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A family of totally ordered groups with some special properties

Elena Olivos

Abstract

Let K be a field with a Krull valuation $|\ |$ and value group $G \neq \{1\}$, and let B_K be the valuation ring. Theories about spaces of countable type and Hilbert-like spaces in [1] and spaces of continuous linear operators in [2] require that all absolutely convex subsets of the base field K should be countably generated as B_K-modules.

By [1] Prop. 1.4.1, the field K is metrizable if and only if the value group G has a cofinal sequence. We prove that for any fixed cardinality $\aleph_\kappa$, there exists a metrizable field K whose value group has cardinality $\aleph_\kappa$. The existence of a cofinal sequence only depends on the choice of some appropriate ordinal α which has cardinality $\aleph_\kappa$ and which has cofinality ω.

By [2] Prop. 1.4.4, the condition that any absolutely convex subset of K be countably generated as a B_K-module is equivalent to the fact that the value group has a cofinal sequence and each element in the completion $G^\#$ is obtained as the supremum of a sequence of elements of G. We prove that for any fixed uncountable cardinal $\aleph_\kappa$ there exists a metrizable field K of cardinality $\aleph_\kappa$ which has an absolutely convex subset that is not countably generated as a B_K-module.

We prove also that for any cardinality $\aleph_\kappa > \aleph_0$ for the value group the two conditions (the whole group has a cofinal sequence and every subset of the group which is bounded above has a cofinal sequence) are logically independent.

1 Preliminaries

In order to obtain, for any cardinality $\aleph_\kappa > \aleph_0$, a metrizable field K whose value group has cardinality $\aleph_\kappa$ we will construct below an abelian totally ordered group as a subset of the direct product of a family of subgroups of $(\mathbb{R}^+, \cdot, \leq)$ indexed by some ordinal α of cardinality $\aleph_\kappa$. In this section we recall the principal features concerning to ordinals and cardinals. Readers may refer to [3] for additional information.

A linear ordering $\leq$ of a set A is a **well-ordering** if every nonempty subset of A has a smallest element. A set T is an **ordinal number** if every element of T is a subset of T and T is well-ordered with respect to the membership-relation ($\in$).

We use small Greeks letters to denote ordinal numbers. The class of all ordinals is denoted by Ord; it is easy to see that Ord is not a set. The relation on Ord defined by $\alpha < \beta$ if and only if $\alpha \in \beta$ is a well-ordering of the class Ord. Thus, $0 = \phi$ is the first ordinal, and for each ordinal number α, $\alpha = \{\beta : \beta < \alpha\}$, we have that $\alpha+1 = \alpha\cup\{\alpha\} = \inf\{\beta : \beta > \alpha\}$ is an ordinal called the **successor** of α. If α is not a successor, then $\alpha = \sup\{\beta : \beta < \alpha\} = \bigcup\limits_{\beta<\alpha} \beta$ and it is called a **limit ordinal**. We also consider 0 as the first limit ordinal. Every well-ordered set is isomorphic to a unique ordinal number. The finite ordinals are denoted by $0, 1, 2, \ldots, n, \ldots$ and they correspond to the order-type of the natural numbers. The order-type of the set of natural numbers is denoted by ω and it is the first limit ordinal different from 0, and the first infinite ordinal. It is important to note that ω, $\omega + 1$ and $\omega + \omega$ represent different order-types, although all of them are isomorphic as sets with the set of natural numbers. For example, $\omega + 1$ represents the order-type of the set $\mathbb{N} \cup \{\infty\}$ such that $n < \infty$ for all $n \in \mathbb{N}$; $\omega + \omega$ represents the order-type of the set $\{\frac{n}{n+1} : n \in \mathbb{N}\} \cup \{1 + \frac{n}{n+1} : n \in \mathbb{N}\}$ with the usual ordering on the rational numbers. That is to say "*two copies of* ω". The lexicographical order on $\mathbb{N}\times\mathbb{N}$ is represented by ω copies of ω. The first uncountable ordinal is denoted by ω_1. Also, ω_1, $\omega_1 + n$, $\omega_1 + \omega$, $\omega_1 + \omega_1$ are different as ordered sets but all of them are isomorphic as sets.

With this fact in mind, cardinals numbers are defined. Two sets have the same cardinality if there exists a bijection between them. The cardinality of a set A is denoted by $|A|$. Thus, for example, $|\omega| = |\omega + \omega|$.

Definition 1.1: An ordinal number κ is a **cardinal number** if $|\lambda| \neq |\kappa|$ for all $\lambda < \kappa$.

The class of all cardinals is denoted by $Card$. Every infinite cardinal is a limit ordinal. Converse is not true. For example $\omega + \omega$ is a limit ordinal but it is not a cardinal number. We use $\aleph_\alpha$ to denote the cardinal number and ω_α to denote its order-type. Thus, $\aleph_0 = \omega_0 = \omega$; $\aleph_{\alpha+1} = \omega_{\alpha+1} = \aleph_\alpha^+$; and $\aleph_\alpha = \omega_\alpha = \sup\{\omega_\beta : \beta < \alpha\}$, if α is a limit. In this case we say that $\aleph_\alpha$ is a limit cardinal. The rules for addition and multiplication of infinite cardinals are quite simple: $\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \max\{\aleph_\alpha, \aleph_\beta\}$.

Definition 1.2: Let $\alpha > 0$ a limit ordinal. The **cofinality** of α, $cf(\alpha)$ is defined as the smallest limit ordinal λ such that there exists an increasing family of ordinals indexed by λ, $\{\alpha_\varepsilon : \varepsilon < \lambda\}$ with $\sup\limits_{\varepsilon<\lambda}\{\alpha_\varepsilon\} = \alpha$.

For example, for any ordinal α, we have $cf(\omega_{\alpha+\omega}) = \omega$ because $\sup\limits_{n<\omega}\{\omega_{\alpha+n}\} = \omega_{\alpha+\omega}$. On the other hand, $cf(\omega_1) = \omega_1$ since there does not exist a countable family of ordinals $\{\alpha_n\}$ such that $\sup\limits_{n<\omega}\{\alpha_n\} = \omega_1$. A limit cardinal $\aleph_\kappa$ is **regular** if $cf(\omega_\kappa) = \omega_\kappa$ and it is **singular** if $cf(\omega_\kappa) < \omega_\kappa$. There are arbitrarily large singular cardinals. Using the axiom of choice, it can be proven that every $\aleph_{\kappa+1}$ is a regular cardinal.

2 The construction of the group Γ_α

Let I be a totally ordered set. For each index $i \in I$, let G_i be a totally ordered multiplicative group with unit element 1_{G_i}. The **direct product** $\prod\limits_{i\in I} G_i$ of the family $\{G_i\}_{i\in I}$ consists of all functions $f : I \to \bigcup\limits_{i\in I} G_i$ such that $f(i) \in G_i$ for all $i \in I$. With respect to the componentwise multiplication $\prod\limits_{i\in I} G_i$ is a group with unit element $\mathbf{1} = (1_{G_i} : i \in I)$. For every $f \in \prod\limits_{i\in I} G_i$, one defines the **support** of f as $supp(f) = \{i \in I : f(i) \neq 1_{G_i}\}$. In [4], the **Hahn product** is defined as the subgroup of the direct product consisting of all functions f such that $supp(f)$ is a well-ordered set and it is denoted by $\mathsf{H}_{i\in I} G_i$. An ordering is introduced by declaring $f < g$ if $f(k) < g(k)$ where k is the first element of I such that $f(k) \neq g(k)$. We will define groups Γ_α as special Hahn products.

Let α be an ordinal, $\{G_\beta\}_{\beta<\alpha}$ a family of abelian multiplicatively written totally ordered groups of rank 1. (This means that each G_β is a subgroup of $\langle(0,\infty), \cdot, \leq\rangle$).

The group Γ_α is a subset of the direct product of the family $\{G_\beta\}_{\beta<\alpha}$ defined by:

$$\Gamma_\alpha = \{f \in \prod_{\beta<\alpha} G_\beta : supp(f)\,\text{is finite}\}$$

with componentwise multiplication. We define the **degree** of f as $deg(f) = \max\{supp(f)\}$. The ordering on Γ_α is defined by $f > 1$ if and only if $f(degf) > 1$. We say that Γ_α is antilexicographically ordered. Furthermore, for every element $b \in G_\beta$, we define the element $\chi_{(b,\beta)} \in \Gamma_\alpha$ by $\chi_{(b,\beta)}(\beta) = b$ and $\chi_{(b,\beta)}(\gamma) = 1$ if $\gamma \neq \beta$.

For example, $\Gamma_\omega = \bigoplus_{n<\omega} G_n$ is used in [1] with a countable family $\{G_n\}$ of cyclic groups. Note that Γ_ω has not a "last copy" of G_n. On the other hand, the group $\Gamma_{\omega+1}$ is also a subgroup of the direct product of a countable family of groups, but in this case we do have a "last copy", the group G_ω.

Consider the set $\langle I, \prec\rangle$ where $I = \{\beta \in Ord : \beta < \alpha\}$ with the inverse ordering on Ord. That is to say $\beta \prec \gamma$ if and only if $\gamma < \beta$ for all $\beta, \gamma \in I$. It is clear that $\langle I, \prec\rangle$ is not well-ordered, but the well-ordered subsets of I are precisely the finite ones. Therefore the groups Γ_α are Hahn products over $\langle I, \prec\rangle$.

A convex subgroup H of a totally ordered group G is called **principal** if there is an element $g \in G$ such that H is the smallest convex subgroup of G containing g. By [4], every convex subgroup H which is not principal is equal to the union of all principal convex subgroups of G contained in H. In this case H is called a **limit** convex subgroup. For every $\beta < \alpha$ we define the sets:

$$H_\beta = \{f \in \Gamma_\alpha : deg(f) \leq \beta\} \quad \text{and} \quad H^*_\beta = \{f \in \Gamma_\alpha : deg(f) < \beta\}$$

For convenience, we put $H^*_0 = \{\mathbf{1}\}$. The next proposition justifies this notation.

Proposition 2.1: *H_β is a principal convex subgroup generated by any element f such that $deg(f) = \beta$. Each principal convex subgroup of Γ_α is equal to H_β for some $\beta < \alpha$.*

Proof:

It is clear that H_β is a convex subgroup. Now, let $f \in H_\beta$ such that $deg(f) = \beta$. This means $f(\beta) \neq 1_{G_\beta}$. Without loss of generality we may suppose that $f(\beta) > 1_{G_\beta}$. Let H be a convex subgroup that contains f and let $h \in H_\beta$, $h > \mathbf{1}$. We shall prove that $h \in H$. If $deg(h) < \beta$ then $\mathbf{1} < h < f$ hence $h \in H$. Suppose $deg(h) = \beta$. Because the order of G_β is archimedean, there exists $n \in \mathbb{N}$ such that $1_{G_\beta} < h(\beta) < f(\beta)^n$ which implies $\mathbf{1} < h < f^n$, hence $h \in H$. Therefore, H_β is a principal convex subgroup because it is the smallest convex subgroup that contains f.

Now let H be a principal convex subgroup generated by some element $g \in H$. Then $g \in H_{deg(g)}$ which means that $H = H_{deg(g)}$. $\square$

Corollary 2.2: *If H is a proper convex subgroup of Γ_α, then there exists $\beta < \alpha$ such that $H = H_\beta$ or $H = H^*_\beta$.*

Proof:

It is immediate because if H is not a principal convex subgroup, from the above proposition $H \neq H_\beta$ for all $\beta < \alpha$, and by [4], H is the union of principal convex subgroups, that is to say $H = \bigcup_{\gamma<\beta} H_\gamma$, for some $\beta < \alpha$. This β exists of course, $\beta = \min\{\gamma : \forall f \in H\,(deg(f) < \gamma)\}$. Therefore, $h \in H$ if and only if $deg(h) = \gamma$ for some $\gamma < \beta$, if and only if $h \in H_\gamma \subseteq H^*_\beta$. $\square$

Remark 2.3: If β is an infinite limit ordinal, then H^*_β is not a principal convex subgroup. Indeed, if H^*_β is generated by f, because $deg(f) = \gamma$ for some $\gamma < \beta$, then $f \in H_\gamma$ from which $H^*_\beta \subseteq H_\gamma$, a contradiction. On the other hand, if $\beta = \gamma + 1$ then $H^*_\beta = H_\gamma$.

Hence, the order-type of the set of all principal convex subgroups of Γ_α, ordered by inclusion, is α.

Proposition 2.4: *Let α be an infinite limit ordinal. Then the following are equivalent:*
i) There exists an increasing sequence of principal convex subgroups $\{H_{\beta_n} : n \in \omega\}$ such that $\Gamma_\alpha = \bigcup_{n<\omega} H_{\beta_n}$.
ii) $cf(\alpha) = \omega$.
iii) Γ_α has a cofinal sequence.

Proof:

($i) \Rightarrow ii)$) If there exists such a sequence then $\sup\{\beta_n : n < \omega\} = \alpha$, hence $cf(\alpha) = \omega$.
($2 \Rightarrow 3$) If $cf(\alpha) = \omega$, there exists an increasing sequence $\{\beta_n : n < \omega\}$ such that $\sup\{\beta_n : n < \omega\} = \alpha$. Because each group G_β is a multiplicative subgroup of $\mathbb{R}^+$, each of them has a cofinal sequence. Let a_{β_n} be the n-th element in a cofinal sequence of G_{β_n}, for all $n \in \mathbb{N}$. Then the sequence $\{\chi_{(a_{\beta_n},\beta_n)} : n < \omega\}$ is cofinal in Γ_α.
($3 \Rightarrow 1$) Finally, if Γ_α has a cofinal sequence $\{f_n : n < \omega\}$, then $\Gamma_\alpha = \bigcup_{n<\omega} H_{deg(f_n)}$. $\square$

Definition 2.5: [2] A totally ordered group G is **quasidiscrete** if $\min\{g \in G : g > 1\}$ exists in G. A group that is not quasidiscrete is called **quasidense**.

In infinite rank it is possible to have a quasidiscrete group which has quasidense subgroups. (See [2], example at the end of 1.2).

Proposition 2.6: *Γ_α/H^*_β is quasidiscrete if and only if G_β is quasidiscrete. In particular, Γ_α is quasidiscrete if and only if G_0 is quasidiscrete.*

Proof:

Suppose Γ_α/H^*_β is quasidiscrete and let $\pi : \Gamma_\alpha \to \Gamma_\alpha/H^*_\beta$ the canonical projection. Remember that for each $f, g \in \Gamma_\alpha$ the ordering in Γ_α/H^*_β is defined by $\pi(f) < \pi(g)$ if and only if $fg^{-1} \notin H^*_\beta$ and $f(deg(fg^{-1})) < g(deg(fg^{-1}))$. Let $f_0 \in \Gamma_\alpha$ such that $\pi(f_0) = \min\{\pi(f) \in \Gamma_\alpha/H^*_\beta : \pi(f) > \pi(\mathbf{1})\}$. We claim that $deg(f_0) = \beta$ and $f_0(\beta) = \min\{g \in G_\beta : g > 1_{G_\beta}\}$. In fact, notice that $f_0 \notin H^*_\beta$ hence $deg(f_0) \geq \beta$ and because it is the minimum, $\deg(f_0) = \beta$. Furthermore, if there exists $a \in G_\beta$ such that $1_{G_\beta} < a < f_0(\beta)$ in G_β, then $\chi_{(a,\beta)} \notin H^*_\beta$ and we have $\pi(1) < \pi(\chi_{(a,\beta)}) < \pi(f_0)$, a contradiction. Therefore, G_β is quasidiscrete.

Conversely, if G_β is quasidiscrete, let $a = \min\{b \in G_\beta : b > 1\}$. It follows immediately that the element $\chi_{(a,\beta)}$ satisfies $\pi(\chi_{(a,\beta)}) = \min\{\pi(f) \in \Gamma_\alpha/H^*_\beta : \pi(f) > \pi(1)\}$, hence Γ_α/H^*_β is quasidiscrete. $\square$

3 Completions of linearly ordered sets

Completions of totally ordered groups are important in order to obtain supremum and infimum of subsets of them. In this section we define the Dedekind completion of arbitrary linearly ordered sets. We study the extension of mappings between two such sets (or groups) to their completions.

A linearly ordered set X is called **Dedekind complete** if each nonempty subset of X that is bounded above has a supremum. A subset A of a linearly ordered set X is called **dense** (in X) if for each $s \in X$:

$$\sup_X\{a \in A : a \leq s\} = \inf_X\{a \in A : a \geq s\} = s$$

Let Z be any set. A function $f : Z \to X$ is called dense if $f(Z)$ is dense in X. A **Dedekind completion** of a linearly ordered set X is a pair $(X^{\#}, i)$

where $X^{\#}$ is a complete linearly ordered set and $i : X \to X^{\#}$ is a strictly increasing and dense mapping. This completion satisfies the following universal property:

Proposition 3.1: *Let $(X^{\#}, i)$ be a Dedekind completion of a linearly ordered set X. Then for every linearly ordered Dedekind complete set Y and every strictly increasing and dense mapping $\varphi : X \to Y$, there is only one strictly increasing function ψ such that $\varphi = \psi \circ i$.*

Proof: Let Y be a linearly ordered complete set and φ a strictly increasing and dense mapping from X to Y. We define $\psi(s) = \sup_Y\{\varphi(x) : x \in X \wedge i(x) \leq s\}$ for each $s \in X^{\#}$. Then for all $x \in X$ we have $\varphi(x) = (\psi \circ i)(x)$. Furthermore, if $s < t$ in $X^{\#}$, by density there exist $a, b \in X$ such that $s \leq i(a) < i(b) \leq t$ and then $\psi(s) \leq \varphi(a) < \varphi(b) \leq \psi(t)$, hence ψ is strictly increasing.

For uniqueness, let $\delta : X^{\#} \to Y$ another strictly increasing mapping such that $\delta \circ i = \varphi$. Then, there is an $s \in X^{\#}$ with $\delta(s) \neq \psi(s)$. By the definition of ψ we have $\psi(s) < \delta(s)$ and by density of φ there is an $a \in X$ such that $\psi(s) \leq \varphi(a) < \delta(s)$ or $\psi(s) < \varphi(a) \leq \delta(s)$. In the first case if $s < i(a)$ then $\delta(s) \leq \delta(i(a)) = \varphi(a)$ a contradiction, therefore $i(a) < s$ from which it follows that $\varphi(a) = \psi(s)$. But $s \notin X$, then there is a $b \in X$ with $i(a) < i(b) < s$ and $\varphi(a) = \varphi(b)$, but that is not possible by injectivity of φ. A similar argument, taking into consideration that $s = \inf_{X^{\#}}\{x \in X : i(x) \geq s\}$ works in the case $\psi(s) < \varphi(a) \leq \delta(s)$.

Therefore, ψ is unique. In particular, for each $s \in X^{\#}$ we have:

$$\sup_Y\{\varphi(x) : x \in X \ \wedge \ i(x) \leq s\} = \inf_Y\{\varphi(x) : x \in X \ \wedge \ i(x) \geq s\}$$

$\square$

The above proposition proves that all completion are order isomorphic and therefore the canonical completion by Dedekind cuts with the natural embedding i is a Dedekind completion. As usual, we shall identify i with the inclusion and we shall write x instead of $i(x)$ for all $x \in X$. From now on we say complete instead Dedekind complete.

Proposition 3.2: *Let $Y \subseteq X$ and $Y^{\#}$, $X^{\#}$ their completions. Then $Y^{\#}$ can be embedded in $X^{\#}$ through a strictly increasing mapping $\tau : Y^{\#} \to X^{\#}$ such that $\tau(y) = y$ for all $y \in Y$.*

Proof: Let $\tau(s) = \sup_{X^{\#}}\{y \in Y : y \leq s\}$ for all $s \in Y^{\#}$. Then $\tau(y) = y$ for all $y \in Y$ and $s < t$ in $Y^{\#}$ implies there are y_1, $y_2 \in Y$ such that

$s \le y_1 < y_2 \le t$. Therefore $\tau(s) \le y_1$ and $\tau(t) \ge y_2$, hence $\tau(s) < \tau(t)$ that is to say τ is strictly increasing. $\square$

Remark 3.3: Actually, the mapping τ of the above proposition is not necessarily unique. For instance, let $X = \mathbb{Q}$, $Y = \mathbb{Q} \cap ([0,1) \cup (2,3])$. Then, $Y^{\#} = [0,1) \cup \{a\} \cup (2,3]$ where $a \in [1,2]$ is arbitrary. Hence, there are infinite mappings which extend $Id\mathbb{Q}$. Uniqueness is obtained if Y is dense in the convex hull of Y in X.

Definition 3.4: [1] Let X be a totally ordered set and let $Y \subseteq X$. The X**-convex hull** of Y is defined by:

$$conv_X Y = \{x \in X : \exists\, y_1,\, y_2 \in Y (y_1 \le x \le y_2)\}$$

Proposition 3.5: *If $Y \subseteq X$ is dense in $conv_X Y$, then the mapping τ of the above proposition is unique.*

PROOF: Let $\delta : Y^{\#} \to X^{\#}$ strictly increasing such that $\delta(y) = y$ for all $y \in Y$. Suppose that $\delta \ne \tau$. Then $\tau(s) < \delta(s)$ for some $s \in Y^{\#} \setminus Y$. This means that there exist $y_1, y_2 \in Y$ such that $y_1 < s < y_2$. Hence, $y_1 < \tau(s) < \delta(s) < y_2$. Therefore there is an element $y \in Y$ such that $\tau(s) \le y < \delta(s)$. But $s < y$ implies $\delta(s) \le y$ and $y < s$ implies $y = \tau(s)$. Hence τ is unique. $\square$

Corollary 3.6: *Let H be a convex subgroup of a totally ordered group G. Then there is only one strictly increasing mapping $\tau : H^{\#} \to G^{\#}$ such that $\tau = Id_H$ and $H^{\#} = conv_{G^{\#}} H$*

Proposition 3.7: *Let G_1, G_2 be totally ordered groups and let $\tau : G_1 \to G_2$ a surjective homomorphism of totally ordered groups. Then there exists an increasing mapping $\tau^{\#} : G_1^{\#} \to G_2^{\#}$ that extends τ.*

PROOF: Again, let $\tau^{\#}$ be defined by $\tau^{\#}(s) = \sup_{G_2^{\#}}\{\tau(g) : g \in G_1 \wedge g \le s\}$. $\square$

Remark 3.8: Since $Ker\,\tau$ is a convex subgroup of G_1 we can restate the proposition 3.7 in the following way:

Let H be a convex subgroup of a totally ordered group G and $\pi : G \to G/H$ the canonical projection. Then there exists an increasing mapping $\pi^{\#} : G^{\#} \to (G/H)^{\#}$ which extends the projection π.

We have uniqueness of $\pi^{\#}$ in the case that G/H is quasidense.

Proposition 3.9: *Let G, H, π as before. If G/H is quasidense then $\pi^{\#}$ is unique. If G/H is quasidiscrete then there are exactly two extensions of π.*

Proof: By proposition 3.7, the mapping $\pi^{\#} : G^{\#} \to (G/H)^{\#}$ is increasing and extends the projection π. Suppose that exists another increasing extension δ of π. Then there is an $s \in G^{\#} \setminus G$ such that $\pi^{\#}(s) < \delta(s)$. By density, there are g_1, $g_2 \in G$ such that $\pi^{\#}(s) \leq \pi(g_1) < \pi(g_2) \leq \delta(s)$. Thus $g_1 < s < g_2$ from which it follows that $\pi(1) < \pi(g_1^{-1}g_2)$ hence $\pi^{\#}(s) = \pi(g_1)$ and $\delta(s) = \pi(g_2)$.

If there exists $a \in G$ with $\pi(1) < \pi(a) < \pi(g_1^{-1}g_2)$ then $\pi(g_1) < \pi(ag_1) < \pi(g_2)$. But $g_1 < ag_1 < s$ implies $a \in H$ and $s < ag_1 < g_2$ implies $\pi(ag_1) = \pi(g_2)$ a contradiction. Therefore if G/H is quasidense, $\pi^{\#}$ is unique. In particular $\sup_{(G/H)^{\#}}\{\pi(g) : g \leq s\} = \inf_{(G/H)^{\#}}\{\pi(g) : g \geq s\}$.

If there does not exist $a \in G$ with $\pi(1) < \pi(a) < \pi(g_1^{-1}g_2)$ then G/H is quasidiscrete and $\pi^{\#}(s) = \sup_{(G/H)^{\#}}\{\pi(g) : g \leq s\}$ and $\tau(s) = \inf_{(G/H)^{\#}}\{\pi(g) : g \geq s\}$ may be different (for instance take $s = \sup H$), but both of them are increasing mappings which extend π. We claim that they are the only ones. In fact if there were another extension δ then, as in the first paragraph, we would have $s < g_2$ which implies $\tau(s) < \delta(s)$, a contradiction. □

4 Cardinality conditions

Spaces of countable type play a fundamental role in the theory of Hilbert-like spaces. Let K be a field with a non-archimedean valuation. If it has the property $(*)$: *Every absolutely convex subset of K is countably generated as a B_K-module*, then all subspaces of spaces of countable type are also of countable type. It is not yet known if this is true when the field K does not satisfy $(*)$. By [1] Proposition 1.4.4, $(*)$ is equivalent to the following: the value group G has a cofinal sequence and every element of $G^{\#}$ is the supremum of a sequence of elements of G.

In this section we show that these conditions are true for the groups Γ_α if and only if $cf(\alpha) = \omega$ and $\alpha < \omega_1$. We give a characterization of those Γ_α in which one and only one of these two conditions are valid, thereby proving that they are logically independent. We describe also those Γ_α in which neither of them hold.

It is well known that for every totally ordered group Γ there exists a valued field K such that Γ is its value group. For our purposes we shall use the construction indicated in [5] of such a K.

Let F be an arbitrary field and let $R = \{f : \Gamma \to F : supp(f) \text{ is finite}\}$. With the operations $+$ and $\cdot$ defined by:

$$(f+g)(\gamma) = f(\gamma) + g(\gamma) \qquad (fg)(\gamma) = \sum_{\gamma_1 \cdot \gamma_2 = \gamma} f(\gamma_1) \cdot g(\gamma_2)$$

R becomes a domain. Let $v : R \to \Gamma$ be the mapping defined by $v(f) = \max\{\gamma \in \Gamma : f(\gamma) \neq 0\}$ and let K be the field of fractions of R. We extend v to K by letting $v(f^{-1}) = v(f)^{-1}$. It is easy to see that v is a Krull valuation of K with $v(K) = \Gamma$.

Theorem 4.1: *For any fixed cardinal $\aleph_\kappa$, there exists a group Γ_α, of cardinality $\aleph_\kappa$ which has a cofinal sequence.*

Proof: Consider the ordinal $\alpha = \omega_\kappa + \omega$. Then $|\alpha| = \aleph_\kappa$ and $cf(\alpha) = \omega$. Therefore, the group Γ_α has cardinality $\aleph_\kappa$ and, by Proposition 2.4, it has a cofinal sequence. $\square$

Corollary 4.2: *For any fixed infinite cardinal $\aleph_\kappa$ there exists a Krull valued field of cardinality $\aleph_\kappa$ which is metrizable.*

Proof: It is enough to consider the group Γ_α of the above proposition which has a cofinal sequence (hence it has a coinitial sequence). Then we use the construction of [5] in order to obtain a field K with value group Γ_α and, by [1] Theorem 1.4.1, the metrizability of this field is guaranteed. $\square$

Theorem 4.3: *For any fixed uncountable cardinal $\aleph_\kappa$, there exists a group Γ_α with $|\alpha| = \aleph_\kappa$ and an element $s \in \Gamma_\alpha^\#$ which is not the supremum of some countable subset of Γ_α.*

Proof: Since $\aleph_\kappa$ is uncountable, we choose α such that $\omega_1 < \alpha$ and $|\alpha| = \aleph_\kappa$. Let $s = \sup H^*_{\omega_1}$. Because $s \notin H^*_{\omega_1}$, $\sup_{\Gamma_\alpha^\#}\{t \in \Gamma_\alpha^\# : t < s\} = s$.

However, if there exists a sequence $\{g_i\}_{i<\omega} \subseteq \Gamma_\alpha$ such that $\sup_{\Gamma_\alpha^\#}\{g_i : i < \omega\} = s$, then the sequence $\{deg(g_i) : i < \omega\}$ would be cofinal in ω_1, a contradiction. $\square$

Corollary 4.4: *For any fixed uncountable cardinal $\aleph_\kappa$ there exists a field K with a Krull valuation which is metrizable and contains absolutely convex subsets which are not countably generated as B_K-modules.*

Proof: Again, by [5], we consider the field K with a Krull valuation $|\ |$ and value group Γ_α where $\omega_1 < \alpha$, $cf(\alpha) = \omega$ and $|\alpha| = \aleph_\kappa$ as the above proposition.

We claim that the set $B(0,s)^-$, where $s = \sup H^*_{\omega_1}$ is not countably generated as a B_K-module. In fact, let $B(0,s)^-$ be generated as a B_K-module by $k_1, k_2, \ldots \in K$. By [1] Proposition 1.4.4, $s = \sup\{|k_i| : i \in \mathbb{N}\}$, a contradiction. $\square$

Let α be an ordinal and consider the group Γ_α. The condition 'Γ_α has a cofinal sequence' will be denoted M and the condition 'every element of $\Gamma_\alpha^\#$ is the supremum of a sequence of elements of Γ_α' will be called S. The above results can be restated as follows:

i) If $\alpha < \omega_1$ then Γ_α satisfies M and S.

ii) If $\alpha = \omega_1$ then Γ_α satisfies S but does not satisfy M.

iii) If $\alpha > \omega_1$ and α is a succesor or $cf(\alpha) = \omega$ then Γ_α satisfies M but does not satisfy S.

iv) If $\alpha > \omega_1$ and $cf(\alpha) \geq \omega_1$ then Γ_α does not satisfy neither M nor S.

Now, let I be an arbitrary linearly ordered set and let $\{G_i\}_{i\in I}$ be a family of totally ordered multiplicative groups of rank 1. Let $\Gamma_I = \{f \in \prod_{i\in I} G_i : supp(f)\,\text{is finite}\}$ with componentwise multiplication and antilexicographically ordered. We prove that Γ_I has a cofinal sequence if and only if I has a cofinal sequence or I has a last element.

With respect to the second one -every $s \in \Gamma_I^\#$ is the supremum of a sequence of elements of Γ_I- we show that it is equivalent to a condition on the index set I, that $\sup_{I^\#}\{deg(f) : f \leq s\}$ should be equal to the supremum of a sequence in I. A straightforward proof shows:

Corollary 4.5: *The group Γ_I has a cofinal sequence if and only if I has a last element or has a cofinal sequence.*

Theorem 4.6: *For every $s \in \Gamma_I^\#$ there exists a sequence $\{g_n\}_{n<\omega} \subseteq \Gamma_I$ such that $s = \sup_{\Gamma_I^\#}\{g_n\}_{n<\omega}$ if and only if for all $k \in I^\#$ there exists a sequence $\{i_n\}_{n<\omega}$ such that $k = \sup_{I^\#}\{i_n\}_{n<\omega}$.*

Proof: $(\rightarrow)$ We only need to prove that if $k \in I^\# \setminus I$ then there exists a sequence $\{i_n\}_{n<\omega}$ such that $k = \sup_{I^\#}\{i_n\}_{n<\omega}$. For each $i \in I$, $i < k$, let $g_i \in G_i$, $g_i > 1$ arbitrary. Let $s = \sup_{\Gamma_{I^\#}}\{\chi_{(g_i,i)} : i < k\}$. By hypothesis, there exists a sequence $\{f_n\} \subseteq \Gamma_I$ such that $s = \sup_{\Gamma_{I^\#}}\{f_n : n < \omega\}$.

Then for each n, there exist i, $j < k$ such that $i < deg(f_n) < j$. Therefore $k = \sup_{I\#}\{deg(f_n) : n < \omega\}$.

($\leftarrow$) Let $s \in \Gamma_I^{\#} \setminus \Gamma_I$. Let $k = \sup_{I\#}\{deg(f) : f \in \Gamma_I \wedge f < s\}$. Then there is a sequence $\{i_n\} \subseteq I$ such that $k = \sup_{I\#}\{i_n\}$. It is immediate that $s = \sup_{\Gamma_{I\#}}\{\chi_{(g_{i_n}, i_n} : n < \omega\}$. $\square$

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