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# A nonlinear Korn inequality in $\mathbb{R}^n$ with an explicitly bounded constant

*Une inégalité de Korn non linéaire dans  $\mathbb{R}^n$  avec une constante majorée explicitement*

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**Abstract.** It is known that the  $W^{1,p}$ -distance between an orientation-preserving mapping in  $W^{1,p}(\Omega; \mathbb{R}^n)$  and another orientation-preserving mapping  $\Theta \in C^1(\bar{\Omega}; \mathbb{R}^n)$ , where  $\Omega$  is a domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , and  $p > 1$  is a real number, is bounded above by the  $L^p$ -distance between the square roots of the metric tensor fields induced by these mappings, multiplied by a constant depending only on  $p$ ,  $\Omega$ , and  $\Theta$ .

The object of this Note is to establish a better inequality of this type, and to provide in addition an explicitly computable upper bound on the constant appearing in it. An essential role is played in our proofs by the notion of geodesic distance inside an open subset of  $\mathbb{R}^n$ .

**Résumé.** Il est connu que la distance dans  $W^{1,p}$  entre une application dans  $W^{1,p}(\Omega; \mathbb{R}^n)$  préservant l'orientation et une autre application  $\Theta \in C^1(\bar{\Omega}; \mathbb{R}^n)$  préservant l'orientation, où  $\Omega$  est un domaine de  $\mathbb{R}^n$ ,  $n \geq 2$ , et  $p > 1$  est un nombre réel, est majorée par la distance dans  $L^p$  entre les racines carrées des champs de tenseurs métriques induits par ces applications, multipliée par une constante dépendant uniquement de  $p$ ,  $\Omega$ , et  $\Theta$ .

L'objet de cette Note est d'établir une meilleure inégalité de ce type, et de fournir en plus une borne supérieure explicitement calculable de la constante qui y apparaît. Un rôle essentiel est joué dans nos preuves par la notion de distance géodésique dans un ouvert de  $\mathbb{R}^n$ .

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### 1. Introduction and main result

Given any integer  $n \geq 2$ , we denote by  $\mathbb{M}^n$  the space of all real  $n \times n$  matrices and we let  $\mathbb{O}_+^n := \{A \in \mathbb{M}^n; AA^T = I \text{ and } \det A = 1\}$ ,  $\mathbb{S}^n := \{A \in \mathbb{M}^n; A = A^T\}$ , and  $\mathbb{S}_>^n := \{A \in \mathbb{S}^n; A \text{ is positive-definite}\}$ . The Euclidean norm in  $\mathbb{R}^n$  and the Frobenius norm in  $\mathbb{M}^n$  are denoted by a same symbol  $|\cdot|$ .

The notation  $|\cdot|_n$  designates the  $n$ -dimensional Lebesgue measure.

A *domain* is a connected and open subset  $\Omega \subset \mathbb{R}^n$  that is bounded and has a Lipschitz-continuous boundary, the set  $\Omega$  being locally on the same side of its boundary (cf. Adams [1] or Maz'ya [9]).

Let any  $1 < p < \infty$ , we denote by  $L^p(\Omega; \mathbb{R}^n)$  the space of vector fields  $v = (v_i) : \Omega \rightarrow \mathbb{R}^n$  such that  $v_i \in L^p(\Omega)$ , and we endow this space with the norm denoted and defined by

$$\|v\|_{L^p(\Omega)} := \left( \int_{\Omega} |v(x)|^p dx \right)^{1/p}.$$

Likewise, we denote by  $L^p(\Omega; \mathbb{M}^n)$  the space of matrix fields  $A = (a_{ij}) : \Omega \rightarrow \mathbb{M}^n$  such that  $a_{ij} \in L^p(\Omega)$ , and we endow this space with the norm denoted and defined by

$$\|A\|_{L^p(\Omega)} := \left( \int_{\Omega} |A(x)|^p dx \right)^{1/p}.$$

Given an open subset  $\Omega$  of  $\mathbb{R}^n$ , the *geodesic distance* between two points  $x, y \in \Omega$  is defined by

$$\text{dist}_{\Omega}(x, y) := \inf \left\{ \ell \in \mathbb{R}; \text{ there exists } \gamma \in \mathcal{C}^1([0, \ell]; \mathbb{R}^n) \text{ such that } \begin{aligned} &\gamma(0) = x, \gamma(\ell) = y, \\ &\gamma(s) \in \Omega \text{ and } |\gamma'(s)| = 1 \text{ for all } s \in [0, \ell] \end{aligned} \right\}.$$

If  $\Omega \subset \mathbb{R}^n$  in a *domain*, there exists a constant  $C_{\Omega}$  such that

$$|x - y| \leq \text{dist}_{\Omega}(x, y) \leq C_{\Omega}|x - y| \text{ for all } x, y \in \bar{\Omega} \tag{1}$$

(see, e.g., Anicic, Le Dret, Raoult [2, Proposition 5.1]).

The notation  $\mathcal{C}^1(\bar{\Omega}; \mathbb{R}^n)$  designates the space of all vector fields  $v = (v_i) : \bar{\Omega} \rightarrow \mathbb{R}^n$  such that  $v_i \in \mathcal{C}^1(\bar{\Omega})$ . The gradient field of a mapping  $\theta \in \mathcal{C}^1(\bar{\Omega}; \mathbb{R}^n)$  is the matrix field  $\nabla \theta \in \mathcal{C}^0(\bar{\Omega}; \mathbb{M}^n)$  whose *column* vectors are the partial derivatives of  $\theta$ . A mapping  $\theta \in \mathcal{C}^1(\bar{\Omega}; \mathbb{R}^n)$  is an *immersion* if  $\det \nabla \theta(x) \neq 0$  at each point of  $\bar{\Omega}$ . An immersion  $\theta \in \mathcal{C}^1(\bar{\Omega}; \mathbb{R}^n)$  is called *orientation-preserving* if  $\det \nabla \theta(x) > 0$  at each point  $x \in \bar{\Omega}$ . The set of all such orientation-preserving immersions is denoted

$$\mathcal{C}_+^1(\bar{\Omega}; \mathbb{R}^n) := \left\{ \theta \in \mathcal{C}^1(\bar{\Omega}; \mathbb{R}^n); \det \nabla \theta > 0 \text{ in } \bar{\Omega} \right\}.$$

Given any domain  $\Omega \subset \mathbb{R}^n$  and any three scalars  $1 \geq \lambda > 0$ ,  $\delta > 0$ , and  $\eta > 0$ , let

$$\mathcal{C}_{\lambda}^1(\bar{\Omega}; \mathbb{R}^n) := \left\{ \theta \in \mathcal{C}^1(\bar{\Omega}; \mathbb{R}^n); \det \nabla \theta \geq \lambda \text{ and } |\nabla \theta| \leq \frac{1}{\lambda} \text{ in } \bar{\Omega} \right\} \tag{2}$$

and

$$\mathcal{C}_{\lambda, \delta, \eta}^1(\bar{\Omega}; \mathbb{R}^n) := \left\{ \theta \in \mathcal{C}_{\lambda}^1(\bar{\Omega}; \mathbb{R}^n); \sup_{\substack{(x, \bar{x}) \in \bar{\Omega} \times \bar{\Omega}, \\ |x - \bar{x}| \leq \delta}} |\nabla \theta(x) - \nabla \theta(\bar{x})| \leq \eta \right\}. \tag{3}$$

Note that the following inclusions hold:

$$\begin{aligned} \lambda > \lambda' &\Rightarrow \mathcal{C}_{\lambda}^1(\bar{\Omega}; \mathbb{R}^n) \subset \mathcal{C}_{\lambda'}^1(\bar{\Omega}; \mathbb{R}^n), \\ \delta > \delta' &\Rightarrow \mathcal{C}_{\lambda, \delta, \eta}^1(\bar{\Omega}; \mathbb{R}^n) \subset \mathcal{C}_{\lambda, \delta', \eta}^1(\bar{\Omega}; \mathbb{R}^n), \\ \eta < \eta' &\Rightarrow \mathcal{C}_{\lambda, \delta, \eta}^1(\bar{\Omega}; \mathbb{R}^n) \subset \mathcal{C}_{\lambda, \delta, \eta'}^1(\bar{\Omega}; \mathbb{R}^n). \end{aligned}$$

Any mapping  $\theta \in \mathcal{C}_+^1(\bar{\Omega}; \mathbb{R}^n)$  belongs to the set  $\mathcal{C}_{\lambda}^1(\bar{\Omega}; \mathbb{R}^n)$  if  $0 < \lambda \leq \lambda_0(\theta)$  for some  $\lambda_0(\theta) > 0$ , and any mapping  $\theta \in \mathcal{C}_{\lambda}^1(\bar{\Omega}; \mathbb{R}^n)$  belongs to the set  $\mathcal{C}_{\lambda, \delta, \eta}^1(\bar{\Omega}; \mathbb{R}^n)$  if  $0 < \delta \leq \delta(\eta, \theta)$  for some  $\delta_0(\eta, \theta) > 0$ . Therefore,

$$\lim_{\lambda \rightarrow 0^+} \mathcal{C}_{\lambda}^1(\bar{\Omega}; \mathbb{R}^n) = \bigcup_{0 < \lambda \leq 1} \mathcal{C}_{\lambda}^1(\bar{\Omega}; \mathbb{R}^n) = \mathcal{C}_+^1(\bar{\Omega}; \mathbb{R}^n)$$

and, for each  $1 \geq \lambda > 0$  and for each  $\eta > 0$ ,

$$\lim_{\delta \rightarrow 0^+} \mathcal{C}_{\lambda, \delta, \eta}^1(\bar{\Omega}; \mathbb{R}^n) = \bigcup_{\delta > 0} \mathcal{C}_{\lambda, \delta, \eta}^1(\bar{\Omega}; \mathbb{R}^n) = \mathcal{C}_{\lambda}^1(\bar{\Omega}; \mathbb{R}^n).$$

The objective of this Note is to indicate how to establish the following *nonlinear Korn inequalities* about mappings from a domain  $\Omega \subset \mathbb{R}^n$  into  $\mathbb{R}^n$ .

**Theorem 1.** *Let  $\Omega$  be a domain in  $\mathbb{R}^n$ ,  $n \geq 2$ . Given any real numbers  $p > 1$ ,  $1 \geq \lambda > 0$ ,  $\delta > 0$ , and  $\eta > 0$ , such that  $4C_{\Omega}\eta \leq \lambda^n$ , there exists a constant  $C_0 = C_0(\Omega, p, \lambda, \delta)$  such that:*

(a) *For all mappings  $\Theta \in \mathcal{C}_{\lambda, \delta, \eta}^1(\bar{\Omega}; \mathbb{R}^n)$  and  $\Phi \in W^{1,p}(\Omega; \mathbb{R}^n)$ ,*

$$\inf_{R \in \mathbb{O}_+^n} \|\nabla \Phi - R \nabla \Theta\|_{L^p(\Omega)} \leq C_0 \left\| \inf_{R \in \mathbb{O}_+^n} |\nabla \Phi - R \nabla \Theta| \right\|_{L^p(\Omega)}.$$

(b) *For all mappings  $\Theta \in \mathcal{C}_{\lambda, \delta, \eta}^1(\bar{\Omega}; \mathbb{R}^n)$  and  $\Phi \in W^{1,p}(\Omega; \mathbb{R}^n)$  such that  $\det \nabla \Phi > 0$  a.e. in  $\Omega$ ,*

$$\inf_{R \in \mathbb{O}_+^n} \|\nabla \Phi - R \nabla \Theta\|_{L^p(\Omega)} \leq C_0 \left\| (\nabla \Phi^T \nabla \Phi)^{1/2} - (\nabla \Theta^T \nabla \Theta)^{1/2} \right\|_{L^p(\Omega)}.$$

Theorem 1(a) constitutes an improvement over a previous generalization by Ciarlet & Mardare [5, Lemma 2] of the *geometric rigidity lemma* of Friesecke, James & Müller [7, Theorem 3.1].

Theorem 1(b) constitutes an improvement over a previous nonlinear Korn inequality in  $\mathbb{R}^n$  established by Ciarlet & Mardare [5, Theorem 1(a)].

The details of the proofs sketched below will be given in a forthcoming paper [8].

## 2. Two preliminary lemmas

Any mapping  $\Theta \in \mathcal{C}_+^1(\bar{\Omega}; \mathbb{R}^n)$  is *locally bi-Lipschitz*, in the sense that each  $x \in \bar{\Omega}$  possesses a neighbourhood such that the restriction of  $\Theta$  to this neighbourhood is Lipschitz-continuous and invertible, with an inverse also Lipschitz-continuous; this property is a consequence of the implicit function theorem applied to the mapping  $\Theta$  (see, e.g., Ciarlet [3] and Ciarlet & Mardare [4]).

The next lemma shows that, if in addition  $\Theta \in \mathcal{C}_{\lambda, \delta, \eta}^1(\bar{\Omega}; \mathbb{R}^n)$ , then the size of this neighbourhood and the Lipschitz constants of the mapping  $\Theta$  restricted to this neighbourhood and of its inverse mapping are all controlled by the constants  $\lambda, \delta, \eta$ , and by the domain  $\Omega$  via the constant  $C_{\Omega}$  appearing in inequality (1).

**Lemma 2.** *Let  $\Omega$  be a domain in  $\mathbb{R}^n$ ,  $n \geq 2$ . Given any real numbers  $p > 1$ ,  $1 \geq \lambda > 0$ ,  $\delta > 0$  and  $\eta > 0$ , and any constant  $C_{\Omega}$  for which inequality (1) is satisfied by the geodesic distance in  $\Omega$ , the following properties hold:*

(a) *For all  $\Theta \in \mathcal{C}_{\lambda}^1(\bar{\Omega}; \mathbb{R}^n)$  and for all  $x, \tilde{x} \in \bar{\Omega}$ ,*

$$\lambda |\Theta(x) - \Theta(\tilde{x})| \leq 2C_{\Omega} |x - \tilde{x}|.$$

(b) *For all  $\Theta \in \mathcal{C}_{\lambda, \delta, \eta}^1(\bar{\Omega}; \mathbb{R}^n)$  and for all  $x, \tilde{x} \in \bar{\Omega}$  such that  $2C_{\Omega} |x - \tilde{x}| \leq \delta$ ,*

$$|\Theta(x) - \Theta(\tilde{x})| \geq (\lambda^n - 2\eta C_{\Omega}) |x - \tilde{x}|.$$

**Sketch of proof.** Property (a) is obtained by writing the difference  $(\Theta(\tilde{x}) - \Theta(x))$  as the integral of the derivative of  $\Theta$  along a curve of class  $\mathcal{C}^1$  joining  $x$  and  $\tilde{x}$  and of length  $\leq 2C_{\Omega} |x - \tilde{x}|$ . That such a curve exists follows from the definition of the geodesic distance in  $\Omega$  and from inequality (1).

Property (b) is obtained by using the same expression of the difference  $(\Theta(\tilde{x}) - \Theta(x))$  as above, combined with the property that the eigenvalues of the symmetric matrices  $(\nabla \Theta(x)^T \nabla \Theta(x))^{1/2}$ ,  $x \in \bar{\Omega}$ , associated with any mapping  $\Theta \in \mathcal{C}_{\lambda}^1(\bar{\Omega}; \mathbb{R}^n)$  all belong to the interval  $[\lambda^n, 1/\lambda]$  (as a

consequence of the definition (2) of the set  $\mathcal{C}_\lambda^1(\bar{\Omega}; \mathbb{R}^n)$ . If in addition  $\Theta \in \mathcal{C}_{\lambda, \delta, \eta}^1(\bar{\Omega}; \mathbb{R}^n)$ , then, for all  $x, \tilde{x} \in \bar{\Omega}$  such that  $2C_\Omega|x - \tilde{x}| \leq \delta$ ,

$$\begin{aligned} \left| \Theta(\tilde{x}) - \Theta(x) \right| &\geq \left| \nabla \Theta(x)(\tilde{x} - x) \right| - 2\eta \operatorname{dist}_\Omega(x, \tilde{x}) \\ &\geq \left( (\tilde{x} - x)^T \nabla \Theta(x)^T \nabla \Theta(x)(\tilde{x} - x) \right)^{1/2} - 2\eta C_\Omega|x - \tilde{x}| \\ &\geq (\lambda^n - 2\eta C_\Omega)|x - \tilde{x}|. \end{aligned}$$

Note that the first inequality above is obtained by using the definition (3) of the set  $\mathcal{C}_{\lambda, \delta, \eta}^1(\bar{\Omega}; \mathbb{R}^n)$ . □

The following lemma is a generalization for  $p \neq 2$  of a previous result by Ciarlet & Mardare [4, Lemma 5]. Its proof is similar to that in the case  $p = 2$  and for this reason is omitted.

**Lemma 3.** *Let  $\Omega$  be a domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , let  $U \subset V \subset \Omega$  be two non-empty open subsets of  $\Omega$ , and let  $1 \geq \lambda > 0$  and  $p \geq 1$  be two real numbers.*

*Given any mappings  $\Theta \in \mathcal{C}_\lambda^1(\bar{\Omega}; \mathbb{R}^n)$  and  $\Phi \in W^{1,p}(\Omega; \mathbb{R}^n)$ , define the matrices*

$$F = F(\Phi, \Theta, U) := \left( \int_U |\det \nabla \Theta| dx \right)^{-1} \int_U \nabla \Phi (\nabla \Theta)^{-1} |\det \nabla \Theta| dx \in \mathbb{M}^n,$$

and

$$Q = Q(\Phi, \Theta, U) \in \mathbb{O}_+^n \text{ such that } |F - Q| = \inf_{R \in \mathbb{O}_+^n} |F - R|.$$

Then

$$\|\nabla \Phi - Q \nabla \Theta\|_{\mathbb{L}^p(V)} \leq \left( 1 + \frac{2|V|_n^{1/p}}{\lambda^{n+1}|U|_n^{1/p}} \right) \inf_{R \in \mathbb{O}_+^n} \|\nabla \Phi - R \nabla \Theta\|_{\mathbb{L}^p(V)}.$$

### 3. Sketch of the proof of Theorem 1

The idea of the proof is to decompose the domain  $\Omega$  into a *finite family of subdomains*  $(\Omega_j)_{j=1}^J$  over which the inequality of part (a) of Theorem 1 is easier to establish, then to add all these inequalities. This strategy uses that we are able to control the number  $J$  of subdomains by an upper bound independent of the mappings  $\Theta$  and  $\Phi$ , and that we are able to estimate the  $\inf_{R \in \mathbb{O}_+^n}$  in inequality (a) established in each subdomain by a single well-chosen matrix in  $\mathbb{O}_+^n$ , independent of the subdomain.

The main steps of the proof are the following:

**(i). Decomposition of the domain  $\Omega$  into a finite family of subdomains.** We show that, given any  $\delta > 0$ , there exist an integer  $J = J(\Omega, \delta)$ , a real number  $\varepsilon = \varepsilon(\Omega, \delta)$ , and  $J$  domains  $\Omega_1, \dots, \Omega_J$  in  $\mathbb{R}^n$  with the following properties:

$$\Omega = \bigcup_{j=1}^J \Omega_j;$$

the closure of each set  $\Omega_j$  is contained in an open ball with diameter  $\leq (2C_\Omega)^{-1}\delta$ , where  $C_\Omega$  denotes the constant appearing in inequality (1) satisfied by the geodesic distance in  $\Omega$ ;

if  $J \geq 2$ , then  $V_k := \bigcup_{j=1}^k \Omega_j$  is connected and satisfies  $|V_k \cap \Omega_{k+1}|_n \geq \varepsilon$ ,  $k \in \{1, \dots, J-1\}$ .

Our proof provides subdomains  $\Omega_j$  that are either open  $n$ -cubes in  $\mathbb{R}^n$  with edges parallel to the axes of coordinates, or intersections with  $\Omega$  of open  $n$ -orthotopes with edges parallel to the axes of coordinates, of one of the local Cartesian frames used to define the Lipschitz-continuous boundary of  $\Omega$  (cf., e.g., Adams [1] or Maz'ya [9]).

(ii). **The inequality of Theorem 1(a) holds with  $\Omega$  replaced by  $\Omega_j$ , for each  $1 \leq j \leq J$ .** Let there be given any  $1 \leq j \leq J$  and any mapping  $\Theta \in \mathcal{C}_{\lambda, \delta, \eta}^1(\bar{\Omega}; \mathbb{R}^n)$ . Since  $\bar{\Omega}_j$  is contained in an open ball with diameter  $\leq (2C_\Omega)^{-1}\delta$ , Lemma 2(b) shows that

$$\Theta|_{\bar{\Omega}_j} \text{ is one-to-one,}$$

where  $\Theta|_{\bar{\Omega}_j}$  denotes the restriction of the mapping  $\Theta$  to the closure of the subset  $\Omega_j$  of  $\Omega$ .

Another application of Lemma 2 combined with the explicit definition of the sets  $\Omega_j$  as either open  $n$ -cubes, or intersections with  $\Omega$  of open  $n$ -orthotopes, shows that the set  $\Theta(\Omega_j)$  coincides with the image of a  $n$ -cube by means of a specific bi-Lipschitz mapping (possibly different from the mapping  $\Theta|_{\Omega_j}$ ) with Lipschitz constants bounded above by a constant  $\mu = \mu(\Omega, \delta, \lambda)$  independent of  $j$  and  $\Theta$ .

This allows us to use the *geometric rigidity lemma* of Friesecke, James & Müller [7] (see also Conti [6] for extensions to  $p \neq 2$ ) for mappings defined on the set  $\Theta(\Omega_j)$  and to infer the existence of a constant  $C(p, \mu)$ , thus in particular independent of  $j$  and  $\Theta$ , such that, for every mapping  $\Phi \in W^{1,p}(\Omega; \mathbb{R}^n)$ ,

$$\inf_{R \in \mathbb{O}_+^n} \left\| \nabla(\Phi \circ (\Theta|_{\Omega_j})^{-1}) - R \right\|_{L^p(\Theta(\Omega_j))} \leq C(p, \mu) \left\| \inf_{R \in \mathbb{O}_+^n} \left| \nabla(\Phi \circ (\Theta|_{\Omega_j})^{-1}) - R \right| \right\|_{L^p(\Theta(\Omega_j))}.$$

This inequality in turn implies that, for every mapping  $\Phi \in W^{1,p}(\Omega; \mathbb{R}^n)$ ,

$$\inf_{R \in \mathbb{O}_+^n} \|\nabla\Phi - R\nabla\Theta\|_{L^p(\Omega_j)} \leq C(p, \mu)\lambda^{-(n+1)(1+1/p)} \left\| \inf_{R \in \mathbb{O}_+^n} |\nabla\Phi - R\nabla\Theta| \right\|_{L^p(\Omega_j)}.$$

(iii). **The inequality of Theorem 1(a) holds.** If  $J = 1$ , then there is nothing to do as  $\Omega = \Omega_1$  and thus the inequality established above coincides with the inequality announced in Theorem 1(a) with  $C_0 := C(p, \mu)\lambda^{-(n+1)(1+1/p)}$ .

So assume that  $J \geq 2$ . Then Lemma 3 allows to use a well-chosen matrix in  $\mathbb{O}_+^n$  to estimate the infimum  $\inf_{R \in \mathbb{O}_+^n}$  appearing in the left-hand sides of the inequalities established in step (ii) for  $j = 1$  and  $j = 2$ . This yields the following inequality:

$$\inf_{R \in \mathbb{O}_+^n} \|\nabla\Phi - R\nabla\Theta\|_{L^p(V_2)} \leq C_{0,2} \left\| \inf_{R \in \mathbb{O}_+^n} |\nabla\Phi - R\nabla\Theta| \right\|_{L^p(V_2)},$$

where

$$C_{0,2} = 2C(p, \mu)\lambda^{-(n+1)(1+1/p)} \left( 1 + \frac{2|\Omega|_n^{1/p}}{\lambda^{n+1}\varepsilon^{1/p}} \right).$$

If  $J > 2$ , we use again Lemma 3 to estimate  $\inf_{R \in \mathbb{O}_+^n}$  in both the above inequality and the inequality established in step (ii) for  $j = 3$  by using another well-chosen matrix in  $\mathbb{O}_+^n$ . This yields the following inequality:

$$\inf_{R \in \mathbb{O}_+^n} \|\nabla\Phi - R\nabla\Theta\|_{L^p(V_3)} \leq C_{0,3} \left\| \inf_{R \in \mathbb{O}_+^n} |\nabla\Phi - R\nabla\Theta| \right\|_{L^p(V_3)},$$

where

$$C_{0,3} = \left( C_{0,2} + C(p, \mu)\lambda^{-(n+1)(1+1/p)} \right) \left( 1 + \frac{2|\Omega|_n^{1/p}}{\lambda^{n+1}\varepsilon^{1/p}} \right).$$

Since  $\Omega = V_J$ , repeating the above argument  $(J - 1)$ -times yields the inequality of Theorem 1(a) with a constant  $C_0 := C_{0,J}$  given as the last iterate of the following recurrence relation:

$$C_{0,1} := C(p, \mu)\lambda^{-(n+1)(1+1/p)} \quad \text{and} \quad C_{0,k+1} := (C_{0,k} + C_{0,1}) \left( 1 + \frac{2|\Omega|_n^{1/p}}{\lambda^{n+1}\varepsilon^{1/p}} \right), \quad 1 \leq k \leq J - 1.$$

That the constant  $C_0$  defined in this fashion depends only on  $\Omega, p, \lambda$ , and  $\delta$ , is clear.

(iv). **The inequality of Theorem 1(b) holds.** Given any two mappings  $\Theta \in \mathcal{C}_{\lambda, \delta, \eta}^1(\bar{\Omega}; \mathbb{R}^n)$  and  $\Phi \in W^{1,p}(\Omega; \mathbb{R}^n)$  such that  $\det \nabla \Phi > 0$  a.e. in  $\Omega$ , let

$$\begin{aligned} P(x) &:= \nabla \Theta(x) (\nabla \Theta(x)^T \nabla \Theta(x))^{-1/2} && \text{for all } x \in \bar{\Omega}, \\ Q(x) &:= \nabla \Phi(x) (\nabla \Phi(x)^T \nabla \Phi(x))^{-1/2} && \text{for almost all } x \in \Omega, \\ R(x) &:= Q(x) P(x)^{-1} && \text{for almost all } x \in \Omega. \end{aligned}$$

Note that all these matrices belongs to the set  $\mathbb{O}_+^n$  for almost all  $x \in \Omega$ . Since the Frobenius norm in  $\mathbb{M}^n$  is invariant under rotations, we then have, for almost all  $x \in \Omega$ ,

$$\begin{aligned} \inf_{R \in \mathbb{O}_+^n} |\nabla \Phi(x) - R \nabla \Theta(x)| &= \inf_{R \in \mathbb{O}_+^n} |Q(x) (\nabla \Phi(x)^T \nabla \Phi(x))^{1/2} - R P(x) (\nabla \Theta(x)^T \nabla \Theta(x))^{1/2}| \\ &= \inf_{R \in \mathbb{O}_+^n} |(\nabla \Phi(x)^T \nabla \Phi(x))^{1/2} - Q(x)^{-1} R P(x) (\nabla \Theta(x)^T \nabla \Theta(x))^{1/2}| \\ &\leq |(\nabla \Phi(x)^T \nabla \Phi(x))^{1/2} - (\nabla \Theta(x)^T \nabla \Theta(x))^{1/2}|. \end{aligned}$$

Besides, the inequality established in step (iii) above asserts that

$$\inf_{R \in \mathbb{O}_+^n} \|\nabla \Phi - R \nabla \Theta\|_{L^p(\Omega)} \leq C_0 \left\| \inf_{R \in \mathbb{O}_+^n} |\nabla \Phi - R \nabla \Theta| \right\|_{L^p(\Omega)},$$

for some specific constant  $C_0 = C_0(\Omega, p, \lambda, \delta)$ .

The inequality of Theorem 1(b) is then obtained by combining the last two inequalities.  $\square$

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