. The set $\mathscr{A}$ of H-connections on $\boldsymbol{E}_H$ is an affine space modelled on $\Omega^1(X, \boldsymbol{E}(\mathfrak{h}))$. Via the Chern correspondence, $\mathscr{A}$ is in one-to-one correspondence with the set $\mathscr{C}$ of holomorphic structures on $\boldsymbol{E}_G$ [1, pp. 191–192, Prop. 5], which is an affine space modelled on $\Omega^{0,1}(X, \boldsymbol{E}_G(\mathfrak{g}))$. Let us denote $\Omega = \Omega^{1,0}(X, \boldsymbol{E}_G(\mathfrak{g}))$. We consider $\mathscr{X} = \mathscr{A} \times \Omega$. Via the identification $\mathscr{A} \cong \mathscr{C}$, we have for $\alpha \in \Omega^{0,1}(X, \boldsymbol{E}_G(\mathfrak{g}))$ and $\psi \in \Omega^{1,0}(X, \boldsymbol{E}_G(\mathfrak{g}))$ the following three complex structures on $\mathscr{X}$:

$$\begin{aligned} J_1(\alpha, \psi) &= (\sqrt{-1}\alpha, \sqrt{-1}\psi)\\ J_2(\alpha, \psi) &= (-\sqrt{-1}\tau(\psi), \sqrt{-1}\tau(\alpha))\\ J_3(\alpha, \psi) &= (\tau(\psi), -\tau(\alpha)), \end{aligned}$$

where τ is the conjugation on $\mathfrak{g}$ defining its compact form $\mathfrak{h}$ (determined fiber-wise by the reduction to $\boldsymbol{E}_H$), combined with the complex conjugation on complex 1-forms.

One has also a Riemannian metric g defined on $\mathscr{X}$: for $\alpha \in \Omega^{0,1}(X, \boldsymbol{E}_G(\mathfrak{g}))$ and $\psi \in \Omega^{1,0}(X, \boldsymbol{E}_G(\mathfrak{g}))$,

$$g((\alpha, \psi), (\alpha, \psi)) = -2\sqrt{-1} \int_X B(\tau(\alpha), \alpha) + B(\psi, \tau(\psi)),$$

where B is the Killing form.

Clearly, J_i, $i = 1, 2, 3$, satisfy the quaternion relations, and define a hyper-Kähler structure on $\mathscr{X}$, with Kähler forms $\omega_i(\cdot\,,\cdot) = g(J_i\cdot\,,\cdot)$, $i = 1, 2, 3$. As shown in [17], the action of the gauge group $\mathscr{H}$ on $\mathscr{X}$ preserves the hyper-Kähler structure and there are moment maps given by

$$\mu_1(A,\varphi) = F_A - [\varphi, \tau(\varphi)], \quad \mu_2(A,\varphi) = \mathrm{Re}(\overline{\partial}_E\varphi), \quad \mu_3(A,\varphi) = \mathrm{Im}(\overline{\partial}_E\varphi).$$

We have that $\boldsymbol{\mu}^{-1}(0)/\mathscr{H}$, where $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$ is the moduli space of solutions to the Hitchin equations (2.1). In particular, if we consider the irreducible solutions (equivalently, smooth) $\boldsymbol{\mu}_*^{-1}(0)$ we have that

$$\boldsymbol{\mu}_*^{-1}(0)/\mathscr{H}$$

is a hyper-Kähler manifold which, by Theorem 2.3, is homeomorphic to the subvariety of smooth points in moduli space $\mathcal{M}(G)$ of stable G-Higgs bundles with the topological class of $\boldsymbol{E}_G$.

Let us now see how the moduli of harmonic flat connections on $\boldsymbol{E}_H$ can be realized as a hyper-Kähler quotient. Let $\mathscr{D}$ be the set of G-connections on $\boldsymbol{E}_G$. This is an affine space modelled on $\Omega^1(X, \boldsymbol{E}_G(\mathfrak{g})) = \Omega^0(X, T^*X \otimes_{\mathbb{R}} \boldsymbol{E}_G(\mathfrak{g}))$. The space $\mathscr{D}$ has a complex structure $I_1 = 1 \otimes \sqrt{-1}$, which comes from the complex structure of the bundle. Using the complex structure of X we have also the complex structure $I_2 = \sqrt{-1} \otimes \tau$. We can finally consider the complex structure $I_3 = I_1 I_2$.

The reduction to H of the G-bundle $\boldsymbol{E}_G$ together with a Riemannian metric in the conformal class of X defines a flat Riemannian metric $g_{\mathscr{D}}$ on $\mathscr{D}$ which is Kähler for the above three complex structures. Hence $(\mathscr{D}, g_{\mathscr{D}}, I_1, I_2, I_3)$ is also a hyper-Kähler manifold. As in the previous case, the action of the gauge group $\mathscr{H}$ on $\mathscr{D}$ preserves the hyper-Kähler structure and there are moment maps

$$\mu_1(D) = d_A^*\psi, \quad \mu_2(D) = \mathrm{Im}(F_D), \quad \mu_3(D) = \mathrm{Re}(F_D),$$

where $D = d_A + \psi$ is the decomposition of D defined by

$$\boldsymbol{E}_G(\mathfrak{g}) = \boldsymbol{E}_H(\mathfrak{h}) \oplus \boldsymbol{E}_H(\sqrt{-1}\mathfrak{h}).$$

Hence the moduli space of solutions to the harmonicity equations (2.2) is the hyper-Kähler quotient defined by

$$\boldsymbol{\mu}^{-1}(0)/\mathscr{H},$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \mu_3)$. The homeomorphism between the moduli spaces of solutions to the Hitchin and the harmonicity equations is induced from the affine map

$$\begin{aligned} \mathscr{A} \times \Omega &\longrightarrow \mathscr{D} \\ (d_A, \varphi) &\longmapsto d_A + \varphi - \tau(\varphi). \end{aligned}$$

One can see easily, for example, that this map sends $\mathscr{A} \times \Omega$ with complex structure J_2 to $\mathscr{D}$ with complex structure I_1 (see [17]).

Now, Theorems 2.3 and 2.6 can be regarded as existence theorems, establishing the non-emptiness of the hyper-Kähler quotient, obtained by focusing on different complex structures. For Theorem 2.3 one gives a special status to the complex structure J_1. Combining the symplectic forms determined by J_2 and J_3 one has the J_1-holomorphic symplectic form $\omega_c = \omega_2 + \sqrt{-1}\omega_3$ on $\mathscr{A} \times \Omega$. The gauge group $\mathscr{G} = \mathscr{H}^{\mathbb{C}}$ acts on $\mathscr{A} \times \Omega$ preserving ω_c. The symplectic quotient construction can also be extended to the holomorphic situation (see e.g. [21]) to obtain the holomorphic symplectic quotient $\{(\overline{\partial}_E, \varphi) \mid \overline{\partial}_E \varphi = 0\}/\mathscr{G}$. What Theorem 2.3 says is that for a class $[(\overline{\partial}_E, \varphi)]$ in this quotient to have a representative (unique up to H-gauge) satisfying $\mu_1 = 0$ it is necessary and sufficient that the pair $(\overline{\partial}_E, \varphi)$ be polystable. This identifies the hyper-Kähler quotient to the set of equivalence classes of polystable G-Higgs bundles on $\boldsymbol{E}_G$. If one now takes J_2 on $\mathscr{A} \times \Omega$ or equivalently $\mathscr{D}$ with I_1 and argues in a similar way, one gets Theorem 2.6 identifying the hyper-Kähler quotient to the set of equivalence classes of reductive flat connections on $\boldsymbol{E}_G$.

3. Real G-Higgs bundles

3.1. Involutions and conjugations of complex Lie groups. —

Let G be a Lie group. We define

$$\operatorname{Int}(G) := \{f \in \operatorname{Aut}(G) \mid f(h) = ghg^{-1}, \text{ for every } h \in G\}.$$

We have that $\operatorname{Int}(G) = \operatorname{Ad}(G)$.

We define the group of outer automorphisms of G as

$$\operatorname{Out}(G) := \operatorname{Aut}(G)/\operatorname{Int}(G).$$

We have a sequence

$$1 \longrightarrow \operatorname{Int}(G) \longrightarrow \operatorname{Aut}(G) \longrightarrow \operatorname{Out}(G) \longrightarrow 1. \tag{3.1}$$

It is well-known that if G is a connected complex reductive group then the extension (3.1) splits (see [22]).

Let G be a complex Lie group and let $G_{\mathbb{R}}$ be the underlying real Lie group. We will say that a real Lie subgroup $G_0 \subset G_{\mathbb{R}}$ is a *real form* of G if it is the fixed point set of a conjugation (anti-holomorphic involution) σ of G.

Now, let G be simple. A compact real form always exists. This follows from the fact that for a simple group there is a maximal compact subgroup $U \subset G$, such that $U^{\mathbb{C}} = G$. From this we can define a conjugation $\tau : G \to G$ such that $G^\tau = U$. Let $\operatorname{Conj}(G)$ be the set of conjugations (i.e., anti-holomorphic involutions) of G. We can define the following equivalence relations in $\operatorname{Conj}(G)$:

$$\sigma \sim \sigma' \text{ if there is } \alpha \in \operatorname{Int}(G) \text{ such that } \sigma' = \alpha\sigma\alpha^{-1},$$

We can define a similar relation $\sim$ in the set $\operatorname{Aut}_2(G)$ of automorphisms of G of order 2.

Remark 3.1. — The equivalence relation $\sim$ for elements in $\operatorname{Aut}_2(G)$ should not be confused with the inner equivalence, meaning the equivalence relation where two elements are equivalent if they map to the same element in $\operatorname{Out}(G)$. It is easy to show that if $\theta \sim \theta'$ then they are inner equivalent.

Cartan [9] shows that there is a bijection

$$\operatorname{Conj}(G)/\sim \;\longleftrightarrow\; \operatorname{Aut}_2(G)/\sim\ .$$

More concretely, one has that given the compact conjugation τ, in each class $\mathrm{Conj}(G)/\sim$ one can find a representative σ commuting with τ so that $\theta:=\sigma\tau$ is an element of $\mathrm{Aut}_2(G)$, and similarly if we start with a class in $\mathrm{Aut}_2(G)/\sim$.

3.2. Pseudo-real principal G-bundles. — We use the notation of Section 3.1. Let G be a semisimple complex affine algebraic group. Let $\tau\in\mathrm{Conj}(G)$ be a compact conjugation of G, and let $\sigma\in\mathrm{Conj}(G)$ commuting with τ, and $\theta=\sigma\tau\in\mathrm{Aut}_2(G)$.

Let $Z^\sigma\subset Z$ be the fixed point locus in the center $Z\subset G$. The subgroup of Z^σ generated by its elements of order two will be denoted by Z_2^σ.

Let X be a compact connected Riemann surface, of genus $g\geqslant 2$, equipped with an anti-holomorphic involution $\alpha : X\to X$.

Definition 3.2. — Let E be a holomorphic principal G-bundle over X. Take any $c\in Z_2^\sigma$. We say that E is (α,σ,c)-*pseudo-real* if E is equipped with an anti-holomorphic map $\widetilde{\alpha}: E\to E$ covering α such that

- $\widetilde{\alpha}(eg)=\widetilde{\alpha}(e)\sigma(g)$, for $e\in E$ and $g\in G$.
- $\widetilde{\alpha}^2(e)=ec$.

If $c=1$, we say that E is (α,σ)-*real*.

Remark 3.3. — An alternative definition of pseudo-real bundle allows for c to be any element of Z. However we can modify $\widetilde{\alpha}$ by the action of an element $a\in Z$ defining a covering map $\widetilde{\alpha}':=\widetilde{\alpha}.a$. By this, the element c gets modified by $c'=a\sigma(a)c$. In particular we can take a lying in Z^σ and the composition is modified by a^2. Therefore if c lies in $(Z^\sigma)^2$, or more generally is of the form $\sigma(a)a$ we can normalize our pseudo-real structure to a real one. But since the natural homomorphism $Z_2^\sigma\to Z^\sigma/(Z^\sigma)^2$ is surjective we can always assume that c is of order 2, as we have done in our definition.

Remark 3.4. — Sometimes to emphasize the pseudo-real structure we will write $(E,\varphi,\widetilde{\alpha})$ for a G-Higgs bundle (E,φ) equipped with a pseudo-real structure $\widetilde{\alpha}$.

Define the quotient
$$G_c:=G/\langle c\rangle.$$
Note that $\langle c\rangle=\mathbb{Z}/2\mathbb{Z}$ if $c\neq 1$. Since c is fixed by σ, the involution σ induced an anti-holomorphic involution of G_c. This anti-holomorphic involution of G_c will be denoted by σ'. Let $(E_G,\widetilde{\alpha})$ be a (α,σ,c)-pseudo-real principal G-bundle on X. Define $E_{G_c}:=E_G/\langle c\rangle$. Note that E_{G_c} is the principal G_c-bundle obtained by extending the structure group of E_G using the quotient homomorphism $G\to G_c$. The above self-map $\widetilde{\alpha}$ of E_G descends to a self-map
$$\widetilde{\alpha}' : E_{G_c}\longrightarrow E_{G_c}.$$
Since $\widetilde{\alpha}^2=c$, we have $\widetilde{\alpha}'\circ\widetilde{\alpha}'=\mathrm{Id}_{E_{G_c}}$. Therefore, $(E_{G_c},\widetilde{\alpha}')$ is a (α,σ')-real principal G_c-bundle.

The pair (X,α) defines a geometrically irreducible smooth projective curve defined over $\mathbb{R}$. This curve defined over $\mathbb{R}$ will be denoted by X'. Assume that $c\neq 1$. Let G'

(respectively, G'_c) be the algebraic group, defined over $\mathbb{R}$, given by the pair (G, σ) (respectively, (G_c, σ')). Consider the short exact sequence of sheaves

$$1 \longrightarrow \langle c\rangle = \mathbb{Z}/2\mathbb{Z} \longrightarrow G' \longrightarrow G'_c \longrightarrow 1$$

on X'. Let

$$H^1_{\text{ét}}(X', G') \longrightarrow H^1_{\text{ét}}(X', G'_c) \xrightarrow{\ \beta\ } H^2_{\text{ét}}(X', \mathbb{Z}/2\mathbb{Z})$$

be the long exact sequence of étale cohomologies corresponding to the above short exact sequence of sheaves on the curve X' defined over $\mathbb{R}$. As noted above, a (α, σ, c)-pseudo-real principal G-bundle on X gives a (α, σ')-real principal G_c-bundle. Note that the isomorphism classes of principal G'_c-bundles on X' are parametrized by the elements of the cohomology $H^1_{\text{ét}}(X', G'_c)$. Indeed, this follows immediately from the fact that any principal G'_c-bundle on X' can be locally trivialized with respect to the étale topology. Therefore, a (α, σ, c)-pseudo-real principal G-bundle on X gives an element of $H^1_{\text{ét}}(X', G'_c)$.

We will give a necessary and sufficient condition for a given (α, σ')-real principal G_c-bundle on X to come from a (α, σ, c)-pseudo-real principal G-bundle.

Let $(E_{G_c}, \widetilde{\alpha}')$ be a (α, σ')-real principal G_c-bundle on X. As explained above, $(E_{G_c}, \widetilde{\alpha}')$ is equivalently a principal G'_c-bundle on X'. This principal G'_c-bundles on X' will be denoted by F_{G_c}. Consider the adjoint action of G on itself. Since c lies in the center of G, this action of G factors through the quotient group G_c. Let

$$E_{G_c}(G) := E_{G_c} \times^{G_c} G \longrightarrow X$$

be the fiber bundle associated to the principal G_c-bundle E_{G_c} for this action of G_c on G. Since the action of G_c on G preserves the group structure on G, each fiber of $E_{G_c}(G)$ is a group isomorphic to G. The action of G_c on G descends to an action of G_c on the quotient $G/\langle c\rangle = G_c$, and this descended action coincides with the adjoint action of G_c on itself. Therefore, the short exact sequence of groups

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow G \longrightarrow G_c \longrightarrow 1$$

produces a short exact sequence of fiber bundles with group structures

$$1 \longrightarrow X \times (\mathbb{Z}/2\mathbb{Z}) \longrightarrow E_{G_c}(G) \longrightarrow \mathrm{Ad}(E_{G_c}) \longrightarrow 1, \tag{3.2}$$

where $\mathrm{Ad}(E_{G_c}) = E_{G_c} \times^{G_c} G_c$ is the adjoint bundle for E_{G_c}.

The involution $\widetilde{\alpha}'$ of E_{G_c} and the involution σ of G together produce an anti-holomorphic involution of $E_{G_c}(G)$ covering α. Similarly, $\widetilde{\alpha}'$ and σ' together produce an anti-holomorphic involution of $\mathrm{Ad}(E_{G_c})$ covering α. Therefore, (3.2) produces a short exact sequence

$$1 \longrightarrow X' \times (\mathbb{Z}/2\mathbb{Z}) \longrightarrow E_{G_c}(G)' \longrightarrow \mathrm{Ad}(E_{G_c})' \longrightarrow 1 \tag{3.3}$$

over the curve X' defined over $\mathbb{R}$. We note that $\mathrm{Ad}(E_{G_c})'$ is the adjoint bundle for the principal G'_c-bundle F_{G_c} over X' defined by the pair $(E_{G_c}, \widetilde{\alpha}')$.

The space of all isomorphism classes of principal G'_c-bundles on X' are parametrized by $H^1_{\text{ét}}(X', \mathrm{Ad}(E_{G_c})')$. This identification is constructed as follows. First recall that $\mathrm{Ad}(E_{G_c})'$ is the adjoint bundle for the principal G'_c-bundle F_{G_c} over X'. Given a principal G'_c-bundle on X', by choosing étale local isomorphisms of it with F_{G_c} we get an element of $H^1_{\text{ét}}(X', \text{Ad}(E_{G_c})')$. Conversely, given a 1–cocycle on X' with values in $\text{Ad}(E_{G_c})'$, by gluing back, using the cocycle, the restrictions of F_{G_c} to the open subsets for the cocycle, we get a principal G'_c-bundle on X'. Note that if F_{G_c} is the trivial principal G'_c-bundle, then $H^1_{\text{ét}}(X', \text{Ad}(E_{G_c})') = H^1_{\text{ét}}(X', G'_c)$.

The set $H^1_{\text{ét}}(X', \text{Ad}(E_{G_c})')$ has a distinguished base point t_0. This point t_0 corresponds to the isomorphism class of the principal G'_c-bundle F_{G_c}.

Consider the short exact sequence of étale cohomologies

$$H^1_{\text{ét}}(X', E_{G_c}(G)') \xrightarrow{\gamma'} H^1_{\text{ét}}(X', \text{Ad}(E_{G_c})') \xrightarrow{\beta'} H^2_{\text{ét}}(X', \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \tag{3.4}$$

associated to (3.3). It can be shown that $(E_{G_c}, \widetilde{\alpha}')$ is given by a (α, σ, c)-pseudo-real principal G-bundle if and only if the base point $t_0 \in H^1_{\text{ét}}(X', \text{Ad}(E_{G_c})')$ lies in the image of the map γ' in (3.4). Indeed, if $(E_G, \widetilde{\alpha})$ is a (α, σ, c)-pseudo-real principal G-bundle on X that gives $(E_{G_c}, \widetilde{\alpha}')$, then the adjoint bundle $\text{Ad}(E_G)$ equipped with the involution constructed using $\widetilde{\alpha}$ and σ produces an element $t' \in H^1_{\text{ét}}(X', E_{G_c}(G)')$ such that $\gamma'(t') = t_0$. Conversely, any $t' \in H^1_{\text{ét}}(X', E_{G_c}(G)')$ produces a (α, σ, c)-pseudo-real principal G-bundle. If $\gamma'(t') = t_0$, then this (α, σ, c)-pseudo-real principal G-bundle gives the pair $(E_{G_c}, \widetilde{\alpha}')$.

Therefore, we have the following.

Proposition 3.5. — *A (α, σ')-real principal G_c-bundle $(E_{G_c}, \widetilde{\alpha}')$ on X comes from a (α, σ, c)-pseudo-real principal G-bundle if and only if $\beta'(t_0) = 0$, where β' is the map in (3.4) and $t_0 \in H^1_{\text{ét}}(X', \text{Ad}(E_{G_c})')$ is the base point.*

The following proposition shows the relation between the reality conditions defined by conjugations of G that are inner equivalent. One has the following.

Proposition 3.6. — *Let $\sigma, \sigma' \in \text{Conj}(G)$ such that $\sigma' = \text{Int}(g_0)\sigma$ for some $g_0 \in G$, i.e., $\sigma'(g) = g_0\sigma(g)g_0^{-1}$. Let E be a G-bundle over X. Then E is (α, σ, c)-pseudo-real if and only if it is (α, σ', c')-pseudo-real, where c and c' are related by g_0 and σ. In fact $c' = c$, if $\sigma(g_0) = g_o^{-1}$.*

Proof. — Let $(E, \widetilde{\alpha})$ be a (α, σ, c)-pseudo-real principal G-bundle on X. Define

$$\widetilde{\alpha}' : E \longrightarrow E, \quad e \longmapsto \widetilde{\alpha}(e)g_0^{-1}.$$

Since $\widetilde{\alpha}$ is anti-holomorphic and covers α, the map $\widetilde{\alpha}'$ is also anti-holomorphic and covers α. For any $e \in E$ and $g \in G$, we have

$$\widetilde{\alpha}'(eg) = \widetilde{\alpha}(eg)g_0^{-1} = \widetilde{\alpha}(e)\sigma(g)g_0^{-1} = \widetilde{\alpha}(e)g_0^{-1}g_0\sigma(g)g_0^{-1} = \widetilde{\alpha}'(e)\sigma'(g).$$

Also,

$$\widetilde{\alpha}'(\widetilde{\alpha}'(e)) = \widetilde{\alpha}'(\widetilde{\alpha}(e)g_0^{-1}) = \widetilde{\alpha}(\widetilde{\alpha}(e)g_0^{-1})g_0^{-1} = \widetilde{\alpha}(\widetilde{\alpha}(e))\sigma(g_0^{-1})g_0^{-1} = ec\sigma(g_0^{-1})g_0^{-1}.$$

Now, $\sigma'^2 = \text{Id}$ implies that $\sigma(g_0^{-1})g_0^{-1} \in Z$, and we can appeal to Remark 3.3 to claim that by modifying $\widetilde{\alpha}'$ by an element of the center $c\sigma(g_0^{-1})g_0^{-1}$ is replaced by an element $c' \in Z_2^\sigma$, and hence E has the structure of a (α, α', c')-pseudo-real principal G-bundle on X. The last claim in the proposition is clear. □

3.3. Pseudo-real G-Higgs bundles. — Let $(E,\widetilde{\alpha})$ be a (α,σ,c)-pseudo-real principal G-bundle on X as defined above. Let

$$\mathrm{ad}(E) := E\times^G \mathfrak{g} =: E(\mathfrak{g})$$

be the adjoint vector bundle for E. The self-map $\widetilde{\alpha}$ of E produces an anti-holomorphic self-map

$$\widetilde{\alpha}_0 : E(\mathfrak{g}) \longrightarrow E(\mathfrak{g}) \tag{3.5}$$

such that $q\circ\widetilde{\alpha}_0 = \alpha\circ q$, where q is the projection of $E(\mathfrak{g})$ to X. Since $c\in Z$, the adjoint action of c on $\mathfrak{g}$ is trivial. This immediately implies that $\widetilde{\alpha}_0$ is an involution. In other words, $(E(\mathfrak{g}),\widetilde{\alpha}_0)$ is a real vector bundle (see [5]).

The real structure of the canonical line bundle K of X given by α and the above real structure $\widetilde{\alpha}_0$ of $E(\mathfrak{g})$ combine to define a real structure on the vector bundle $E(\mathfrak{g})\otimes K$. For notational convenience, this real structure on $E(\mathfrak{g})\otimes K$ will also be denoted by $\widetilde{\alpha}$. So

$$\widetilde{\alpha} : E(\mathfrak{g})\otimes K \longrightarrow E(\mathfrak{g})\otimes K$$

is an anti-holomorphic involution over α.

Definition 3.7. — Let (E,φ) be a G-Higgs bundle. We say that (E,φ) is $(\alpha,\sigma,c,+)$-*pseudo-real* (respectively, $(\alpha,\sigma,c,-)$-*pseudo-real*) if E is (α,σ,c)-pseudo-real, and φ satisfies

$$\widetilde{\alpha}(\varphi)=\varphi \quad \text{(respectively, } \widetilde{\alpha}(\varphi)=-\varphi).$$

The concept of $(\alpha,\sigma,c,+)$-*pseudo-real* Higgs bundle was introduced in [5], where notions of (semi)stability and polystability for these objects were defined. These notions are identical for the $(\alpha,\sigma,c,-)$-pseudo-real case. For the benefit of the reader we recall the basic definitions and facts (see [5] for details).

Let $\mathrm{Ad}(E) := E\times^G G$ be the group-scheme over X associated to E for the adjoint action of G on it self. The bundle $\mathrm{Ad}(E)$ is equipped with an anti-holomorphic involution

$$\widetilde{\alpha} : \mathrm{Ad}(E) \longrightarrow \mathrm{Ad}(E) \tag{3.6}$$

(abusing notation again) covering α. Note that $\widetilde{\alpha}^2 = \mathrm{Id}_{\mathrm{Ad}(E)}$ since the adjoint action of Z^σ on G is trivial.

A *parabolic subgroup scheme* of $\mathrm{Ad}(E)$ is a Zariski closed analytically locally trivial subgroup scheme $\underline{P}\subset\mathrm{Ad}(E)$ such that $\mathrm{Ad}(E)/\underline{P}$ is compact. For such a parabolic subgroup scheme $\underline{P}$ let $\underline{\mathfrak{p}}\subset\mathrm{ad}(E)$ be the corresponding bundle of Lie algebras.

A $(\alpha,\sigma,c,\pm)$-*pseudo-real* G-Higgs bundle $(E,\varphi,\widetilde{\alpha})$ is *semistable* (respectively *stable*) if for every proper parabolic subgroup scheme $\underline{P}\subset\mathrm{Ad}(E)$ such that $\widetilde{\alpha}(\underline{P})\subset\underline{P}$, where $\widetilde{\alpha}$ is given by (3.6), and $\varphi\in H^0(X,\underline{\mathfrak{p}}\otimes K)$,

$$\deg(\underline{\mathfrak{p}})\leqslant 0 \quad \text{(respectively, } \deg(\underline{\mathfrak{p}})<0),$$

where $\underline{\mathfrak{p}}$ is the vector bundle associated to $\underline{P}$ defined above.

One has the following (see [5]).

Proposition 3.8. — *Let* $(E,\varphi,\widetilde{\alpha})$ *be a* $(\alpha,\sigma,c,\pm)$-pseudo-real *G-Higgs bundle.*

(1) *If* (E,φ) *is semistable (respectively stable), in the sense of Section* 2.1*, then* $(E,\varphi,\widetilde{\alpha})$ *is semistable (respectively stable).*

(2) *If* $(E,\varphi,\widetilde{\alpha})$ *is semistable then* (E,φ) *is semistable.*

(3) *If* $(E,\varphi,\widetilde{\alpha})$ *is stable then* (E,φ) *is polystable (in the sense given in Section* 2.1.

To define polystability for a pseudo-real G-Higgs bundle let $\underline{\mathfrak{p}}\subset \mathrm{ad}(E)$ be a parabolic subalgebra bundle such that $\widetilde{\alpha_0}(\underline{\mathfrak{p}})=\underline{\mathfrak{p}}$, where $\widetilde{\alpha_0}$ is the involution defined in (3.5).

Let $R_u(\underline{\mathfrak{p}})\subset\underline{\mathfrak{p}}$ be the holomorphic subbundle over X whose fiber over a point $x\in X$ is the nilpotent radical of the parabolic subalgebra $\mathfrak{p}_x$. Therefore, the quotient $\underline{\mathfrak{p}}/R_u(\underline{\mathfrak{p}})$ is a bundle of reductive Lie algebras. Note that $\widetilde{\alpha_0}(R_u(\underline{\mathfrak{p}}))=R_u(\underline{\mathfrak{p}})$. A *Levi subalgebra bundle* of $\underline{\mathfrak{p}}$ is a holomorphic subbundle

$$\ell(\underline{\mathfrak{p}})\subset\underline{\mathfrak{p}}$$

such that for each $x\in X$, the fiber $\ell(\underline{\mathfrak{p}})_x$ is a Lie subalgebra of $\underline{\mathfrak{p}}_x$, and the composition

$$\ell(\underline{\mathfrak{p}})\hookrightarrow\underline{\mathfrak{p}}\longrightarrow\underline{\mathfrak{p}}/R_u(\underline{\mathfrak{p}})$$

is an isomorphism, where $\underline{\mathfrak{p}}\to\underline{\mathfrak{p}}/R_u(\underline{\mathfrak{p}})$ is the quotient map.

A semistable $(\alpha,\sigma,c,\pm)$-pseudo-real G-Higgs bundle $(E,\varphi,\widetilde{\alpha})$ is *polystable* if either is stable, or there is a proper parabolic subalgebra bundle $\underline{\mathfrak{p}}\subsetneq \mathrm{ad}(E)$, and a Levi subalgebra bundle $\ell(\underline{\mathfrak{p}})\subset\underline{\mathfrak{p}}$, such that

$$\widetilde{\alpha_0}(\underline{\mathfrak{p}})=\underline{\mathfrak{p}},\quad \widetilde{\alpha_0}(\ell(\underline{\mathfrak{p}}))=\ell(\underline{\mathfrak{p}}),\quad \varphi\in H^0(X,\ell(\underline{\mathfrak{p}})\otimes K),$$

and for every parabolic subalgebra bundle $\underline{\mathfrak{q}}\subset\ell(\underline{\mathfrak{p}})$ with $\widetilde{\alpha_0}(\underline{\mathfrak{q}})=\underline{\mathfrak{q}}$ we have

$$\deg(\underline{\mathfrak{q}})<0.$$

We have the following (see [5]).

Proposition 3.9. — *A* $(\alpha,\sigma,c,\pm)$*-pseudo-real G-Higgs bundle* $(E,\varphi,\widetilde{\alpha})$ *is polystable if and only if* (E,φ) *is polystable.*

We can thus define the moduli space $\mathcal{M}(G,\alpha,\sigma,c,\pm)$ of isomorphism classes of polystable $(\alpha,\sigma,c,\pm)$-pseudo-real G-Higgs bundles, and, as a consequence of Proposition 3.9, define maps

$$\mathcal{M}(G,\alpha,\sigma,c,\pm)\longrightarrow\mathcal{M}(G) \tag{3.7}$$

that forget the pseudo-real structure.

4. Involutions of moduli spaces

4.1. Involutions of Higgs bundle moduli spaces. — As before, let $\alpha : X \to X$ and $\sigma : G \to G$ be anti-holomorphic involutions. For a holomorphic principal G-bundle E on X, let $\sigma(E)$ be the C^∞ principal G-bundle on X obtained by extending the structure group of E using the homomorphism σ. So the total space of $\sigma(E)$ is identified with that of E, but the action of $g \in G$ on $e \in E$ coincides with the action of $\sigma(g)$ on $e \in \sigma(E)$. Consequently, the pullback $\alpha^*\sigma(E)$ has a holomorphic structure given by the holomorphic structure of E. Let

$$\widetilde{\sigma} : E(\mathfrak{g}) \longrightarrow E(\mathfrak{g})$$

be the conjugate linear isomorphism that sends the equivalence class of any $(e, v) \in E \times \mathfrak{g}$ to the equivalence class of $(e, d\sigma(v))$, where $d\sigma$ is the automorphism of $\mathfrak{g}$ corresponding to σ. Let φ be a Higgs field on E. Let $\sigma(\varphi)$ be the C^∞ section of $E(\mathfrak{g}) \otimes K$ defined by $\widetilde{\sigma}$ and the C^∞ isomorphism $K \to \overline{K}$ defined by $df \mapsto d\overline{f}$, where f is any locally defined holomorphic function on X.

We have involutions

$$\begin{aligned} \iota(\alpha,\sigma)^{\pm} : \mathcal{M}(G) &\longrightarrow \mathcal{M}(G) \\ (E,\varphi) &\longmapsto (\alpha^*\sigma(E), \pm\alpha^*\sigma(\varphi)). \end{aligned} \tag{4.1}$$

Proposition 4.1. — *The image of the map*

$$\mathcal{M}(G,\alpha,\sigma,c,+) \longrightarrow \mathcal{M}(G)$$

in (3.7) *is contained in the fixed point locus of the involution $\iota(\alpha,\sigma)^+$. Moreover, the fixed point locus of $\iota(\alpha,\sigma)^+$ in the smooth locus $\mathcal{M}(G)^{\mathrm{sm}} \subset \mathcal{M}(G)$ is the intersection of $\mathcal{M}(G)^{\mathrm{sm}}$ with the union of the images of $\mathcal{M}(G,\alpha,\sigma,c,+)$ as c runs over Z_2^σ, where Z_2^σ as before is the subgroup of Z^σ generated by the order two points.*

Similarly, the fixed point locus of $\iota(\alpha,\sigma)^-$ in $\mathcal{M}(G)^{\mathrm{sm}}$ is the intersection of $\mathcal{M}(G)^{\mathrm{sm}}$ with the union of the images of $\mathcal{M}(G,\alpha,\sigma,c,-)$ as c runs over Z_2^σ.

Proof. — From the definition of $\iota(\alpha,\sigma)^+$ (respectively, $\iota(\alpha,\sigma)^-$) it follows immediately that $\mathcal{M}(G,\alpha,\sigma,c,+)$ (respectively, $\mathcal{M}(G,\alpha,\sigma,c,-)$) is contained in the fixed point locus of $\iota(\alpha,\sigma)^+$ (respectively, $\iota(\alpha,\sigma)^-$).

A G-Higgs bundle (E,φ) lies in $\mathcal{M}(G)^{\mathrm{sm}}$ if (E,φ) is stable and the automorphism group of (E,φ) coincides with the center Z of G, i.e., if the Higgs bundle is simple as defined in [14, 13, 6] (we recall that such bundles are called regularly stable). Suppose that $(E,\varphi) \in \mathcal{M}(G)^{\mathrm{sm}}$ is fixed under the involution $\iota(\alpha,\sigma)^+$ (respectively $\iota(\alpha,\sigma)^-$). This means that there exists an isomorphism

$$f : E \longrightarrow \alpha^*\sigma(E)$$

such that $\alpha^*\sigma(f)\circ f \in \operatorname{Aut}(E,\varphi)$, but since $\operatorname{Aut}(E,\varphi) = Z$, we have that $\alpha^*\sigma(f)\circ f = c \in Z$. We can interpret f as a map $f' : E \to \sigma(E)$ such that $\sigma(f')\circ f' = c \in Z$. Identifying $\sigma(E)$ with E with multiplication on the right defined by $e \cdot g = e\sigma(g)$, where $g \in G$ and $e \in E$, we are indeed defining a $(\alpha,\sigma,c,+)$ (respectively $(\alpha,\sigma,c,-)$) pseudo-real structure on (E,φ), since we can always assume that $c \in Z_2^\sigma$, as explained in Remark 3.3. In other words, (E, φ) lies in the image of $\mathcal{M}(G, \alpha, \sigma, c, +)$ (respectively $\mathcal{M}(G, \alpha, \sigma, c, -)$). □

Remark 4.2. — In Definition 3.2 we could have defined a pseudo-real structure replacing $\widetilde{\alpha}$ by an anti-holomorphic map $\widetilde{\alpha}' : E \to \sigma(E)$ of G–bundles covering α. Although $\sigma(E)$ is no longer a holomorphic bundle, its total space is a complex manifold because it is identified with the total space of E, and hence the anti-holomorphicity condition makes sense. The condition $\widetilde{\alpha}(eg) = \widetilde{\alpha}(e)\sigma(g)$ in Definition 3.2 is now automatic since $\widetilde{\alpha}'$ is a G–bundle map.

Proposition 4.3. — *Let σ and σ' be inner equivalent elements in* $\mathrm{Conj}(G)$*, i.e., they define the same element in* $\mathrm{Out}_2(G)$*. Then*

$$\iota(\alpha,\sigma)^+ = \iota(\alpha,\sigma')^+ \ (\textit{respectively},\ \iota(\alpha,\sigma)^- = \iota(\alpha,\sigma')^-).$$

Proof. — If we replace σ by $\sigma' := g_0\sigma g_0^{-1}$, where $g_0 \in G$, then the corresponding anti-holomorphic involution of the moduli space is replaced by its composition with the holomorphic automorphism of the moduli space corresponding to the automorphism of G defined by $g \mapsto g_0 g g_0^{-1}$. But this automorphism of G produces the identity map of the moduli space. Therefore, the anti-holomorphic involution of the moduli space is unchanged if σ is replaced by σ'. □

Remark 4.4. — Consider the identification between the (α, σ, c)-pseudo-real principal G-bundles and the (α, σ', c')-pseudo-real principal G-bundles on X given by Proposition 3.6 when σ and σ' are inner equivalent. Note that a Higgs field on a (α, σ', c')-pseudo-real principal G-bundle produces a Higgs field on the corresponding (α, σ, c)-pseudo-real principal G-bundle, and vice versa. We thus have that by Proposition 3.6 $\mathcal{M}(G, \alpha, \sigma, c, +)$ is isomorphic to $\mathcal{M}(G, \alpha, \sigma', c', +)$ (respectively $\mathcal{M}(G, \alpha, \sigma, c, -)$ is isomorphic to $\mathcal{M}(G, \alpha, \sigma', c', -)$) giving the same image under the corresponding maps to $\mathcal{M}(G)$.

4.2. Correspondence with representations for $\iota(\alpha,\sigma)^+$.

— We have the orbifold fundamental group of (X, α) that we will denote $\Gamma(X, \alpha)$ (see [5] for example). This fits into an exact sequence

$$1 \longrightarrow \pi_1(X, x_0) \longrightarrow \Gamma(X, x_0) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 1. \tag{4.2}$$

Let $\mathrm{Map}'(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z}))$ be the space of all maps

$$\delta : \Gamma(X, x_0) \longrightarrow G \times (\mathbb{Z}/2\mathbb{Z})$$

such that the following diagram is commutative:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(X, x_0) & \longrightarrow & \Gamma(X, x_0) & \xrightarrow{\eta} & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \delta & & \| & & \\ 1 & \longrightarrow & G & \longrightarrow & G \times (\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \mathbb{Z}/2\mathbb{Z} & \longrightarrow & 1. \end{array} \tag{4.3}$$

Take an element $c \in Z_2^\sigma$ in the subgroup generated by the elements of Z^σ order two. Using it, we will define another group structure on $G \times (\mathbb{Z}/2\mathbb{Z})$. The group operation is given by

$$(g_1, e_1) \cdot (g_2, e_2) = (g_1(\sigma)^{e_1}(g_2)c^{e_1 e_2}, e_1 + e_2).$$

Note that when $c = 1$ we obtain a semidirect product.

Let $\mathrm{Hom}_c(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z})$ be the space of all maps

$$\delta \in \mathrm{Map}'(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z}))$$

such that δ is a homomorphism with respect to this group structure.

Two elements $\delta, \delta' \in \mathrm{Hom}_c(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z}))$ are called *equivalent* if there is an element $g \in G$ such that $\delta'(z) = g^{-1}\delta(z)g$ for all $z \in \pi_1(X, \alpha)$.

THEOREM 4.5. — *The moduli space $\mathcal{M}(G, \alpha, \sigma, c, +)$ is identified with the space of equivalence classes of reductive elements of* $\mathrm{Hom}_c(\Gamma(X, x_0), G \times (\mathbb{Z}/2\mathbb{Z}))$.

Proof. — This follows from Proposition 5.6 of [5]. □

THEOREM 4.6. — *Consider the involution $\iota(\alpha, \sigma)^+$ of $\mathcal{M}(G)$. It is anti-holomorphic with respect to the almost complex structures J_1 and J_2, and it is holomorphic with respect to J_3.*

Proof. — The almost complex structure J_1 is the almost complex structure of the Dolbeault moduli space (the moduli space of Higgs bundles). Therefore, $\iota(\alpha, \sigma)^+$ is anti-holomorphic with respect to J_1.

The almost complex structure J_2 is the almost complex structure of the Betti moduli space (the representation space $\mathcal{R}(G)$). Note that the almost complex structure of the Betti moduli space coincides with that of the de Rham moduli space.

As before, fix a base point $x_0 \in X$. The involution α of X produces an isomorphism

$$\alpha' : \pi_1(X, x_0) \longrightarrow \pi_1(X, \alpha(x_0)).$$

This in turn gives a biholomorphism

$$\alpha'' : \mathrm{Hom}^+(\pi_1(X, x_0), G)/G \longrightarrow \mathrm{Hom}^+(\pi_1(X, \alpha(x_0)), G)/G.$$

As noted before, $\mathcal{R}(G) = \mathrm{Hom}^+(\pi_1(X, x_0), G)/G$ is independent of the choice of the base point. So α'' is a biholomorphism

$$\alpha'' : \mathcal{R}(G) \longrightarrow \mathcal{R}(G). \tag{4.4}$$

Since α is an involution, it follows that α'' is also an involution.

Let

$$b : \mathcal{R}(G) = \mathrm{Hom}^+(\pi_1(X, x_0), G)/G \longrightarrow \mathcal{R}(G)$$

be the anti-holomorphic involution defined by $\rho \mapsto \sigma \circ \rho$. In other words, b sends a homomorphism $\rho : \pi_1(X) \to G$ to the composition

$$\pi_1(X, x_0) \xrightarrow{\rho} G \xrightarrow{\sigma} G.$$

Clearly b commutes with the above involution α'' in (4.4). Therefore, $b \circ \alpha''$ is also an involution. Note that $b \circ \alpha''$ is anti-holomorphic because α'' is holomorphic and b is anti-holomorphic.

The above involution $b\circ\alpha''$ of $\mathcal{R}(G)$ coincides with the involution $\iota(\alpha,\sigma)^+$ of $\mathcal{M}(G)$ under the correspondence $\mathcal{M}(G)\cong\mathcal{R}(G)$. Therefore, $\iota(\alpha,\sigma)^+$ is anti-holomorphism with respect to J_2.

We recall that $J_3=J_1J_2$. Since $\iota(\alpha,\sigma)^+$ is anti-holomorphic with respect to both J_1 and J_2, from the above identity it follows immediately that $\iota(\alpha,\sigma)^+$ is holomorphic with respect to J_3. □

Since $\mathcal{R}(G)$ is hyper-Kähler, the holomorphic symplectic form Ω_2 on it is flat with respect to the Kähler structure ω_2 corresponding to J_2. Similarly, the holomorphic symplectic form Ω_1 with respect to J_1 is flat with respect to the Kähler structure ω_1 corresponding to J_1. In particular, $\mathcal{R}(G)$ and $(\mathcal{M}(G),J_1,\omega_1,\Omega_1)$ are Calabi-Yau.

Theorem 4.7. — *The moduli space $\mathcal{M}(G,\alpha,\sigma,c,+)$ is a special Lagrangian subspace of $\mathcal{R}(G)$. Similarly, it is special Lagrangian with respect to $(\mathcal{M}(G),J_1,\omega_1,\Omega_1)$. Also, it is complex Lagrangian with respect to (J_3,Ω_3).*

Proof. — Since the involution $\iota(\alpha,\sigma)^+$ is holomorphic with respect to J_3, it follows that $\mathcal{M}(G,\alpha,\sigma,c,+)$ is a holomorphic subspace with respect to J_3. Recall that $\Omega_3=\omega_1+\sqrt{-1}\omega_2$. The involution $\iota(\alpha,\sigma)^+$ is anti-holomorphic with respect to J_1 and J_2 and it is an isometry. Hence $\iota(\alpha,\sigma)^+$ takes ω_1 and ω_2 to $-\omega_1$ and $-\omega_2$ respectively. Hence $\iota(\alpha,\sigma)^+$ takes Ω_3 to $-\Omega_3$. This immediately implies that $\mathcal{M}(G,\alpha,\sigma,c,+)$ is Lagrangian with respect to Ω_3.

Since $\mathcal{M}(G,\alpha,\sigma,c,+)$ is the fixed point locus of an isometric anti-holomorphic involution of the Calabi-Yau space $\mathcal{R}(G)$, it follows that $\mathcal{M}(G,\alpha,\sigma,c,+)$ is a special Lagrangian subspace of $\mathcal{R}(G)$. For a similar reason, $\mathcal{M}(G,\alpha,\sigma,c,+)$ is a special Lagrangian subspace of $(\mathcal{M}(G),J_1,\omega_1)$. □

4.3. Correspondence with representations for $\iota(\alpha,\sigma)^-$. — Next we consider the involution $\iota(\alpha,\sigma)^-$.

Consider the holomorphic involution $\theta=\sigma\tau$ of G as defined earlier in Section 3.1. Using $c\in Z_2^\sigma$, we will define yet another group structure on $G\times(\mathbb{Z}/2\mathbb{Z})$. The group operation is given by

$$(g_1,e_1)\cdot(g_2,e_2)=(g_1(\theta)^{e_1}(g_2)c^{e_1e_2},e_1+e_2).$$

Let $\mathrm{Hom}_c^-(\pi_1(X,\alpha),G\times(\mathbb{Z}/2\mathbb{Z}))$ be the space of all maps

$$\delta\in\mathrm{Map}'(\pi_1(X,\alpha),G\times(\mathbb{Z}/2\mathbb{Z}))$$

such that δ is a homomorphism with respect to this new group structure.

Two elements $\delta',\delta'\in\mathrm{Hom}_c^-(\pi_1(X,\alpha),G\times(\mathbb{Z}/2\mathbb{Z}))$ are called *equivalent* if there is an element $g\in G$ such that $\delta'(z)=g^{-1}\delta(z)g$ for all $z\in\pi_1(X,\alpha)$.

Theorem 4.8. — *The moduli space $\mathcal{M}(G,\alpha,\sigma,c,-)$ is identified with the space of equivalence classes of reductive elements of* $\mathrm{Hom}_c^-(\pi_1(X,\alpha),G\times(\mathbb{Z}/2\mathbb{Z}))$.

Proof. — This follows from Proposition 5.6 of [5]. □

Theorem 4.9. — *Consider the involution $\iota(\alpha,\sigma)^-$ of $\mathcal{M}(G)$. It is anti-holomorphic with respect to the almost complex structures J_1 and J_3, and it is holomorphic with respect to J_2.*

Proof. — The involution $\iota(\alpha,\sigma)^-$ is clearly anti-holomorphic with respect to J_1 because J_1 coincides with the complex structure of the Dolbeault moduli space.

Let

$$\widetilde{b} : \mathcal{R}(G) = \mathrm{Hom}^+(\pi_1(X, x_0), G)/G \longrightarrow \mathcal{R}(G)$$

be the holomorphic involution defined by $\rho \mapsto \theta \circ \rho$. In other words, $\widetilde{b}$ sends a homomorphism $\rho : \pi_1(X) \to G$ to the composition

$$\pi_1(X) \xrightarrow{\ \rho\ } G \xrightarrow{\ \theta\ } G.$$

Clearly $\widetilde{b}$ commutes with the above involution α'' in (4.4). Therefore, $\widetilde{b} \circ \alpha''$ is also an involution. The composition $\widetilde{b} \circ \alpha''$ is holomorphic because both α'' and $\widetilde{b}$ are holomorphic.

The above involution $\widetilde{b}\circ\alpha''$ of $\mathcal{R}(G)$ coincides with $\iota(\alpha,\sigma)^-$, and the complex structure of the Betti moduli space $\mathcal{R}(G)$ is given by J_2. Therefore, $\iota(\alpha,\sigma)^-$ is holomorphic with respect to J_2.

Since $J_3 = J_1 J_2$, and $\iota(\alpha,\sigma)^-$ is anti-holomorphic with respect to J_1 and holomorphic with respect to J_2, we conclude that $\iota(\alpha,\sigma)^-$ is anti-holomorphic with respect to J_3. □

Consider the complex structure J_3 and the corresponding holomorphic symplectic form Ω_3. Since $\mathcal{R}(G)$ is hyper-Kähler, Ω_3 is flat with respect to the Kähler structure for J_3. Now we have following analog of Theorem 4.7.

Theorem 4.10. — *The moduli space $\mathcal{M}(G,\alpha,\sigma,c,-)$ is a special Lagrangian subspace of $(\mathcal{M}(G), J_1, \omega_1, \Omega_1)$. Similarly, it is special Lagrangian with respect to $(\mathcal{M}(G), J_3, \omega_3, \Omega_3)$. Also, it is a complex Lagrangian subspace with respect to $(\mathcal{R}(G), J_2, \Omega_2)$.*

Corollary 4.11. — *The fixed point locus of the involution $\iota(\alpha,\sigma)^-$ is a complex subspace of $\mathcal{M}(G)$ with the complex structure induced by J_2, i.e., the natural complex structure of the moduli space of representations $\mathcal{R}(G)$.*

Remark 4.12. — Corollary 4.11 is obtained by Baraglia–Schaposnik [3] in the case when σ is the compact conjugation τ.

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Indranil Biswas, School of Mathematics, Tata Institute of Fundamental Research
Homi Bhabha Road, Bombay 400005, India
E-mail : `indranil@math.tifr.res.in`

Oscar García-Prada, Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM
Nicolás Cabrera, 13–15, 28049 Madrid, Spain
E-mail : `oscar.garcia-prada@icmat.es`
Url : `http://www.icmat.es/miembros/garcia-prada/index.html`