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On the global existence for the axisymmetric Euler equations

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Abstract

This paper deals with the global well-posedness of the 3D axisymmetric Euler equations for initial data lying in critical Besov spaces $B_{p,1}^{1+\frac{3}{p}}$. In this case the BKM criterion is not known to be valid and to circumvent this difficulty we use a new decomposition of the vorticity .

1. Introduction

The evolution of homogeneous inviscid incompressible fluid flows in $\mathbb{R}^3$ is governed by the Euler system

$$(\mathrm{E}) \begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla\pi = 0, \\ \operatorname{div} u = 0, \\ u_{|t=0} = u_0. \end{cases}$$

Here, $u = u(t,x) \in \mathbb{R}^3$ denotes the velocity of the fluid, $\pi = \pi(t,x)$ the scalar pressure which is determined through the incompressibility assumption, that is, $\operatorname{div} u = 0$.

The local well-posedness theory of the system (E) seems to be in a satisfactory state and several results are obtained by numerous authors in many standard function spaces. In [9], Kato proved the local existence and uniqueness for initial data $u_0 \in H^s(\mathbb{R}^3)$ with $s > 5/2$ and Chemin [5] gave similar results for initial data lying in Hölderian spaces C^r with $r > 1$.

Other local results are recently obtained by Chae [4] in critical Besov spaces $B_{p,1}^{\frac{3}{p}+1}$, with $p \in]1,\infty[$ and by Pak and Park [11] for the space $B_{\infty,1}^1$. Notice that these spaces have the same scaling as Lipschitz functions (the space which is relevant for the hyperbolic theory) and in this sens they are called critical.

The question of global existence is still open and continues to be one of the most leading problem in mathematical fluid mechanics. Nevertheless there are many criteria of finite time blowup. One of them is the well-known BKM criterion [1] which ensures that the development of finite time singularities is related to the blowup of the L^∞ norm of the vorticity near the maximal ime existence. A direct consequence of this result is the global well-posedness of two-dimensional Euler solutions for smooth initial data since the vorticity is only advected and then does not grow. We emphasize that new geometric blowup criteria are recently discovered by Constantin, Fefferman and Majda [6].

Let us recall what the vorticity is? In space dimension three it is defined by the vector $\omega = \nabla \times u$ and satisfies the equation

$$\partial_t \omega + (u \cdot \nabla)\omega - (\omega \cdot \nabla)u = 0.$$

The main difficulty for establishing global regularity is to understand how the vortex stretching term $(\omega \cdot \nabla)u$ affects the dynamic of the fluid. While global existence is not proved for arbitrary initial smooth data, there are partial results in the case of the so-called axisymmetric flows without swirl. We say that a vector field u is axisymmetric if it has the form:

$$u(x,t) = u^r(r,z,t)e_r + u^z(r,z,t)e_z, \quad x = (x_1, x_2, z), \quad r = \sqrt{x_1^2 + x_2^2},$$

where $\Big(e_r, e_\theta, e_z\Big)$ is the cylindrical basis of $\mathbb{R}^3$ and the components u^r and u^z do not depend on the angular variable. The main feature of axisymmetric flows arises in the vorticity which takes the form (more precise discussion will be done in Proposition 3.1 and 3.2),

$$\omega = (\partial_z u^r - \partial_r u^z)e_\theta$$

and satisfies

$$\partial_t \omega + (u \cdot \nabla)\omega = \frac{u^r}{r}\omega. \tag{1}$$

Consequently the quantity $\alpha := \omega/r$ is only advected by the flow, that is

$$\partial_t \alpha + (u \cdot \nabla)\alpha = 0. \tag{2}$$

This fact induces the conservation of all the norms $\|\alpha\|_{L^p}, 1 \leq p \leq \infty$. In [15], Ukhovskii and Yudovich took advantage of these conservation laws to prove the global existence for axisymmetric initial data with finite energy and satisfying in addition $\omega_0 \in L^2 \cap L^\infty$ and $\frac{\omega_0}{r} \in L^2 \cap L^\infty$. In terms of Sobolev regularity these assumptions are satisfied if the velocity u_0 belongs to H^s with $s > \frac{7}{2}$. This is far from the critical regularity of local existence theory $s = \frac{5}{2}$. The optimal result in Sobolev spaces is done by Shirota and Yanagisawa in [14] who proved global existence in H^s, with $s > \frac{5}{2}$. Their proof is based on the boundness of the quantity $\|\frac{u^r}{r}\|_{L^\infty}$ by using Biot-Savart law. We mention also the reference [13] where similar results are given in different function spaces. In a recent work [7], Danchin has weakened the Ukhoviskii and Yudovich conditions. More precisely, he obtains global existence and uniqueness for initial data $\omega_0 \in L^{3,1} \cap L^\infty$ and $\frac{\omega_0}{r} \in L^{3,1}$. Here, we denote by $L^{3,1}$ the Lorentz space.

As we have seen above the global strong solutions are constructed in H^s, with $s > \frac{5}{2}$, but the best known spaces for local well-posedness are $B_{p,1}^{1+\frac{3}{p}}$, with $p \in [1, \infty]$. In this paper we address the question of global existence in these spaces. Comparing to the sub-critical spaces this problem is extremely hard to deal with because we are deprived of an important tool which is the BKM criterion. Even in space dimension two we encounter the same problem. Although the quantity $\|\omega(t)\|_{L^\infty}$ is conserved, this is not sufficient to propagate for all time the initial regularity. As it was pointed by Vishik in [16] the significant quantity is $\|\omega(t)\|_{B^0_{\infty,1}}$ and its control needs the use of the special structure of the vorticity, which is only transported by the flow. The key estimate is the following,

$$\|\omega(t)\|_{B^0_{\infty,1}} \leq C\|\omega_0\|_{L^\infty}\Big(1+\int_0^t \|\nabla v(\tau)\|_{L^\infty}d\tau\Big).$$

Owing to the streching term $\omega\, u^r/r$, the estimate of $\|\omega(t)\|_{B^0_{\infty,1}}$ for axisymmetric flows is more complicated and needs as we shall see a refined analysis of the geometric structure of the vorticity.
The main result of this paper can be stated as follows (for the definition of function spaces see next section).

Theorem 1.1. *Let u_0 be an axisymmetric divergence free vector field belonging to $B_{p,1}^{\frac{3}{p}+1}$, with $p\in[1,\infty]$. We assume in addition that its vorticity satisfies $\dfrac{\omega_0}{r}\in L^{3,1}$. Then the system* (E) *has a unique global solution u belonging to the space $\mathcal{C}(\mathbb{R}_+;\, B_{p,1}^{1+\frac{3}{p}})$. Besides, there exists a constant C_0 depending on the initial data norms such that for every $t\in\mathbb{R}_+$*

$$\|u(t)\|_{B_{p,1}^{1+\frac{3}{p}}} \leq C_0 e^{e^{\exp C_0 t}}. \tag{3}$$

The rest of this paper is organized as follows. In section 2 we recall some function spaces and gather some preliminary estimates. Section 3 is devoted to the study of some geometric properties of any solution to a vorticity equation model. The proof of Theorem 1.1 is done in several steps in section 4.

2. Notations and preliminaries

Throughout this paper, C stands for some real positive constant which may be different in each occurrence. We shall sometimes alternatively use the notation $X\lesssim Y$ for an inequality of type $X\leq CY$.
Let us start with a classical dyadic decomposition of the full space (see for instance [5]): there exist two radial functions $\chi\in\mathcal{D}(\mathbb{R}^3)$ and $\varphi\in\mathcal{D}(\mathbb{R}^3\backslash\{0\})$ such that

i) $\chi(\xi)+\sum_{q\geq 0}\varphi(2^{-q}\xi)=1 \ \ \forall\xi\in\mathbb{R}^3,$

ii) $\sum_{q\in\mathbb{Z}}\varphi(2^{-q}\xi)=1 \ \text{ if } \ \xi\neq 0,$

iii) $\text{supp}\ \varphi(2^{-p}\cdot)\cap\text{supp}\ \varphi(2^{-q}\cdot)=\varnothing$, if $|p-q|\geq 2$,

iv) $q\geq 1\Rightarrow \text{supp}\chi\cap\text{supp}\ \varphi(2^{-q})=\varnothing.$

For every $u\in\mathcal{S}'(\mathbb{R}^3)$ one defines the nonhomogeneous Littlewood-Paley operators by,

$$\Delta_{-1}u=\chi(\mathrm{D})u;\ \forall q\in\mathbb{N},\ \Delta_q u=\varphi(2^{-q}\mathrm{D})u \quad\text{and}\quad S_q u=\sum_{-1\leq j\leq q-1}\Delta_j u.$$

One can easily prove that for every tempered distribution u,

$$u=\sum_{q\geq -1}\Delta_q\, u. \tag{4}$$

The homogeneous operators are defined as follows

$$\forall q \in \mathbb{Z}, \quad \dot{\Delta}_q u = \varphi(2^{-q}\mathrm{D})v \quad \text{and} \quad \dot{S}_q u = \sum_{j \leq q-1} \dot{\Delta}_j u.$$

We notice that these operators can be written as a convolution. For example for $q \in \mathbb{Z}$, $\dot{\Delta}_q u = 2^{3q} h(2^q \cdot) \star u$, where $h \in \mathcal{S}$ and $\widehat{h}(\xi) = \varphi(\xi)$.
For the homogeneous decomposition, the identity (4) is not true due to the polynomials but we have,

$$u = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u \quad \forall\, u \in \mathcal{S}'(\mathbb{R}^3)/\mathcal{P}[\mathbb{R}^3],$$

where $\mathcal{P}[\mathbb{R}^3]$ is the whole of polynomials (see [12]).
We will make continuous use of Bernstein inequalities (see for example [5]).

Lemma 2.1. *There exists a constant C such that for $k \in \mathbb{N}$, $1 \leq a \leq b$ and for $u \in L^a(\mathbb{R}^d)$,*

$$\begin{aligned}
\sup_{|\alpha|=k} \|\partial^\alpha S_q u\|_{L^b} &\leq C^k\, 2^{q(k+d(\frac{1}{a}-\frac{1}{b}))} \|S_q u\|_{L^a}, \\
C^{-k} 2^{qk} \|\dot{\Delta}_q u\|_{L^a} &\leq \sup_{|\alpha|=k} \|\partial^\alpha \dot{\Delta}_q u\|_{L^a} \leq C^k 2^{qk} \|\dot{\Delta}_q u\|_{L^a}.
\end{aligned}$$

Let us now introduce the basic tool of the paradifferential calculus which is Bony's decomposition [3]. It distinguishes in a product uv three parts as follows:

$$uv = T_u v + T_v u + \mathcal{R}(u,v),$$

where

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad \text{and} \quad \mathcal{R}(u,v) = \sum_q \Delta_q u \widetilde{\Delta}_q v,$$

$$\text{with} \quad \widetilde{\Delta}_q = \sum_{i=-1}^{1} \Delta_{q+i}.$$

$T_u v$ is called paraproduct of v by u and $\mathcal{R}(u,v)$ the remainder term.
Let $(p,r) \in [1,+\infty]^2$ and $s \in \mathbb{R}$, then the nonhomogeneous Besov space $B^s_{p,r}$ is the set of tempered distributions u such that

$$\|u\|_{B^s_{p,r}} := \Big(2^{qs} \|\Delta_q u\|_{L^p} \Big)_{\ell^r} < +\infty.$$

We remark that we have the identification $B^s_{2,2} = H^s$. Also, by using the Bernstein inequalities (see [5] Lemma 2.1.1) we get easily

$$B^s_{p_1,r_1} \hookrightarrow B^{s+3(\frac{1}{p_2}-\frac{1}{p_1})}_{p_2,r_2}, \qquad p_1 \leq p_2 \quad \textit{and} \quad r_1 \leq r_2.$$

Let us now recall the Lorentz spaces that will be used here. For a measurable function f we define its nonincreasing rearrangement by

$$f^*(t) := \inf \Big\{ s,\ \mu\Big(\{x,\ |f(x)| > s\} \Big) \leq t \Big\},$$

where μ denotes the usual Lebesgue measure. For $(p,q) \in [1,+\infty]^2$, the Lorentz space $L^{p,q}$ is the set of functions f such that $\|f\|_{L^{p,q}} < \infty$, with

$$\|f\|_{L^{p,q}} := \begin{cases} \Big(\displaystyle\int_0^\infty [t^{\frac{1}{p}} f^*(t)]^q \frac{dt}{t} \Big)^{\frac{1}{q}}, & \text{for } 1 \leq q < \infty \\ \displaystyle\sup_{t>0} t^{\frac{1}{p}} f^*(t), & \text{for } q = \infty. \end{cases}$$

Notice that we can also define Lorentz spaces by real interpolation from Lebesgue spaces:

$$(L^{p_0}, L^{p_1})_{(\theta,q)} = L^{p,q},$$

where $1 \le p_0 < p < p_1 \le \infty$, θ satisfies $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ and $1 \le q \le \infty$. We have the classical properties:

$$\|uv\|_{L^{p,q}} \le \|u\|_{L^\infty} \|v\|_{L^{p,q}}. \tag{5}$$

$$L^{p,q} \hookrightarrow L^{p,q'}, \forall\, 1 \le p \le \infty; 1 \le q \le q' \le \infty \quad \text{and} \quad L^{p,p} = L^p. \tag{6}$$

The following result will be needed.

Proposition 2.2. *Given $(p,q) \in [1,\infty]^2$ and a smooth divergence free vector field u.Let f be a smooth solution of the transport equation*

$$\partial_t f + u \cdot \nabla f = 0,\ f_{|t=0} = f_0.$$

1. *For $f_0 \in L^{p,q}$, we have for every $t \in \mathbb{R}_+$,*

$$\|f(t)\|_{L^{p,q}} \le \|f_0\|_{L^{p,q}}.$$

2. *For $f_0 \in B^s_{p,r}(\mathbb{R}^3)$ and $g \in L^1_{\text{loc}}(\mathbb{R}_+; B^s_{p,r})$. we have for every $t \in \mathbb{R}_+$,*

$$\|f(t)\|_{B^s_{p,r}} \le Ce^{CU_1(t)} \Big(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-CU_1(\tau)} \|g(\tau)\|_{B^s_{p,r}} d\tau \Big),$$

where $U_1(t) = \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau$ and C is a constant depending on s.

The above estimate holds also true in the limiting cases:

$$s = -1, r = \infty, p \in [1,\infty]) \quad or \quad s = 1, r = 1, p \in [1,\infty],$$

provided that we change U_1 by $U(t) := \|u\|_{L^1_t B^1_{\infty,1}}$.

In addition, if $f = \operatorname{curl} u$, then the above estimate (2) holds true for all $s \in [1,+\infty[$.

3. Special structure of the vorticity

In this section we will describe some special geometric properties of axisymmetric flows.

Proposition 3.1. *Let $u = (u^1, u^2, u^3)$ be a smooth axisymmetric vector field. Then we have*

1) *the vector $\omega = \nabla \times u = (\omega^1, \omega^2, \omega^3)$ satsifies $\omega \times e_\theta = (0,0,0)$. In particular, we have for every $(x_1, x_2, z) \in \mathbb{R}^3$,*

$$\omega^3 = 0, \quad x_1\omega^1(x_1,x_2,z) + x_2\omega^2(x_1,x_2,z) = 0 \quad and$$

$$\omega^1(x_1,0,z) = \omega^2(0,x_2,z) = 0.$$

2) *for every $q \ge -1$, $\Delta_q u$ is axisymmetric and*

$$(\Delta_q u^1)(0,x_2,z) = (\Delta_q u^2)(x_1,0,z) = 0.$$

In this last part we study a vorticity equation type in which no relations between the vector field u and the solution Ω are supposed.

$$(\mathrm{V}) \begin{cases} \partial_t\Omega + (u\cdot\nabla)\Omega = \Omega\cdot\nabla u, \\ \operatorname{div} u = 0, \\ \Omega_{|t=0} = \Omega_0. \end{cases}$$

We will assume that u is axisymmetric and the unknown function $\Omega = (\Omega^1, \Omega^2, \Omega^3)$ is a vector field. The following result describes the preservation of some initial geometric conditions of the solution Ω.

Proposition 3.2. *Let u be an axisymmetric vector field belonging to the space $L^1_{\text{loc}}(\mathbb{R}_+, \text{Lip}(\mathbb{R}^3))$ with zero divergence and Ω be the unique global solution of (V) corresponding to smooth initial data Ω_0. Then we have the following properties:*

1) *If* $\operatorname{div}\Omega_0 = 0$, *then* $\operatorname{div}\Omega(t) = 0$ *for every* $t \in \mathbb{R}_+$.

2) *If* $\Omega_0 \times e_\theta = (0,0,0)$, *then we have*

$$\forall t\in\mathbb{R}_+, \qquad \Omega(t)\times e_\theta = (0,0,0).$$

Consequently, $\Omega^1(t, x_1, 0, z) = \Omega^2(t, 0, x_2, z) = 0$, *and*

$$\partial_t\Omega + (u\cdot\nabla)\Omega = \frac{u^r}{r}\Omega.$$

Proof. 1) We apply the divergence operator to the equation (V) leading under the assumption $\operatorname{div} u = 0$, to

$$\partial_t \operatorname{div}\Omega + u\cdot\nabla \operatorname{div}\Omega = 0.$$

Thus, the quantity $\operatorname{div}\Omega$ is transported by the flow and consequently the incompressibility of Ω remains true for every time.
2) We denote by $(\Omega^r, \Omega^\theta, \Omega^z)$ the coordinates of Ω in cylindrical basis. It is obvious that $\Omega^r = \Omega\cdot e_r$. Recall that in cylindrical coordinates the operator $u\cdot\nabla$ has the form

$$u\cdot\nabla = u^r\partial_r + \frac{1}{r}u^\theta\partial_\theta + u^z\partial_z = u^r\partial_r + u^z\partial_z.$$

We have used in the last equality the fact that for axisymmetric flows the angular component is zero. Hence we get

$$\begin{aligned}(u\cdot\nabla\Omega)\cdot e_r &= u^r\partial_r\Omega\cdot e_r + u^z\partial_z\Omega\cdot e_r \\ &= (u^r\partial_r + u^z\partial_z)(\Omega\cdot e_r) \\ &= u\cdot\nabla\Omega^r,\end{aligned}$$

Where we use $\partial_r e_r = \partial_z e_r = 0$. Now it remains to compute $(\Omega\cdot\nabla u)\cdot e_r$. By a straightforward computations we get,

$$\begin{aligned}(\Omega\cdot\nabla u)\cdot e_r &= \Omega^r\,\partial_r u\cdot e_r + \frac{1}{r}\Omega^\theta\,\partial_\theta u\cdot e_r + \Omega^3\,\partial_3 u\cdot e_r \\ &= \Omega^r\partial_r u^r + \Omega^3\partial_3 u^r.\end{aligned}$$

Thus the component Ω^r obeys to the equation

$$\partial_t\Omega^r + u\cdot\nabla\Omega^r = \Omega^r\partial_r u^r + \Omega^3\partial_3 u^r.$$

From the maximum principle we deduce

$$\|\Omega^r(t)\|_{L^\infty} \le \int_0^t \Big(\|\Omega^r(\tau)\|_{L^\infty} + \|\Omega^3(\tau)\|_{L^\infty}\Big)\|\nabla u(\tau)\|_{L^\infty} d\tau.$$

On the other hand the component Ω^3 satisfies the equation

$$\partial_t \Omega^3 + u\cdot\nabla\Omega^3 \;=\; \Omega^3\partial_3 u^3 + \Omega^r\partial_r u^3.$$

This leads to

$$\|\Omega^3(t)\|_{L^\infty} \le \int_0^t \Big(\|\Omega^3(\tau)\|_{L^\infty} + \|\Omega^r(\tau)\|_{L^\infty}\Big)\|\nabla u(\tau)\|_{L^\infty} d\tau.$$

Combining these estimates and using Gronwall's inequality we obtain for every $t\in\mathbb{R}_+$, $\Omega^3(t) = \Omega^r(t) = 0$, which is the desired result.
Under these assumptions the stretching term becomes

$$\begin{aligned}\Omega\cdot\nabla u &= \frac{1}{r}\Omega^\theta\partial_\theta(u^r e_r)\\ &= \frac{1}{r}u^r\Omega^\theta e_\theta = \frac{1}{r}u^r\Omega.\end{aligned}$$

□

4. Proof of Theorem 1.1

The proof of Theorem 1.1 will be done in several steps and it suffices to establish the *a priori* estimates. The hard part of the proof will be the Lipschitz bound of the velocity.

4.1. Some a priori estimates

We start with the following estimates:

Proposition 4.1. *Let u be an axisymmetric solution of (E), then we have for every $t\in\mathbb{R}_+$,*

1) *Biot-Savart law:*

$$\Big\|u^r(t)/r\Big\|_{L^\infty} \lesssim \Big\|\omega_0/r\Big\|_{L^{3,1}}.$$

2) *Vorticity bound:*

$$\|\omega(t)\|_{L^\infty} \lesssim \|\omega_0\|_{L^\infty} e^{Ct\|\omega_0/r\|_{L^{3,1}}}$$

3) *Velocity bound:*

$$\|u(t)\|_{L^\infty} \lesssim \Big(\|u_0\|_{L^\infty} + \|\omega_0\|_{L^\infty}\Big) e^{\exp Ct\|\omega_0/r\|_{L^{3,1}}}.$$

Proof. 1) According to Lemma 1 in [14] (see also [7]) one has,

$$|u^r(t,x)| \lesssim \int_{|y-x|\le r}\frac{|\omega(t,y)|}{|x-y|^2}dy + r\int_{|y-x|\ge r}\frac{|\omega(t,y)|}{|x-y|^3}dy,$$

with $r=\sqrt{x_1^2+x_2^2}$. Thus, one can estimate

$$\begin{aligned}|u^r(t,x)| &\lesssim \int_{|y-x|\le r}\frac{|\omega(t,y)|}{r'}\frac{r'}{|x-y|^2}dy + r\int_{|y-x|\ge r}\frac{|\omega(t,y)|}{r'}\frac{r'}{|x-y|^3}dy\\ &\lesssim r\int_{|y-x|\le r}\frac{|\omega(t,y)|}{r'}\frac{1}{|x-y|^2}dy + r\int_{|y-x|\ge r}\frac{|\omega(t,y)|}{r'}\frac{r'-r+r}{|x-y|^3}dy,\end{aligned}$$

where we use the notation $r'=\sqrt{y_1^2+y_2^2}$. In the first term of the right-hand side we have used the general fact $|r'-r|\le|x-y|$ which leads to

$$|x-y|\le r \Rightarrow r'\le 2r.$$

Since $|r'-r|\le|x-y|$, then we get easily

$$|u^r(t,x)|\lesssim r\int_{\mathbb{R}^3}\frac{|\omega(t,y)|}{r'}\frac{1}{|x-y|^2}dy.$$

It follows that

$$|u^r/r|\lesssim\frac{1}{|\cdot|^2}\star|\omega/r|.$$

As $\frac{1}{|\cdot|^2}\in L^{\frac{3}{2},\infty}(\mathbb{R}^3)$, then Young inequalities on $L^{q,p}$ spaces imply

$$\big\|u^r/r\big\|_{L^\infty}\lesssim\big\|\omega/r\big\|_{L^{3,1}}.$$

Since ω/r satisfies (2) then applying Proposition 2.2 gives

$$\big\|u^r/r\big\|_{L^\infty}\lesssim\big\|\omega_0/r\big\|_{L^{3,1}} \tag{7}$$

2) From the maximum principle applied to (1) one has

$$\|\omega(t)\|_{L^\infty}\le\|\omega_0\|_{L^\infty}+\int_0^t\big\|u^r(\tau)/r\big\|_{L^\infty}\|\omega(\tau)\|_{L^\infty}d\tau.$$

Using Gronwall's lemma and inequality (7) gives the desired result.
3) To estimate L^∞ norm of the velocity we write

$$\|u(t)\|_{L^\infty}\le\|\dot S_{-N}u\|_{L^\infty}+\sum_{q\ge -N}\|\dot\Delta_q u\|_{L^\infty},$$

where N is an arbitrary positive integer that will be fixed later. By Bernstein inequality we infer

$$\sum_{q\ge -N}\|\dot\Delta_q u\|_{L^\infty}\lesssim 2^N\|\omega\|_{L^\infty}.$$

On the other hand using the integral equation we get

$$\begin{aligned}\|\dot S_{-N}u\|_{L^\infty}&\le\|\dot S_{-N}u_0\|_{L^\infty}+\int_0^t\|\dot S_{-N}\Big(\mathbb{P}(u\cdot\nabla)u\Big)\|_{L^\infty}d\tau\\ &\lesssim\|u_0\|_{L^\infty}+\sum_{j<-N}\int_0^t\|\dot\Delta_j\Big(\mathbb{P}(u\cdot\nabla)u\Big)\|_{L^\infty}d\tau,\end{aligned}$$

where $\mathbb{P}$ denotes the Leray's projector over divergence free vector fields. Since $\dot\Delta_j\mathbb{P}$ maps L^p to itself uniformly in $j\in\mathbb{Z}$, we get

$$\|\dot S_{-N}u\|_{L^\infty}\lesssim\|u_0\|_{L^\infty}+2^{-N}\int_0^t\|u(\tau)\|_{L^\infty}^2d\tau.$$

Hence we obtain

$$\|u(t)\|_{L^\infty} \lesssim \|u_0\|_{L^\infty} + 2^N \|\omega(t)\|_{L^\infty} + 2^{-N} \int_0^t \|u(\tau)\|_{L^\infty}^2 d\tau.$$

If we choose N such that

$$2^{2N} \approx 1 + \|\omega(t)\|_{L^\infty}^{-1} \int_0^t \|u(\tau)\|_{L^\infty}^2 d\tau,$$

then we obtain

$$\|u(t)\|_{L^\infty}^2 \lesssim \|u_0\|_{L^\infty}^2 + \|\omega(t)\|_{L^\infty}^2 + \|\omega(t)\|_{L^\infty} \int_0^t \|u(\tau)\|_{L^\infty}^2 d\tau.$$

Thus Gronwall's lemma and the L^∞ bound of the vorticity yield

$$\begin{aligned} \|u(t)\|_{L^\infty} &\lesssim \Big(\|u_0\|_{L^\infty} + \|\omega\|_{L^\infty_t L^\infty}\Big) e^{Ct\|\omega\|_{L^\infty_t L^\infty}} \\ &\lesssim \Big(\|u_0\|_{L^\infty} + \|\omega_0\|_{L^\infty}\Big) e^{\exp Ct\|\omega\|_{L^\infty_t L^\infty}}. \end{aligned}$$

□

4.2. Lipschitz estimate of the velocity

The Lipschitz estimate of the velocity is heavily related to the following interpolation result which is the heart of this work:

Theorem 4.2. *There exists a decomposition $(\tilde{\omega}_q)_{q\geq -1}$ of the vorticity ω such that*

1) *For every $t \in \mathbb{R}_+$, $\omega(t,x) = \sum_{q\geq -1} \tilde{\omega}_q(t,x)$.*

2) *For every $t \in \mathbb{R}_+, \operatorname{div} \tilde{\omega}_q(t,x) = 0$.*

3) *For every $q \geq -1$ we have $\|\tilde{\omega}_q(t)\|_{L^\infty} \leq \|\Delta_q \omega_0\|_{L^\infty} e^{Ct\|\omega_0/r\|_{L^{3,1}}}$.*

4) *For all $j, q \geq -1$ we have*

$$\|\Delta_j \tilde{\omega}_q(t)\|_{L^\infty} \leq C 2^{-|j-q|} e^{CU(t)} \|\Delta_q \omega_0\|_{L^\infty},$$

with $U(t) := \|u\|_{L^1_t B^1_{\infty,1}}$ and C an absolute constant.

Proof. We will use for this purpose a new approach similar to [8]. Let $q \geq -1$ and denote by $\tilde{\omega}_q$ the unique global vector-valued solution of the problem

$$\begin{cases} \partial_t \tilde{\omega}_q + (u \cdot \nabla)\tilde{\omega}_q = \tilde{\omega}_q \cdot \nabla u \\ \tilde{\omega}_{q|t=0} = \Delta_q \omega_0. \end{cases} \tag{8}$$

Since $\operatorname{div} \Delta_q \omega_0 = 0$, then it follows from Proposition 3.2 that $\operatorname{div} \tilde{\omega}_q(t,x) = 0$. On the other hand we have by linearity and uniqueness

$$\omega(t,x) = \sum_{q\geq -1} \tilde{\omega}_q(t,x). \tag{9}$$

We will now rewrite the equation (8) under a suitable form.
As $\Delta_q \omega_0 = \operatorname{curl} \Delta_q u_0$ and $\Delta_q u_0$ is axisymmetric then we obtain from Proposition 3.1 that $(\Delta_q \omega_0) \times e_\theta = (0,0,0)$. This leads in view of Proposition 3.2 to $\tilde{\omega}_q(t) \times e_\theta = (0,0,0)$ and

$$\begin{cases} \partial_t \tilde{\omega}_q + (u \cdot \nabla)\tilde{\omega}_q = \frac{u^r}{r} \tilde{\omega}_q \\ \tilde{\omega}_{q|t=0} = \Delta_q \omega_0. \end{cases} \tag{10}$$

Applying the maximum principle and using Proposition 4.1 we obtain

$$\begin{aligned}\|\tilde\omega_q(t)\|_{L^\infty} &\leq \|\Delta_q\omega_0\|_{L^\infty}e^{\int_0^t\|u^r(\tau)/r\|_{L^\infty}d\tau}\\ &\leq \|\Delta_q\omega_0\|_{L^\infty}e^{Ct\|\omega_0\|_{L^{3,1}}}. \qquad (11)\end{aligned}$$

Let us move to the proof of the last point which is more technical. From the fact $\tilde\omega_q(t)\times e_\theta=(0,0,0)$ we see that the solution $\tilde\omega_q$ has two components in the cartesian basis, $\tilde\omega_q=(\tilde\omega_q^1,\tilde\omega_q^2,0)$. The analysis will be exactly the same for both components, so we will deal only with the first component $\tilde\omega_q^1$.
From the identity $\frac{u^r}{r}=\frac{u^1}{x_1}=\frac{u^2}{x_2}$, which is an easy consequence of $u^\theta=0$, it is plain that the functions $\tilde\omega_q^1$ is solution of

$$\begin{cases}\partial_t\tilde\omega_q^1+(u\cdot\nabla)\tilde\omega_q^1=u^2\frac{\tilde\omega_q^1}{x_2},\\ \tilde\omega^1_{q|t=0}=\Delta_q\omega_0^1.\end{cases}$$

Remark that the desired estimate is equivalent to

$$\|\tilde\omega_q(t)\|_{B^{\pm1}_{\infty,\infty}}\lesssim\|\Delta_q\omega_0\|_{B^{\pm1}_{\infty,\infty}}e^{CU(t)}.$$

We start with the propagation for the positive sign. Unfortunately, we are not able to close the estimate in Besov space $B^1_{\infty,\infty}$ due to the invalidity of a commutator estimate, see Proposition 2.2 for the limiting case $s=1$. Nevertherless we will be able to do it for Besov space $B^1_{\infty,1}$. From Proposition 2.2 we have

$$e^{-CU(t)}\|\tilde\omega_q^1(t)\|_{B^1_{\infty,1}}\lesssim\|\tilde\omega_q^1(0)\|_{B^1_{\infty,1}}+\int_0^t e^{-CU(\tau)}\Big\|u^2\frac{\tilde\omega_q^1}{x_2}(\tau)\Big\|_{B^1_{\infty,1}}d\tau. \qquad (12)$$

For the last term we write from Bony's decomposition,

$$\Big\|u^2\frac{\tilde\omega_q^1}{x_2}\Big\|_{B^1_{\infty,1}}\leq\Big\|T_{\frac{\tilde\omega_q^1}{x_2}}u^2\Big\|_{B^1_{\infty,1}}+\Big\|T_{u^2}\frac{\tilde\omega_q^1}{x_2}\Big\|_{B^1_{\infty,1}}+\Big\|\mathcal{R}(u^2,\tilde\omega_q^1/x_2)\Big\|_{B^1_{\infty,1}}.$$

To estimate the first paraproduct we write by definition,

$$\begin{aligned}\Big\|T_{\frac{\tilde\omega_q^1}{x_2}}u^2\Big\|_{B^1_{\infty,1}} &\lesssim \sum_j 2^j\|S_{j-1}(\tilde\omega_q^1/x_2)\|_{L^\infty}\|\Delta_j u^2\|_{L^\infty}\\ &\lesssim \|u\|_{B^1_{\infty,1}}\|\tilde\omega_q^1/x_2\|_{L^\infty}. \qquad (13)\end{aligned}$$

The remainder term is estimated as follows,

$$\begin{aligned}\|\mathcal{R}(u^2,\tilde\omega_q^1/x_2)\|_{B^1_{\infty,1}} &\lesssim \sum_{k\geq j-3}2^j\|\Delta_k u^2\|_{L^\infty}\|\widetilde\Delta_k(\tilde\omega_q^1/x_2)\|_{L^\infty}\\ &\lesssim \|u\|_{B^1_{\infty,1}}\|\tilde\omega_q^1/x_2\|_{L^\infty}. \qquad (14)\end{aligned}$$

The treatment of the second term is more subtle and needs the axisymmetry of the vector field u,

$$\Big\|T_{u^2}\frac{\tilde\omega_q^1}{x_2}\Big\|_{B^1_{\infty,1}}\lesssim\sum_{j\in\mathbb{N}}2^j\|S_{j-1}u^2(x)\Delta_j(\tilde\omega_q^1(x)/x_2)\|_{L^\infty}.$$

Now we write

$$\begin{aligned}S_{j-1}u^2(x)\Delta_j(\tilde\omega_q^1(x)/x_2) &= S_{j-1}u^2(x)\Delta_j\tilde\omega_q^1(x)/x_2+S_{j-1}u^2(x)\Big[\Delta_j,\frac{1}{x_2}\Big]\tilde\omega_q^1\\ &:= \mathrm{I}_j(x)+\mathrm{II}_j(x).\end{aligned}$$

Since $S_{j-1}u$ est axisymmetric then it follows from Proposition 3.1 that $S_{j-1}u^2(x_1,0,z)=0$. Thus from Taylor formula we get

$$\|\mathrm{I}_j\|_{L^\infty}\lesssim\|\nabla u\|_{L^\infty}\|\Delta_j\tilde\omega_q^1\|_{L^\infty}.$$

This yields

$$\sum_j 2^j\|\mathrm{I}_j\|_{L^\infty}\lesssim\|\nabla u\|_{L^\infty}\|\tilde\omega_q^1\|_{B^1_{\infty,1}}. \tag{15}$$

For the commutator term II_j we write by definition

$$\begin{aligned}\mathrm{II}_j(x) &= S_{j-1}u^2(x)/x_2\; 2^{3j}\int_{\mathbb{R}^3}h(2^j(x-y))(x_2-y_2)\tilde\omega_q^1(y)/y_2dy\\ &= 2^{-j}(S_{j-1}u^2(x)/x_2)\,2^{3j}\tilde h(2^j\cdot)\star(\tilde\omega_q^1/y_2)(x),\end{aligned}$$

with $\tilde h(x)=x_2h(x)$. Now we claim that for every $f\in\mathcal{S}'$ we have

$$2^{3j}\tilde h(2^j\cdot)\star f=\sum_{|j-k|\le1}2^{3j}\tilde h(2^j\cdot)\star\Delta_kf.$$

Indeed, we have $\widehat{\tilde h}(\xi)=i\partial_{\xi_2}\widehat h(\xi)=i\partial_{\xi_2}\varphi(\xi)$. It follows that $supp\,\widehat{\tilde h}\subset supp\,\varphi$. So we get $2^{3j}\tilde h(2^j\cdot)\star\Delta_kf=0$, for $|j-k|\ge2$. This leads to

$$\begin{aligned}\sum_{j\in\mathbb{N}}2^j\|\mathrm{II}_j\|_{L^\infty} &\lesssim \sum_{|j-k|\le1}\|S_{j-1}u^2/x_2\|_{L^\infty}\|\Delta_k(\tilde\omega_q^1/x_2)\|_{L^\infty}\\ &\lesssim \|\nabla u\|_{L^\infty}\|\tilde\omega_q^1/x_2\|_{B^0_{\infty,1}}.\end{aligned} \tag{16}$$

Using (15) et (16) one obtains

$$\Big\|T_{u^2}\tilde\omega_q^1/x_2\Big\|_{B^1_{\infty,1}}\lesssim\|\nabla u\|_{L^\infty}\Big(\|\tilde\omega_q^1\|_{B^1_{\infty,1}}+\|\tilde\omega_q^1/x_2\|_{B^0_{\infty,1}}\Big). \tag{17}$$

Putting together (13) (14) and (17) we find

$$\Big\|u^2\frac{\tilde\omega_q^1}{x_2}\Big\|_{B^1_{\infty,1}}\lesssim\|u\|_{B^1_{\infty,1}}\Big(\|\tilde\omega_q^1\|_{B^1_{\infty,1}}+\|\tilde\omega_q^1/x_2\|_{B^0_{\infty,1}}\Big)$$

Therefore we get from (12),

$$\begin{aligned}e^{-CU(t)}\|\tilde\omega_q^1(t)\|_{B^1_{\infty,1}} &\lesssim \|\tilde\omega_q^1(0)\|_{B^1_{\infty,1}}+\int_0^te^{-CU(\tau)}\|\tilde\omega_q^1(\tau)\|_{B^1_{\infty,1}}\|u(\tau)\|_{B^1_{\infty,1}}d\tau\\ &+ \int_0^te^{-CU(\tau)}\|u(\tau)\|_{B^1_{\infty,1}}\|\tilde\omega_q^1(\tau)/x_2\|_{B^0_{\infty,1}}d\tau.\end{aligned}$$

According to Gronwall's inequality we have

$$\|\tilde\omega_q^1(t)\|_{B^1_{\infty,1}}\lesssim e^{CU(t)}\Big(\|\tilde\omega_q^1(0)\|_{B^1_{\infty,1}}+\|\tilde\omega_q^1/x_2\|_{L^\infty_tB^0_{\infty,1}}\Big). \tag{18}$$

It is easy to check that $\tilde\omega_q^1/x_2$ is advected by the flow, that is

$$\begin{cases}(\partial_t+u\cdot\nabla)\frac{\tilde\omega_q^1}{x_2}=0\\ \frac{\tilde\omega_q^1}{x_2}|_{t=0}=\frac{\Delta_q\omega_0^1}{x_2}.\end{cases}$$

Thus we deduce from Proposition 2.2,

$$\Big\|\tilde\omega_q^1(t)/x_2\Big\|_{B^0_{\infty,1}}\le\Big\|\Delta_q\omega_0^1/x_2\Big\|_{B^0_{\infty,1}}e^{CU(t)}, \tag{19}$$

At this stage we claim the following estimate:

$$\Big\|\Delta_q\omega_0^1/x_2\Big\|_{B^0_{\infty,1}} \lesssim 2^q\|\Delta_q\omega_0\|_{L^\infty}.$$

In fact, since u_0 is axisymmetric then according to Proposition 3.1, $\Delta_q u_0$ is too. Consequently $\Delta_q\omega_0$ is the curl of an axisymmetric vector field and then by Proposition 3.1 and Taylor expansion

$$\Delta_q\omega_0^1(x_1,x_2,z) = x_2\int_0^1(\partial_{x_2}\Delta_q\omega_0^1)(x_1,\tau x_2,z)d\tau.$$

Hence we get in view of Proposition 5.1

$$\begin{aligned}\Big\|\Delta_q\omega_0^1/x_2\Big\|_{B^0_{\infty,1}} &\leq \int_0^1\|(\partial_{x_2}\Delta_q\omega_0^1)(\cdot,\tau\cdot,\cdot)\|_{B^0_{\infty,1}}d\tau\\ &\lesssim \|\partial_{x_2}\Delta_q\omega_0^1\|_{B^0_{\infty,1}}\int_0^1(1-\log\tau)d\tau\\ &\lesssim 2^q\|\Delta_q\omega_0^1\|_{L^\infty},\end{aligned}\tag{20}$$

as claimed. Thus combining (18) with (19) and (20) we obtain

$$\|\tilde\omega_q^1(t)\|_{B^1_{\infty,1}} \leq C2^q\|\Delta_q\omega_0\|_{L^\infty}e^{CU(t)}.$$

This can be written as

$$\|\Delta_j\tilde\omega_q^1(t)\|_{L^\infty} \leq C2^{q-j}e^{CU(t)}\|\Delta_q\omega_0\|_{L^\infty}.\tag{21}$$

The same calculus gives a similar estimate for $\tilde\omega_q^2$. It remains to show the following estimate

$$\|\Delta_j\tilde\omega_q(t)\|_{L^\infty} \leq C2^{j-q}e^{CU(t)}\|\Delta_q\omega_0\|_{L^\infty}.$$

We will use for this purpose the alternative equation (8). According to Proposition 2.2 one has

$$\begin{aligned}e^{-CU(t)}\|\tilde\omega_q(t)\|_{B^{-1}_{\infty,\infty}} &\lesssim \|\Delta_q\omega_0\|_{B^{-1}_{\infty,\infty}}\\ &+ \int_0^t e^{-CU(\tau)}\|\tilde\omega_q\cdot\nabla u(\tau)\|_{B^{-1}_{\infty,\infty}}d\tau.\end{aligned}\tag{22}$$

To estimate the last term we write in view of Bony's decomposition

$$\begin{aligned}\|\tilde\omega_q\cdot\nabla u\|_{B^{-1}_{\infty,\infty}} &\leq \|T_{\tilde\omega_q}\cdot\nabla u\|_{B^{-1}_{\infty,\infty}} + \|T_{\nabla u}\cdot\tilde\omega_q\|_{B^{-1}_{\infty,\infty}}\\ &+ \|\mathcal{R}\big(\tilde\omega_q\cdot\nabla,u\big)\|_{B^{-1}_{\infty,\infty}}\\ &\lesssim \|\nabla u\|_{L^\infty}\|\tilde\omega_q\|_{B^{-1}_{\infty,\infty}} + \|\mathcal{R}\big(\tilde\omega_q\cdot\nabla,u\big)\|_{B^{-1}_{\infty,\infty}}.\end{aligned}$$

Since $\operatorname{div}\tilde\omega_q = 0$, then the remainder term can be treated as follows

$$\begin{aligned}\|\mathcal{R}\big(\tilde\omega_q\cdot\nabla,u\big)\|_{B^{-1}_{\infty,\infty}} &= \|\operatorname{div}\mathcal{R}\big(\tilde\omega_q\otimes,u\big)\|_{B^{-1}_{\infty,\infty}}\\ &\lesssim \sup_k\sum_{j\geq k-3}\|\Delta_j\tilde\omega_q\|_{L^\infty}\|\widetilde\Delta_j u\|_{L^\infty}\\ &\lesssim \|\tilde\omega_q\|_{B^{-1}_{\infty,\infty}}\|u\|_{B^1_{\infty,1}}.\end{aligned}$$

Il follows that

$$\|\tilde\omega_q\cdot\nabla u\|_{B^{-1}_{\infty,\infty}} \lesssim \|u\|_{B^1_{\infty,1}}\|\tilde\omega_q\|_{B^{-1}_{\infty,\infty}}$$

Inserting this estimate into (22) we get

$$\begin{aligned} e^{-CU(t)}\|\tilde\omega_q(t)\|_{B^{-1}_{\infty,\infty}} &\lesssim \|\Delta_q\omega_0\|_{B^{-1}_{\infty,\infty}}\\ &+ \int_0^t \|u(\tau)\|_{B^1_{\infty,1}} e^{-CU(\tau)}\|\tilde\omega_q(\tau)\|_{B^{-1}_{\infty,\infty}}d\tau \end{aligned}$$

Hence we obtain by Gronwall's inequality

$$\begin{aligned} \|\tilde\omega_q(t)\|_{B^{-1}_{\infty,\infty}} &\le C\|\Delta_q\omega_0\|_{B^{-1}_{\infty,\infty}} e^{CU(t)}\\ &\le C2^{-q}\|\Delta_q\omega_0\|_{L^\infty} e^{CU(t)}. \end{aligned}$$

This gives by definition

$$\|\Delta_j\tilde\omega_q(t)\|_{L^\infty} \le C2^{j-q}\|\Delta_q\omega_0\|_{L^\infty} e^{CU(t)},$$

which achieves the proof of the theorem. □

In the next proposition we give some precise estimates of the velocity.

Proposition 4.3. *The Euler solution with initial data* $u_0 \in B^{1+\frac{3}{p}}_{p,1}$ *such that* $\frac{\omega_0}{r} \in L^{3,1}$ *satisfies for every* $t \in \mathbb{R}_+$,

1) *Case* $p = \infty$,

$$\|\omega(t)\|_{B^0_{\infty,1}} + \|u(t)\|_{B^1_{\infty,1}} \le C_0 e^{\exp C_0 t}.$$

2) *Case* $1 \le p < \infty$,

$$\|u(t)\|_{B^{1+\frac{3}{p}}_{p,1}} \le C_0 e^{e^{\exp C_0 t}},$$

with C_0 *depends on the norms of* u_0.

Proof. 1) Let N be a fixed positive integer that will be carefully chosen later. Then we have from (9)

$$\begin{aligned} \|\omega(t)\|_{B^0_{\infty,1}} &\le \sum_j \|\Delta_j \sum_q \tilde\omega_q(t)\|_{L^\infty}\\ &\le \sum_{|j-q|\ge N} \|\Delta_j\tilde\omega_q(t)\|_{L^\infty} + \sum_{|j-q|<N} \|\Delta_j\tilde\omega_q(t)\|_{L^\infty}\\ &:= \mathrm{I} + \mathrm{II}. \end{aligned} \tag{23}$$

To estimate the first term we use Theorem (4.2) and the convolution inequality for the series

$$\mathrm{I} \lesssim 2^{-N}\|\omega_0\|_{B^0_{\infty,1}} e^{CU(t)}. \tag{24}$$

To estimate the term II we use two facts: the first one is that the operator Δ_j maps uniformly L^∞ into itself while the second is the L^∞ estimate (11),

$$\begin{aligned} \mathrm{II} &\lesssim \sum_{|j-q|<N} \|\tilde\omega_q(t)\|_{L^\infty}\\ &\lesssim e^{C_0 t} \sum_{|j-q|<N} \|\Delta_q\omega_0\|_{L^\infty}\\ &\lesssim e^{C_0 t} N\|\omega_0\|_{B^0_{\infty,1}}. \end{aligned} \tag{25}$$

Combining this estimate with (25), (24) and (23) we obtain

$$\|\omega(t)\|_{B^0_{\infty,1}} \lesssim 2^{-N}e^{CU(t)} + Ne^{C_0 t}.$$

Putting

$$N = \Big[CU(t)\Big] + 1,$$

we obtain

$$\|\omega(t)\|_{B^0_{\infty,1}} \lesssim \Big(U(t)+1\Big)e^{C_0 t}.$$

On the other hand we have

$$\|u\|_{B^1_{\infty,1}} \lesssim \|u\|_{L^\infty} + \|\omega\|_{B^0_{\infty,1}},$$

which yields in view of Proposition 4.1,

$$\begin{aligned}\|u(t)\|_{B^1_{\infty,1}} &\lesssim \|u(t)\|_{L^\infty} + \|\omega(t)\|_{B^0_{\infty,1}}\\ &\le C_0 e^{\exp C_0 t} + C_0 e^{C_0 t}\int_0^t \|u(\tau)\|_{B^1_{\infty,1}} d\tau.\end{aligned}$$

Hence we obtain by Gronwall's inequality

$$\|u(t)\|_{B^1_{\infty,1}} \le C_0 e^{\exp C_0 t},$$

which gives in turn

$$\|\omega(t)\|_{B^0_{\infty,1}} \le C_0 e^{\exp C_0 t}.$$

This concludes the first part of Proposition 4.3.
2) Applying Proposition 2.2 to the vorticity equation we get

$$e^{-CU_1(t)}\|\omega(t)\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \|\omega_0\|_{B^{\frac{3}{p}}_{p,1}} + \int_0^t e^{-CU_1(\tau)}\|\omega\cdot\nabla u(\tau)\|_{B^{\frac{3}{p}}_{p,1}} d\tau. \tag{26}$$

As $\omega = \text{curl } u$, we have

$$\|\omega\cdot\nabla u\|_{B^{\frac{3}{p}}_{p,1}} \;\lesssim\; \|\omega\|_{B^{\frac{3}{p}}_{p,1}}\|\nabla u\|_{L^\infty}. \tag{27}$$

Indeed, from Bony's decomposition we write

$$\begin{aligned}\|\omega\cdot\nabla u\|_{B^{\frac{3}{p}}_{p,1}} &\le \|T_{\nabla u}\cdot\omega\|_{B^{\frac{3}{p}}_{p,1}} + \|T_\omega\cdot\nabla u\|_{B^{\frac{3}{p}}_{p,1}} + \|\mathcal{R}(\omega,\nabla u)\|_{B^{\frac{3}{p}}_{p,1}}\\ &\lesssim \|\nabla u\|_{L^\infty}\|\omega\|_{B^{\frac{3}{p}}_{p,1}} + \|T_\omega\cdot\nabla u\|_{B^{\frac{3}{p}}_{p,1}}.\end{aligned}$$

From the definition we write

$$\begin{aligned}\|T_\omega\cdot\nabla u\|_{B^{\frac{3}{p}}_{p,1}} &\lesssim \sum_{q\in\mathbb{N}} 2^{q\frac{3}{p}}\|S_{q-1}\omega\|_{L^\infty}\|\nabla\Delta_q u\|_{L^p}\\ &\lesssim \|\omega\|_{L^\infty}\sum_{q\in\mathbb{N}} 2^{q\frac{3}{p}}\|\Delta_q\omega\|_{L^p}\\ &\lesssim \|\nabla u\|_{L^\infty}\|\omega\|_{B^{\frac{3}{p}}_{p,1}}.\end{aligned}$$

We have used here the fact that for $p\in[1,\infty]$ and $q\in\mathbb{N}$ the composition operator $\Delta_q R : L^p \to L^p$ is continuous uniformly with respect to p and q, where R denotes Riesz transform. Combining (26) and (27) we find,

$$e^{-CU_1(t)}\|\omega(t)\|_{B^{\frac{3}{p}}_{p,1}} \lesssim \|\omega_0\|_{B^{\frac{3}{p}}_{p,1}} + \int_0^t e^{-CU_1(\tau)}\|\omega(\tau)\|_{B^{\frac{3}{p}}_{p,1}}\|\nabla u(\tau)\|_{L^\infty} d\tau.$$

Gronwall's inequality yields

$$\|\omega(t)\|_{B^{\frac{3}{p}}_{p,1}} \le \|u_0\|_{B^{\frac{3}{p}+1}_{p,1}} e^{C\int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau} \le C_0 e^{e^{\exp C_0 t}}.$$

Let us estimate the velocity. We write

$$\begin{aligned}\|u(t)\|_{B^{1+\frac{3}{p}}_{p,1}} &\lesssim \|\Delta_{-1}u\|_{L^p}+\sum_{q\in\mathbb{N}}2^{q\frac{3}{p}}2^q\|\Delta_q u\|_{L^p}\\ &\lesssim \|u(t)\|_{L^p}+\|\omega(t)\|_{B^{\frac{3}{p}}_{p,1}}\end{aligned}$$

Thus it remains to estimate $\|u\|_{L^p}$. For $1<p<\infty$, since Riesz transforms acts continuously on L^p, we get

$$\begin{aligned}\|u(t)\|_{L^p} &\leq \|u_0\|_{L^p}+C\int_0^t\|u\cdot\nabla u(\tau)\|_{L^p}d\tau\\ &\lesssim \|u_0\|_{L^p}+\int_0^t\|u(\tau)\|_{L^p}\|\nabla u(\tau)\|_{L^\infty}d\tau.\end{aligned}$$

It suffices now to use Gronwall's inequality.
For the case $p=1$, we write

$$\begin{aligned}\|u(t)\|_{L^1} &\leq \|\dot{S}_0u(t)\|_{L^1}+\sum_{q\geq 0}\|\dot{\Delta}_q u(t)\|_{L^1}\\ &\lesssim \|\dot{S}_0u(t)\|_{L^1}+\sum_{q\geq 0}2^{-q}\|\dot{\Delta}_q\nabla u(t)\|_{L^1}\\ &\lesssim \|\dot{S}_0u(t)\|_{L^1}+\|\omega(t)\|_{L^1}.\end{aligned}$$

However, it is easy to see that

$$\|\omega(t)\|_{L^1}\leq\|\omega_0\|_{L^1}e^{\int_0^t\|\nabla u(\tau)\|_{L^\infty}d\tau}.$$

Concerning $\dot{S}_0u$ we use the equation on u leading to

$$\begin{aligned}\|\dot{S}_0u(t)\|_{L^1} &\lesssim \|\dot{S}_0u_0\|_{L^1}+\sum_{q\leq -1}\|\dot{\Delta}_q((u\cdot\nabla)u(t))\|_{L^1}\\ &\lesssim \|u_0\|_{L^1}+\sum_{q\leq -1}2^q\|\dot{\Delta}_q(u\otimes u(t))\|_{L^1}\\ &\lesssim \|u_0\|_{L^1}+\|u(t)\|^2_{L^2}\\ &\lesssim \|u_0\|_{L^1}+\|u_0\|^2_{L^2}.\end{aligned}$$

This yields

$$\|u(t)\|_{L^1}\leq C_0e^{e^{\exp C_0t}}.$$

The proof is now achieved.

□

5. Appendix

The following result describes the anisotropic dilatation in Besov spaces.

Proposition 5.1. *Let $f:\mathbb{R}^3\to\mathbb{R}$ be a function belonging to $B^0_{\infty,1}$ and denote by $f_\lambda(x_1,x_2,x_3)=f(\lambda x_1,x_2,x_3)$. Then, there exists an absolute constant $C>0$ such that for all $\lambda\in]0,1[$*

$$\|f_\lambda\|_{B^0_{\infty,1}}\leq C(1-\log\lambda)\|f\|_{B^0_{\infty,1}}.$$

Proof. Let $q \geq -1$, we denote by $f_{q,\lambda} = (\Delta_q f)_\lambda$. From the definition we have

$$\begin{aligned}\|f_\lambda\|_{B^0_{\infty,1}} &= \|\Delta_{-1} f_\lambda\|_{L^\infty} + \sum_{j\in\mathbb{N}} \|\Delta_j f_\lambda\|_{L^\infty}\\ &\leq C\|f\|_{L^\infty} + \sum_{\substack{j\in\mathbb{N}\\ q\geq -1}} \|\Delta_j f_{q,\lambda}\|_{L^\infty}.\end{aligned}$$

For $j, q \in \mathbb{N}$, the Fourier transform of $\Delta_j f_{q,\lambda}$ is supported in the set

$$\Big\{|\xi_1| + |\xi'| \approx 2^j \quad \text{and} \quad \lambda^{-1}|\xi_1| + |\xi'| \approx 2^q\Big\},$$

where $\xi' = (\xi_2, \xi_3)$. A direct consideration shows that this set is empty if $2^q \lesssim 2^j$ or $2^{j-q} \lesssim \lambda$. For $q = -1$ the set is empty if $j \geq n_0$, this last number is absolute. Thus we get for an integer n_1

$$\begin{aligned}\|f_\lambda\|_{B^0_{\infty,1}} &\lesssim \|f\|_{L^\infty} + \sum_{\substack{q-n_1+\log\lambda\leq j\\ j\leq q+n_1}} \|\Delta_j f_{q,\lambda}\|_{L^\infty}\\ &\lesssim \|f\|_{L^\infty} + (n_1 - \log\lambda)\sum_q \|f_{q,\lambda}\|_{L^\infty}\\ &\lesssim \|f\|_{L^\infty} + (n_1 - \log\lambda)\sum_q \|f_q\|_{L^\infty}\\ &\lesssim (1-\log\lambda)\|f\|_{B^0_{\infty,1}}.\end{aligned}$$

□

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