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Global boundedness of solutions to a chemotaxis consumption model with signal dependent motility and logistic source

Limite globale des solutions d'un modèle de consommation de chimiotaxie avec motilité dépendante du signal et source logistique

Khadijeh Baghaei

Abstract. This paper deals with the following chemotaxis system:

$$\begin{cases} u_t = \nabla\cdot\big(\gamma(v)\nabla u - u\xi(v)\nabla v\big) + \mu u(1-u), & x\in\Omega,\ t>0,\\ v_t = \Delta v - uv, & x\in\Omega,\ t>0, \end{cases}$$

under homogeneous Neumann boundary conditions in a bounded domain $\Omega\subset\mathbb{R}^n, n\geq 2$, with smooth boundary. Here, the functions $\gamma(v)$ and $\xi(v)$ are as:

$$\gamma(v) = (1+v)^{-k} \quad \text{and} \quad \xi(v) = -(1-\alpha)\gamma'(v),$$

where $k>0$ and $\alpha\in(0,1)$.

For the above system, we prove that the corresponding initial boundary value problem admits a unique global classical solution which is uniformly-in-time bounded. This result is obtained under some conditions on initial value v_0 and μ and without any restriction on k and α. The obtained result extends the recent results obtained for this problem.

Résumé. Cet article porte sur le système de chimiotaxie suivant :

$$\begin{cases} u_t = \nabla\cdot\big(\gamma(v)\nabla u - u\xi(v)\nabla v\big) + \mu u(1-u), & x\in\Omega,\ t>0,\\ v_t = \Delta v - uv, & x\in\Omega,\ t>0, \end{cases}$$

sous des conditions aux limites homogènes de Neumann dans un domaine borné $\Omega\subset\mathbb{R}^n, n\geq 2$, avec une frontière lisse. Ici, les fonctions $\gamma(v)$ et $\xi(v)$ sont les suivantes :

$$\gamma(v) = (1+v)^{-k} \quad \text{and} \quad \xi(v) = -(1-\alpha)\gamma'(v),$$

où $k>0$ and $\alpha\in(0,1)$.

Pour le système ci-dessus, nous prouvons que le problème de valeur limite initiale correspondant admet une unique solution classique globale qui est uniformément bornée en temps. Ce résultat est obtenu sous certaines conditions sur la valeur initiale v_0 et μ et sans restriction sur k et α. Le résultat obtenu étend les résultats récents obtenus pour ce problème.

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1. Introduction

In this paper, we consider the following initial boundary value problem:

$$\begin{cases} u_t = \nabla\cdot\big(\gamma(v)\nabla u - u\xi(v)\nabla v\big) + \mu u(1-u), & x\in\Omega,\ t>0,\\ v_t = \Delta v - uv, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0)=u_0,\ v(x,0)=v_0, & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n, n\geq 2$, is a bounded domain with smooth boundary, ν denotes the unit outward normal vector to $\partial\Omega$ and u_0 and v_0 are initial functions. In the above problem, $u=u(x,t)$ is the cell density and $v=v(x,t)$ denotes the nutrient consumed chemical concentrations.

In mathematical biology, systems such as (1.1) describe the mechanism of chemotaxis. The chemotaxis is the movement of cells towards a higher concentration a chemical signal substance produced by cells. We first state the results related to the classical chemotaxis system, which has been introduced by Keller and Segel in [15]. The classical chemotaxis system can be written as follows:

$$\begin{cases} u_t = \nabla\cdot\big(\gamma(v)\nabla u - u\xi(v)\nabla v\big) + \mu u(1-u), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases} \tag{1.2}$$

where $\tau\in\{0,1\}$. In the following, we write some important results for this problem in the absence and presence of the logistic source, respectively. In the absence of logistic source, when $\gamma\in C^3((0,\infty))$ and $\xi=-\gamma'$ as well as

$$\limsup_{s\to\infty}\gamma(s) < \frac{1}{\tau},$$

then the problem (1.2) admits a unique global classical solution in any dimension [33]. Also, for $n\geq 1$, when the function γ has strictly positive upper and lower bounds, then the classical solutions are uniformly-in-time bounded [33]. This result is also proved when $n\geq 2$ and the function γ decays at a certain slow rate at infinity [33]. Among the special functions that have been studied as γ are:

$$\gamma(v) = c_0\, v^{-k} \quad \text{with} \quad k>0 \quad \text{and} \quad c_0>0$$

and

$$\gamma(v) = \mathrm{e}^{-\chi v} \quad \text{with} \quad \chi>0.$$

For $\gamma(v)=c_0\,v^{-k}$, for all $k>0$, the global existence and boundedness of the solution is proved under a smallness assumption on c_0 in any dimension [34]. Also, for this function, by removing the assumption on c_0, when $n\geq 2$ and $k\in(0,\frac{2}{n-2})$, the same result is proved in cases $\tau=0$ [1] and $\tau=1$ [10]. In the other special case $\gamma(v)=\mathrm{e}^{-\chi v}$, if $n=2$ and $\int_\Omega u_0\,\mathrm{d}x<\frac{4\pi}{\chi}$, then the classical solutions are global and bounded whereas for $\int_\Omega u_0\,\mathrm{d}x>\frac{4\pi}{\chi}$, blow up occurs either in finite or infinite time [14]. But, in the case of $\tau=0$, the blow up occurs in infinite time [11]. Also, for $n=2$, the classical solutions are globally bounded if the positive function γ decreases slower than an exponential speed at high signal concentrations [9]. For $n\geq 3$, the same result is true when γ decreases at certain algebraically speed [9]. We now state the result obtained in [8] which is the global existence of very weak solution to the problem (1.2) when $\gamma(v)=\frac{1}{c+v^k}$ with $c\geq 0$ and $k>0$. This result was obtained without any smallness assumption on the initial data provided that

$$k\in\begin{cases} (0,\frac{7}{3}), & \text{if } n=1,\\ (0,2), & \text{if } n=2,\\ (0,\frac{4}{3}), & \text{if } n=3. \end{cases}$$

Also, for $n \geq 1$ with $\gamma(v) = \mathrm{e}^{-v}$, the existence of weak-strong solutions is proved in [6]. We now state the results to the problem (1.2) in the presence of logistic source. If the decreasing function $\gamma \in C^3([0,\infty))$ satisfies $\lim_{s\to\infty}\gamma(s) = 0$ and $\lim_{s\to\infty}\frac{\gamma'(s)}{\gamma(s)}$ exists, then the classical solutions are global and bounded for $n = 2$ [13]. For $n \geq 3$, the same result is true when $\mu > 0$ is large and the last condition is replaced with $|\gamma'(s)| \leq m$, where m is some positive constant [28]. Also, in two dimensional case, the same result holds when $\gamma \in C^3((0,\infty))$ and $|\gamma'| + |\gamma''| \in L^\infty((0,\infty))$ [31].
Now, we write some important results related to the problem (1.1). The origin of the definition of this problem comes from the following chemotaxis-Navier–Stokes system which describes the motion of oxygen-driven swimming bacteria in an in-compressible fluid

$$\begin{cases} u_t + \omega\cdot\nabla u = \nabla\cdot\big(\nabla u - u\xi(v)\nabla v\big), & x\in\Omega,\ t>0,\\ v_t + \omega\cdot\nabla v = \Delta v - ug(v), & x\in\Omega,\ t>0,\\ \omega_t + (\omega\cdot\nabla)\omega = \Delta\omega - \nabla P + u\nabla\phi, & x\in\Omega,\ t>0,\ t>0,\\ \nabla\cdot\omega = 0, & x\in\Omega,\ t>0,\ t>0. \end{cases}$$

Here, u denotes the bacteria density and v is the oxygen concentration. Also, ω and P are the velocity and pressure of the fluid, respectively. The function ξ measures the chemotactic sensitivity, g is the consumption rate of the oxygen by the bacteria, and ϕ is a given potential function [27]. For the related results with the chemotaxis-Navier–Stokes systems, we refer the interested readers to [5, 7, 12, 30] and references therein. We see that the problem (1.1) can be obtained from the preceding chemotaxis-Navier–Stokes system in the case of $\gamma(v) \equiv 1$ with the choice $\omega \equiv 0$ and $g(v) = v$. For the problem (1.1), in the absence of logistic source, when $\gamma(v) \equiv 1$, $\xi(v) \equiv \chi$, where χ is some positive constant, for $n = 2$, the classical solutions are global and bounded in bounded convex domains with a smooth boundary [25]. Also, for $n \geq 3$, this result holds in bounded domains with a smooth boundary provided that $\|v_0\|_{L^\infty(\Omega)} \leq \frac{1}{6(n+1)\chi}$ [24]. Later, this condition was extended to $\|v_0\|_{L^\infty(\Omega)} \leq \frac{\pi}{\chi\sqrt{2(n+1)}}$ and the same result was obtained in the absence of logistic source [3] and the presence of logistic source [4].
We now state the related results to the problem (1.1) in the absence of logistic source, when the function γ is not constant. We begin by stating the results obtained in [20] which is for the positive function $\gamma \in C^0([0,\infty))$ and $\xi = -\gamma'$. For non-negative initial data from $(C^0(\overline{\Omega}))^* \times L^\infty(\Omega)$, this problem admits a global very weak solution in all dimensions, also, for $\gamma \in C^1([0,\infty))$, the solutions stabilize toward a semi-trivial spatially homogeneous steady state in the large time limit. This result is obtained for $n \leq 3$ in [20] and later for $n \geq 1$ in [18]. If the decreasing function γ belongs to $C^3([0,\infty))$ and $\xi = -\gamma'$, then for non-negative initial data from $C^0(\overline{\Omega}) \times W^{1,\infty}(\overline{\Omega})$ under a smallness assumption on $\|v_0\|_{L^\infty(\Omega)}$, there exists a unique global classical solution that is bounded [19]. If the initial data (u_0, v_0) belongs to $(W^{1,\infty}(\Omega))^2$ and $\gamma \in C^3([0,\infty))$, then classical solutions are globally bounded when $n \leq 2$, and weak solutions are global when $n \geq 3$, in particular, such weak solutions become eventually smooth if $n = 3$ [21]. If $\gamma \in C^0([0,\infty)) \cap C^3([0,\infty))$, $\gamma(s) > 0$ for all $s > 0$ as well as $\xi = -\gamma'$, and γ satisfies:

$$\liminf_{s\searrow 0}\frac{\gamma(s)}{s^\alpha} > 0 \quad \text{and} \quad \limsup_{s\searrow 0} s^\beta|\gamma'(s)| < \infty$$

with $\alpha > 0$ and $\beta > 0$, then this problem admits a global generalized solution for all reasonably regular initial data [32]. In [26], the function γ is assumed to satisfy:

$$c_1 s^{-k} \leq \gamma(s) \leq c_2 s^{-k} \quad \text{for all} \quad s > 0, \quad (*)$$

where k, c_1 and c_2 are some positive constants. Under this assumption for $\gamma \in C^3([0,\infty))$ and the initial data belongs to $(W^{1,\infty}(\Omega))^2$, the problem (1.1) admits a global classical solution when

$n = 1$, and a global weak-strong solution when $n \geq 2$. Also, this result is true if $2 \leq n \leq 5$, $k > \frac{n-2}{6-n}$ and γ satisfies $(*)$ and

$$|\gamma'(s)| \leq c_3\, s^{-k-1} \quad \text{for all} \quad s > 0,$$

where c_3 is some positive constant [26].

Finally, we state the important results related to the problem (1.1) in the presence of logistic source. If $\gamma \in C^3([0,\infty))$ and $\gamma'(s) < 0$ for all $s \geq 0$ as well as $\xi = -\gamma'$, then for $n = 2$ and $\mu > 0$, the solutions are global and bounded [22]. Also, the same result is obtained when $n \geq 3$ and μ is suitably large [22]. Moreover, the solution converges exponentially to $(1,0)$ when t tends to infinity [22]. For this problem, when the logistic source is as $f(u) = a\,u - \mu\,u^{\kappa}$ with $a > 0$, $\mu > 0$ and $\kappa > 1$, the classical solutions are global and bounded if one of the cases $\big(n \leq 2, \kappa > 1;\ n \geq 3, \kappa > 2$ or $n \geq 3, \kappa = 2$ and μ is large$\big)$ holds [29]. If the positive decreasing function γ belongs to $C^2([0,\infty))$ and $\gamma'' \geq 0$ as well as $\xi = -(1-\alpha)\,\gamma'$ with $\alpha \in (0,1)$, in [23], it is proved that the problem (1.1) admits a unique global classical solution that is uniformly in time bounded provided that:

$$\frac{(\gamma'(s))^2}{\gamma''(s)} \leq \frac{n}{2(n+1)^3}, \qquad 0 < \|v_0\|_{L^\infty(\Omega)} \leq \gamma^{-1}\left(\frac{1}{n+1}\right)$$

and

$$\mu > \max_{0<s\leq \|v_0\|_{L^\infty(\Omega)}} \frac{-\gamma'(s)\,\|v_0\|_{L^\infty(\Omega)}}{\gamma(s)}.$$

We see that the above conditions in the case of $\gamma(s) = (1+s)^{-k}\,(k > 0)$ are as follows:

$$k < \frac{n}{2(n+1)^3 - n}, \qquad \mu > k\|v_0\|_{L^\infty(\Omega)} \qquad \text{and} \qquad \|v_0\|_{L^\infty(\Omega)} \leq \gamma^{-1}\left(\frac{1}{n+1}\right).$$

Because of $\gamma' < 0$, the last condition is written as:

$$0 < \|v_0\|_{L^\infty(\Omega)} \leq (n+1)^{\frac{1}{k}} - 1.$$

In [2], we studied the special case $\gamma(s) = (1+s)^{-k}\,(k > 0)$ and we were able to improve the conditions in [23]. In fact, we proved the same result in [23] under the following conditions:

$$k(1-\alpha) < \frac{4}{n+5}, \qquad 0 < \|v_0\|_{L^\infty(\Omega)} \leq \left[\frac{4\big[l-1\big]}{n+1}\right]^{\frac{1}{k}} - 1$$

and

$$\mu > \frac{n\|v_0\|_{L^\infty(\Omega)}}{l\,(n+1)(1+\|v_0\|_{L^\infty(\Omega)})}$$

with $l = \frac{1}{k(1-\alpha)}$. In this paper, we focus again on the functions γ and ξ as follows:

$$\gamma(s) = (1+s)^{-k} \quad \text{and} \quad \xi(s) = -(1-\alpha)\gamma'(s), \tag{1.3}$$

where $k > 0$ and $\alpha \in (0,1)$ and extend our recent result. In fact, we remove the condition on $k(1-\alpha)$ and prove the solutions are uniformly in time bounded under some conditions on $\|v_0\|_{L^\infty(\Omega)}$ and μ.

2. Our results

Here, we state the standard well-posedness and classical solvability result.

Lemma 1. *Let $u_0 \geq 0$ and $v_0 \geq 0$ satisfy $(u_0, v_0) \in (W^{1,r}(\Omega))^2$ for some $r > n$. Then problem* (1.1) *has a unique local in time classical solution*

$$(u,v) \in \Big(C\big([0,T_{\max}); W^{1,r}(\Omega)\big) \cap C^{2,1}\big(\overline{\Omega} \times (0,T_{\max})\big)^2$$

where $T_{\max}$ denotes the maximal existence time. In addition, if $T_{\max} < +\infty$, then:

$$\limsup_{t\to T_{\max}} \|u(\,.\,,t)\|_{L^\infty(\Omega)} = +\infty.$$

Moreover, u and v satisfy the following inequalities:

$$u \ge 0 \quad \textit{and} \quad 0 \le v \le \|v_0\|_{L^\infty(\Omega)} \quad \textit{in} \quad \Omega \times (0, T_{\max}), \tag{2.1}$$

also,

$$\int_\Omega u(\,.\,,t)\,\mathrm{d}x \le c, \tag{2.2}$$

where c is some positive constant.

For details of the proof, we refer the reader to [13, 23].
Based on main ideas in [3, 4, 16, 17], we write the following key lemma similar to [2, Lemma 2.2].

Lemma 2. *Let (u,v) be the solution of problem* (1.1). *If there exists a smooth positive function $\varphi(s)$ such that for $0 \le s \le \|v_0\|_{L^\infty(\Omega)}$, the following inequality holds:*

$$(B(s))^2 - 4\,A(s)\,C(s) \le 0, \tag{2.3}$$

where for $p \ge 2$, the functions A, B and C are defined as:

$$\begin{cases} A(s) = (p-1)\,\varphi(s)\,\gamma(s), \\ B(s) = (p-1)\,\varphi(s)\,\xi(s) - \varphi'(s)\,(\gamma(s)+1), \\ C(s) = \frac{1}{p}\,\varphi''(s) - \varphi'(s)\,\xi(s), \end{cases} \tag{2.4}$$

then:

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x \le -\int_\Omega \left[\mu\,\varphi(v) + \frac{1}{p}\,v\,\varphi'(v)\right] u^{p+1}\,\mathrm{d}x + \mu\int_\Omega u^p\varphi(v)\,\mathrm{d}x.$$

Proof. We assume that there exists a smooth positive function $\varphi(s)$ such that for $0 \le s \le \|v_0\|_{L^\infty(\Omega)}$ and $p \ge 2$, (2.3) holds. We take this function and use (1.1) and integration by parts to write:

$$\begin{aligned} \frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x &= \int_\Omega u^{p-1}\varphi(v)\,u_t\,\mathrm{d}x + \frac{1}{p}\int_\Omega u^p\,\varphi'(v)\,v_t\,\mathrm{d}x \\ &= -(p-1)\int_\Omega u^{p-2}\varphi(v)\,\gamma(v)\,|\nabla u|^2\,\mathrm{d}x \\ &\qquad + \int_\Omega u^{p-1}\left[(p-1)\,\varphi(v)\,\xi(v) - \varphi'(v)\,(\gamma(v)+1)\right]\left(\nabla u\cdot\nabla v\right)\mathrm{d}x \\ &\qquad + \int_\Omega u^p\left[\varphi'(v)\,\xi(v) - \frac{1}{p}\,\varphi''(v)\right]|\nabla v|^2\,\mathrm{d}x \\ &\qquad - \int_\Omega \left[\mu\,\varphi(v) + \frac{1}{p}\,v\,\varphi'(v)\right]u^{p+1}\,\mathrm{d}x + \mu\int_\Omega u^p\varphi(v)\,\mathrm{d}x. \end{aligned} \tag{2.5}$$

For convenience in calculations, we write (2.5) as follows:

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x = \int_\Omega J(u,v)\,\mathrm{d}x - \int_\Omega\left[\mu\,\varphi(v) + \frac{1}{p}\,v\,\varphi'(v)\right]u^{p+1}\,\mathrm{d}x + \mu\int_\Omega u^p\varphi(v)\,\mathrm{d}x \tag{2.6}$$

with

$$\begin{aligned} J(u,v) &= -(p-1)\,u^{p-2}\varphi(v)\,\gamma(v)\,|\nabla u|^2 \\ &\qquad + u^{p-1}\left[(p-1)\,\varphi(v)\,\xi(v) - \varphi'(v)\,(\gamma(v)+1)\right]\left(\nabla u\cdot\nabla v\right) \\ &\qquad + u^p\left[\varphi'(v)\,\xi(v) - \frac{1}{p}\,\varphi''(v)\right]|\nabla v|^2 \\ &= -u^{p-2}\,|\nabla u|^2 A(v) + u^{p-1}\left(\nabla u\cdot\nabla v\right)B(v) - u^p\,|\nabla v|^2\,C(v), \end{aligned} \tag{2.7}$$

where A, B and C are defined in (2.4). Now, by considering (2.7), we can write

$$\begin{aligned} J(u,v) &= -\left(\sqrt{u^{p-2}A(v)}\,\nabla u - \frac{u^{p-1}B(v)}{2\sqrt{u^{p-2}A(v)}}\,\nabla v\right)\cdot\left(\sqrt{u^{p-2}A(v)}\,\nabla u - \frac{u^{p-1}B(v)}{2\sqrt{u^{p-2}A(v)}}\,\nabla v\right) \\ &\qquad + u^p\left[\frac{(B(v))^2}{4\,A(v)} - C(v)\right]|\nabla v|^2 \\ &\le u^p\left[\frac{(B(v))^2 - 4\,A(v)\,C(v)}{4\,A(v)}\right]|\nabla v|^2. \end{aligned}$$

In view of the condition (2.3), we see that $J \le 0$. Thus, the equality (2.6) becomes

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x \le -\int_\Omega\left[\mu\,\varphi(v) + \frac{1}{p}\,v\,\varphi'(v)\right]u^{p+1}\,\mathrm{d}x + \mu\int_\Omega u^p\varphi(v)\,\mathrm{d}x.$$

This completes our proof. □

Before presenting a smooth positive function φ such that the relation (2.3) holds, we present the following preliminary lemma.

Lemma 3. *Let the function γ for $s \ge 0$ is defined as $\gamma(s) = (1+s)^{-k}$ with $k > 0$. If the function H for $z > 0$ is defined as*

$$H(z) = d^2\,p\,z^{2\lambda} - 2\,d^2(p-2)\,z^{2\lambda-1} + d^2 p\,z^{2\lambda-2} - 2\,d\big[2\,d\,(p-1) - p + 2\,l\big]z^{\lambda-1} - 2\,d\,p\,z^\lambda + p, \quad (2.8)$$

where for $\alpha \in (0,1)$ and $p > 2$, the parameters λ, l and d are as follows:

$$\lambda = d\,(p-1)\,(1-\alpha), \qquad l = \frac{1}{k\,(1-\alpha)}$$

and

$$d > \frac{-l + \sqrt{l^2 + p\,(p-2)}}{2\,(p-2)}. \quad (2.9)$$

Then there exists $\delta_0 > 0$ such that for $0 < s \le \delta_0$, $H(\gamma(s)) \le 0$.

Proof. At first, we see that

$$\begin{aligned} H(1) &= d^2\,p - 2\,d^2(p-2) + d^2\,p - 2\,d\big[2\,\mathrm{d}\,(p-1) - p + 2\,l\big] - 2\,d\,p + p \\ &= -4\,(p-2)\,d^2 - 4\,l\,d + p. \end{aligned}$$

It is not difficult to see that the choice of d as (2.9) implies that:

$$-4\,(p-2)\,d^2 - 4\,l\,d + p < 0.$$

Thus, $H(1) < 0$. By considering the continuity of the function H on $(0,\infty)$ and γ on $[0,\infty)$, and also $H(\gamma(0)) = H(1) < 0$, we conclude that there exists $\delta_0 > 0$ such that for $0 < s \le \delta_0$, $H(\gamma(s)) \le 0$. □

We now present a smooth positive function φ and show that for this function, the relation (2.3) holds. We must state that the following lemma is the only place where the special choice of γ is used.

Lemma 4. *Assume that the initial values u_0 and v_0 are non-negative and satisfy:*

$$(u_0, v_0) \in (W^{1,r}(\Omega))^2 \quad \textit{for some} \quad r > n,$$

and

$$0 < \|v_0\|_{L^\infty(\Omega)} \le \delta_0,$$

which δ_0 is introduced in Lemma 3. Also, assume that the functions $\gamma(v)$ and $\xi(v)$ are defined as (1.3) *and for $p > n$, the following condition holds:*

$$\mu > \frac{d\,(p-1)\,\|v_0\|_{L^\infty(\Omega)}}{l\,p\,(1 + \|v_0\|_{L^\infty(\Omega)})}, \quad (2.10)$$

where $l = \frac{1}{k(1-\alpha)}$ *and* d *is chosen as* (2.9). *Then, there exists some positive constant* c *which also depends on* δ_0 *and* p *such that the following estimate holds:*

$$\|u(\,.\,,t)\|_{L^p(\Omega)} \le c, \qquad \forall\, t \in (0, T_{\max}). \tag{2.11}$$

Proof. We prove this lemma in two steps. In the first step, we present a smooth positive function φ and show that for this function (2.3) holds. In the second step, we prove (2.11) holds.

Step 1. As a starting point, in Lemma 2, we take $p > n$ and define the smooth positive function φ as follows:

$$\varphi(v) = \mathrm{e}^{(\gamma(v))^{\lambda}}$$

with $\lambda = d\,(p-1)\,(1-\alpha)$, where d is taken as (2.9). For this function, we have:

$$\varphi'(v) = \lambda\,(\gamma(v))^{\lambda-1}\gamma'(v)\,\varphi(v) \tag{2.12}$$

and

$$\varphi''(v) = \lambda\left[\,(\lambda-1)(\gamma'(v))^2 + \gamma(v)\gamma''(v) + \lambda\,(\gamma(v))^{\lambda}(\gamma'(v))^2\,\right](\gamma(v))^{\lambda-2}\varphi(v).$$

Because of

$$\gamma'(v) = -k\,(1+v)^{-k-1} \quad \text{and} \quad \gamma''(v) = k\,(k+1)\,(1+v)^{-k-2},$$

we see that $\gamma(v)\gamma''(v) = \left(1+\frac{1}{k}\right)(\gamma'(v))^2$. Thus,

$$\varphi''(v) = \lambda\left[\lambda + \frac{1}{k} + \lambda\,(\gamma(v))^{\lambda}\right](\gamma(v))^{\lambda-2}\,(\gamma'(v))^2\varphi(v). \tag{2.13}$$

By considering the values of A, B and C from (2.4), we can write:

$$\begin{aligned}(B(v))^2 - 4\,A(v)\,C(v) &= (p-1)^2(\varphi(v))^2\,(\xi(v))^2 + (\varphi'(v))^2\,(\gamma(v)+1)^2 \\ &\quad - \frac{4\,(p-1)}{p}\,\varphi(v)\,\varphi''(v)\,\gamma(v) - 2\,(p-1)\,\varphi(v)\,\varphi'(v)\,\xi(v)\,(1-\gamma(v)).\end{aligned}$$

Making use of (2.12), (2.13) and $\xi(v) = -(1-\alpha)\gamma'(v)$, we obtain:

$$\begin{aligned}(B(v))^2 - 4\,A(v)\,C(v) &= (\gamma'(v))^2(\varphi(v))^2\left\{(p-1)^2(1-\alpha)^2 + \lambda^2(\gamma(v))^{2\lambda-2}(\gamma(v)+1)^2\right. \\ &\quad \left. - \frac{4\,\lambda\,(p-1)}{p}\left[\lambda + \frac{1}{k} + \lambda(\gamma(v))^{\lambda}\right](\gamma(v))^{\lambda-1} + 2\,\lambda\,(p-1)(1-\alpha)(\gamma(v))^{\lambda-1}(1-\gamma(v))\right\}.\end{aligned}$$

Because of $\lambda = d\,(p-1)(1-\alpha)$, we replace $(p-1)(1-\alpha)$ with $\frac{\lambda}{d}$ to have:

$$\begin{aligned}
&(B(v))^2 - 4\,A(v)\,C(v)\\
&\quad= (\gamma'(v))^2(\varphi(v))^2\Big\{\frac{\lambda^2}{d^2} + \lambda^2(\gamma(v))^{2\lambda-2}(\gamma(v)+1)^2\\
&\qquad - \frac{4\,\lambda^2}{d\,p\,(1-\alpha)}\left[\lambda\left(1+(\gamma(v))^{\lambda}\right)+\frac{1}{k}\right](\gamma(v))^{\lambda-1} + \frac{2\,\lambda^2}{d}(\gamma(v))^{\lambda-1}(1-\gamma(v))\Big\}\\
&\quad= \frac{\lambda^2(\gamma'(v))^2(\varphi(v))^2}{d^2\,p}\Big\{p + d^2\,p\,(\gamma(v))^{2\lambda-2}(\gamma(v)+1)^2\\
&\qquad - 4\,\mathrm{d}\left[d\,(p-1)(1+(\gamma(v))^{\lambda}) + \frac{1}{k\,(1-\alpha)}\right](\gamma(v))^{\lambda-1} + 2\,\mathrm{d}\,p\,(\gamma(v))^{\lambda-1}(1-\gamma(v))\Big\} \qquad (2.14)\\
&\quad= \frac{\lambda^2(\gamma'(v))^2(\varphi(v))^2}{d^2\,p}\Big\{p + \mathrm{d}^2 p\,(\gamma(v))^{2\lambda} - 2\,\mathrm{d}^2(p-2)(\gamma(v))^{2\lambda-1}\\
&\qquad + d^2\,p\,(\gamma(v))^{2\lambda-2} - 2\,\mathrm{d}\big[2\,\mathrm{d}\,(p-1) - p + 2\,l\big](\gamma(v))^{\lambda-1} - 2\,\mathrm{d}\,p\,(\gamma(v))^{\lambda}\Big\}\\
&\quad:= \frac{\lambda^2}{d^2\,p}(\gamma'(v))^2(\varphi(v))^2 H(\gamma(v)),
\end{aligned}$$

where the function H is defined by (2.8). By considering Lemma 3, we see that there exists $\delta_0 > 0$ such that for $0 < v \le \delta_0$, $H(\gamma(v)) \le 0$. Because of $0 < \|v_0\|_{L^\infty(\Omega)} \le \delta_0$, we conclude that $H(\gamma(v))$ is non-positive in the interval $(0, \|v_0\|_{L^\infty(\Omega)}]$. Therefore, by combining this with (2.14), we conclude that the relation (2.3) holds.

Step 2. In order to prove (2.11), we apply Lemma 2 and write:

$$\begin{aligned}
&\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x + \mu\int_\Omega u^p\varphi(v)\,\mathrm{d}x\\
&\qquad\le -\int_\Omega\left[\mu\varphi(v) + \frac{1}{p}\,v\,\varphi'(v)\right]u^{p+1}\,\mathrm{d}x + 2\,\mu\int_\Omega u^p\varphi(v)\,\mathrm{d}x. \qquad (2.15)
\end{aligned}$$

We now apply the Young inequality to the second term on the right hand side of (2.15) to have:

$$2\,\mu\int_\Omega u^p\,\varphi(v)\,\mathrm{d}x \le \epsilon\int_\Omega u^{p+1}\varphi(v)\,\mathrm{d}x + c_\epsilon\int_\Omega \varphi(v)\,\mathrm{d}x,$$

where ϵ is chosen as follows:

$$0 < \epsilon < \mu - \frac{k\,\lambda\,\|v_0\|_{L^\infty(\Omega)}}{p\,(1+\|v_0\|_{L^\infty(\Omega)})} \qquad (2.16)$$

and:

$$c_\epsilon = \frac{1}{p+1}\left[\frac{p}{\epsilon\,(p+1)}\right]^p (2\mu)^{p+1}.$$

Because of $\gamma'(v) < 0$, (2.12) implies that $\varphi'(v) < 0$. Thus,

$$\varphi(\|v_0\|_{L^\infty(\Omega)}) < \varphi(v) < \varphi(0). \qquad (2.17)$$

Making use of this, we can write:

$$2\,\mu\int_\Omega u^p\varphi(v)\,\mathrm{d}x \le \epsilon\int_\Omega u^{p+1}\varphi(v)\,\mathrm{d}x + c_0 \qquad (2.18)$$

with $c_0 = c_\epsilon\,|\Omega|\,\varphi(0)$. We now combine the inequality (2.18) with (2.15) to obtain:

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x + \mu\int_\Omega u^p\varphi(v)\,\mathrm{d}x \le \int_\Omega\left[(\epsilon-\mu)\,\varphi(v) - \frac{1}{p}\,v\,\varphi'(v)\right]u^{p+1}\,\mathrm{d}x + c_0. \qquad (2.19)$$

We use $\gamma'(v) = -\frac{k}{1+v}\gamma(v)$, $\gamma(v) \le 1$ and (2.16) to have

$$\begin{aligned}(\epsilon-\mu)\,\varphi(v) - \frac{1}{p}\,v\,\varphi'(v) &= \left[\epsilon-\mu-\frac{\lambda}{p}\,v\,(\gamma(v))^{\lambda-1}\gamma'(v)\right]\varphi(v)\\ &= \left[\epsilon-\mu+\frac{k\,\lambda\,v}{p\,(1+v)}\,(\gamma(v))^{\lambda}\right]\varphi(v)\\ &\le \left[\epsilon-\mu+\frac{k\,\lambda\,\|v_0\|_{L^\infty(\Omega)}}{p\,(1+\|v_0\|_{L^\infty(\Omega)})}\right]\varphi(v)\\ &\le 0.\end{aligned}$$

This along with (2.19) yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^p\varphi(v)\,\mathrm{d}x + \mu\,p\int_\Omega u^p\varphi(v)\,\mathrm{d}x \le p\,c_0.$$

We put:

$$y(t) = \int_\Omega u^p\varphi(v)\,\mathrm{d}x.$$

Thus,

$$y'(t) + \mu\,p\,y(t) \le p\,c_0.$$

This yields:

$$y(t) \le \max\left\{y(0), \frac{c_0}{\mu}\right\}. \tag{2.20}$$

This along with (2.17) allows us to write:

$$\int_\Omega u^p\,\mathrm{d}x \le \left(\varphi(\|v_0\|_{L^\infty(\Omega)})\right)^{-1}\max\left\{y(0), \frac{c_0}{\mu}\right\} := c.$$

By considering the value of c_0, we see that c also depends on δ_0 and p. This completes our proof. □

The proof of the following lemma is the same as [24, Lemma 3.2] or [2, Lemma 2.4]. But, we write it to complement our content.

Lemma 5. *Assume that for $p > n$, the following estimate holds:*

$$\|u(\,.\,,t)\|_{L^p(\Omega)} \le C_1, \qquad \forall\, t \in (0, T_{\max}), \tag{2.21}$$

where C_1 is some positive constant, then there exists some positive constant C_2 which also depends on C_1 and $\|v_0\|_{L^\infty(\Omega)}$ such that

$$\|\nabla v(\,.\,,t)\|_{L^\infty(\Omega)} \le C_2 \tag{2.22}$$

for all $t \in (0, T_{\max})$.

Proof. By considering Lemma 1, we see that it is sufficient to prove for any $\tau \in (0, T_{\max})$,

$$\|\nabla v(\,.\,,t)\|_{L^\infty(\Omega)} \le C_2 \quad \text{for all } t \in (\tau, T_{\max}). \tag{2.23}$$

We use the representation formula for the second equation (1.1) to have:

$$v(\,.\,,t) = \mathrm{e}^{t(\Delta-1)}v_0 + \int_0^t \mathrm{e}^{(t-s)(\Delta-1)}(1-u(\,.\,,s))\,v(\,.\,,s)\,\mathrm{d}s, \quad t \in (0, T_{\max}).$$

We now take $p > n$ and use $0 \le v \le \|v_0\|_{L^\infty(\Omega)}$ to write:

$$
\begin{aligned}
\|(1-u(\,.\,,s))\,v(\,.\,,s)\|_{L^p(\Omega)} &\le \|v(\,.\,,s)\|_{L^\infty(\Omega)} \left(\int_\Omega |1-u(\,.\,,s))|^p\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\le \|v(\,.\,,s)\|_{L^\infty(\Omega)} \left(\int_\Omega \big(1+|u(\,.\,,s))|\big)^p\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\le \|v(\,.\,,s)\|_{L^\infty(\Omega)} \left(2^{p-1}\int_\Omega \big(1+|u(\,.\,,s))|^p\big)\,\mathrm{d}x\right)^{\frac{1}{p}} \\
&\le 2^{\frac{p-1}{p}}\,\|v(\,.\,,s)\|_{L^\infty(\Omega)} \left(|\Omega|^{\frac{1}{p}} + \|u(\,.\,,s))\|_{L^p(\Omega)}\right) \\
&\le c,
\end{aligned} \tag{2.24}
$$

where we have used the inequality $(a+b)^m \le 2^{m-1}(a^m+b^m)$ with $a,b \ge 0$ and $m>1$, also $(a+b)^{m'} \le (a^{m'}+b^{m'})$ with $0<m'<1$. We note that the constant c in the last estimate, it also depends on C_1 and $\|v_0\|_{L^\infty(\Omega)}$. In order to prove (2.23), we take $\tau \in (0,\min\{1,T_{\max}\})$ and $\theta \in (\frac{p+n}{2p},1)$ and use the estimates (3.16) and (3.17) in [24], also (2.24) to obtain:

$$
\begin{aligned}
\|v(\,.\,,t)\|_{W^{1,\infty}(\Omega)} &\le c\,\|(-\Delta+1)^\theta v(\,.\,,t)\|_{L^p(\Omega)} \\
&\le c\,t^{-\theta}\mathrm{e}^{-\delta t}\,\|v_0\|_{L^p(\Omega)} + c\int_0^t (t-s)^{-\theta}\mathrm{e}^{-\delta(t-s)}\,\|(1-u(\,.\,,s))\,v(\,.\,,s)\|_{L^p(\Omega)}\,\mathrm{d}s \\
&\le c\,t^{-\theta} + c\int_0^t (t-s)^{-\theta}\mathrm{e}^{-\delta(t-s)}\,\mathrm{d}s \\
&\le c\,t^{-\theta} + c\int_0^{+\infty} \sigma^{-\theta}\mathrm{e}^{-\delta\sigma}\,\mathrm{d}\sigma \\
&\le c\,(\tau^{-\theta}+1), \quad t \in (\tau, T_{\max}),
\end{aligned}
$$

where the constants can vary from line to line. In this estimate, from the third line on-wards, the constants are also dependent on C_1 and $\|v_0\|_{L^\infty(\Omega)}$. This completes our proof. □

The proof of the following lemma is the same as [2, Lemma 2.5] and similar to [24, Lemma 3.2]. But, we write it to complement our content.

Lemma 6. *Assume that the initial values u_0 and v_0 are non-negative and satisfy:*

$$(u_0, v_0) \in (W^{1,r}(\Omega))^2 \quad \textit{for some} \quad r > n.$$

Also, assume that (2.22) *holds. Then there exists some positive constant c such that for all $t \in (0, T_{\max})$, the following estimate holds*

$$\|u(\,.\,,t)\|_{L^\infty(\Omega)} \le c.$$

Proof. We take $q \ge 2$ and use from (1.1) and integration by parts to obtain:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q\,\mathrm{d}x &= q\int_\Omega u^{q-1}\left[\nabla\cdot(\gamma(v)\nabla u - u\xi(v)\nabla v) + \mu\,u(1-u)\right]\mathrm{d}x \\
&= -q(q-1)\int_\Omega \gamma(v)\,u^{q-2}|\nabla u|^2\,\mathrm{d}x + q(q-1)\int_\Omega u^{q-1}\xi(v)\big(\nabla u\cdot\nabla v\big)\,\mathrm{d}x \\
&\qquad + \mu q\int_\Omega u^q(1-u)\,\mathrm{d}x.
\end{aligned} \tag{2.25}
$$

Because of $0 \le v \le \|v_0\|_{L^\infty(\Omega)}$, we have:

$$
\begin{aligned}
\gamma(v) &= (1+v)^{-k} \ge (1+\|v_0\|_{L^\infty(\Omega)})^{-k} := c_1, \\
\xi(v) &= k(1-\alpha)(1+v)^{-k-1} \le k(1-\alpha) := c_2.
\end{aligned}
$$

Making use of these, (2.22) and Young's inequality, we can write (2.25) as follows:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q\,\mathrm{d}x &\le -c_1\,q\,(q-1)\int_\Omega u^{q-2}|\nabla u|^2\,\mathrm{d}x + C_2\,c_2\,q\,(q-1)\int_\Omega u^{q-1}|\nabla u|\,\mathrm{d}x + \mu\,q\int_\Omega u^q\,\mathrm{d}x\\ &= -\frac{4\,c_1\,(q-1)}{q}\int_\Omega \left|\nabla u^{\frac{q}{2}}\right|^2\mathrm{d}x + 2\,C_2\,c_2\,(q-1)\int_\Omega u^{\frac{q}{2}}\cdot\left|\nabla u^{\frac{q}{2}}\right|\mathrm{d}x + \mu\,q\int_\Omega u^q\,\mathrm{d}x \qquad (2.26)\\ &\le -\frac{2\,c_1\,(q-1)}{q}\int_\Omega \left|\nabla u^{\frac{q}{2}}\right|^2\mathrm{d}x + q\left(\frac{(q-1)\,(C_2\,c_2)^2}{2\,c_1} + \mu\right)\int_\Omega u^q\,\mathrm{d}x.\end{aligned}$$

We now add $q\int_\Omega u^q\,\mathrm{d}x$ on both sides of (2.26) to have:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q\,\mathrm{d}x + q\int_\Omega u^q\,\mathrm{d}x \le -\frac{2\,c_1\,(q-1)}{q}\int_\Omega |\nabla u^{\frac{q}{2}}|^2\,\mathrm{d}x + c_3\int_\Omega u^q\,\mathrm{d}x \qquad (2.27)$$

with

$$c_3 = q\left(\frac{(q-1)\,(C_2\,c_2)^2}{2\,c_1} + \mu + 1\right).$$

We first apply the Gagliardo–Nirenberg inequality and then use the Young inequality with exponents $s = \frac{n+2}{2}$ and $s' = \frac{n+2}{n}$ to obtain:

$$\begin{aligned}c_3\int_\Omega u^q\,\mathrm{d}x = c_3\,\|u^{\frac{q}{2}}\|^2_{L^2(\Omega)} &\le c_3\,(C_{GN})^2\left(\|\nabla u^{\frac{q}{2}}\|^{\frac{n}{n+2}}_{L^2(\Omega)}\|u^{\frac{q}{2}}\|^{\frac{2}{n+2}}_{L^1(\Omega)} + \|u^{\frac{q}{2}}\|_{L^1(\Omega)}\right)^2\\ &\le 2\,c_3\,(C_{GN})^2\left(\|\nabla u^{\frac{q}{2}}\|^{\frac{2n}{n+2}}_{L^2(\Omega)}\|u^{\frac{q}{2}}\|^{\frac{4}{n+2}}_{L^1(\Omega)} + \|u^{\frac{q}{2}}\|^2_{L^1(\Omega)}\right)\\ &\le \frac{2\,c_1\,(q-1)}{q}\|\nabla u^{\frac{q}{2}}\|^2_{L^2(\Omega)} + \left(c_4 + 2\,c_3\,(C_{GN})^2\right)\|u^{\frac{q}{2}}\|^2_{L^1(\Omega)}\\ &= \frac{2\,c_1(q-1)}{q}\int_\Omega |\nabla u^{\frac{q}{2}}|^2\,\mathrm{d}x + c_5\left(\int_\Omega u^{\frac{q}{2}}\,\mathrm{d}x\right)^2\end{aligned}$$

with

$$c_4 = \frac{1}{s}\left(\frac{2\,c_1\,s'\,(q-1)}{q}\right)^{-\frac{s}{s'}}\left(2\,c_3\,(C_{GN})^2\right)^s \quad \text{and} \quad c_5 = c_4 + 2\,c_3\,(C_{GN})^2,$$

where C_{GN} is the constant in the Gagliardo–Nirenberg inequality. Combining the last inequality with (2.27) yields:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u^q\,\mathrm{d}x + q\int_\Omega u^q\,\mathrm{d}x \le c_5\left(\int_\Omega u^{\frac{q}{2}}\,\mathrm{d}x\right)^2.$$

For $0 \le t \le T_{\max}$, we can write:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{q\,t}\int_\Omega u^q\,\mathrm{d}x\right) \le c_5\,\mathrm{e}^{q\,t}\left(\int_\Omega u^{\frac{q}{2}}\,\mathrm{d}x\right)^2.$$

Now, we integrate and use $\mathrm{e}^{-q\,t} \le 1$ to get:

$$\begin{aligned}\int_\Omega u^q\,\mathrm{d}x &\le \int_\Omega u_0^q\,\mathrm{d}x + \frac{c_5}{q}\sup_{0\le t\le T_{\max}}\left(\int_\Omega u^{\frac{q}{2}}\,\mathrm{d}x\right)^2\\ &\le |\Omega|\,\|u_0\|^q_{L^\infty(\Omega)} + \frac{c_5}{q}\sup_{0\le t\le T_{\max}}\left(\int_\Omega u^{\frac{q}{2}}\,\mathrm{d}x\right)^2.\end{aligned}$$

Thus,

$$\begin{aligned}\left(\int_\Omega u^q\,\mathrm{d}x\right)^{\frac{1}{q}} &\le \left[|\Omega|\,\|u_0\|^q_{L^\infty(\Omega)} + \frac{c_5}{q}\sup_{0\le t\le T_{\max}}\left(\int_\Omega u^{\frac{q}{2}}\,\mathrm{d}x\right)^2\right]^{\frac{1}{q}}\\ &\le |\Omega|^{\frac{1}{q}}\,\|u_0\|_{L^\infty(\Omega)} + \left(\frac{c_5}{q}\right)^{\frac{1}{q}}\sup_{0\le t\le T_{\max}}\left(\int_\Omega u^{\frac{q}{2}}\,\mathrm{d}x\right)^{\frac{2}{q}}. \qquad (2.28)\end{aligned}$$

We note that

$$\begin{aligned}
c_5 &= c_4 + 2\,c_3\,(C_{GN})^2\\
&= \frac{1}{s}\left(\frac{2\,c_1\,s'\,(q-1)}{q}\right)^{-\frac{s}{s'}} \left(2\,c_3\,(C_{GN})^2\right)^s + 2\,c_3\,(C_{GN})^2\\
&= \frac{1}{s}\left(2\,c_1\,s'\right)^{-\frac{s}{s'}} \left(2\,(C_{GN})^2\right)^s \Big(\frac{q}{q-1}\Big)^{\frac{n}{2}} (c_3)^s + 2\,c_3\,(C_{GN})^2\\
&\le m\Big[\Big(\frac{q}{q-1}\Big)^{\frac{n}{2}} (c_3)^s + c_3\Big]\\
&\le m\Big[\Big(\frac{q}{q-1}\Big)^{\frac{n}{2}} + 1\Big](c_3)^s
\end{aligned}$$

with

$$m = \max\Big\{\frac{1}{s}\left(2\,c_1\,s'\right)^{-\frac{s}{s'}}\left(2\,(C_{GN})^2\right)^s, 2\,(C_{GN})^2\Big\}.$$

Here, we have used from $c_3 > 1$ and $s > 1$. By inserting c_3 and using $q \ge 2$, we obtain:

$$\begin{aligned}
\frac{c_5}{q} &\le m\left[\Big(\frac{q}{q-1}\Big)^{\frac{n}{2}} + 1\right]\left(\frac{(q-1)\,(C_2\,c_2)^2}{2\,c_1} + \mu + 1\right)^{\frac{n}{2}+1} q^{\frac{n}{2}}\\
&\le 2\,m\left(\frac{(C_2\,c_2)^2}{2\,c_1} + \mu + 1\right)^{\frac{n}{2}+1} \Big(\frac{q}{q-1}\Big)^{\frac{n}{2}} (q-1)^{\frac{n}{2}+1}\, q^{\frac{n}{2}}\\
&= c_6\,(q-1)\,q^n\\
&\le c_6\,q^{n+1}
\end{aligned} \tag{2.29}$$

with

$$c_6 = 2\,m\left(\frac{(C_2\,c_2)^2}{2\,c_1} + \mu + 1\right)^{\frac{n}{2}+1}.$$

Making use of (2.29) and $q^{\frac{n+1}{q}} > 1$, we can write (2.28) as follows:

$$\begin{aligned}
\left(\int_\Omega u^q\,\mathrm{d}x\right)^{\frac{1}{q}} &\le |\Omega|^{\frac{1}{q}}\,\|u_0\|_{L^\infty(\Omega)} + \left(c_6\,q^{n+1}\right)^{\frac{1}{q}} \sup_{0\le t\le T_{\max}} \left(\int_\Omega u^{\frac{q}{2}}\,\mathrm{d}x\right)^{\frac{2}{q}}\\
&\le c_7^{\frac{1}{q}}\,q^{\frac{n+1}{q}}\left(\|u_0\|_{L^\infty(\Omega)} + \sup_{0\le t\le T_{\max}}\left(\int_\Omega u^{\frac{q}{2}}\,\mathrm{d}x\right)^{\frac{2}{q}}\right)
\end{aligned} \tag{2.30}$$

with $c_7 = |\Omega| + c_6$. We now define:

$$M(q) = \max\left\{\|u_0\|_{L^\infty(\Omega)}, \sup_{0\le t\le T_{\max}}\left(\int_\Omega u^q\,\mathrm{d}x\right)^{\frac{1}{q}}\right\}.$$

This allows us to write (2.30) as:

$$M(q) \le 2\,c_7^{\frac{1}{q}}\,q^{\frac{n+1}{q}}\,M\Big(\frac{q}{2}\Big).$$

We now take $q = 2^i$ $(i \in \mathbb{N})$ to obtain:

$$\begin{aligned}
M(2^i) &\le 2\,c_7^{2^{-i}}\,2^{\frac{(n+1)i}{2^i}}\,M(2^{i-1})\\
&\le 2\,c_7^{2^{-i}+2^{-i+1}}\,2^{(n+1)\left(\frac{i}{2^i}+\frac{i-1}{2^{i-1}}\right)}\,M(2^{i-2})\\
&\le \cdots\\
&\le 2\,c_7^{2^{-i}+2^{-i+1}+\cdot+2^{-1}}\,2^{(n+1)\left(\frac{i}{2^i}+\frac{i-1}{2^{i-1}}+\cdot+\frac{1}{2}\right)}\,M(1),
\end{aligned} \tag{2.31}$$

and compute the following elementary series:

$$S := \sum_{i=1}^{\infty}\frac{i}{2^i} = \sum_{i=0}^{\infty}\frac{i+1}{2^{i+1}} = \sum_{i=0}^{\infty}\Big(\frac{i}{2^{i+1}} + \frac{1}{2^{i+1}}\Big) = \frac{1}{2}\sum_{i=1}^{\infty}\frac{i}{2^i} + \sum_{i=0}^{\infty}\frac{1}{2^{i+1}} = \frac{1}{2}S + 1.$$

Thus, $S = 2$. Making use of this, $\lim_{i\to\infty} \|u(\,.\,,t)\|_{L^{2^i}(\Omega)} = \|u(\,.\,,t)\|_{L^\infty(\Omega)}$ and (2.2), by letting $i \to \infty$ in (2.31), we obtain the desired result. □

We now can write our main theorem.

Theorem 7. *Let the initial values u_0 and v_0 are non-negative and satisfy:*

$$(u_0, v_0) \in (W^{1,r}(\Omega))^2 \quad \textit{for some} \quad r > n$$

and

$$0 < \|v_0\|_{L^\infty(\Omega)} \le \delta_0,$$

which δ_0 is introduced in Lemma 3. Assume that the condition (2.10) *holds. Then, the solution of the problem* (1.1) *with the functions $\gamma(v)$ and $\xi(v)$ defined by* (1.3) *is global and bounded.*

Proof. By considering the extensibility criterion provided by Lemma 1, the proof is a consequence of (2.3) and Lemma 6.

□

Declaration of interests

The authors do not work for, advise, own shares in, or receive funds from any organization that could benefit from this article, and have declared no affiliations other than their research organizations.

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