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On Sharpness of $L\log L$ Criterion for Weak Type $(1,1)$ boundedness of rough operators

Sur la netteté du critère $L\log L$ pour les faibles de type $(1,1)$ continuité des opérateurs rugueux

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Abstract. In this note, we show that the $\Omega \in L\log L$ hypothesis is the strongest size condition on a function Ω on the unit sphere with mean value zero, which ensures that the corresponding singular integral T_Ω defined by

$$T_\Omega f(x) = p.v.\int \frac{1}{|x-y|^d}\Omega\Big(\frac{x-y}{|x-y|}\Big) f(y)\,\mathrm{d}y,$$

maps $L^1(\mathbb{R}^d)$ to weak $L^1(\mathbb{R}^d)$, provided T_Ω is bounded in $L^2(\mathbb{R}^d)$.

Résumé. Dans cette note, nous montrons que l'hypothèse $\Omega \in L\log L$ est la condition de taille la plus forte sur une fonction Ω sur la sphère unitaire de valeur moyenne zéro, qui assure que l'intégrale singulière correspondante T_Ω définie par

$$T_\Omega f(x) = p.v.\int \frac{1}{|x-y|^d}\Omega\Big(\frac{x-y}{|x-y|}\Big) f(y)\,\mathrm{d}y,$$

est borné de $L^1(\mathbb{R}^d)$ dans $L^1(\mathbb{R}^d)$ faibles, à condition que T_Ω soit bornée dans $L^2(\mathbb{R}^d)$.

Keywords. Singular Integrals, Orlicz spaces.

Mots-clés. Intégrales singulières, espaces d'Orlicz.

2020 Mathematics Subject Classification. 42B20.

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1. Introduction

Let $\Omega \in L^1(\mathbb{S}^{d-1})$ with $\int_{\mathbb{S}^{d-1}}\Omega(\theta)\mathrm{d}\theta = 0$, where $\mathrm{d}\theta$ is the surface measure on $\mathbb{S}^{d-1}$. Calderón and Zygmund [2] considered the rough singular integrals defined as,

$$T_\Omega f(x) = p.v.\int \frac{1}{|x-y|^d}\Omega\Big(\frac{x-y}{|x-y|}\Big) f(y)\,\mathrm{d}y,$$

They showed that $\Omega \in L\log L(\mathbb{S}^{d-1})$ i.e. $\int_{\mathbb{S}^1}|\Omega(\theta)|\log(e+|\Omega(\theta)|) < \infty$ implies that T_Ω is bounded on $L^p(\mathbb{R}^d)$ for $1 < p < \infty$. The singular integral T_Ω was shown to be of weak type $(1,1)$ using

TT^* arguments by Christ and Rubio de Francia [3] in dimension $d = 2$ (and independently by Hofmann [10]). The case of general dimensions was resolved by Seeger [16] by showing that T_Ω is of weak type $(1,1)$ for $\Omega \in L\log L(\mathbb{S}^{d-1})$.

It is of interest to know other sufficient conditions on Ω that ensures the weak type boundedness of the operator T_Ω. In fact, during the inception of this problem, Calderón and Zygmund [2] showed that $\Omega \in L\log L$ is "almost" a necessary size condition for T_Ω to be L^2 bounded. If we drop the condition that $\Omega \in L\log L$, then Calderón and Zygmund [2] pointed out that T_Ω may even fail to be L^2 bounded. Infact, the examples of Ω constructed in [18] lies outside the space $L\log L$ and the corresponding operator T_Ω is unbounded on $L^2(\mathbb{R}^d)$. Later on, it was shown in [5, 13] that $\Omega \in H^1(\mathbb{S}^1)$ in the sense of Coifman and Weiss [4] implies $T_\Omega : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$, $1 < p < \infty$. For a detailed proof, we refer to [8, 9, 14]. It is still an open problem if T_Ω is of weak type $(1,1)$ for $\Omega \in H^1(\mathbb{S}^1)$. A partial result assuming additional conditions on H^1-atoms in dimension two was obtained by Stefanov [17].

In [7, 11], it was shown that T_Ω distinguishes L^p spaces by considering a suitable quantity based on the Fourier transform of Ω. However, we would like to know if there exists an Orlicz space $X \supsetneq L\log L$ which would ensure that the L^2 boundedness of T_Ω implies the weak $(1,1)$ boundeness of T_Ω when $\Omega \in X$. We will show that no such X exists. To state our main result, we introduce the Orlicz spaces and discuss some of its basic properties.

Definition 1 ([1]). *Let $\Phi : [0,\infty) \to [0,\infty)$ be a Young's function i.e. there exists an increasing and left continuous function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that $\Phi(t) = \int_0^t \phi(u)\,\mathrm{d}u$. We say $\Omega \in \Phi(L)(\mathbb{S}^1)$, if the quantity*

$$\|\Omega\|_{\Phi(L)} = \int_{\mathbb{S}^1} \Phi(|\Omega(\theta)|)\,\mathrm{d}\theta \tag{1}$$

is finite.

We note that the function $\frac{\Phi(t)}{t}$ is non-decreasing.

The quantity in (1) fails to be a norm and $\Phi(L)(\mathbb{S}^1)$ is not even a linear space. To remedy that, we define the set

$$L^\Phi(\mathbb{S}^1) = \{\Omega : \mathbb{S}^1 \longrightarrow \mathbb{R} : \exists k > 0 \text{ such that } \|k^{-1}\Omega\|_{\Phi(L)} < \infty\}.$$

We define the Luxemburg norm as

$$|||\Omega|||_{\Phi(L)} = \inf\{k > 0 : \|k^{-1}\Omega\|_{\Phi(L)} \le 1\}.$$

It is well-known that the Orlicz space $L^\Phi(\mathbb{S}^1)$ forms a Banach space with this norm. For details, we refer to [1].

2. Main result

We state our main result for dimension two but the same also holds for higher dimensions using the methods in [7, 18]. Our main result is the following,

Theorem 2. *Let Φ be a Young's function such that*

$$\Psi(t) = \frac{t\log(e+t)}{\Phi(t)} \longrightarrow \infty, \quad \text{as } t \longrightarrow \infty, \tag{2}$$

Then there exists an $\Omega \in \Phi(L)(\mathbb{S}^1)$ such that T_Ω is L^p bounded iff $p = 2$. In particular, T_Ω does not map $L^1(\mathbb{R}^2)$ to $L^{1,\infty}(\mathbb{R}^2)$.

We note that using the geometric construction in [11], one can obtain the above theorem for the space $L(\log L)^{1-\epsilon}(\mathbb{S}^1)$, $0 < \epsilon \le 1$. To obtain the general case, we will employ the construction in [7] with a suitable modification to ensure that the resulting Ω lies in the required Orlicz space.

The proof of Theorem 2 is contained in Section 3. We will require the following notations throughout the paper. We say $X \lesssim Y$ if there exists an absolute constant $C > 0$ (not depending on X and Y) such that $X \leq CY$. Similarly, we say $X \gtrsim Y$ if there exists an absolute constant $C > 0$ (not depending on X and Y) such that $X \geq CY$. We say $X \sim Y$ if $X \lesssim Y$ and $X \gtrsim Y$.

3. Proof of Theorem 2

To prove Theorem 2, we will construct a sequence of even functions $\{\Omega_n\} \in \Phi(L)$ with mean value zero such that the L^p norm of T_{Ω_n} is large for $p \neq 2$ while having bounded $\Phi(L)-$Orlicz norm uniformly in n. Moreover, the quantity $\|m(\Omega_n)\|_{L^\infty}$ grows slowly in terms of n. More precisely, we will show that Ω_n satisfies,

$$\|T_{\Omega_n}\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)} \gtrsim n^{|\frac{1}{2}-\frac{1}{p}|},$$

and

$$|||\Omega_n|||_{\Phi(L)(\mathbb{S}^1)} + \|m(\Omega_n)\|_{L^\infty(\mathbb{S}^1)} \lesssim \log n.$$

This will lead to a contradiction by an application of uniform boundedness principle. The proof is divided into four crucial steps described below.

Step 1. The geometric construction of functions w_k and Ω_n. We will construct even functions w_k and a sequence of even functions Ω_n on the unit circle $\mathbb{S}^1$ with mean value zero in this step.

We fix a large $N \in \mathbb{N}$. Let $n \in \mathbb{N}$ be a number depending on N to be chosen later (see (6)).

Let $s_n \in \mathbb{N}$ and $t_1, t_2, \ldots, t_{2n} \in \mathbb{Z}$ be such that,

- The numbers t_k are in arithmetic progression, i.e. $t_{k+1} - t_k = t_k - t_{k-1}$.
- Let $x_k = (t_k, s_n) \in \mathbb{R}^2$. Then x_k, $k = 1, \ldots, 2n$, lies in the second quadrant between the lines y-axis and $y = -x$.
- $|\frac{x_{k+1}}{|x_{k+1}|} - \frac{x_k}{|x_k|}| \sim \frac{1}{n}$.

(We note that the points $x_k = (-kn, 10n^2)$, $k = 1, \ldots, 2n$, satisfies the above properties.)

We denote $\widetilde{x}_k$ to be the point on $\mathbb{S}^1$ obtained by rotating the point $\frac{x_k}{|x_k|}$ by $\frac{\pi}{2}$ radians clockwise. We consider I_k, $k = 1, \ldots, 2n$, to be the arc on $\mathbb{S}^1$ with centre $\widetilde{x}_k$ and arc length N^{-1} and denote $\mathfrak{R}_\alpha(I_k)$ to be the arc obtained by rotating I_k by α radians counterclockwise. We note that the arcs I_k, $k = 1, \ldots, 2n$, are disjoint for our choice of n; we will justify this in Step 3.

We define w_k as

$$w_k(\theta) = c_{I_k}(-\chi_{I_k}(\theta) + \chi_{\mathfrak{R}_{\frac{\pi}{2}}(I_k)}(\theta) - \chi_{\mathfrak{R}_\pi(I_k)}(\theta) + \chi_{\mathfrak{R}_{\frac{3\pi}{2}}(I_k)}(\theta)),$$

where the constants c_{I_k} are determined in Step 2.

We now set

$$\Omega_n = \sum_{k=1}^{2n} (-1)^k \epsilon_{\left[\frac{k+1}{2}\right]} w_k,$$

where [ ] denotes the integer part and the coefficients $\epsilon_{[.]}$ are as in Lemma 4 in Step 3. It is easy to see that w_k and Ω_n are even functions with mean value zero for all $k = 1, \ldots, 2n$.

Step 2 Auxiliary properties of $m(w_k)$. In this step, we will obtain some basic estimates for the quantity $m(w_k)$ and the Fourier transform of w_k. We recall that the Fourier transform of the kernel in T_Ω for any even Ω with mean value zero is given by

$$\widehat{K}_\Omega(\xi) = \int_{\mathbb{S}^1} \Omega(\theta) \log \frac{1}{|\langle \xi, \theta \rangle|} \, d\theta.$$

We define the larger quantity $m(\Omega)$ which will be useful for our purpose.

$$m(\Omega)(\xi) := \int_{\mathbb{S}^1} |\Omega(\theta)| \log \frac{1}{|\langle \xi, \theta\rangle|} \,\mathrm{d}\theta.$$

Clearly, $|\widehat{K}_\Omega(\xi)| \le m(\Omega)(\xi)$.

We choose c_{I_k} such that $m(w_k)(\frac{x_k}{|x_k|}) = 1$.

It is not difficult to see that c_{I_k} and $\widehat{K}_{w_k}(\frac{x_k}{|x_k|})$ are independent of k. Moreover, we have the following estimates,

Proposition 3. *For $k = 1, \dots, 2n$, the following holds true,*

(1) *There exists an absolute constant $c > 0$ such that $\frac{N}{c \log N} \le c_{I_k} \le \frac{cN}{\log N}$.*

(2) $1 \lesssim \sup_x |\widehat{K}_{w_k}(x)| = \left|\widehat{K}_{w_k}\left(\frac{x_k}{|x_k|}\right)\right| \le \sup_x m(w_k)(x) = 1$.

(3) *Let J_k be the arc centered at the point $\frac{x_k}{|x_k|}$ and of length $\frac{1}{100n}$. Then for $x \in \mathbb{S}^1$ lying in second quadrant between the lines y-axis and $y = -x$ with $x \notin \bigcup_{i=0}^3 \mathfrak{R}_{\frac{i\pi}{2}}(J_k)$, we have*

$$m(w_k)(x) \lesssim \frac{\log n}{\log N}. \tag{3}$$

(4) *For $1 \le k \le n$ and $x \in \mathbb{S}^1$ lying in second quadrant between the lines y-axis and $y = -x$ with $x \notin \left(\bigcup_{i=0}^3 \mathfrak{R}_{\frac{i\pi}{2}}(J_{2k})\right) \cup \left(\bigcup_{i=0}^3 \mathfrak{R}_{\frac{i\pi}{2}}(J_{2k-1})\right)$, we have*

$$|\widehat{K}_{w_{2k}}(x) - \widehat{K}_{w_{2k-1}}(x)| \lesssim \left(n \log N \left|\frac{x}{|x|} - \frac{x_{2k}}{|x_{2k}|}\right|\right)^{-1}. \tag{4}$$

Proof. First, we observe that it is enough to prove (2) for $x \in \mathbb{S}^1$ as $\int_0^{2\pi} w_k(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta = 0$. Since, w_k is even, we have that for any $0 \le \gamma < 2\pi$,

$$\begin{aligned}
\widehat{K}_{w_k}(\mathrm{e}^{\mathrm{i}\gamma}) &= \int_0^{2\pi} w_k(\mathrm{e}^{\mathrm{i}\theta}) \log \frac{1}{|\mathrm{e}^{\mathrm{i}\theta} \cdot \mathrm{e}^{\mathrm{i}\gamma}|} \,\mathrm{d}\theta \\
&= \int_0^{2\pi} w_k(\mathrm{e}^{\mathrm{i}\theta}) \log \frac{1}{|\cos(\theta - \gamma)|} \,\mathrm{d}\theta \\
&= c_{I_k} \left[-\int_{A_k} + \int_{A_k + \frac{\pi}{2}} - \int_{A_k + \pi} + \int_{A_k + \frac{3\pi}{2}} \right] \log \frac{1}{|\cos(\theta - \gamma)|} \,\mathrm{d}\theta \\
&= -2 c_{I_k} \int_{-\frac{|A_k|}{2}}^{\frac{|A_k|}{2}} \log |\tan(\theta + \theta_k - \gamma)| \,\mathrm{d}\theta,
\end{aligned} \tag{5}$$

where $\mathrm{e}^{\mathrm{i}\theta_k} = \frac{x_k}{|x_k|}$ and A_k be the interval in $(0, \frac{\pi}{4})$ such that $I_k - \frac{x_k}{|x_k|} = \{\mathrm{e}^{\mathrm{i}\theta} : \theta \in A_k\}$. Similarly, we obtain that,

$$\begin{aligned}
c_{I_k}^{-1} &\sim -\int_{-\frac{|A_k|}{2}}^{\frac{|A_k|}{2}} \log|\sin\theta| \,\mathrm{d}\theta \\
&= -2 \int_0^{\frac{|A_k|}{2}} \log \sin\theta \,\mathrm{d}\theta \\
&\sim -\int_0^{\frac{|A_k|}{2}} \log t \,\mathrm{d}t \\
&\sim |A_k| |\log |A_k|| \sim \frac{\log N}{N},
\end{aligned}$$

where we used the fact that $\sin\theta \sim \theta$ for $\theta \in (0, \frac{\pi}{4})$. Thus, we obtain (1) and the estimate (2) follows similarly from (5).

The estimate (3) follows from the fact that for $\gamma \in (2I_k)^c \cap (0, \frac{\pi}{4})$, we have

$$|m(w_{I_k})(e^{i\gamma})| \lesssim \frac{|\log|\gamma - \widetilde{x}_k||}{|\log|I_k||}.$$

Indeed, for $\theta \in I_k$, we have $|\gamma - \widetilde{x}_k| < |\theta - \widetilde{x}_k| + |\theta - \gamma| < \frac{|I_k|}{2} + |\theta - \gamma| < |\gamma - \widetilde{x}_k|/2 + |\theta - \gamma|$. Thus $\frac{|\tilde{x}_k - \gamma|}{2} < |\theta - \gamma|$ and it follows that

$$\begin{aligned}
|m(w_{I_k})(\mathrm{e}^{\mathrm{i}\gamma})| &\lesssim -c_{I_k} \int_{I_k} \log|\sin(\theta - \gamma)|\,\mathrm{d}\theta \\
&\le c_{I_k} |I_k| |\log \left| \sin\left(\frac{|\gamma - \widetilde{x}_k|}{2}\right)\right| \\
&\lesssim \frac{|\log|\gamma - \widetilde{x}_k||}{|\log|I_k||}.
\end{aligned}$$

We now prove the estimate (4). Let $\mathrm{e}^{\mathrm{i}\theta_{2k}} = \frac{x_{2k}}{|x_{2k}|}$, $\mathrm{e}^{\mathrm{i}\gamma} = \frac{x}{|x|}$ and A_{2k} be the interval in $(-\frac{\pi}{4}, \frac{\pi}{4})$ such that $I_{2k} - \frac{x_{2k}}{|x_{2k}|} = \{\mathrm{e}^{\mathrm{i}\theta} : \theta \in A_{2k}\}$. By using mean value theorem twice and the fact that $|\theta_{2k} - \theta_{2k-1}|$ is small, we have

$$\begin{aligned}
|\widehat{K}_{w_{2k}}(x) - \widehat{K}_{w_{2k-1}}(x)| &\lesssim c_{I_{2k}} \int_{A_{2k}} \left(\log \frac{1}{|\tan(\theta + \theta_{2k} - \gamma)|} - \log \frac{1}{|\tan(\theta + \theta_{2k-1} - \gamma)|}\right) \mathrm{d}\theta \\
&\lesssim c_{I_{2k}} \int_{A_{2k}} \frac{|\tan(\theta + \theta_{2k} - \gamma) - \tan(\theta + \theta_{2k-1} - \gamma)|}{|\tan(\theta + \theta_{2k} - \gamma)|}\,\mathrm{d}\theta \\
&\lesssim c_{I_{2k}} \int_{A_{2k}} \frac{|\theta_{2k} - \theta_{2k-1}|}{|\theta + \theta_{2k} - \gamma|}\,\mathrm{d}\theta \\
&\lesssim \frac{c_{I_{2k}}}{n} \int_{A_{2k}} \frac{1}{|\gamma - \theta_{2k}|}\,\mathrm{d}\theta \\
&\lesssim \left(n \log N \left|\frac{x}{|x|} - \frac{x_{2k}}{|x_{2k}|}\right|\right)^{-1},
\end{aligned}$$

where we have used $|\gamma - \theta_{2k}| \le 2|\theta + \theta_{2k} - \gamma|$ and $\tan\theta \sim \theta$ away from odd multiples of $\frac{\pi}{2}$. □

Step 3. The calculation of the $|||\Omega_n|||_{\Phi(L)(\mathbb{S}^1)}$ and the L^p-norms of T_{Ω_n}. In this step, we compute the $\Phi(L)$−Orlicz norm of Ω_n and the L^p− norm of the corresponding operator T_{Ω_n}. We begin by choosing n as follows,

$$n = \left[\frac{N}{16\Phi\left(\frac{cN}{\log N}\right)}\right] + 1, \tag{6}$$

where $c > 0$ is as in Proposition 3(1). By hypothesis (2), we have $n \to \infty$ as $N \to \infty$. Moreover, we have $N^{-1} \lesssim n^{-1}$ as Φ is an increasing function. This implies that the corresponding arcs I_k, $k = 1, \ldots, 2n$, are disjoint. Hence, we have

$$\begin{aligned}
\|\Omega_n\|_{\Phi(L)(\mathbb{S}^1)} &= \sum_{k=1}^{2n} \sum_{l=0}^{3} \int_{A_k + \frac{l\pi}{2}} \Phi\left(|\epsilon_{\left[\frac{k+1}{2}\right]} c_{I_k}|\right) \mathrm{d}\theta \\
&\le \frac{8n}{N} \Phi\left(\frac{cN}{\log N}\right) \\
&\le 1.
\end{aligned}$$

Thus, by the definition of the Luxemburg norm $|||\cdot|||_{\Phi(L)}$, we have

$$|||\Omega_n|||_{\Phi(L)(\mathbb{S}^1)} \le 1. \tag{7}$$

We estimate the quantity $\|m(\Omega_n)\|_{L^\infty(\mathbb{S}^1)}$ by employing Proposition 3(3). Indeed, we have

$$\|m(\Omega_n)\|_{L^\infty(\mathbb{S}^1)} \lesssim 1 + \frac{n\log n}{\log N} \le \frac{\log n}{8c} \frac{\frac{cN}{\log N}}{\Phi\left(\frac{cN}{\log N}\right)} \lesssim \log n, \tag{8}$$

where we used that $\frac{\Phi(t)}{t}$ is a non-decreasing function in the last step.

We now compute the L^p-norms of the corresponding operator T_{Ω_n}. The space of L^p multipliers $M^p(\mathbb{T})$ and $M^p(\mathbb{R}^2)$ are defined as

$$M^p(\mathbb{T}) = \left\{ \mathbf{a} = \{a_n\} \in l^\infty(\mathbb{Z}) : \ T_{\mathbf{a}} f(x) = \sum_{n\in\mathbb{Z}} a_n \widehat{f}(n)\, \mathrm{e}^{2\pi \mathrm{i} n x} \text{ is bounded on } L^p(\mathbb{T}) \right\},$$
$$M^p(\mathbb{R}^2) = \left\{ \gamma \in L^\infty(\mathbb{R}^2) : \ T_\gamma f(x) = \int_{\mathbb{R}^2} \gamma(\xi) \widehat{f}(\xi)\, \mathrm{e}^{2\pi \mathrm{i} x\cdot\xi}\, \mathrm{d}\xi \text{ is bounded on } L^p(\mathbb{R}^2) \right\}.$$

We define $\|\mathbf{a}\|_{M^p(\mathbb{T})} = \|T_{\mathbf{a}}\|_{L^p(\mathbb{T})\to L^p(\mathbb{T})}$ and $\|\gamma\|_{M^p(\mathbb{R}^2)} = \|T_\gamma\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)}$.

We state two lemmas from [7] that will be useful in estimating the L^p-norms of T_{Ω_n}. The first lemma states that there exist a sequence of multipliers $\{\{\ldots,0,\epsilon_1,\epsilon_2,\ldots,\epsilon_n,0,\ldots\} : \ n \in \mathbb{N}\}$ on $\mathbb{T}$ whose L^p-norm blows up as n tends to infinity for $p \neq 2$. This was achieved in [7] by employing the fact that $\{\mathrm{e}^{2\pi \mathrm{i} kx}, \ k \in \mathbb{Z}\}$ is not an unconditional basis for $L^p(\mathbb{T})$, $p \neq 2$. Moreover, the quantity $\|\{\ldots,0,\epsilon_1,\epsilon_2,\ldots,\epsilon_n,0,\ldots\}\|_{M^p(\mathbb{T})}$ grows atleast of the order $n^{|\frac{1}{2}-\frac{1}{p}|}$. To justify this growth, we invoke Theorem 1 from [15],

For $n \in \mathbb{N}$, there exists $\{\epsilon_k\}_{k=1}^n$ with $\epsilon_k = \pm 1$ such that $\|\sum_{k=1}^n \epsilon_k \mathrm{e}^{2\pi \mathrm{i} kx}\|_{L^\infty(\mathbb{T})} \le 5n^{\frac{1}{2}}$,

and the well-known fact (Exercise 3.1.6 from [6]) that the L^p-norm of the Dirichlet kernel satisfies the following estimate:

$$\left\| \sum_{k=1}^n \mathrm{e}^{2\pi \mathrm{i} kx} \right\|_{L^p(\mathbb{T})} \sim n^{1-\frac{1}{p}} \text{ for } 1 < p < \infty.$$

Thus, we have,

$$\begin{aligned}\|\{\ldots,0,\epsilon_1,\epsilon_2,\ldots,\epsilon_n,0,\ldots\}\|_{M^p(\mathbb{T})} &\ge \frac{\left\|\sum_{k=1}^n \epsilon_k^2 \mathrm{e}^{2\pi \mathrm{i} kx}\right\|_{L^p(\mathbb{T})}}{\left\|\sum_{k=1}^n \epsilon_k \mathrm{e}^{2\pi \mathrm{i} kx}\right\|_{L^p(\mathbb{T})}} \\ &\ge \frac{\left\|\sum_{k=1}^n \epsilon_k^2 \mathrm{e}^{2\pi \mathrm{i} kx}\right\|_{L^p(\mathbb{T})}}{\left\|\sum_{k=1}^n \epsilon_k \mathrm{e}^{2\pi \mathrm{i} kx}\right\|_{L^\infty(\mathbb{T})}} \gtrsim n^{\frac{1}{2}-\frac{1}{p}}.\end{aligned}$$

The inequality $\|\{\ldots,0,\epsilon_1,\epsilon_2,\ldots,\epsilon_n,0,\ldots\}\|_{M^p(\mathbb{T})} \gtrsim n^{\frac{1}{p}-\frac{1}{2}}$ follows from

$$\|\{\ldots,0,\epsilon_1,\epsilon_2,\ldots,\epsilon_n,0,\ldots\}\|_{M^p(\mathbb{T})} = \|\{\ldots,0,\epsilon_1,\epsilon_2,\ldots,\epsilon_n,0,\ldots\}\|_{M^{\frac{p}{p-1}}(\mathbb{T})}$$

for $1 < p < \infty$.

Lemma 4 ([7]). *For $p \neq 2$ and fixed $n \in \mathbb{N}$, there exists finite sequences $\{a_k\}_{k=1}^n$ and $\{\epsilon_k\}_{k=1}^n$ (depending on n) with $\epsilon_k \in \{-1,1\}$ such that*

$$\left\| \sum_{k=1}^n \epsilon_k a_k \mathrm{e}^{2\pi \mathrm{i} kx} \right\|_{L^p(\mathbb{T})} \ge c_p n^{|\frac{1}{2}-\frac{1}{p}|} \left\| \sum_{k=1}^n a_k \mathrm{e}^{2\pi \mathrm{i} kx} \right\|_{L^p(\mathbb{T})},$$

where $c_p > 0$ depends only on p. Consequently, $\|\{\ldots,0,\epsilon_1,\epsilon_2,\ldots,\epsilon_n,0,\ldots\}\|_{M^p(\mathbb{T})} \gtrsim n^{|\frac{1}{2}-\frac{1}{p}|}$. Moreover, we can choose ϵ_k such that

$$\|\{\ldots,0,\epsilon_1,\epsilon_2,\ldots,\epsilon_n,0,\ldots\}\|_{M^p(\mathbb{T})} = \sup\left\{ \|\{\ldots,0,\delta_1,\delta_2,\ldots,\delta_n,0,\ldots\}\|_{M^p(\mathbb{T})} : \ |\delta_k| \le 1 \right\}.$$

The second lemma (stated below) along with an application of Lemma 4 provides us with a sequence of multipliers on the plane such that their L^p-norm blows up as n tends to infinity for $p \neq 2$. This lemma is based on a classical transference result of de Leeuw [12]. For a proof of the lemma, we refer to [7].

Lemma 5 ([7]). *Let $1 < p < \infty$ and $\gamma \in M^p(\mathbb{R}^2)$ be continuous on an arithmetic progression $\{x_k\}_{k=1}^n$ in $\mathbb{R}^2$ (i.e. there exists vector $v \in \mathbb{R}^2$ such that $x_k - x_{k-1} = v$). Then there exists a constant $C_p > 0$ such that*

$$\|\gamma\|_{M^p(\mathbb{R}^2)} \geq C_p \|\{\dots,0,\gamma(x_1),\gamma(x_2),\dots,\gamma(x_n),0,\dots\}\|_{M^p(\mathbb{T})}.$$

Now we turn to the estimate of L^p-bounds of T_{Ω_n}. We claim that

$$\|T_{\Omega_n}\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)} \gtrsim n^{|\frac{1}{2}-\frac{1}{p}|}. \tag{9}$$

For $1 \leq k \leq n$, we have

$$\widehat{K}_{\Omega_n}(x_{2k}) = (-1)^{2k}\widehat{K}_{w_{2k}}(x_{2k})\epsilon_k + \sum_{1\leq i\neq 2k\leq 2n} (-1)^i \epsilon_{\left[\frac{i+1}{2}\right]}\widehat{K}_{w_i}(x_{2k}) = D\epsilon_k + \delta_k,$$

where $D = \widehat{K}_{w_{2k}}(x_{2k})$ and $\delta_k = \sum_{1\leq i\neq 2k\leq 2n}(-1)^i\epsilon_{\left[\frac{i+1}{2}\right]}\widehat{K}_{w_i}(x_{2k})$.

Using Proposition 3(3) for the term $i = 2k-1$ and Proposition 3(4) for the remaining terms (in pair), we get

$$|\delta_k| \leq C\left(\frac{\log n}{\log N} + \frac{1}{\log N}\sum_{i=1}^{2n}\frac{1}{i}\right) \leq \frac{C'\log n}{\log N} \leq \frac{|D|}{4} \text{ (for large } n).$$

Hence, by the choice of Lemma 4, we have

$$\frac{1}{2}\|\{\dots,0,\epsilon_1,\epsilon_2,\dots,\epsilon_n,0,\dots\}\|_{M^p(\mathbb{T})} \geq \left\|\left\{\dots,0,\frac{\delta_1}{D},\frac{\delta_2}{D},\dots,\frac{\delta_n}{D},0,\dots\right\}\right\|_{M^p(\mathbb{T})}.$$

Since $\widehat{K}_{\Omega_n}(\theta)$ is a circular convolution of a $L^1(\mathbb{S}^1)$ and $L^\infty(\mathbb{S}^1)$, it is continuous at the points $x_{2k}, k = 1,\dots,n$, and applying Lemma 5, we have

$$\begin{aligned}
\|T_{\Omega_n}\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)} &= \|\widehat{K}_{\Omega_n}\|_{M^p(\mathbb{R}^2)} \\
&\gtrsim \|\{\dots,0,\widehat{K}_{\Omega_n}(x_2),\widehat{K}_{\Omega_n}(x_4),\dots,\widehat{K}_{\Omega_n}(x_{2n}),0,\dots\}\|_{M^p(\mathbb{T})} \\
&\gtrsim |D|\left(\|\{\dots,0,\epsilon_1,\epsilon_2,\dots,\epsilon_n,0,\dots\}\|_{M^p(\mathbb{T})} - \left\|\left\{\dots,0,\frac{\delta_1}{D},\frac{\delta_2}{D},\dots,\frac{\delta_n}{D},0,\dots\right\}\right\|_{M^p(\mathbb{T})}\right) \\
&\geq \frac{|D|}{2}\|\{\dots,0,\epsilon_1,\epsilon_2,\dots,\epsilon_n,0,\dots\}\|_{M^p(\mathbb{T})} \\
&\gtrsim n^{|\frac{1}{2}-\frac{1}{p}|},
\end{aligned}$$

where we used Lemma 4 in the last step.

Step 4. The uniform boundedness principle and the conclusion. We conclude the proof by an application of uniform boundedness principle. Indeed, We define the space,

$$\mathfrak{B} := \left\{\Omega : \mathbb{S}^1 \longrightarrow \mathbb{R} \text{ is even} : \int \Omega = 0 \text{ and } \|\Omega\|_{\mathfrak{B}'} = \|\|\Omega\|\|_{\Phi(L)(\mathbb{S}^1)} + \|m(\Omega)\|_{L^\infty(\mathbb{S}^1)} < \infty\right\}.$$

The space $\mathfrak{B}$ forms a Banach space.

Fix $p \neq 2$. For $\mathfrak{F} = \{f \in L^p(\mathbb{R}^2) : \|f\|_p = 1\}$, we define a collection of operators $\Theta_f : \mathfrak{B} \to L^p$ as $\Theta_f(\Omega) = T_\Omega(f)$. Suppose we have

$$\|T_\Omega\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)} = \sup_{f\in\mathfrak{F}'}\|T_\Omega f\|_{L^p(\mathbb{R}^2)} < \infty, \ \forall \Omega \in \mathfrak{B}.$$

Then by uniform boundedness principle, there exists $M > 0$ such that

$$\|T_\Omega\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)} = \sup_{f\in\mathfrak{F}}\|\Theta_f(\Omega)\|_{L^p(\mathbb{R}^2)} < M\|\Omega\|_{\mathfrak{B}},$$

which along with (7), (8) and (9) implies that

$$\begin{aligned}n^{\left|\frac{1}{2}-\frac{1}{p}\right|} &\lesssim \|T_{\Omega_n}\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)}\\ &\lesssim |||\Omega_n|||_{\Phi(L)(\mathbb{S}^1)} + \|m(\Omega_n)\|_{L^\infty(\mathbb{S}^1)}\\ &\lesssim \log n.\end{aligned}$$

This is a contradiction for large n and $p \neq 2$ and that concludes the proof of Theorem 2.

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