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Partial differential equations / *Équations aux dérivées partielles*

Ground state solution for a non-autonomous 1-Laplacian problem involving periodic potential in $\mathbb{R}^N$

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Abstract. In this paper, we consider the following 1-Laplacian problem

$$-\Delta_1 u + V(x)\frac{u}{|u|} = f(x,u),\ x\in\mathbb{R}^N,\ u>0,\ u\in BV\left(\mathbb{R}^N\right),$$

where $\Delta_1 u = \operatorname{div}(\frac{Du}{|Du|})$, V is a periodic potential and f is periodic and verifies the super-primary condition at infinity. By the Nehari type manifold method, the concentration compactness principle and some analysis techniques, we show the 1-Laplacian equation has a ground state solution.

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1. Introduction and main result

In this paper, we consider the following 1-Laplace with potentials:

$$\begin{cases} -\Delta_1 u + V(x)\frac{u}{|u|} = f(x,u), & x\in\mathbb{R}^N, \\ u>0 \\ u\in BV\left(\mathbb{R}^{\mathbb{N}}\right), \end{cases} \tag{1}$$

where $\Delta_1 u = \operatorname{div}(\frac{Du}{|Du|})$. Assume that V and f satisfies the following conditions:

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(V_1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$, $\inf_{\mathbb{R}^N} V(x) \geq V_0 > 0$, $V(x)$ is 1-periodic in each of $x_1, x_2, \dots, x_N$.

(f_1) $f \in C^1$ is 1-periodic in each of $x_1, x_2, \dots, x_N$, and there exists $p \in (1, 1^*)(1^* = \frac{N}{N-1}$ if $N \geq 2$ and $1^* = +\infty$ if $N = 1)$ such that

$$\lim_{|s| \to \infty} \frac{\left|f(x,s)\right|}{|s|^{p-1}} = 0, \text{ uniformly in } x \in \mathbb{R}^N.$$

(f_2) $\lim_{|s| \to 0} f(x,s) = 0$ uniformly in $x \in \mathbb{R}^N$.

(f_3) $\lim_{|s| \to \infty} \frac{F(x,s)}{s} = \infty$, a. e. in $x \in \mathbb{R}^N$.

(f_4) $f(x,s)$ is strictly increasing in $s \in \mathbb{R} \setminus \{0\}$ for every $x \in \mathbb{R}^N$.

The 1-Laplace problem like (1) derived from image denoising and restoration is of crucial importance for many mathematical and physical fields introduced in [5]-[12]. Recently, more and more attention has been paid to p-Laplace operator $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$, $p \in [1, +\infty)$. For more details about the applications, see [2]-[7] and the references therein. However, there are few results about the 1-Laplacian problem, partially because of the compactness of sequences as $p \to 1$ occurs in weak norms, and partially because the associated energy functional is no longer smooth and strictly convex.

For the equation like (1), there are some results. For the case of $V(x) = 1$ and $f(x,s) = s$, in [8], the versions of the Radial Lemma of Strauss and the Lions Lemma in $BV(\mathbb{R}^N)$ are proved and applied to obtain existence of solutions for 1-Laplacian problem. Furthermore, the authors in [1, 11] obtained the existence of nontrivial bounded variation solution in $BV(\mathbb{R}^N)$ under the differential Berestycki–Lions' type conditions. For the case of $V(x) = 1$ and $f(x,s) = Q(x)g(s)$, Zhou and Shen in [13] obtained the existence of nontrivial radial bounded variation solution under the suitable Q and f where f satisfies the (AR) type condition. For the case of $V(x) \neq$ constant and $f(x,s) = K(x)g(s)$, in [6], the existence of ground state bounded variation solution is obtained when V is vanishing potential and suitable conditions under of K, g. Moreover, Alves, Figueiredo and Pimenta in [2] got the similar results when V is steep potential and autonomous $f(x,s) = f(s)$.

From the above discussion, we find that if V is the periodic potential function and f is periodic and verifies the super-primary condition at infinity, there are no results for 1-Laplacian problem (1). In this paper, inspired by [9], we shall study this case. In order to overcome difficulties derived from nonsmoothness of convex functional and compactness of the embeddings $BV(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)(p \in (1, 1^*))$ of problem (1), we use the subdifferential and concentration compactness principle to obtain our result. Here is the main result of this paper.

Theorem 1. *Suppose that* (V_1) *and* (f_1) -(f_4) *hold. Then there exists a nontrivial bounded variation ground state solution $u \in X$ (function space X please see the next Section 2) of* (1) *such that $\Phi(u) = c > 0$, c is defined by*

$$c = \inf_{\mathcal{N}} \Phi(u),$$

where

$$\mathcal{N} := \left\{u \in X : u \neq 0, \gamma(u) = 0\right\},$$

Φ and γ are the same as in the following (2) *and* (4).

Remark 2. Let

$$F(x,s) = \sin(2\pi x_1)\, s \ln(1 + |s|).$$

Then

$$f(x,s) = \sin(2\pi x_1) \left(\ln(1 + |s|) + \frac{|s|}{1 + |s|}\right).$$

It is easy to see that f satisfies (f_1)-(f_4).

In this paper, denote L^∞ norm by $|\cdot|_\infty$. $B_r(x)$ represents a ball with the center at x and radius R. Using C and $C(\varepsilon)$ to denote by various positive constants and functions with ε, which may vary from line to line.

2. Preliminaries

We shall work with the space of functions of bounded variation denoted by

$$BV\left(\mathbb{R}^N\right) := \left\{u \in L^1\left(\mathbb{R}^N\right);\ Du \in \mathscr{M}\left(\mathbb{R}^N, \mathbb{R}^N\right)\right\},$$

where Du represents the distributional derivative of u and $\mathscr{M}$ denotes the set of vectorial Radon measures. In [8], $BV_{rad}(\mathbb{R}^N) = \{u \in BV(\mathbb{R}^N); u(x) = u(|x|)\}$ compactly embedded into $L^q(\mathbb{R}^N)$ is proved for $1 < q < 1^*$. It can be proved that $u \in BV(\mathbb{R}^N)$ if and only if $u \in L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} |Du| = \sup\left\{\int_{\mathbb{R}^N} u \operatorname{div}\phi dx\,;\ \phi \in C_c^1\left(\mathbb{R}^N, \mathbb{R}^N\right), \text{s.t.} |\phi|_\infty \leq 1\right\} < +\infty.$$

$BV(\mathbb{R}^N)$ is a Banach space with the norm

$$\|u\| := \int_{\mathbb{R}^N} |Du| + \int_{\mathbb{R}^N} |u| dx.$$

Denote the subspace of $BV(\mathbb{R}^N)$ by X where

$$X := \left\{u \in BV\left(\mathbb{R}^N\right) : \int_{\mathbb{R}^N} V(x)|u| dx < \infty\right\}$$

endowed with the following norm

$$\|u\|_X := \int_{\mathbb{R}^N} |Du| + \int_{\mathbb{R}^N} V(x)|u| dx.$$

Note that the embedding $X \hookrightarrow BV(\mathbb{R}^N)$ is continuous in such a way that X is a Banach space that is continuously embedded into $L^r(\mathbb{R}^N)$ for $r \in [1, 1^*]$. As one can see in [4], $C_0^\infty(\mathbb{R}^N)$ is not dense in X with respect to the strong convergence, but is dense with respect to the intermediate convergence, where $\{u_n\} \subset X$ converges to $u \in X$ in the sense of intermediate convergence if $u_n \to u$ in $L^1(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \left|Du_n\right| \to \int_{\mathbb{R}^N} |Du|$$

as $n \to \infty$.

For a vectorial Radon measure $\mu \in \mathscr{M}(\mathbb{R}^N, \mathbb{R}^N)$, we denote by the usual decomposition by $\mu = \mu^a + \mu^s$ according to the Radon Nikodyn Theorem, where μ^a and μ^s are the absolute continuous and the singular parts with respect to the N−dimensional Lebesgue measure $\mathscr{L}^N$, respectively. With $|\mu|$ being the scalar Radon measure, the usual Lebesgue derivative of μ with respect to $|\mu|$ is defined by

$$\frac{\mu}{|\mu|}(x) = \lim_{r \to 0} \frac{\mu\left(B_r(x)\right)}{|\mu|\left(B_r(x)\right)}.$$

We can define the energy functional corresponding to (1) in X as

$$\Phi(u) = \int_{\mathbb{R}^N} |Du| + \int_{\mathbb{R}^N} V(x)|u| dx - \int_{\mathbb{R}^N} F(x, u) dx. \tag{2}$$

For convenience, we let

$$\Phi(u) = \mathscr{J}_V(u) - \mathscr{H}(u),$$

where

$$\mathscr{J}_V(u) = \int_{\mathbb{R}^N} |Du| + \int_{\mathbb{R}^N} V(x)|u| dx,\ \mathscr{H}(u) = \int_{\mathbb{R}^N} F(x, u) dx.$$

Clearly, $\mathscr{J}_V(u)$ is convex and Lipschitz continuous in X. Furthermore, $\mathscr{J}_V$ is lower semicontinuous with respect to the $L^r(\mathbb{R}^N)$ topology for $r \in [1,1^*]$. Although nonsmooth, the functional $\mathscr{J}_V$ admits some directional derivatives. From [3], we know that, given $u \in X$, for all $v \in X$ such that $(Dv)^s$ is absolutely continuous with respect to $(Dv)^s$ and $v \equiv 0$ a.e. in the set where u vanishes,

$$\mathscr{J}_V'(u)v = \int_{\mathbb{R}^N} \frac{(Du)^a(Dv)^a}{|(Du)^a|}dx + \int_{\mathbb{R}^N} \frac{Du}{|Du|}(x)\frac{Dv}{|Dv|}(x)\left|(Dv)^s\right| + \int_{\mathbb{R}^N} V(x)\,\mathrm{sgn}(u)v\,dx. \tag{3}$$

where $\mathrm{sgn}(u(x)) = 0$ if $u(x)=0$ and $\mathrm{sgn}(u(x)) = \frac{u(x)}{|u(x)|}$ if $u(x) \neq 0$. In particular, note that $\mathscr{J}_V'(u)u = \mathscr{J}_V(u)$ for all $u \in X$. Especially, from (f_1)-(f_3), it can be proved that $\mathscr{H} \in C^1(X,\mathbb{R}))$. Taking $v = u$ in (3), it follows that

$$\gamma(u) := \Phi'(u)u = \mathscr{J}_V'(u)u - \mathscr{H}'(u)u = \|u\|_X - \int_{\mathbb{R}^N} f(x,u)u\,dx. \tag{4}$$

Since Φ can be written as the difference between the Lipschitz functional $\mathscr{J}_V$ and a smooth functional $\mathscr{H}$, we say that $u \in X$ is a bounded variation solution of (1) if $0 \in \partial\Phi(u)$, where $\partial\Phi(u)$ denote the generalized gradient (subdifferential) of Φ in u. This is equivalent to $\mathscr{H}'(u) \in \partial\mathscr{J}_V'u)$. By the convexity of $\mathscr{J}_V$, we have

$$\|w\|_X - \|u\|_X \geq \int_{\mathbb{R}^N} f(x,u)(w-u)\,dx,\ \forall\ w \in X. \tag{5}$$

Hence, all $u \in X$ such that (5) holds is called a bounded variation solution of (1).

Note that (1) is not well-defined wherever $x \in \mathbb{R}^N \setminus \{0\}$ such that $\nabla u(x) = 0$ or $u(x) = 0$. So (1) is only a formal version of the precise Euler–Lagrange equation associated to Φ. Arguing similarly to that in [8], we can prove that, if $u \in X$ is a bounded variation critical point of Φ, then it satisfies the following version of (1):

$$\begin{cases} \exists\, \mathbf{z} \in L^\infty\left(\mathbb{R}^N,\mathbb{R}^N\right), |\mathbf{z}|_\infty \leq 1, \mathrm{div}\mathbf{z} \in L_{\infty,N}\left(\mathbb{R}^N\right),\ \text{s.t.} -\int_{\mathbb{R}^N} u\,\mathrm{div}\mathbf{z}\,dx = \int_{\mathbb{R}^N}|Du| \\ \exists\, z_* \in L_{\infty,N}\left(\mathbb{R}^N\right),\ \text{s.t. } z_*|u| = V(x)u \quad \text{a.e. in } \mathbb{R}^N \\ -\mathrm{div}\mathbf{z} + z_* = f(x,u), \quad \text{a.e. in } \mathbb{R}^N, \end{cases}$$

where

$$L_{\infty,N}\left(\mathbb{R}^N\right) = \left\{f : \mathbb{R}^N \to \mathbb{R} \text{ measurable}, \left|f\right|_{\infty,N} < \infty\right\}$$

and

$$\left|f\right|_{\infty,N} = \sup_{|\phi|+|\phi|_{1^*}\leq 1}\left|\int_{\mathbb{R}^N} f\phi\,dx\right|.$$

3. Proof of Theorem 1

In order to prove Theorem 1, we need the following lemmas.

Lemma 3. *For each $u \in X\setminus\{0\}$, there exists a unique $t_u = t(u) > 0$, such that $t_u u \in \mathcal{N}$ and $\max_{t>0}\Phi(tu) = \Phi(t_u u)$.*

Proof. The proof can be seen in [7]. We omit its proof. □

Lemma 4. *Let $\{u_n\}$ be a minimizing sequence for c. Then*

(i) *There is $\beta > 0$ such that* $\liminf_{n\to\infty}\|u_n\|_X \geq \beta$.
(ii) *$\{u_n\}$ is bounded in X.*
(iii) *For a subsequence, up to translations, $u_n \rightharpoonup u \neq 0$.*

Proof.
(i) By (f_1) and (f_2), for each $\varepsilon \in (0,1)$, there exists $C(\varepsilon) > 0$ such that

$$\left|f(x,s)\right| \le \varepsilon + C(\varepsilon)|s|^{p-1}\left(p \in \left(1,1^*\right)\right). \tag{6}$$

Together with $\gamma(u_n) = 0$ and the continuity of embedding $BV(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, we get

$$\|u_n\|_X = \int_{\mathbb{R}^N} f(x,u_n)\,u_n dx \le \varepsilon \int_{\mathbb{R}^N} |u_n|\,dx + C_\varepsilon \int_{\mathbb{R}^N} |u_n|^p\,dx \le \varepsilon \|u_n\|_X + C_\varepsilon \|u_n\|_X^p. \tag{7}$$

Then, it yields that there exists $\beta > 0$ such that $\liminf_{n\to\infty} \|u_n\|_X \ge \beta$.
(ii) Let $\{u_n\}$ be a minimizing sequence for c. If $\{u_n\}$ is not bounded, we define $v_n = u_n/\|u_n\|_X$, so $\|v_n\|_X = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in X, $v_n \to v$ in $L^p_{loc}(\mathbb{R}^N)$ ($p \in (1,1^*)$), $v_n \to v$ a. e. on $\mathbb{R}^N$.

If $v \ne 0$, by

$$\int_{\mathbb{R}^N} \left|Du_n\right| + \int_{\mathbb{R}^N} V(x)|u_n|\,dx - \int_{\mathbb{R}^N} F(x,u_n)\,dx = c + o(1),$$

we have

$$1 - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|_X} dx = \frac{c+o(1)}{\|u_n\|_X} > 0$$

because $c > 0$, its yields that

$$1 - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{u_n} v_n dx = \frac{c+o(1)}{\|u_n\|_X} > 0.$$

Together with Fatou's lemma and the second limit of (f_2), we get a contraction as follows,

$$1 \ge \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x,u_n)}{u_n} v_n dx \ge \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x,u_n)}{u_n} v_n dx = \infty.$$

If $v = 0$, we take $y_n = (y_n^1, y_n^2, \dots, y_n^N) \in \mathbb{N}^N$ with all $y_n^i (1 \le i \le N)$ being integers. Define translations of v_n by $w_n(x) = v_n(x+y_n)$. Since $V(x)$ and $f(x,u)$ are 1-periodic in each of $x_1, x_2, \dots, x_N$, we have $\|w_n\|_X = \|v_n\|_X = 1$, $|w_n|_p = |v_n|_p$ and $\Phi(w_n) = \Phi(v_n)$. Passing to a subsequence, we have $w_n \rightharpoonup w$ in X, $w_n \to w$ in $L^p_{loc}(\mathbb{R}^N)$ ($p \in (1,1^*)$), $w_n \to w$ a. e. on $\mathbb{R}^N$.

(a) If there exists y_n such that $w_n \rightharpoonup w \ne 0$ in X, then $v_n \rightharpoonup v \ne 0$ in X, this contradicts to $v = 0$.
(b) If for any y_n, $w_n \to 0$ in X. It yields that $v_n \to 0$ in $L^p_{loc}(\mathbb{R}^N)$. We have the following claim

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} |v_n|^p\,dx = 0, \text{ for all } p \in (1,1^*).$$

If this claim is not true, there exists $p \in (1,1^*)$ and $\delta > 0$ such that

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_2(y)} |v_n|^p\,dx \ge \delta > 0.$$

Then there exists $z_n \in \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \int_{B_2(z_n)} |v_n|^p\,dx \ge \frac{\delta}{2} > 0.$$

Choosing $y_n \in \mathbb{N}^N$ and $y_n \in B_2(z_n)$ such that $B_1(y_n) \subset B_2(z_n)$ and

$$\lim_{n\to\infty} \int_{B_1(y_n)} |v_n|^p\,dx \ge \frac{\delta}{2} > 0,$$

this reduces that

$$\lim_{n\to\infty} \int_{B_1(0)} |w_n|^p\,dx \ge \frac{\delta}{2} > 0.$$

It yields $w_n \to w \ne 0$ in X. This is a contradiction because of the assumption of (b).

By Lions' Lemma in $BV(\mathbb{R}^N)$([8, Theorem 1.3]), we get $v_n \to 0$ in $L^p(\mathbb{R}^N)$. Fix $p \in (1, 1^*)$, by (6), we have $|F(x,s)| \le \varepsilon|s| + C(\varepsilon)|s|^p$. Then, by fixing an $R > c$ and using Lebesgue Dominated Convergence theorem, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x, Rv_n)\, dx = \int_{\mathbb{R}^N} \lim_{n\to\infty} F(x, Rv_n)\, dx = 0. \tag{8}$$

Since by Lemma 3, $\Phi(tu_n) \le \Phi(u_n)$ for $t > 0$, we thus have

$$c + o(1) = \Phi(u_n) \ge \Phi\left(\frac{R}{\|u_n\|_X} u_n\right) = \Phi\big(Rv_n\big) = R - \int_{\mathbb{R}^N} F\big(x, Rv_n\big) dx.$$

Together with (8), passing to limit, we have $R \le c$, which is a contradiction. Thus $\{u_n\}$ is bounded.

(iii) Assume $u_n \rightharpoonup u$. To show $u \ne 0$, again we define translations of $\{u_n\}$ as above, assume $y_n = (y_n^1, y_n^2, \dots, y_n^N) \in \mathbb{N}^N$ with all $y_n^i (1 \le i \le N)$ being integers. $y_n = (y_n^1, y_n^2, \dots, y_n^N) \in \mathbb{N}^N$ with all $y_n^i (1 \le i \le N)$ being integers.$u_n^{y_n}(x) = u_n(x + y_n)$ are all possible translation of u_n. If for some $y_n \in \mathbb{N}^N$, $u_n^{y_n} \rightharpoonup u \ne 0$ we are done. If for any $y_n \in \mathbb{N}^N$, $u_n^{y_n} \rightharpoonup 0$, by similar argument as above we can prove $u_n \to 0$ in $L^p(\mathbb{R}^N)$ $(p \in (1, 1^*))$. Then we have $\lim_{n\to\infty} \int_{\mathbb{R}^N} u_n f(x, u_n) dx = 0$. Therefore, we get $0 < \beta \le \|u_n\|_X = \int_{\mathbb{R}^N} u_n f(x, u_n) dx \to 0$ as $n \to \infty$. Thus this yields that $\|u_n\|_X = 0$, which contradicts to (i). Therefore (iii) holds. □

Lemma 5. *Let $\{u_n\} \in X \setminus \{0\}$ be a sequence such that $\gamma(u_n) \to 0$ and $\int_{\mathbb{R}^N} u_n f(x, u_n) dx \to 0$ as $n \to \infty$. Then exist $t_n > 0$ such that $t_n u_n \in \mathcal{N}$, $t_n \to 1$, as $n \to \infty$.*

Proof. Since $u_n \ne 0$, by Lemma 3, there exists unique $t_n > 0$ such that $t_n u \in \mathcal{N}$. Together with (4), we have

$$t_n \|u_n\|_X - \int_{\mathbb{R}^N} f\big(x, t_n u_n\big) t_n u_n dx = 0. \tag{9}$$

By (6), we have $|f(x,s)s| \le \varepsilon|s| + C(\varepsilon)|s|^p$ for any $\varepsilon > 0$, together with the above equality, (V_1) and the continuity of embedding, we have

$$t_n \|u_n\|_X \le \varepsilon t_n \|u_n\|_X + C(\varepsilon) t_n^p \|u_n\|_X^p.$$

It yields that $t_n \to 0$ by Lemma 4 (ii). Thus there exists $T > 0$ such that $t_n \ge T > 0$. By (f_4), we have

$$F(x,s) = \int_0^s f(x,t)dt \le \int_0^s f(x,s)dt = f(x,s)s,$$

that is $f(x,s)s \ge F(x,s)$. Together with (9), (f_3) and Lemma 4 (ii), if $t_n \to \infty$, we have

$$a + o(1) = \int_{\mathbb{R}^N} u_n f(x, u_n)\, dx = \|u_n\|_X = \int_{\mathbb{R}^N} \frac{f\big(x, t_n u_n\big) t_n u_n}{t_n} dx \ge \int_{\mathbb{R}^N} \frac{F\big(x, t_n u_n\big)}{t_n u_n} u_n dx \to \infty.$$

This is a contradiction. Thus $0 < T \le t_n < C$ for some $C > 0$. Assume that $t_n \to t_0$. By $t_n u_n \in \mathcal{N}$, the condition of $\gamma(u_n) \to 0$ and $t_n \to t_0$ as $n \to \infty$, we have

$$t_0 \|u_n\|_X - \int_{\mathbb{R}^N} f\big(x, t_0 u_n\big) t_0 u_n dx = o(1)$$

and

$$\|u_n\|_X - \int_{\mathbb{R}^N} f\big(x, u_n\big) u_n dx = o(1).$$

Two equalities are subtracted, we obtain

$$o(1) = \int_{\mathbb{R}^N} \left(f\big(x, t_0 u_n\big) u_n - \int_{\mathbb{R}^N} f\big(x, u_n\big) u_n dx \right) dx = \int_{\mathbb{R}^N} \big(f\big(x, t_0 u_n\big) - f\big(x, u_n\big)\big) u_n dx. \tag{10}$$

From Lemma 4, we know that there exists a subsequence, still denoted by u_n, such that $u_n \to u$ in $L^p_{loc}(\mathbb{R}^N)$ for $p \in (1, 1^*)$. According to Lemma 4 (iii), we may assume $u \ne 0$. By Fatou's lemma, passing the limit to (10), we obtain that

$$\int_{\mathbb{R}^N} \big(f\big(x, t_0 u\big) - f(x, u)\big) u dx = 0.$$

Together with (f_4), we deduce $t_0 = 1$. □

In order to prove $\mathscr{H}(u_n) \to \mathscr{H}(u)$ as $n \to \infty$, consider equation (1) on $B_R(0)$

$$\begin{cases} -\Delta_1 u + V(x)\frac{u}{|u|} = f(x,u), & x \in B_R(0), \\ u = 0, & x \in \partial B_R(0), \\ u \in BV\left(\mathbb{R}^{\mathbb{N}}\right), \end{cases} \tag{11}$$

We can similarly define $\mathcal{N}_R$, C_R. By the similar method of [7] about 1-Laplacian problem with $V(x) = 0$ and $f(x,s) = f(s)$ satisfying (f_1)-(f_4) in the bounded domain, there exists $u_R \in \mathcal{N}_R$ such that u_R is a positive solution of (11). It is easy to check that $c_R > c$ and $c_R \to c$ as $R \to \infty$. This implies that u_R minimizes c as $R \to \infty$. Let $R_n \to \infty$, $u_n := u_{R_n}$. Then the following lemma holds.

Lemma 6. *For $p \in (1, 1^*)$, we have:*

(i) $\int_{\mathbb{R}^N} |u_n|^p dx \to A > 0$.
(ii) *There exist $\{x_n\} \subset \mathbb{R}^N$ such that for $\forall\, \varepsilon > 0, \exists\, R$.* $\liminf \int_{B_R(x_n)} |u_n|^p \geq A - \varepsilon$.

Proof.
(i) Since $\gamma(u_n) = 0$, (7) and Lemma 4, there exists $A > 0$ such that $\int_{\mathbb{R}^N} |u_n|^p dx \to A$ as $n \to \infty$.
(ii) We shall apply the concentration compactness principle [10] to $\int_{\mathbb{R}^N} |u_n|^p dx$ to prove (ii). By (i), vanishing doesn't occur for $\int_{\mathbb{R}^N} |u_n|^p dx$. Then there exist $\alpha \in (0,1]$, $\{x_n\} \subset \mathbb{R}^N$ such that for $\forall\, \varepsilon > 0$, $\exists\, R > 0$, $\forall\, r > R$, $r' > R$, we have

$$\liminf \int_{B_r(x_n)} |u_n|^p \geq \alpha A - \varepsilon \quad \text{and} \quad \liminf \int_{B^C_{r'}(x_n)} |u_n|^p \geq (1-\alpha)A - \varepsilon. \tag{12}$$

Next, we shall prove that $\alpha \in (0,1)$(dichotomy) cannot occur, but $\alpha = 1$(compactness) holds.

Assume that $\alpha \in (0,1)$. Choosing $\varepsilon_n \to 0$, $r_n \to \infty$ and $r'_n = 4r_n$. Take $\phi(x) = \xi(|x - x_n|/r_n)u_n$, where the cut-off function $\xi \in C_0^\infty(\mathbb{R}^N)$ such that

$$\xi(x) = \begin{cases} 0, \ |x| \leq 1 \ \text{ or } \ |x| \geq 4, \\ 1, \ 2 \leq |x| \leq 3, \end{cases}$$

and $|\xi'(x)| \leq 2$. By (3), consider the following equation

$$\begin{aligned} 0 &= \mathscr{J}'_V(u_n)\phi - \mathscr{H}'(u_n)\phi \\ &= \int_{\mathbb{R}^N} \frac{(Du_n)^a (D\phi)^a}{\left|(Du_n)^a\right|} dx + \int_{\mathbb{R}^N} \frac{Du_n}{\left|Du_n\right|}(x)\frac{D\phi}{\left|D\phi\right|}(x)\left|(D\phi)^s\right| \\ &\quad + \int_{\mathbb{R}^N} V(x)\,\mathrm{sgn}(u_n)\phi dx - \int_{\mathbb{R}^N} f(x,u_n)\phi dx. \end{aligned}$$

Since u_n is a solution of (11), one has

$$\int_{B_{R_n}} \left(\left|Du_n\right| + V(x)\right)\xi dx - \int_{B_{R_n}} f(x,u_n)\,\xi dx = o(1).$$

Together with the definition of ξ, we have

$$\int_{B_{3r_n}(x_n)\setminus B_{2r_n}(x_n)} \left(\left|Du_n\right| + V(x)u_n\right) dx - \int_{B_{3r_n}(x_n)\setminus B_{2r_n}(x_n)} f(x,u_n)\,u_n dx = o(1).$$

Take another cut-off function $\eta \in C_0^\infty(\mathbb{R}^N)$ such that

$$\eta(x) = \begin{cases} 0, \ |x| \geq 3, \\ 1, \ |x| \leq 2, \end{cases}$$

and $|\eta'(x)| \leq 2$ for $2 \leq |x| \leq 3$. Let

$$w_n(x) = \eta\left(|x - x_n|/r_n\right)u_n, \quad v_n(x) = \left(1 - \xi\left(|x - x_n|/r_n\right)\right)u_n.$$

Together with (2) and (12), by the same method, we have

$$\Phi(u_n) = \Phi(w_n) + \Phi(v_n) + o(1),$$

$$\int_{\mathbb{R}^N} |w_n|^p \geq \alpha A - \varepsilon_n$$

and

$$\int_{\mathbb{R}^N} |v_n|^p \geq (1-\alpha) A - \varepsilon_n.$$

From $\mathscr{J}_V'(u_n) w_n - \mathscr{H}'(u_n) w_n = 0$ and $\mathscr{J}_V'(u_n) v_n - \mathscr{H}'(u_n) v_n = 0$, we have

$$\gamma(w_n) = \mathscr{J}_V'(u_n) w_n - \mathscr{H}'(u_n) w_n + o(1) = o(1),$$
$$\gamma(v_n) = \mathscr{J}_V'(u_n) v_n - \mathscr{H}'(u_n) v_n + o(1) = o(1).$$

By Lemma 5, there exist $t_n \to 1$ and $s_n \to 1$ such that $t_n w_n \in \mathcal{N}$ and $s_n v_n \in \mathcal{N}$. Then

$$c + o(1) = \Phi(u_n) = \Phi(w_n) + \Phi(v_n) + o(1) = \Phi(t_n w_n) + \Phi(s_n v_n) + o(1) \geq 2c + o(1).$$

This is a contradiction. Thus $\alpha = 1$. □

Proof of Theorem 1. Let $\{u_n\} \subset \mathcal{N}$ be the minimizing sequence for c defined as in Theorem 1. By Lemma 4, $\{u_n\}$ is bounded in X and weak convergent to $u \neq 0$. By Lemma 6, we can obtain $\mathscr{H}(u_n) = \mathscr{H}(u) + o(1)$. Moreover, since $\mathscr{J}_V$ is lower semicontinuous and convex, take the liminf both sides of $\Phi(u_n) = \mathscr{J}_V(u_n) + \mathscr{H}(u_n) = c + o(1)$, we have $\Phi(u) \leq c$. If $u \in \mathcal{N}$, we have $\Phi(u) = c$. If $u \notin \mathcal{N}$, by Lemma 3, there exists $t > 0$ such that $tu \in \mathcal{N}$. Then

$$c \leq \Phi(tu) \leq \liminf_{n\to\infty} \Phi(tu_n) \leq \liminf_{n\to\infty} \Phi(u_n) = c.$$

Now, by [7, Theorem 2], the minimizer is a critical point of Φ. □

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