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On the series solutions of integral equations in scattering

Sur les solutions en série des équations intégrales en diffusion

Faouzi Triki [*, a] **and Mirza Karamehmedović** [b]

[a] Laboratoire Jean Kuntzmann, Université Grenoble-Alpes, Grenoble, France

[b] Department of Applied Mathematics and Computer Science, Technical University of Denmark, Kgs. Lyngby, Denmark

E-mails: Faouzi.Triki@univ-grenoble-alpes.fr, mika@dtu.dk

Abstract. We study the validity of the Neumann or Born series approach in solving the Helmholtz equation and coefficient identification in related inverse scattering problems. Precisely, we derive a sufficient and necessary condition under which the series is strongly convergent. We also investigate the rate of convergence of the series. The obtained condition is optimal and it can be much weaker than the traditional requirement for the convergence of the series. Our approach makes use of reduction space techniques proposed by Suzuki [21]. Furthermore we propose an interpolation method that allows the use of the Neumann series in all cases. Finally, we provide several numerical tests with different medium functions and frequency values to validate our theoretical results.

Résumé. Nous étudions la validité de l'approche série de Neumann ou de Born pour résoudre l'équation de Helmholtz, et pour l'identification de coefficients dans des problèmes inverses de diffusion. Plus précisément, nous obtenons une condition nécessaire et suffisante sous laquelle la série converge fortement. Cette condition est beaucoup plus faible que celle utilisée traditionnellement. Nous examinons également le taux de convergence de la série. Notre approche utilise des techniques d'espace de réduction proposées par Suzuki [21]. De plus, nous proposons une méthode d'interpolation qui permet l'utilisation de la série de Neumann dans tous les cas. Enfin, nous fournissons plusieurs tests numériques avec différentes fonctions de milieu et valeurs de fréquence pour valider nos résultats théoriques.

Keywords. Helmholtz equation, Born series, scattering.

Mots-clés. Équation de Helmholtz, série de Born, diffusion.

2020 Mathematics Subject Classification. 35R30, 34L25, 78A46.

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* Corresponding author

1. Introduction and main results

Let $d = 1,2,\dots$, fix a positive k_0 and $\widehat{k} \in S^{d-1}$, let $q \in L^\infty(\boldsymbol{R}^d)$ be compactly supported with $q(x) > -1$, and consider the following Helmholtz equation in $\boldsymbol{R}^d$ with the Sommerfeld radiation condition:

$$\begin{cases}(\Delta + k_0^2(1+q(x)))u = -k_0^2 q(x)\,\mathrm{e}^{\mathrm{i}k_0\hat{k}.x} & \text{in } \boldsymbol{R}^d,\\ \lim_{|x|\to\infty}|x|^{(d-1)/2}\left(\partial_{|x|} - \mathrm{i}k_0\right)u = 0 & \text{uniformly in } x/|x| \in S^{d-1}.\end{cases} \tag{1}$$

The system (1) is satisfied by the scattered field arising from the interaction of the incident uniform plane wave $\mathrm{e}^{\mathrm{i}k_0\hat{k}.x}$ with the medium $q(x)$. Convolving the PDE in (1) with the outgoing fundamental solution[1] of the Helmholtz operator $\Delta + k_0^2$ in $\boldsymbol{R}^d$,

$$\Phi_d(x) = \begin{cases}(-2\pi|x|)^{(1-d)/2}(\mathrm{i}/2k_0)\partial_{|x|}^{(d-1)/2}\mathrm{e}^{\mathrm{i}k_0|x|}, & x \in \boldsymbol{R}^d\setminus\{0\},\ d \text{ odd},\\ (-2\pi|x|)^{(2-d)/2}(\mathrm{i}/4)\partial_{|x|}^{(d-2)/2}H_0^{(1)}(k_0|x|), & x \in \boldsymbol{R}^d\setminus\{0\},\ d \text{ even},\end{cases}$$

and integrating by parts, we get the Lippmann–Schwinger equation

$$(I - V_q(k_0))u = V_q(k_0)\,\mathrm{e}^{\mathrm{i}k_0\hat{k}(\cdot)} \quad \text{in } \boldsymbol{R}^d, \tag{2}$$

where

$$V_q(k_0)u(x) = k_0^2\int_{y\in\operatorname{supp}q}\Phi_d(x-y)q(y)u(y)\,\mathrm{d}y$$

exists as an improper integral for each $x \in \boldsymbol{R}^d$. It is well-known that the integral equation (2) is equivalent with (1), and that it suffices to solve (2) in, say, a bounded open ball $B \subset \boldsymbol{R}^d$ that includes $\operatorname{supp}q$, followed by the continuous extension $u(x) = V_q(k_0)[u(\cdot) + \exp(\mathrm{i}k_0\widehat{k}(\cdot))](x)$ for $x \in \boldsymbol{R}^d \setminus B$. The mapping $V_q(k_0) : L^2(B) \to L^2(B)$ is compact, and we shall in the following consider only the restriction of the Lippmann–Schwinger equation in (2) to B. The objective of the paper is to study the successive approximations for solving the integral equation (2):

$$u_0 = V_q(k_0)\,\mathrm{e}^{\mathrm{i}k_0\hat{k}(\cdot)};\ u_{n+1} = u_0 + V_q(k_0)u_n,\ \ n \in \boldsymbol{N}. \tag{3}$$

The computational advantage of this iterative method is that it does not need to solve the partial differential equation (1) in the whole space and deal with the radiation conditions. Instead, one can obtain a good approximation u_n of the solution u by applying successively the integral operator $V_q(k_0)$ if the sequence converges.

On the other hand the strong convergence of the sequence $(u_n)_{n\in\boldsymbol{N}}$ to the solution u of the integral equation (2) is equivalent to the convergence of the Neumann series:

$$\lim_{n\to\infty}u_n = \sum_{j=0}^{\infty}V_q(k_0)^{j+1}\,\mathrm{e}^{(\cdot)k_0\hat{k}(\cdot)} = (I - V_q(k_0))^{-1}V_q(k_0)\,\mathrm{e}^{(\cdot)k_0\hat{k}(\cdot)}. \tag{4}$$

In inverse scattering problems the Neumann series approach known more under the name of *Born approximation* was initially employed to study the quantum mechanical inverse backscattering problem in one dimension (see for instance [18] and references therein). The principal advantage of using this technique in inverse medium problem is that it requires solving a linear equation instead of an oscillatory nonlinear one [4, 5]. It has also been applied to various other inverse problems, including optical and electrical impedance tomographies, acoustic and electromagnetic parameters identification [1–3, 6, 10, 11, 13, 15–17, 20]. However, it is important to note, that the strategies considered in these works are based on purely formal analysis or assume strong conditions on the targeted physical parameters.

It is well known that a sufficient condition for the convergence of the Neumann series (4) is that the spectral radius $\operatorname{Spr}(V_q(k_0))$ of the compact operator $V_q(k_0)$ is strictly less than one, that is $\operatorname{Spr}(V_q(k_0)) < 1$. But this latter condition while it is optimal for the expansion of the operator $(I - V_q(k_0))^{-1}$ in $L^2(B)$, it is obviously too restrictive for the convergence of (4). Then is it possible

[1] Here $H_0^{(1)}$ the Hankel function of the first kind and order zero.

to derive a necessary and sufficient condition for the convergence of only (4)? On the other hand the strong convergence

$$V_q(k_0)^j\, \mathrm{e}^{\mathrm{i}k_0\hat{k}(\cdot)} \longrightarrow 0,\;\; j \longrightarrow \infty, \tag{5}$$

in $L^2(B)$, is evidently a necessary condition for the convergence of the series (4). Suzuki in his seminal work [21] wondered if this condition is also sufficient. Surprisingly, it turns out that this condition also guarantees the convergence of the series. The main idea of the proof is to derive a minimal invariant space where the expansion of the restriction of $(I - V_q(k_0))^{-1}$ to that space is equivalent to the convergence of the series (4).

Let

$$L^2_{k_0,\hat{k}}(B) = \operatorname{Span}\left(V_q(k_0)\,\mathrm{e}^{\mathrm{i}k_0\hat{k}(\cdot)}, V_q(k_0)^2\,\mathrm{e}^{\mathrm{i}k_0\hat{k}(\cdot)},\dots, V_q(k_0)^{j+1}\,\mathrm{e}^{\mathrm{i}k_0\hat{k}(\cdot)},\dots\right). \tag{6}$$

By construction $L^2_{k_0,\hat{k}}(B)$ is invariant by $V_q(k_0)$. Denote $\widetilde{V}_q(k_0)$ the restriction of $V_q(k_0)$ to $L^2_{k_0,\hat{k}}(B)$. Suzuki showed that condition (5) indeed implies $\operatorname{Spr}(\widetilde{V}_q(k_0)) < 1$, and hence ensures the convergence of the Neumann series to the unique solution.

Proposition 1. *The convergence of the Neumann series* (4) *is equivalent to the condition* (5).

Remark 2. Since $V_q(k_0)$ is a compact operator the strong convergence (5) can be replaced by a weak convergence of a subsequence. Notice that $L^2_{k_0,\hat{k}}(B)$ can also be generated by finite sums of the sequence

$$L^2_{k_0,\hat{k}}(B) = \operatorname{Span}\left(\sum_{j=0}^{J} V_q(k_0)^{j+1}\,\mathrm{e}^{\mathrm{i}k_0\hat{k}(\cdot)};\;\; J \in \boldsymbol{N}\right).$$

Recall that the traditional condition to ensure the convergence of the Neumann series is [4]

$$\|V_q(k_0)\| \le C_{k_0,q} = \left(\int_B\int_B |k_0^2\Phi_d(x-y)q(x)|^2\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2} < 1. \tag{7}$$

This condition occurs in the situation for weak scattering, and is not valid for high wave number k_0, or large magnitude of the medium function q. But since $\mathrm{e}^{\mathrm{i}k_0\hat{k}(\cdot)}$ is sparse we expect that $L^2_{k_0,\hat{k}}(B)$ has a lower dimensionality than the whole space $L^2(B)$, and consequently the convergence of the Neumann series (4) may happen beyond the conventional limitation (7). In other words $\operatorname{Spr}(\widetilde{V}_q(k_0)) < 1$ can be satisfied by a larger class of wave numbers and medium functions not necessary within the weak scattering regime. This was also observed in many numerical experiments in the past, has fueled many discussions and was the origin of several investigations [2, 10, 11, 15–17, 20]. This pattern is clearly confirmed by many numerical examples in Section 4.

Theorem 3. *Assume that the condition* (5) *is satisfied, that is*

$$\lim_{n\to\infty}\left\| V_q(k_0)^n\,\mathrm{e}^{\mathrm{i}k_0\hat{k}(\cdot)}\right\| = 0.$$

Then there exists a constant $C > 0$ independent of n such that the following error estimate

$$\|u - u_n\| \le C\left\| V_q(k_0)^n\,\mathrm{e}^{\mathrm{i}k_0\hat{k}(\cdot)}\right\|, \tag{8}$$

holds for all $n \in \boldsymbol{N}$.

The rest of the paper is organized as follows. In Section 2, we provide the proofs for Proposition 1, and Theorem 3. Section 3 is devoted to the construction of a preconditioner for the integral equation (2). Precisely, we propose an interpolation method that allows the use of the Neumann series independently of the fact that the condition (5) is fulfilled or not. We present then several numerical experiments to show the effectiveness of the derived theoretical results in Section 4.

2. Proof of the main results

In this section we shall prove the main results of the paper. To ease the notation we set

$$A = V_q(k_0);\;\; \psi = V_q(k_0)\,\mathrm{e}^{\mathrm{i}k_0\hat{k}(\cdot)};\;\; \mathfrak{H} = L^2(B);\;\; \mathfrak{H}_0 = L^2_{k_0,\hat{k}}(B).$$

Proof of Proposition 1. If the series

$$\sum_{j=0}^{\infty} A^j\Psi = (I-A)^{-1}\Psi, \tag{9}$$

is convergent then obviously we will have $A^j\Psi \to 0$ strongly in $\mathfrak{H}$. Now assume that $A^j\Psi$ converges strongly to zero in $\mathfrak{H}$, and focus on the nontrivial opposite direction.

The main observation of Suzuki is that the convergence of the series (9) depends more on the specific local behavior of the operator A relative to the given vector ψ rather than its global properties on the whole space $\mathfrak{H}$ which requires that $\mathrm{Spr}(A) < 1$. Indeed consider the Hilbert subspace $\mathfrak{H}_0 \subset \mathfrak{H}$ space generated by the vectors $A^j\Psi$, $j \in \boldsymbol{N}$, that is

$$\mathfrak{H}_0 = \mathrm{Span}\Big(\Psi, A\Psi, \ldots, A^j\Psi, \ldots\Big). \tag{10}$$

Clearly $\mathfrak{H}_0$ is invariant by A, and since Ψ lies in $\mathfrak{H}_0$ to prove that the series (9) strongly in $\mathfrak{H}_0$ it is sufficient to show that A_0 the restriction of A to $\mathfrak{H}_0$ verifies $\mathrm{Spr}(A_0) < 1$. Remark that since $\mathfrak{H}_0 \subset \mathfrak{H}$ we have $\mathrm{Spr}(A_0) \le \mathrm{Spr}(A)$.

Let $\mathfrak{M}$ be the linear manifold formed by the vectors $v \in \mathfrak{H}_0$ satisfying $A^j v$ tends strongly to zero as $j \to \infty$. We first remark that the fact $j \to A^j\Psi$ tends strongly to zero, $\mathfrak{M}$ contains all the vectors $A^j\Psi$, $j \in \boldsymbol{N}$, and consequently is dense in $\mathfrak{H}_0$.

Let $\sigma(A_0)$ denotes the spectrum of A_0, and set $\sigma_-(A_0) = \{\lambda \in \Sigma(A_0);\ |\lambda| < 1\}$ and $\sigma_+(A_0) = \{\lambda \in \Sigma(A_0);\ |\lambda| \ge 1\}$. Since A_0 is compact $\sigma_+(A_0)$ is finite, in addition there exists a rectifiable Jordan curve $\mathscr{C}_+$ in the resolvent set surrounding $\sigma_+(A_0)$ and does not contain other eigenvalues of $\sigma(A_0)$. Similarly there exists a rectifiable Jordan curve $\mathscr{C}_-$ in the resolvent set surrounding only $\sigma_-(A_0)$. Then following [9], the spectral projections

$$P_\pm = \frac{1}{2\mathrm{i}\pi}\int_{\mathscr{C}_\pm} (\lambda I - A_0)^{-1}\mathrm{d}\lambda, \tag{11}$$

verify the following identities

$$P_- + P_+ = I;\ P_-P_+ = P_+P_- = 0;\ P_\pm A = AP_\pm. \tag{12}$$

Recalling that $\mathrm{Spr}(A_0) = \sup_{\lambda\in\sigma(A_0)}|\lambda|$. Since $\sigma(A_0)$ is a sequence of complex values that may converge to zero, proving that $\mathrm{Spr}(A_0) < 1$ is equivalent to show that $\sigma_+(A_0)$ is an empty set. Let now $v \in P_+\mathfrak{H}_0$. By the density of the set $\mathfrak{M}$ in $\mathfrak{H}_0$, there exists a sequence $(v_n)_{n\in\boldsymbol{N}_0} \in \mathfrak{M}$ that converges strongly to v. $\boldsymbol{N}_0$ here is the set $\boldsymbol{N}\setminus\{0\}$. Denote $v_{n,\pm} = P_\pm v_n$. Therefore $v_n = v_{n,+} + v_{n,-}$. Remarking that $Av_{n,-} = AP_-v_n = P_-Av_n$ converges strongly to $P_-v = 0$ leads to $v_{n,-} \in \mathfrak{M}$. Hence $v_{n,+} = v_n - v_{n,-}$ lies in fact in $\mathfrak{M}\cap P_+\mathfrak{H}_0$. This shows that $\mathfrak{M}\cap P_+\mathfrak{H}_0$ is indeed dense in $P_+\mathfrak{H}_0$. But since $\sigma_+(A_0)$ is finite $P_+\mathfrak{H}_0$ is finite dimensional space and consequently $\mathfrak{M}\cap P_+\mathfrak{H}_0 = P_+\mathfrak{H}_0$. This is clear not correct if $P_+\mathfrak{H}_0$ is not trivial (take any eigenvector of A_0 associated to $\lambda \in \sigma_+(A_0)$, it obviously does not belong to $\mathfrak{M}$). Then $\sigma_+(A_0)$ is an empty set, and finally $\mathrm{Spr}(A_0) < 1$, which achieves the proof.

□

Proof of Theorem 3. Since $\psi \in \mathfrak{M}$ we deduce from Proposition 1 that the Neumann series (9) is convergent. On the other hand we deduce from (12) $A^j\psi = A^jP_-\psi = A_0^j\psi$. Therefore

$$\left\|\sum_{j=0}^{\infty} A^j\Psi\right\| = \left\|\sum_{j=0}^{\infty} A_0^j\Psi\right\| = \left\|(I-A_0)^{-1}\Psi\right\| \le \left\|(I-A_0)^{-1}\right\|\,\|\Psi\|. \tag{13}$$

Let $n \in \boldsymbol{N}$ be fixed. The fact that $\psi \in \mathfrak{M}$ implies that $A^{n+1}\psi \in \mathfrak{M}$. Applying inequality (13) to the vector $A^{n+1}\psi$ leads to

$$\left\| \sum_{j=n+1}^{\infty} A^j \Psi \right\| = \left\| \sum_{j=0}^{\infty} A^j A^{n+1}\Psi \right\| \leq \left\| (I-A_0)^{-1} \right\| \left\| A^{n+1}\Psi \right\| \leq C \left\| A^n \Psi \right\|, \tag{14}$$

with $C = \|(I-A_0)^{-1}\| \|A_0\|$, which finishes the proof of the Theorem. □

Remark 4. The proofs stay valid for any general compact operator A and even if $\mathfrak{H}$ is a Banach space. In the particular case where A is in addition normal, that is $AA^* = A^*A$, the obtained results are straightforward. Indeed if $\sigma(A) = \{\lambda_k; k \in \boldsymbol{N}_0\}$ the eigenvalues of A, and P_k is the orthogonal spectral projection associated to λ_k, we have

$$A = \sum_{k=1}^{\infty} \lambda_k P_k,$$

and it is clear that the condition $\psi \in \mathfrak{M}$ is equivalent to

$$\psi = \sum_{|\lambda_k|<1} P_k \psi.$$

Therefore

$$\left\| \sum_{j=n+1}^{\infty} A^j \Psi \right\|^2 = \left\| \sum_{|\lambda_k|<1} \frac{\lambda_k^{n+1}}{1-\lambda_k} P_k \Psi \right\|^2 = \sum_{|\lambda_k|<1} \frac{\lambda_k^{2(n+1)}}{(1-\lambda_k)^2} \|P_k\Psi\|^2 \leq \frac{r_0^2}{(1-r_0)^2} \|A^n \Psi\|^2,$$

where $r_0 = \max_{|\lambda_k|<1} |\lambda_k| = \|A_0\| = \mathrm{Spr}(A_0)$. One can verify that $C = \|(I-A_0)^{-1}\| \|A_0\| = \frac{r_0}{1-r_0}$. Finally it is easy to find examples of A such that the inequalities

$$\mathrm{Spr}(A_0) \ll 1 \ll \mathrm{Spr}(A) = \|A\|,$$

hold, and where the benefit of considering the reduced space $\mathfrak{H}_0$ is indeed remarkable.

3. Preconditioning

By "preconditioning" we here mean the transformation of the original Lippmann–Schwinger equation $(I - V_q(k_0))u = \psi$ to an integral equation solvable by a convergent Neumann series regardless of the value of $\|V_q(k_0)\|_{L^2(B)\to L^2(B)}$ and of whether or not the sequence $(\|V_q(k_0)^j\psi\|_{L^2})_{j\in\boldsymbol{N}}$ converges to zero. See [8, 12–14, 19] for related approaches. Throughout this section we assume the problem dimension $d \in \{1,2,3\}$.

Lemma 5. *If $q(x) \geq 0$, $q \not\equiv 0$, then there is a complex constant γ such that the solution of the equation $(I - V_q(k_0))u = \psi$ in $L^2(B)$ is expressible in terms of the convergent Neumann series*

$$u = \sum_{j=0}^{\infty} M^j \gamma \psi,$$

where $M = (1-\gamma)I + \gamma V_q(k_0)$.

Proof. Let $V_q(k_0)\varphi = \lambda\varphi$ in B with nonzero λ. Then

$$\begin{cases} \varphi'' + k_0^2(1+q(x)/\lambda)\varphi = 0, & x \in]-R,R[, \\ -\varphi'(-R) = \mathrm{i}k_0\varphi(-R), \\ \varphi'(R) = \mathrm{i}k_0\varphi(R) \end{cases} \tag{15}$$

for $d = 1$, and

$$\begin{cases} (\Delta + k_0^2(1+q(x)/\lambda))\varphi = 0 & \text{in } \boldsymbol{R}^d, \\ \lim_{|x|\to\infty} |x|^{(d-1)/2}\left(\partial_{|x|} - \mathrm{i}k_0\right)\varphi = 0 & \text{uniformly in } x/|x| \in S^{d-1}, \end{cases}$$

for $d \in \{2,3\}$. Thus, for sufficiently large $R > 0$ we have

$$0 = \int_{|x|<R} (\overline{\varphi}\Delta\varphi + k_0^2|\varphi|^2 + k_0^2\lambda^{-1}q(x)|\varphi|^2)$$
$$= \int_{|x|<R} (k_0^2|\varphi|^2 - |\nabla\varphi|^2) + \int_{|x|=R} \overline{\varphi}\partial_r\varphi + k_0^2\lambda^{-1}\int_{x\in\operatorname{supp} q} q(x)|\varphi|^2, \tag{16}$$

as well as $(\Delta + k_0^2)\varphi = 0$ in $\{|x| > R\}$. In the case $d = 1$ we readily find that

$$\Im \int_{|x|=R} \overline{\varphi}\partial_r\varphi = k_0(|\varphi(-R)|^2 + |\varphi(R)|^2) > 0,$$

while in the case $d \in \{2,3\}$ we can follow the argument in, e.g., [7, Theorem 2.13, p. 38] to find

$$\Im \int_{|x|=R} \overline{\varphi}\partial_r\varphi > 0.$$

Hence

$$\int_{x\in\operatorname{supp} q} q(x)|\varphi|^2\,\mathrm{d}x > 0,$$

and this in conjunction with (16) gives

$$\Im(\lambda^{-1}) = -\frac{\Im\int_{|x|=R}\overline{\varphi}\partial_r\varphi}{k_0^2\int_{x\in\operatorname{supp} q} q(x)|\varphi|^2} < 0,$$

so $\Im\lambda > 0$ and finally $\Re(\mathrm{e}^{\mathrm{i}\alpha}(1-\lambda)) > 0$ if $\alpha \in \,]-\pi/2, \pi/2[$ satisfies

$$\tan\alpha > \max_{\lambda'\in\sigma(V_q(k_0))} \frac{\Re\lambda' - 1}{\Im\lambda'}, \tag{17}$$

where $\sigma(V_q(k_0))$ is the spectrum of $V_q(k_0)$. The existence of the maximum in (17) follows from the fact that the eigenvalues of the compact operator $V_q(k_0) : L^2(B) \to L^2(B)$ reside in the closed ball $\{\lambda' \in \boldsymbol{C},\ |\lambda'| \le \|V_q(k_0)\|_{L^2(B)\to L^2(B)}\}$ and can accumulate only at zero. These facts also imply that there exists $\varepsilon > 0$ such that $|\gamma(1-\lambda') - 1| < 1$ for all $\lambda' \in \sigma(V_q(k_0))$, where $\gamma = \varepsilon\mathrm{e}^{\mathrm{i}\alpha}$. It remains to notice that each eigenvalue μ of M is of the form $\mu = 1 - \gamma + \gamma\lambda'$, where λ' is some eigenvalue of $V_q(k_0)$, and finally that the equation $(I - M)u = \gamma\psi$ is equivalent with the equation $(I - V_q(k_0))u = \psi$. □

We can in fact be more specific in a special case in dimension one. Let L be a positive constant, set $B = \,]0, L[$, and let $q(x) \equiv q_0 = \text{const.} > 0$ for $x \in \overline{B}$.

Lemma 6. *If* $\varepsilon' > k_0Lq_0/2$,

$$\alpha = \arctan\frac{1 + \frac{k_0Lq_0}{2}\varepsilon'}{\varepsilon' - \frac{k_0Lq_0}{2}}, \qquad 0 < \varepsilon < \frac{1}{1 + k_0Lq_0/2}\,\frac{|\tan(2\max\{\alpha, \arctan\varepsilon'\})|}{\tan(\max\{\alpha, \arctan\varepsilon'\})},$$

and $\gamma = \varepsilon\,\mathrm{e}^{\mathrm{i}\alpha}$, *then each eigenvalue* μ *of* $M = (1-\gamma)I + \gamma V_q(k_0)$ *satisfies* $|\mu| < 1$.

Proof. Assume $\lambda \in \boldsymbol{C}\setminus\{0\}$ and $\varphi \in L^2(]0, L[)$, $\varphi \not\equiv 0$, satisfy

$$V_q(k_0)\varphi(x) = \lambda\varphi(x), \quad x \in (0, L). \tag{18}$$

Integration by parts readily shows the equivalence of the Lippmann–Schwinger equation (18) with the Helmholtz system

$$\begin{cases} \varphi''(x) + k_0^2 s^2\varphi(x) = 0, & x \in \,]0, L[, \\ -\varphi'(0) = \mathrm{i}k_0\varphi(0), \\ \varphi'(L) = \mathrm{i}k_0\varphi(L), \end{cases} \tag{19}$$

where $s = \left(1 + q_0/\lambda\right)^{1/2}$. The eigenvectors of the Laplacian on $]0, L[$ are generally of the form

$$\varphi(x) = A\exp(\mathrm{i}k_0 sx) + B\exp(-\mathrm{i}k_0 sx), \tag{20}$$

and we then readily find that (19) is equivalent with (20) together with

$$s \neq 1, \quad B = A(s+1)/(s-1), \quad e^{2ik_0Ls} = (s+1)^2/(s-1)^2. \tag{21}$$

Next define the constant $T(k_0L) > 0$ by $T(k_0L)\sinh(k_0LT(k_0L)) = 1$. Using the last condition in (21), we find that, necessarily,

$$s \in S_\pm(k_0L) = \left\{ \frac{-\cosh k_0Lt \pm \sqrt{1 - t^2\sinh^2 k_0Lt}}{\sinh k_0Lt} + it,\ t \in (0, T(k_0L)] \right\},$$

which in turn implies

$$\lambda \in \Lambda(q_0, k_0L) = \left\{ \frac{q_0}{s^2-1},\ s \in S_-(k_0L) \cup S_+(k_0L) \right\}.$$

As an example, Figure 1 shows the set $\Lambda(q_0, k_0L)$, as well as the numerically computed spectrum, for two sets of parameter values for k_0, L, and q_0.

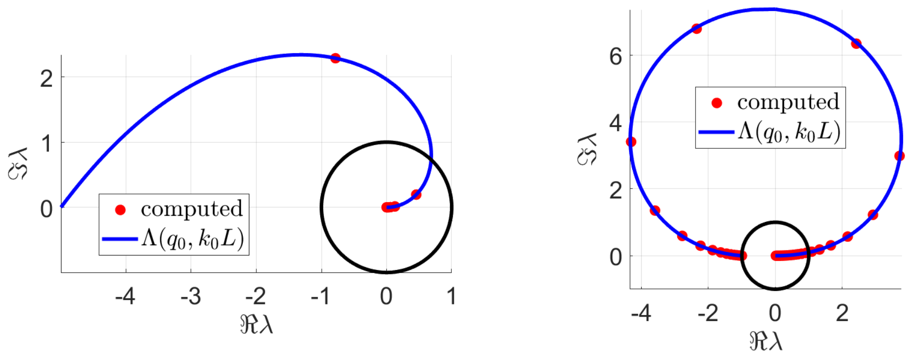


Figure 1. The theoretically predicted set $\Lambda(q_0, k_0L)$ that includes the eigenvalues of $V_q(k_0)$, plotted against the numerically computed eigenvalues. The parameter values are: top, $k_0 = 1$, $L = 1$, and $q_0 = 5$; bottom, $k_0 = 50$, $L = 1$, and $q_0 = 1$. The reference circle is centered at the origin and has radius one.

Since $\Im\lambda > 0$ and since the eigenvalues of $V_q(k_0)$ accumulate precisely at zero, we have

$$\lim_{t\to 0} \frac{\Re\omega(t) - 1}{\Im\omega(t)} = -\infty$$

for

$$\omega(t) = \frac{q_0}{s(t)^2 - 1}, \qquad t \in (0, T(k_0L)];$$

here $s(t)$ is given by the above definition of $S_\pm(k_0L)$. Furthermore, if $\Re\omega(t) < 1$ then $(\Re\omega(t) - 1)/\Im\omega(t) < 0$, while $\Re\omega(t) \geq 1$ implies

$$\frac{1 - t^2\sinh^2 k_0Lt + \cosh(k_0Lt)\sqrt{1 - t^2\sinh^2 k_0Lt}}{\sinh^2 k_0Lt} \leq \frac{q_0}{2};$$

the latter can be seen by rewriting $\Re\omega - 1 \geq 0$ as

$$((\Re s)^2 - (\Im s)^2 - 1)((\Re s)^2 - (\Im s)^2 - 1 - q_0) + 4(\Re s)^2(\Im s)^2 \leq 0,$$

and noting that $(\Re s)^2-(\Im s)^2-1>0$ and $(\Re s)^2(\Im s)^2>0$. Now since also

$$\frac{1-t^2\sinh(k_0Lt)^2-\cosh(k_0Lt)\sqrt{1-t^2\sinh(k_0Lt)^2}}{t\sinh(k_0Lt)(\cosh(k_0Lt)-\sqrt{1-t^2\sinh(k_0Lt)^2})} \le \frac{1-t^2\sinh(k_0Lt)^2+\cosh(k_0Lt)\sqrt{1-t^2\sinh(k_0Lt)^2}}{t\sinh(k_0Lt)(\cosh(k_0Lt)+\sqrt{1-t^2\sinh(k_0Lt)^2})},$$

we have

$$\frac{\Re\omega(t)-1}{\Im\omega(t)}<\frac{\Re\omega(t)}{\Im\omega(t)}<\frac{1-t^2\sinh^2 k_0Lt+\cosh(k_0Lt)\sqrt{1-t^2\sinh^2 k_0Lt}}{t\sinh(k_0Lt)(\cosh k_0Lt+\sqrt{1-t^2\sinh^2 k_0Lt})},$$

so $\Re\omega(t)\ge 1$ implies

$$\begin{aligned}\frac{\Re\omega(t)-1}{\Im\omega(t)}&<\frac{q_0}{2}\frac{\sinh k_0Lt}{t(\cosh k_0Lt+\sqrt{1-t^2\sinh^2 k_0Lt})}\le\frac{q_0}{2}\frac{\tanh k_0Lt}{t}\\&\le\frac{q_0}{2}\lim_{\tau\searrow 0}\frac{\tanh k_0L\tau}{\tau}=\frac{q_0k_0L}{2},\end{aligned}$$

that is, we get a similar estimate on $(\Re\omega(t)-1)/\Im\omega(t)$ as we do on $\|V_q(k_0)\|$. Next, define

$$\xi_+=\arctan\sup_{\omega\in\Lambda(q_0,k_0L)}\frac{\Im\left(\mathrm{e}^{\mathrm{i}\alpha}(1-\omega)\right)}{\Re\left(\mathrm{e}^{\mathrm{i}\alpha}(1-\omega)\right)} \tag{22}$$

and

$$\xi_-=\arctan\inf_{\omega\in\Lambda(q_0,k_0L)}\frac{\Im\left(\mathrm{e}^{\mathrm{i}\alpha}(1-\omega)\right)}{\Re\left(\mathrm{e}^{\mathrm{i}\alpha}(1-\omega)\right)}. \tag{23}$$

With $\alpha\in(0,\pi/2)$, $\varepsilon>0$, and $\gamma=\varepsilon\,\mathrm{e}^{\mathrm{i}\alpha}$, we have $\arg(\gamma(1-\omega))\in[\xi_-,\xi_+]$ for all $\omega\in\Lambda(q_0,k_0L)$, and if

$$\varepsilon<\frac{1}{1+k_0Lq_0/2}\frac{|\tan(2\max\{|\xi_+|,|\xi_-|\})|}{\tan(\max\{|\xi_+|,|\xi_-|\})} \tag{24}$$

then $|\gamma(1-\omega)-1|<1$ for all $\omega\in\Lambda(q_0,k_0L)$, and specifically $|\gamma(1-\lambda')-1|<1$ for all eigenvalues λ' of $V_q(k_0)$; the condition (24) can be deduced by requiring $\Im(\gamma(1-\omega))<|\tan(\pi-2\arg(\gamma(1-\omega)))|$ and using $|1-\omega|\le 1+\|V_q(k_0)\|$. Note that we must choose $\alpha<\pi/2$ rather than $\alpha=\pi/2$ since

$$\frac{\Im(\mathrm{e}^{\mathrm{i}\alpha}(1-\omega(t)))}{\Re(\mathrm{e}^{\mathrm{i}\alpha}(1-\omega(t)))}\xrightarrow[t\to 0]{}\tan\alpha,$$

which for $\alpha=\pi/2$ forces $\xi_+=\pi/2$ and thus $\varepsilon<0$. Note furthermore that we can bound (22)–(23) analytically for

$$\alpha=\arctan\frac{1+\frac{k_0Lq_0}{2}\varepsilon'}{\varepsilon'-\frac{k_0Lq_0}{2}},$$

with $\varepsilon'>k_0Lq_0/2$, since

$$-\cot\alpha\le\frac{\Im\left(\mathrm{e}^{\mathrm{i}\alpha}(1-\omega)\right)}{\Re\left(\mathrm{e}^{\mathrm{i}\alpha}(1-\omega)\right)}\le\tan\alpha,\quad \omega\in\Lambda(q_0,k_0L),\quad \Re\omega\le 1,$$

since furthermore (recall that $(\Re\omega-1)/\Im\omega<k_0Lq_0/2<\tan\alpha$)

$$-\frac{\frac{k_0Lq_0}{2}\tan\alpha+1}{\tan\alpha-\frac{k_0Lq_0}{2}}\le\frac{\Im\left(\mathrm{e}^{\mathrm{i}\alpha}(1-\omega)\right)}{\Re\left(\mathrm{e}^{\mathrm{i}\alpha}(1-\omega)\right)}<0,\quad \omega\in\Lambda(q_0,k_0L),\quad \Re\omega>1,$$

and finally since

$$-\cot\alpha>-\frac{\frac{k_0Lq_0}{2}\tan\alpha+1}{\tan\alpha-\frac{k_0Lq_0}{2}}=-\varepsilon'. \qquad \square$$

Remark 7. A drawback of preconditioning is that, with increasing $k_0 L q_0$, the acceptable values of α and of $\arctan\varepsilon'$ tend to $\pi/2$, and ε therefore tends to zero. Thus, while the Neumann series remains convergent, the equation

$$(I-M)u=\gamma\psi$$

may be said, especially in a numerical context, to lose information about the operator $V_q(k_0)$ and about the original inhomogeneity ψ, as both are multiplied with $\gamma=\varepsilon\,\mathrm{e}^{\mathrm{i}\alpha}$ there.

Remark 8. Instead of using the bound on ε stated in Lemma 6, we can estimate ξ_+ and ξ_- from (22)–(23) numerically and arrive at a larger sufficiently small ε using (24).

We show numerical examples of the use of preconditioning in Section 4.

4. Numerical examples

We here present several numerical examples in dimension one. Fix a positive wavenumber k_0, obstacle size $L>0$, medium function $q\in L^2(]0,L[)$, $q(x)>-1$, and consider the following system for the scattered wave $u(x)$ corresponding to the left excitation $\exp(\mathrm{i}k_0x)$ in dimension one:

$$\begin{cases}\psi''(x)+k_0^2(1+q(x))\psi(x)=-k_0^2q(x)\exp(\mathrm{i}k_0x), & x\in\,]0,L[,\\ -\psi'(0)=\mathrm{i}k_0\psi(0),\\ \psi'(L)=\mathrm{i}k_0\psi(L).\end{cases}\tag{25}$$

The function $G(x,y)=(i/2k_0)\exp(\mathrm{i}k_0|x-y|)$, $x,y\in[0,L]$, is the free-space Green's function associated with the boundary problem (25), since

$$\begin{cases}(\partial_y^2+k_0^2)G(x,y)=-\delta(x-y), & x,y\in\,]0,L[,\\ -\partial_yG(x,0)=\mathrm{i}k_0G(x,0), & x\in\,]0,L[,\\ \partial_yG(x,L)=\mathrm{i}k_0G(x,L), & x\in[0,L].\end{cases}$$

Multiplying the differential equation in (25) with $G(x,y)$ and integrating by parts, we get the Lippmann–Schwinger equation

$$(I-V_q(k_0))\psi(x)=V_q(k_0)\exp(\mathrm{i}k_0(\cdot))(x),\quad x\in\,]0,L[,\tag{26}$$

where

$$V_q(k_0)u(x)=\frac{\mathrm{i}k_0}{2}\int_{y=0}^{L}\mathrm{e}^{\mathrm{i}k_0|x-y|}\,q(y)u(y)\,\mathrm{d}y,\quad x\in\,]0,L[.$$

The operator $V_q(k_0):L^2(]0,L[)\to L^2(]0,L[)$ is compact, with norm satisfying

$$\|V_q(k_0)\|^2\le\int_{x=0}^{L}\int_{y=0}^{L}\left|\frac{\mathrm{i}k_0}{2}\,\mathrm{e}^{\mathrm{i}k_0|x-y|}\,q(y)\right|^2\mathrm{d}y\,\mathrm{d}x=\frac{k_0^2L\|q\|_2^2}{4}.$$

Now let $q_0\in\,]-1,+\infty[$, and assume that $q|_{]0,L[}\equiv q_0$. One can easily prove

$$\mathfrak{H}_0=\operatorname{Span}\left(V_q(k_0)^j\,\mathrm{e}^{\mathrm{i}k_0(\cdot)};\ j\in\mathbb{N}\right)=\operatorname{Span}\left(V_1(k_0)^j\,\mathrm{e}^{\mathrm{i}k_0(\cdot)};\ j\in\mathbb{N}\right).$$

On the other hand, since $V_q(k_0)=q_0V_1(k_0)$, we have

$$\operatorname{Spr}\left((V_q)_0(k_0)\right)=|q_0|\operatorname{Spr}\left((V_1)_0(k_0)\right),$$

where $(V_q)_0(k_0)$ and $(V_1)_0(k_0)$ are the restrictions of $V_q(k_0)$ and $V_1(k_0)$, respectively, to the space $\mathfrak{H}_0$. Following the proof of Proposition 1, the Born series

$$\sum_{j=1}^{\infty}\left(V_q(k_0)\right)^j\mathrm{e}^{\mathrm{i}k_0(\cdot)}$$

is convergent if and only if $\operatorname{Spr}\big((V_q)_0(k_0)\big) < 1$, that is, if and only if

$$|q_0| < \frac{1}{\operatorname{Spr}((V_1)_0(k_0))}.$$

To illustrate this, we show in Figure 2 two cases of repeated application of $V_q(k_0)$ on the original right-hand side $V_q(k_0)\,\mathrm{e}^{\mathrm{i}k_0(\cdot)}$.

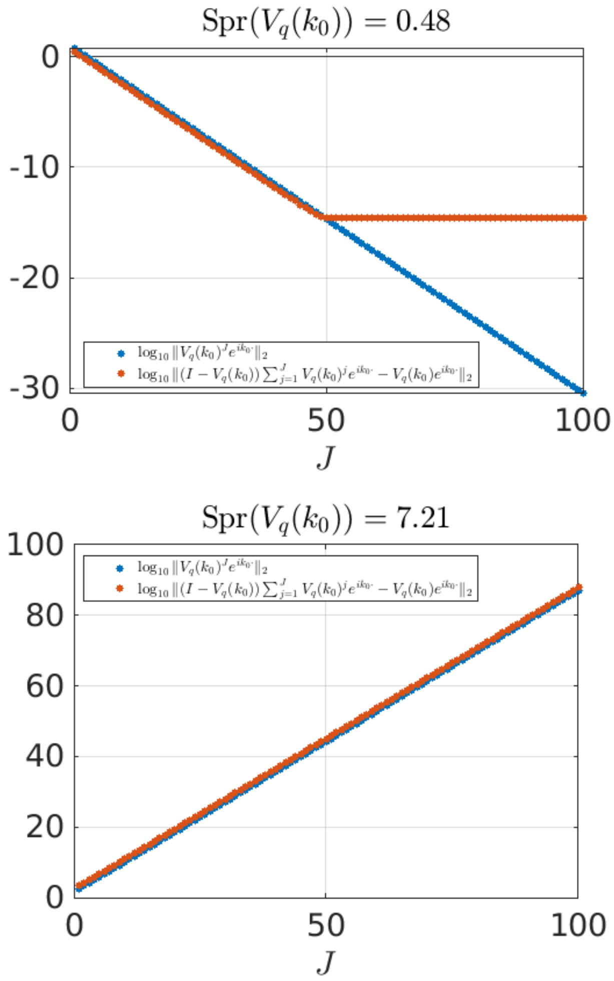
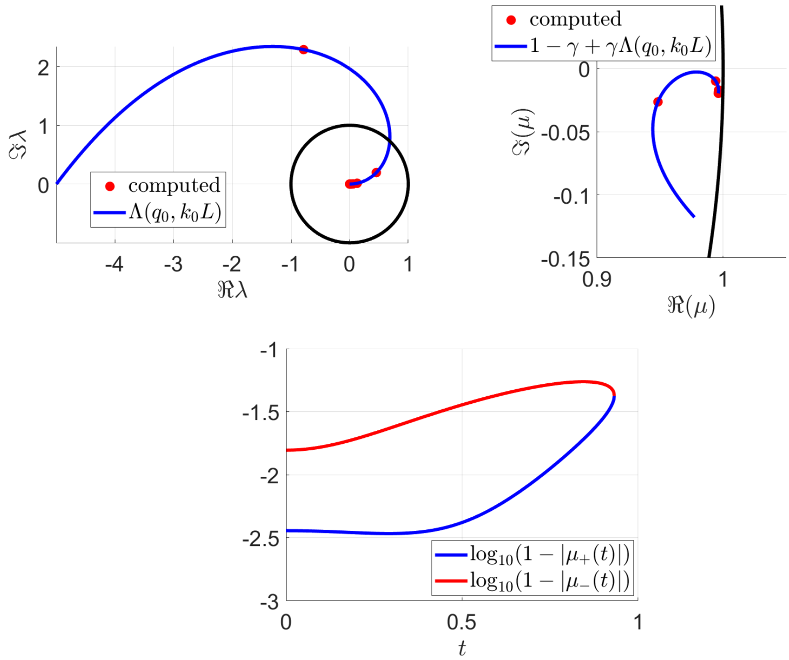


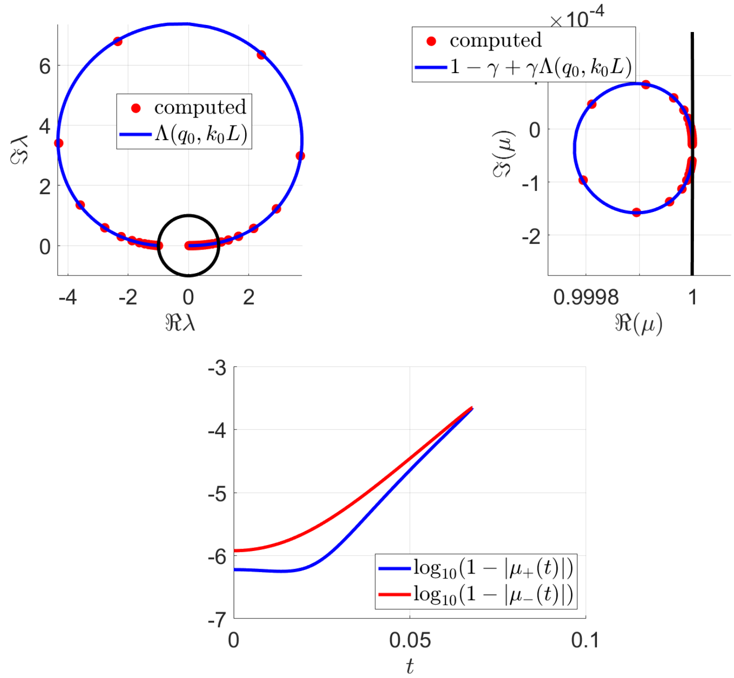
Figure 2. Top: $k_0 = 1$, $L = 1$, $q_0 = 1$. Bottom: $k_0 = 50$, $L = 1$, $q_0 = 1$.

In the case where $\operatorname{Spr}((V_q)_0(k_0)) < 1$, the series $(V_q(k_0)^J\,\mathrm{e}^{\mathrm{i}k_0(\cdot)})$ converges strongly to zero, while $(I - V_q(k_0))\sum_{j=1}^{J} V_q(k_0)^j\,\mathrm{e}^{\mathrm{i}k_0(\cdot)}$ converges strongly to $V_q(k_0)\,\mathrm{e}^{\mathrm{i}k_0(\cdot)}$, The second of the two convergence processes plateaus for large values of J due to the effect of numerical errors. In contrast, neither sequence converges in the case where $\operatorname{Spr}((V_q)_0(k_0)) \geq 1$.

Finally, we illustrate Lemma 6 numerically in Figures 3 and 4, picking parameter values that violate the strong (norm) condition (7): the computed values of $\operatorname{Spr}(V_q(k_0))$ in the examples in Figures 3 and 4 are 2.42 and 7.21. In spite of this, we indeed do produce the operator $M = (1-\gamma)I + \gamma V_q(k_0)$ with $\|M\| < 1$, and we specifically show in Figure 4 that we get a convergent Neumann series solution for the case $k_0 = 50$, $L = 1$, $q_0 = 1$, $\operatorname{Spr}(V_q(k_0)) = 7.21$, albeit the convergence is rather slow.



a) k_0=1, $L = 1$, $q_0 = 5$, $\mathrm{Spr}(V_q(k_0)) = 2.42$, $\varepsilon' = 5.2$, $\alpha = 1.3803$, $\varepsilon = 0.02$, $T = 0.9320$.



b) $k_0 = 50$, $L = 1$, $q_0 = 1$, $\mathrm{Spr}(V_q(k_0)) = 7.21$, $\varepsilon' = 50$, $\alpha = 1.5508$, $\varepsilon = 3 \cdot 10^{-5}$, $T = 0.0677$.

Figure 3. The original theoretically predicted and numerically computed spectra of $V_q(k_0)$; the transformed curve $\{z = 1-\gamma(1-\omega),\ \omega \in \Lambda(q_0, k_0L)\} \subset \boldsymbol{C}$ is included in the unit open disk centered at 1.

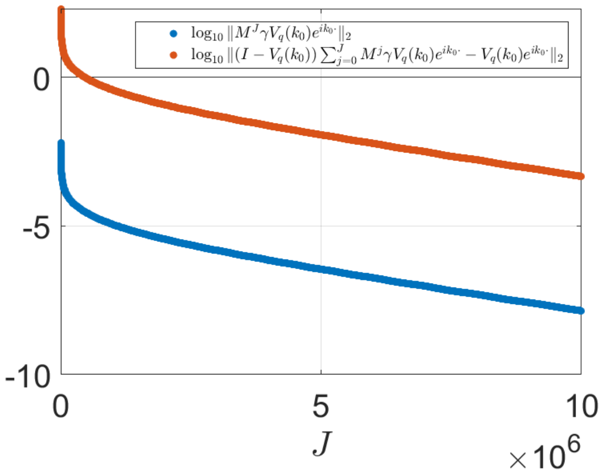


Figure 4. Preconditioning of the equation $(I - V_q(k_0))u = V_q(k_0)\,e^{ik_0(\cdot)}$ results in a convergent Neumann series solution. The parameters here are as in Figure 2 (bottom) and Figure 3 b), and specifically $\mathrm{Spr}(V_q(k_0)) = 7.21$.

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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