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Around the bounded L^2 curvature conjecture in general relativity

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Abstract

We report on recent progress obtained on the construction and control of a parametrix to the homogeneous wave equation $\square_{\mathbf{g}}\phi = 0$, where \mathbf{g} is a rough metric satisfying the Einstein vacuum equations. Controlling such a parametrix as well as its error term when one only assumes L^2 bounds on the curvature tensor \mathbf{R} of \mathbf{g} is a major step towards the proof of the bounded L^2 curvature conjecture.

1. Introduction

1.1. Einstein vacuum equations

We start by introducing Einstein vacuum equations. We consider a Lorentzian manifold $(\mathcal{M}, \mathbf{g})$, i.e. \mathcal{M} is four-dimensional, and g is a bilinear form on the tangent space of \mathcal{M} with signature $(-, +, +, +)$. To \mathbf{g} one can associate its Levi-Civita connection \mathbf{D} . Its curvature tensor \mathbf{R} is then defined as:

$$\mathbf{R}(X, Y)Z = \mathbf{D}_X \mathbf{D}_Y Z - \mathbf{D}_Y \mathbf{D}_X Z - \mathbf{D}_{[X, Y]}Z,$$

where X, Y, Z denote three vectorfields on \mathcal{M} . The curvature tensor is a four-tensor and satisfies several algebraic identities, one of them being the Bianchi identity:

$$\mathbf{D}_\tau \mathbf{R}_{\alpha\beta\gamma\delta} + \mathbf{D}_\beta \mathbf{R}_{\tau\alpha\gamma\delta} + \mathbf{D}_\alpha \mathbf{R}_{\beta\tau\gamma\delta} = 0. \quad (1)$$

(1) will be particularly useful in situations where we have better control of derivatives in certain 'good' directions. When we encounter derivatives in 'bad' directions, we will try to use (1) to trade them against derivatives in 'good' directions.

Finally, taking the trace of the curvature tensor, we obtain the Ricci tensor:

$$\mathbf{R}_{\alpha\beta} = \mathbf{R}^\gamma{}_{\alpha\gamma\beta}.$$

This allows us to define Einstein vacuum equations:

$$\mathbf{R}_{\alpha\beta} = 0. \quad (2)$$

Since the Ricci tensor is a symmetric 2-tensor on \mathcal{M} , (2) is a system of 10 equations. Also, the Ricci tensor expressed in any coordinate system takes the form:

$$\phi_1(\mathbf{g})\partial^2\mathbf{g} + \phi_2(\mathbf{g})(\partial\mathbf{g})^2$$

for some nonlinear functions ϕ_1, ϕ_2 , so that (2) is a system of 10 quasilinear second order partial differential equations.

Let us conclude this section by mentioning the simplest explicit solution to (2) which is the Minkowski space-time (\mathbb{R}^{1+3}, m) , with m given by:

$$m = -(dx_0)^2 + (dx_1)^2 + (dx_2)^2 + (dx_3)^2. \quad (3)$$

1.2. The Cauchy problem for Einstein vacuum equations

To get a better sense of (2), it is instructive to consider a particular coordinate system on \mathcal{M} called the wave coordinates. These coordinates satisfy wave equations:

$$\square_{\mathbf{g}}x^\alpha = \frac{1}{\sqrt{|\mathbf{g}|}}\partial_\beta(\mathbf{g}^{\beta\gamma}\sqrt{|\mathbf{g}|}\partial_\gamma)x^\alpha = 0, \alpha = 0, 1, 2, 3. \quad (4)$$

In these coordinates, (2) becomes a system of quasilinear wave equations:

$$\square_{\mathbf{g}}\mathbf{g}_{\alpha\beta} = \mathcal{N}_{\alpha\beta}(\mathbf{g}, \partial\mathbf{g}), \alpha, \beta = 0, 1, 2, 3, \quad (5)$$

where $\mathcal{N}_{\alpha\beta}$ is quadratic with respect to $\partial\mathbf{g}$. The Cauchy data for (5) consist of $\mathbf{g}(0, \cdot)$ and $\partial_t\mathbf{g}(0, \cdot)$. In general, it is preferable to work with a coordinate invariant definition: the Cauchy data for Einstein vacuum equations (2) consist of a Riemannian three dimensional metric g_{ij} and a symmetric 2-tensor k_{ij} on the space-like hypersurface Σ . The Cauchy problem then consists in finding a metric \mathbf{g} satisfying (2) such that the metric induced by \mathbf{g} on Σ coincides with g and the 2-tensor k is the second fundamental form of the hypersurface Σ (k corresponds to $\partial_t\mathbf{g}(0, \cdot)$).

Remark 1. *In the case of the Minkowski space-time (\mathbb{R}^{1+3}, m) , the initial data set is given by $(\mathbb{R}^3, \delta, 0)$ where δ denotes the Euclidean metric.*

Now, it is well known that the Einstein equations form an overdetermined system. As a consequence, the initial data set (Σ, g, k) cannot be prescribed arbitrarily and, in fact, must satisfy the system of constraint equations:

$$\begin{cases} \nabla^j k_{ij} - \nabla_i \text{Tr}k = 0, \\ R - |k|^2 + (\text{Tr}k)^2 = 0, \end{cases} \quad (6)$$

where the covariant derivative ∇ is defined with respect to the metric g , and R is the scalar curvature of g .

In this paper, we would like to consider the local existence for (2). In particular, we are interested in the minimal regularity properties of the initial data set (Σ, g, k) which guarantee the existence and uniqueness of local developments. In view of (5), we will start by recalling well-known facts about the local existence theory for nonlinear wave equations.

2. Review of local existence theory for nonlinear wave equations

2.1. Local existence theory for semilinear wave equations

We consider the semilinear wave equation in \mathbb{R}^{1+3} :

$$\begin{cases} \square\phi = \mathcal{N}(\phi, \partial\phi), (t, x) \in \mathbb{R}^{1+3}, \\ \phi(0, \cdot) = \phi_0 \in H^s(\mathbb{R}^3), \partial_t\phi(0, \cdot) = \phi_1 \in H^{s-1}(\mathbb{R}^3), \end{cases} \quad (7)$$

where \mathcal{N} is quadratic with respect to $\partial\phi$, and we investigate for which s it is locally well posed.

Using the usual energy estimate, we obtain:

$$\|\phi(t)\|_{H^s(\mathbb{R}^3)} + \|\partial_t\phi(t)\|_{H^{s-1}(\mathbb{R}^3)} \lesssim \exp\left(\int_0^t \|\partial\phi\|_{L^\infty(\mathbb{R}^3)}\right) (\|\phi_0\|_{H^s(\mathbb{R}^3)} + \|\phi_1\|_{H^{s-1}(\mathbb{R}^3)}),$$

and one is left with controlling the quantity:

$$\|\partial\phi\|_{L^1_{[0,T]}L^\infty(\mathbb{R}^3)}. \quad (8)$$

Using that $H^{3/2+\epsilon}(\mathbb{R}^3)$ embeds in $L^\infty(\mathbb{R}^3)$ immediately yields well-posedness for $s > 5/2$.

In order to go beyond $s > 5/2$, one has to exploit the time integration in (8). This can be achieved by the use of the following Strichartz estimate for the three-dimensional linear wave equation:

$$\|\partial\phi\|_{L^2_{[0,T]}L^\infty(\mathbb{R}^3)} \lesssim (\|\phi_0\|_{H^s(\mathbb{R}^3)} + \|\phi_1\|_{H^{s-1}(\mathbb{R}^3)} + \|\square\phi\|_{L^1_{[0,T]}H^{s-1}(\mathbb{R}^3)}), \quad \forall s > 2, \quad (9)$$

which allows to get well-posedness for $s > 2$ as was first seen in [19].

In general, this result is optimal. In fact, explicit counterexamples of solutions to (7) that are ill-posed for $s = 2$ can be constructed (see [18]). However, one can go below $s > 2$ provided the nonlinearity has the so-called null structure. In this case, one can rely on bilinear estimates to prove well-posedness for $s > 3/2$ (see [8]). This result is optimal since $s = 3/2$ is at the level of the scaling of (7). For completeness, we give one example of nonlinearity exhibiting the null structure:

$$\mathcal{N} = Q_{\alpha\beta} \text{ with } Q_{\alpha\beta}(\phi, \psi) = \partial_\alpha\phi\partial_\beta\psi - \partial_\alpha\psi\partial_\beta\phi, 0 \leq \alpha, \beta \leq 3. \quad (10)$$

For this nonlinearity, one obtains the following bilinear estimate (see [7]):

$$\|Q_{\alpha\beta}(\phi, \psi)\|_{L^2(\mathbb{R}^{1+3})} \lesssim (\|\phi_0\|_{H^2(\mathbb{R}^3)} + \|\phi_1\|_{H^1(\mathbb{R}^3)})(\|\psi_0\|_{H^1(\mathbb{R}^3)} + \|\psi_1\|_{L^2(\mathbb{R}^3)}), \quad (11)$$

where ϕ and ψ are solutions to the flat wave equation $\square\phi = \square\psi = 0$.

2.2. Local existence theory for quasilinear wave equations

We consider the following quasilinear wave equation in \mathbb{R}^{1+3} :

$$\begin{cases} \square_{g(\phi)}\phi = \mathcal{N}(\phi, \partial\phi), (t, x) \in \mathbb{R}^{1+3}, \\ \phi(0, \cdot) = \phi_0 \in H^s(\mathbb{R}^3), \partial_t\phi(0, \cdot) = \phi_1 \in H^{s-1}(\mathbb{R}^3), \end{cases} \quad (12)$$

where \mathcal{N} is quadratic with respect to $\partial\phi$, and we investigate for which s it is locally well posed.

The method using the energy method together with the Sobolev embedding is still valid and yields well-posedness for $s > 5/2$. Together with (5), this has been used to

prove well-posedness for the Einstein vacuum equations (2) in the wave coordinates (see [5] for the case $s \geq 4$ and [6] for the improvement $s > 5/2$).

To go beyond $s > 5/2$, one is faced with the difficult task of deriving Strichartz estimates for the wave equation on a curved background $\square_g \phi = 0$ where g is only assumed to be in $L^\infty_{[0,T]} H^s(\mathbb{R}^3)$ and ∂g in $L^2_{[0,T]} L^\infty(\mathbb{R}^3)$. The first result in this direction was obtained in [20] where a precise analog of (9) was obtained under the condition that g is C^2 . This result is optimal (see [21]) and requires too much regularity to tackle the case $s \leq 5/2$. The breakthrough then came from [2] where the authors realized that one does not need the precise analog of (9) to go beyond $s > 5/2$. Instead they prove a Strichartz estimate with a loss which allows them to reach $s > 2 + 1/4$ (see also [24]). Subsequent improvements were made in [1], [25], [9] until the optimal result $s > 2$ was finally reached for the Einstein vacuum equations in the sequence of papers [17] [16] [13], and for general quasilinear wave equations in [22].

Motivated by the semilinear case, it is natural to consider whether it is possible to go below $s > 2$ for quasilinear wave equations provided the nonlinearity satisfies the null structure. However, it is not clear what this null structure should be in this case. In fact, in the semilinear case, the null structure is connected to the geometry of the light cones of the Minkowski metric (3) which are known explicitly. In the quasilinear case (12), the metric g depends on the solution and is therefore an unknown of the problem. Thus, we have no a priori knowledge on the light cones. A nice way to circumvent this problem is the following: instead of trying to guess what should be the null structure for quasilinear wave equations, let us take a quasilinear wave equation which has a rich structure, and is therefore likely to have the null structure. In view of (5) and the rich structure provided by the Bianchi identities (1), Einstein vacuum equations stand out as a natural candidate. Thus, we will consider whether one can prove well-posedness for (2) in H^2 .

3. The bounded L^2 curvature conjecture

The following conjecture has been first stated in [12]:

Conjecture. *Let (Σ, g, k) be asymptotically flat and satisfying the constraint equations (6), with $R \in L^2(\Sigma)$, $\nabla k \in L^2(\Sigma)$ and perhaps some weaker geometric characteristics of Σ . Then, Einstein vacuum equations are locally well-posed.*

Remark 2. *The assumptions on the regularity $R \in L^2(\Sigma)$, $\nabla k \in L^2(\Sigma)$ are at the level of two derivatives of \mathbf{g} in L^2 , and therefore consistent with H^2 .*

Let us mention the following motivations for attacking this problem:

- This problem is clearly motivated by the local well-posedness theory reviewed in the previous section. Such a result would be the first well-posedness result for a quasilinear wave equation below $s > 2$. Note also that going from $s > 2$ to $s = 2$ will not result from a technical improvement of previous methods. One will have to abandon Strichartz estimates, and rely instead on bilinear estimates for quasilinear wave equations.

- The assumptions $R \in L^2(\Sigma)$, $\nabla k \in L^2(\Sigma)$ are natural from the point of view of geometry, since all quantities are invariantly defined (in the contrary to stating a result in H^2 which would depend on an a priori choice of coordinates).
- Rather than a well-posedness result, it can be viewed as a continuation argument. As long as $R \in L^2$ and $\nabla k \in L^2$ along a spacelike hypersurface, one may extend the solution of Einstein equations at least for a little longer.

4. Strategy for a proof

In light of the results obtained for the semilinear wave equation (see section 2.1 and the discussion in [12]), to prove the bounded L^2 curvature conjecture one needs the following ingredients:

- A** *Provide a system of coordinates relative to which (2) verifies an appropriate version of the null condition.*
- B** *Construct a parametrix for solutions to the homogeneous wave equations $\square_{\mathbf{g}}\phi = 0$ on a fixed Einstein vacuum background, and obtain control of the parametrix and of its error term only using the fact that the curvature is bounded in L^2 .*
- C** *Prove appropriate bilinear estimates for solutions to $\square_{\mathbf{g}}\phi = 0$, on a fixed Einstein vacuum background (endowed with the coordinate system indicated in **A**) using the parametrix constructed in **B**.*

As far as step **C** is concerned, a bilinear estimate which is a precise analog of (11) has been proved in [14]. The authors rely on the following plane wave parametrix:

$$Tf(t, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} f(\lambda \omega) \lambda^2 d\lambda d\omega, \quad (13)$$

where $u(\cdot, \cdot, \omega)$ is a solution to the Eikonal equation $\mathbf{g}^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$ on \mathcal{M} such that $u(0, x, \omega) \sim x \cdot \omega$ when $|x| \rightarrow +\infty$ on Σ .

We would like to carry out step **B** with the parametrix defined in (13). Via the energy estimates for the wave equation, it suffices to control the parametrix at $t = 0$ (i.e. restricted to Σ):

$$Tf(0, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(0, x, \omega)} f(\lambda \omega) \lambda^2 d\lambda d\omega, \quad (14)$$

and the error term:

$$Ef(t, x) = \square_{\mathbf{g}} Tf(t, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} \square_{\mathbf{g}} u(t, x, \omega) f(\lambda \omega) \lambda^3 d\lambda d\omega. \quad (15)$$

This requires the following ingredients, the two first being related to the control of the parametrix restricted to Σ (14), and the two others being related to the control of the error term (15):

- B1** *Make an appropriate choice for the equation satisfied by $u(0, x, \omega)$ on Σ , and control the geometry of the foliation of Σ by the level surfaces of $u(0, x, \omega)$.*
- B2** *Prove that the parametrix at $t = 0$ given by (14) is bounded in $\mathcal{L}(L^2(\mathbb{R}^3), L^2(\Sigma))$ using the estimates for $u(0, x, \omega)$ obtained in **B1**.*

B3 Control the geometry of the foliation of \mathcal{M} given by the level hypersurfaces of u .

B4 Prove that the error term (15) satisfies the estimate $\|Ef\|_{L^2(\mathcal{M})} \leq C\|\lambda f\|_{L^2(\mathbb{R}^3)}$ using the estimates for u and $\square_{\mathbf{g}}u$ proved in **B3**.

Step **B1** and **B3** are more geometric, while step **B2** and **B4** blend geometric estimates with harmonic analysis decompositions. In order to keep the exposition simple while at the same time giving a flavor of both the geometric and the harmonic analysis part, we will focus on step **B1** and **B4**.

Remark 3. Step **B3** has already been initiated in the series of papers [15] [10] [11] where the authors prove the crucial estimate $\square_{\mathbf{g}}u \in L^\infty(\mathcal{M})$.

Remark 4. This paper reports on recent progress obtained for step **B**. To complete the proof of the conjecture, one still needs to address step **A**, and some bilinear estimates of step **C** in addition to the one already proved in [14].

5. Step B1: control of the foliation at initial time

5.1. Geometry of the foliation of \mathcal{M} by u and of Σ by $u(0, x, \omega)$

Remember that u is solution to the Eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ on \mathcal{M} . Let $L = -\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta$ be the corresponding null generator vectorfield and s its affine parameter, i.e. $L(s) = 1$. Let us introduce the level hypersurfaces of u

$$\mathcal{H}_u = \{(t, x) \text{ in } \mathcal{M} \text{ such that } u(t, x, \omega) = u\}$$

which form a foliation of \mathcal{M} . The level surfaces $P_{s,u}$ of s generate the geodesic foliation on \mathcal{H}_u .

The geometry of \mathcal{H}_u depends in particular on the null second fundamental form

$$\chi(X, Y) = \langle \mathbf{D}_X L, Y \rangle \quad (16)$$

where X, Y are arbitrary vectorfields tangent to the s -foliation $P_{s,u}$ and where \mathbf{D} is the covariant differentiation with respect to \mathbf{g} . We denote by $\text{tr}\chi$ the trace of χ , i.e. $\text{tr}\chi = \delta^{AB}\chi_{AB}$ where χ_{AB} are the components of χ relative to an orthonormal frame $(e_A)_{A=1,2}$ on the leaves of the s -foliation. An easy computation yields:

$$\square_{\mathbf{g}}u = \text{tr}\chi \quad (17)$$

so that ones needs to prove enough regularity for $\text{tr}\chi$ to control the error term of the parametrix (15). It satisfies the well known Raychadhouri equation

$$\frac{d}{ds}\text{tr}\chi + \frac{1}{2}(\text{tr}\chi)^2 = -|\hat{\chi}|^2 \quad (18)$$

with $\hat{\chi}_{AB} = \chi_{AB} - 1/2\text{tr}\chi\delta_{AB}$ the traceless part of χ . This transport equation is used in [15] to prove the crucial estimate $\text{tr}\chi \in L^\infty(\mathcal{M})$ provided that $\text{tr}\chi$ is in $L^\infty(\Sigma)$ at $t = 0$.

Remark 5. It is useful to remember what are the corresponding objects in the case of the Minkowski space-time (\mathbb{R}^{1+3}, m) . One has $u(t, x, \omega) = t + x \cdot \omega$, so that \mathbb{R}^{1+3} is foliated by parallel hyperplanes $\mathcal{H}_u = \{(t, x) / t + x \cdot \omega = u\}$. We also have $L = \partial_t - \omega \partial_x$, $s = (t - x \cdot \omega)/2$, and $P_{s,u} = \{(t, x) / t = u/2 + s \text{ and } x \cdot \omega = u/2 - s\}$

are planes so that $\chi \equiv 0$. In particular, the fact that $u(t, x, \omega) = t + x \cdot \omega$ implies that the parametriz (13) is the usual plane wave representation of the flat wave equation. Also, $\text{tr}\chi = 0$ and (17) imply that the error term (15) vanishes making step **B4** trivial in this case.

Let us now recall the link between $u(0, x, \omega)$ and $\text{tr}\chi(0, x, \omega)$. We define the lapse $a = |\nabla u(0, x, \omega)|^{-1}$, and the unit vector N such that $\nabla u(0, x, \omega) = a^{-1}N$. We also define the level surfaces $P_u = \{x / u(0, x, \omega) = u\}$ so that N is the normal to P_u . The second fundamental form θ of P_u is defined by

$$\theta(X, Y) = \langle \nabla_X N, Y \rangle \quad (19)$$

where X, Y are arbitrary vectorfields tangent to the u -foliation P_u of Σ and where ∇ denotes the covariant differentiation with respect to g . We denote by $\text{tr}\theta$ the trace of θ , i.e. $\text{tr}\theta = \delta^{AB}\theta_{AB}$ where θ_{AB} are the components of θ relative to an orthonormal frame $(e_A)_{A=1,2}$ on P_u .

Remark 6. *Again, let us precise these objects in the case of the Minkowski space-time (\mathbb{R}^{1+3}, m) . One has $\Sigma = \mathbb{R}^3$, $g = \delta$ and $u(0, x, \omega) = x \cdot \omega$, so that \mathbb{R}^3 is foliated by parallel planes $P_u = \{x / x \cdot \omega = u\}$. We also have $a = 1$, $N = \omega$ and $\theta \equiv 0$. In particular, the fact that $u(0, x, \omega) = x \cdot \omega$ implies that the parametriz at initial time (14) is the inverse Fourier transform which certainly satisfies step **B2**.*

We have the following equality on Σ :

$$\text{tr}\chi = \text{tr}\theta + \text{tr}k. \quad (20)$$

Now, $\text{Tr}k = \text{tr}k + k_{NN}$. Furthermore, in addition to the constraint equations (6), we may impose $\text{Tr}k = 0$ which corresponds to a maximal foliation (see [3]). Thus, we obtain the following relation between $u(0, x, \omega)$ and $\text{tr}\chi(0, x, \omega)$ on Σ :

$$\text{tr}\chi = \text{tr}\theta - k_{NN} \text{ on } \Sigma. \quad (21)$$

So $\text{tr}\chi$ is in $L^\infty(\Sigma)$ at $t = 0$ if and only if

$$\text{tr}\theta - k_{NN} \in L^\infty(\Sigma). \quad (22)$$

Finally, we recall the structure equations of the $u(0, x, \omega)$ foliation:

$$\begin{cases} \nabla_A N = \theta_A, \\ \nabla_N N = -\nabla \log(a), \end{cases} \quad (23)$$

and

$$\begin{cases} a^{-1} \Delta(a) = -\nabla_N \text{tr}\theta - |\theta|^2 - R_{NN}, \\ \nabla^B \hat{\theta}_{AB} = \frac{1}{2} \nabla_A \text{tr}\theta + R_{NA}, \\ a^{-1} \nabla_A \nabla_B a + \nabla_N \theta_{AB} + \theta_A^C \theta_{CB} + K \gamma_{AB} = R_{AB}, \end{cases} \quad (24)$$

where $\hat{\theta}_{AB} = \theta_{AB} - 1/2 \text{tr}\theta \delta_{AB}$ is the traceless part of θ , K is the Gauss curvature of P_u , γ is the metric on P_u induced by g , and ∇ is the intrinsic covariant derivative on P_u . Taking the trace of the last equality of (24) and using the first equality yields:

$$2K - \text{tr}\theta^2 + |\theta|^2 = R - 2R_{NN}. \quad (25)$$

5.2. Control of the foliation of Σ

In view of (22), we look for $u(0, x, \omega)$ satisfying the following three conditions:

B1a $u(0, x, \omega) \sim x \cdot \omega$ when $|x| \rightarrow +\infty$ on Σ

B1b $\text{tr}\theta - k_{NN} \in L^\infty(\Sigma)$

B1c $u(0, x, \omega)$ has as enough regularity with respect to x and ω to achieve step **B2**, i.e. to control the parametrix at $t = 0$ given by (14)

where the initial data set (Σ, g, k) satisfies

$$\begin{cases} \nabla^j k_{ij} = 0, \\ R = |k|^2, \\ \text{Tr}k = 0, \end{cases} \quad (26)$$

and where R and ∇k are in $L^2(\Sigma)$.

In order to motivate our choice of $u(0, x, \omega)$, we investigate the regularity of the lapse a , which by (24) satisfies the following equation:

$$a^{-1} \Delta(a) = -\nabla_N \text{tr}\theta - |\theta|^2 - R_{NN}. \quad (27)$$

Since R is in $L^2(\Sigma)$, (27) implies that a has at most two derivatives in $L^2(\Sigma)$. Thus, $u(0, x, \omega)$ has at most three derivatives with respect to x in $L^2(\Sigma)$. This is not enough to satisfy **B1c** (i.e. to obtain the boundnessness of the parametrix at $t = 0$ on L^2). In fact, the classical T^*T argument (see for example [23]) relies on integrations by parts in x and would require at least one more derivative.

Alternatively, we could try to use the TT^* argument which relies on integration by parts in ω . Indeed, R being independent of ω , one would expect the regularity of $u(0, x, \omega)$ with respect to ω to be better. Differentiating (27) with respect to ω , we obtain:

$$a^{-1} \Delta(\partial_\omega a) = 2\nabla \nabla_N a + \dots, \quad (28)$$

where the term on the right-hand side comes from the commutator $[\partial_\omega, \Delta]$. Thus, obtaining an estimate for $\partial_\omega a$ from (28) requires to control $\nabla_N a$. Unfortunately, (27) seems to provide control of tangential derivatives of a only. This is where the specific choice of $u(0, x, \omega)$ comes into play.

Having in mind the equation of minimal surfaces (i.e. $\text{tr}\theta = 0$), condition **B1b** suggest the choice $\text{tr}\theta - k_{NN} = 0$. Unfortunately, this equation together with (27) does not provide any control of $\nabla_N a$. We propose as a second guess to take instead $\text{tr}\theta - k_{NN} = \nabla_N a$. Plugging into (27) yields an elliptic equation for a : $\nabla_N^2 a + a^{-1} \Delta(a) = -|\theta|^2 - \nabla_N k_{NN} - R_{NN}$. This allows us to control $\nabla_N^2 a$ in $L^2(\Sigma)$. However, $\nabla_N a$ is at most in $H^1(\Sigma)$ which does not embed in $L^\infty(\Sigma)$ so that condition **B1b** is not satisfied. To sum up, the first guess $\text{tr}\theta - k_{NN} = 0$ satisfies **B1b**, but not **B1c**, whereas the second guess $\text{tr}\theta - k_{NN} = \nabla_N a$ might satisfy **B1c**, but does not satisfy **B1b**.

The correct choice sits in the middle:

$$\text{tr}\theta - k_{NN} = 1 - a. \quad (29)$$

Plugging (29) in (27) yields:

$$\nabla_N a - a^{-1} \Delta(a) = |\theta|^2 + \nabla_N k_{NN} + R_{NN}. \quad (30)$$

This parabolic equation together with the Bianchi identity for R allows us to control not only tangential but also normal derivatives of a . In particular, we are able to obtain:

$$\nabla^2 a \in L^2(\Sigma), \nabla \nabla_N a \in L^2(\Sigma) \text{ and } \nabla_N^2 a \in L_u^2 H^{-1}(P_u). \quad (31)$$

One can deduce from (31) that $a - 1$ belongs to $L^\infty(\Sigma)$ so that **B1b** is satisfied.

Then, differentiating (30) several times with respect to ω and using (31), we obtain:

$$\partial_\omega^{3-\epsilon} u \in L^\infty(\Sigma) \text{ for all } \epsilon > 0. \quad (32)$$

In fact, due to the parabolic nature of (30), we control less normal derivatives than tangential ones as shown by the last estimate of (31). Now, since each differentiation with respect to ω introduces a normal derivative of a as shown by (28), one expects to be able to differentiate $u(0, x, \omega)$ at most three times, which is confirmed by (32).

Let us mention some of the difficulties that we have to face in the course of the proof of (31) and (32):

- To close the estimates for a , we have to obtain corresponding estimates for θ and N using (23) and (24).
- From (30), we control in particular Δa in $L^2(S)$. To obtain that $\nabla^2 a$ belongs to $L^2(\Sigma)$, one uses the Bochner identity for scalars:

$$\int_{P_u} |\nabla^2 a|^2 \mu_u = \int_{P_u} |\Delta a|^2 \mu_u - \int_{P_u} K |\nabla a|^2 \mu_u, \quad (33)$$

which forces us to obtain a good enough control of the Gauss curvature K .

- To prove our a priori estimates, we rely on Sobolev embeddings on the foliation of Σ given by $u(0, x, \omega)$. Once $u(0, x, \omega)$ is constructed, we have to prove a posteriori that these embeddings hold. The constants appearing in the various embeddings are then incorporated in a bootstrap.
- Some a priori estimates require the use of Littlewood-Paley projections on P_u . We use the geometric approach derived in [10] for 2-dimensional manifolds.

Finally, let us mention briefly some difficulties we have to face in step **B2**. We are not able to obtain the estimate $\partial_\omega a \in L^\infty(\Sigma)$ with the choice (29). In turn, this implies that we are not able to get the estimate $\partial_\omega \partial_x u \in L^\infty(\Sigma)$. This is a serious difficulty in view of satisfying step **B2** (i.e. the boundnessness of the parametrix at $t = 0$ on L^2). In fact, the classical T^*T and TT^* arguments (see for example [23]) prove boundnessness on L^2 by using several integrations by parts. Without the assumption that $\partial_\omega \partial_x u$ is in L^∞ , one can not perform even one of these integrations by parts. Anyway, even if these integrations by parts could be performed, recall that we don't have enough regularity in x to apply the T^*T method. Alternatively, we could try the TT^* method which relies on integration by parts in ω . But (32) is also not enough and we would need at least one more derivative in ω . Nevertheless, we are able to prove that the regularity of u given by (31) and (32) is enough to achieve step **B2**. To this end, we use both the regularity in x and ω , and take advantage of the geometry (i.e. the fact that we have more regularity along the tangential directions).

6. Step B4: control of the error term

Recall from section 5.1 that we have associated to u its null generator vectorfield L , the affine parameter s , its level hypersurfaces \mathcal{H}_u , surfaces $P_{s,u}$, and the second fundamental form χ . Following [4], we also introduce a null frame $L, \underline{L}, e_A, A = 1, 2$, where $e_A, A = 1, 2$ are arbitrary orthonormal vectors tangent to $P_{s,u}$, and \underline{L} completes the frame according to the following relations:

$$\mathbf{g}(L, L) = \mathbf{g}(\underline{L}, \underline{L}) = 0, \mathbf{g}(\underline{L}, L) = -2, \mathbf{g}(L, e_A) = \mathbf{g}(\underline{L}, e_A) = 0, j = 1, 2, \mathbf{g}(e_A, e_B) = \delta_A^B.$$

While we do not give details for step **B3**, let us still mention that we obtain less control for the derivatives in the \underline{L} direction than for the derivatives in tangential directions $e_A, A = 1, 2$ and in the direction L (in the same spirit that we have less control for the normal derivatives than for the tangential ones at initial time).

Using (17), the error term (15) can be rewritten as follows:

$$Ef(t, x) = i \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} \text{tr}\chi(t, x, \omega) f(\lambda\omega) \lambda^3 d\lambda d\omega, \quad (34)$$

and our goal is to prove:

$$\|Ef\|_{L^2(\mathcal{M})} \leq C\|\lambda f\|_{L^2(\mathbb{R}^3)}. \quad (35)$$

The following computation is instructive:

$$\begin{aligned} \|Ef\|_{L^2(\mathcal{M})} &\leq \int_{\mathbb{S}^2} \left\| \text{tr}\chi(t, x, \omega) \int_0^{+\infty} e^{i\lambda u} f(\lambda\omega) \lambda^3 d\lambda \right\|_{L^2(\mathcal{M})} d\omega \\ &\leq \int_{\mathbb{S}^2} \|\text{tr}\chi\|_{L^\infty(\mathcal{M})} \left\| \int_0^{+\infty} e^{i\lambda u} f(\lambda\omega) \lambda^3 d\lambda \right\|_{L^2} d\omega \leq \|\lambda^2 f\|_{L^2(\mathbb{R}^3)}, \end{aligned} \quad (36)$$

where we have used Plancherel with respect to λ , Cauchy-Schwarz with respect to ω and the fact that $\text{tr}\chi$ is in $L^\infty(\mathcal{M})$. (36) misses (35) by a power of λ . Now, assume for a moment that we may replace a power of λ by a derivative on $\text{tr}\chi$. Then, the same computation yields:

$$\begin{aligned} &\left\| \int_{\mathbb{S}^2} \int_0^{+\infty} \nabla \text{tr}\chi(t, x, \omega) e^{i\lambda u} f(\lambda\omega) \lambda^2 d\lambda d\omega \right\|_{L^2(\mathcal{M})} \\ &\leq \int_{\mathbb{S}^2} \|\nabla \text{tr}\chi\|_{L_u^\infty L^2(\mathcal{H}_u)} \left\| \int_0^{+\infty} e^{i\lambda u} f(\lambda\omega) \lambda^3 d\lambda \right\|_{L_u^2} d\omega \\ &\leq \|\nabla \text{tr}\chi\|_{L_{\omega, u}^\infty L^2(\mathcal{H}_u)} \|\lambda f\|_{L^2(\mathbb{R}^3)}. \end{aligned} \quad (37)$$

(37) would yield (35) since we are able to prove that $\nabla \text{tr}\chi \in L_{\omega, u}^\infty L^2(\mathcal{H}_u)$ in step **B3**. This suggests a strategy which consists in making integrations by parts to trade powers of λ against derivatives of $\text{tr}\chi$. We proceed in three steps.

6.1. Decomposition in frequency

Let φ and ψ two smooth compactly supported functions on \mathbb{R} such that:

$$\varphi(\lambda) + \sum_{j \geq 0} \psi(2^{-j}\lambda) = 1 \text{ for all } \lambda \in \mathbb{R}. \quad (38)$$

We use (38) to decompose the error term as follows:

$$Ef(t, x) = \sum_{j \geq -1} E_j f(t, x), \quad (39)$$

where for $j \geq 0$:

$$E_j f(t, x) = i \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} \text{tr} \chi(t, x, \omega) \psi(2^{-j} \lambda) f(\lambda \omega) \lambda^3 d\lambda d\omega, \quad (40)$$

and

$$E_{-1} f(t, x) = i \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} \text{tr} \chi(t, x, \omega) \varphi(\lambda) f(\lambda \omega) \lambda^3 d\lambda d\omega. \quad (41)$$

This decomposition is classical and is known as the first dyadic decomposition (see [23]). The goal of this section is to prove that:

$$\left| \int_{\mathcal{M}} E_j(t, x) E_k(t, x) d\mathcal{M} \right| \leq C 2^{-|k-j|} \|\psi(2^{-j} \lambda) \lambda f\|_{L^2(\mathbb{R}^3)} \|\psi(2^{-k} \lambda) \lambda f\|_{L^2(\mathbb{R}^3)} \text{ for } j \neq k, \quad (42)$$

which in turn will imply:

$$\|E f\|_{L^2(\mathcal{M})}^2 \leq \sum_{j \geq -1} \|E_j f\|_{L^2(\mathcal{M})}^2. \quad (43)$$

To obtain (42), we integrate by parts twice with respect to \underline{L} . Assume for instance that $j > k$. The worst term is then the one where the two \underline{L} derivatives fall on the same $\text{tr} \chi$ corresponding to E_j . One is then able to obtain (42) provided $\underline{L} \underline{L} \text{tr} \chi$ satisfies the following decomposition:

$$\underline{L} \underline{L} \text{tr} \chi = \nabla h + \dots, \text{ where } h \in L_u^\infty L^2(\mathcal{H}_u). \quad (44)$$

In fact, the two integrations by parts with respect to \underline{L} gain 2^{-2j} . We then integrate by parts by ∇ to get rid of ∇ in front of h at the expense of 2^k . Now, 2^{-j} is used to absorb the excess of λ in computation (36), and we finally obtain a gain of 2^{k-j} which yields (42).

Obtaining the decomposition (44) requires a lot of work. Let us just mention the basic idea behind it. Since $\underline{L} \text{tr} \chi$ is in $L_u^\infty L^2(\mathcal{H}_u)$ by step **B3**, (44) essentially means that we may trade one \underline{L} derivative against a ∇ derivative. The key ingredient of this trade of derivatives is the use of the Bianchi identity (1).

6.2. Decomposition in angle

Here we perform a second dyadic decomposition (see [23]). We introduce a smooth partition of unity on the sphere \mathbb{S}^2 :

$$\sum_{\nu} \eta_j^{\nu}(\omega) = 1 \text{ for all } \omega \in \mathbb{S}^2, \quad (45)$$

where the support of η_j^{ν} is a patch on \mathbb{S}^2 of diameter $\sim 2^{-j/2}$. We use (45) to decompose E_j as follows:

$$E_j f(t, x) = \sum_{\nu} E_j^{\nu} f(t, x), \quad (46)$$

where:

$$E_j^{\nu} f(t, x) = i \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} \text{tr} \chi(t, x, \omega) \psi(2^{-j} \lambda) \eta_j^{\nu}(\omega) f(\lambda \omega) \lambda^3 d\lambda d\omega. \quad (47)$$

The goal of this section is to obtain an estimate for

$$\left| \int_{\mathcal{M}} E_j^{\nu}(t, x) E_j^{\nu'}(t, x) d\mathcal{M} \right|, \nu \neq \nu', \quad (48)$$

that will in turn will imply:

$$\|E_j f\|_{L^2(\mathcal{M})}^2 \leq \sum_{\nu} \|E_j^{\nu} f\|_{L^2(\mathcal{M})}^2. \quad (49)$$

In order to estimate (48), we integrate by parts twice with respect to ∇ and obtain:

$$\begin{aligned} & \left| \int_{\mathcal{M}} E_j^{\nu}(t, x) E_j^{\nu'}(t, x) d\mathcal{M} \right| \\ & \leq C(2^{j/2}|\nu - \nu'|)^{-2} \|\psi(2^{-j}\lambda)\eta_j^{\nu}(\omega)\lambda f\|_{L^2(\mathbb{R}^3)} \|\psi(2^{-j}\lambda)\eta_j^{\nu'}(\omega)\lambda f\|_{L^2(\mathbb{R}^3)} \text{ for } \nu \neq \nu'. \end{aligned} \quad (50)$$

Unfortunately, (50) does not imply (49) since:

$$\sup_{\nu'} \sum_{\nu} (2^{j/2}|\nu - \nu'|)^{-2} \simeq j. \quad (51)$$

Beating the log-loss in (51) requires a lot of effort. Let us just mention that the key ingredient is to make a further decomposition using the geometric Littlewood-Paley projections P_k introduced in [10]. In fact, we decompose $\text{tr}\chi$ in the following way:

$$\text{tr}\chi = \sum_{k \geq 0} P_k \text{tr}\chi, \quad (52)$$

which in turns yields a decomposition for E_j^{ν} :

$$E_j^{\nu} = \sum_{k \geq 0} E_j^{\nu, k}, \quad (53)$$

where $E_j^{\nu, k}$ is defined by:

$$E_j^{\nu, k} f(t, x) = i \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} P_k \text{tr}\chi(t, x, \omega) \psi(2^{-j}\lambda) \eta_j^{\nu}(\omega) f(\lambda\omega) \lambda^3 d\lambda d\omega. \quad (54)$$

6.3. Control of $\|E_j^{\nu}\|_{L^2(\mathcal{M})}$

Using (43) and (49), we have reduced the proof of (35) to the proof of:

$$\|E_j^{\nu}\|_{L^2(\mathcal{M})} \leq C \|\psi(2^{-j}\lambda)\eta_j^{\nu}(\omega)\lambda f\|_{L^2(\mathbb{R}^3)}. \quad (55)$$

Unfortunately, the computation (36) only yields:

$$\|E_j^{\nu}\|_{L^2(\mathcal{M})} \leq C 2^{j/2} \|\psi(2^{-j}\lambda)\eta_j^{\nu}(\omega)\lambda f\|_{L^2(\mathbb{R}^3)}, \quad (56)$$

where the $2^{j/2}$ gain with respect to (36) comes from the fact that taking Cauchy-Schwarz in ω gains the square root of the volume of the support of the cut-off η_j^{ν} . By comparing (55) and (56), we see that we still need to gain $2^{-j/2}$.

Let us reintroduce the parametrix (13) localized both in frequency and angle:

$$T_j^{\nu} f(t, x) = \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} \psi(2^{-j}\lambda) \eta_j^{\nu}(\omega) f(\lambda\omega) \lambda^2 d\lambda d\omega. \quad (57)$$

The key is to use the wave equation satisfied by T_j^{ν} :

$$\square_{\mathbf{g}} T_j^{\nu} f(t, x) = E_j^{\nu} f(t, x), \quad (58)$$

which is the analog of (15). Now:

$$\begin{aligned} E_j^{\nu} f(t, x) &= i \text{tr}\chi(t, x, \nu) \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} \psi(2^{-j}\lambda) \eta_j^{\nu}(\omega) f(\lambda\omega) \lambda^3 d\lambda d\omega \\ &+ i \int_{\mathbb{S}^2} \int_0^{+\infty} e^{i\lambda u(t, x, \omega)} (\text{tr}\chi(t, x, \omega) - \text{tr}\chi(t, x, \nu)) \psi(2^{-j}\lambda) \eta_j^{\nu}(\omega) f(\lambda\omega) \lambda^3 d\lambda d\omega. \end{aligned} \quad (59)$$

The last term in (59) gains $2^{-j/2}$ with respect to (56) thanks to $\operatorname{tr}\chi(t, x, \omega) - \operatorname{tr}\chi(t, x, \nu)$ and the fact that $|\nu - \omega| \leq 2^{-j/2}$, so that it is bounded in $L^2(\mathcal{M})$. Thus we may rewrite (59) as:

$$E_j^\nu f \simeq \operatorname{tr}\chi(\nu, t, x) dT_j^\nu f(t, x) + L^2(\mathcal{M}). \quad (60)$$

(58) and (60) yield:

$$\square_{\mathbf{g}} T_j^\nu f(t, x) \simeq \operatorname{tr}\chi(\nu, t, x) dT_j^\nu f(t, x) + L^2(\mathcal{M}), \quad (61)$$

and (55) follows from the energy estimate for the wave equation, the fact that $\operatorname{tr}\chi \in L^\infty(\mathcal{M})$ and (60). Finally, (43), (49) and (55) imply (35) which concludes the proof of step **B4**.

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