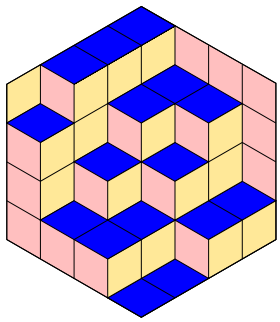

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Dual filtered graphs

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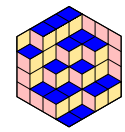

Dual filtered graphs

Rebecca Patrias & Pavlo Pylyavskyy

Abstract We define a K-theoretic analogue of Fomin's dual graded graphs, which we call dual filtered graphs. The key formula in the definition is $DU - UD = D + I$. Our major examples are K-theoretic analogues of Young's lattice, of shifted Young's lattice, and of the Young–Fibonacci lattice. We suggest notions of tableaux, insertion algorithms, and growth rules whenever such objects are not already present in the literature. (See the table below.) We also provide a large number of other examples. Most of our examples arise via two constructions, which we call the Pieri construction and the Möbius construction. The Pieri construction is closely related to the construction of dual graded graphs from a graded Hopf algebra, as described in [1, 19, 16]. The Möbius construction is more mysterious but also potentially more important, as it corresponds to natural insertion algorithms.

	Tableaux	Insertion	Growth
Young	Standard Young tableaux [33]	RSK insertion [13, 23, 25]	[9]
Shifted Young	Standard shifted Young tableaux with and without circles [24]	Shifted Robinson–Schensted insertion [24, 32]	[11]
Young–Fibonacci	Young–Fibonacci tableaux [9, 10]	Young–Fibonacci insertion [9]	[11]
Möbius Young	Increasing and set-valued tableaux [5, 29]	Hecke insertion [6]	Section 4.4
Möbius shifted Young	Definitions 5.9 and 5.16	Section 5.3	Section 5.4
Möbius Young–Fibonacci	Definitions 6.3 and 6.9	Section 6.2	Section 6.3

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Keywords. dual graded graphs, insertion algorithms, K-theory, symmetric functions.

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1. Introduction

Fomin's *dual graded graphs* [10], as well as their predecessors - Stanley's *differential posets* [26], were invented as a tool to better understand the Robinson–Schensted insertion algorithm. Dual graded graphs are significant in the areas where the Robinson–Schensted correspondence appears, for example in Schubert calculus or in the study of representations of towers of algebras [1, 4]. A recent work of Lam and Shimozono [17] associates a dual graded graph to any Kac–Moody algebra, bringing their study to a new level of generality.

1.1. Weyl algebra and its deformations.

One way to view the theory is as a study of certain *combinatorial representations* of the *first Weyl algebra*. Let us briefly recall the definitions. The *Weyl algebra*, or the first Weyl algebra, is an algebra over some field K (usually $K = \mathbb{R}$) generated by two generators U and D with a single relation $DU - UD = 1$. It was originally introduced by Hermann Weyl in his study of quantum mechanics. We refer the reader to [2], for example, for more background on the Weyl algebra.

A *graded graph* is a triple $G = (P, \rho, E)$, where P is a discrete set of vertices, $\rho : P \to \mathbb{Z}$ is a rank function, and E is a multiset of edges/arcs (x, y), where $\rho(y) = \rho(x) + 1$. In other words, each vertex is assigned a rank, and edges can only join vertices in successive ranks. For the set of vertices P, let P_n denote the subset of vertices of rank n. For any field K of characteristic zero, the formal linear combinations of vertices of G form the vector space KP.

Let $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ be a pair of graded graphs with a common vertex set and rank function. From this pair, define an *oriented graded graph* $G = (P, \rho, E_1, E_2)$ by orienting the edges of G_1 in the direction of increasing rank and the edges of G_2 in the direction of decreasing rank. Let $a_i(x, y)$ denote the number of edges of E_i joining x and y or the multiplicity of edge (x, y) in E_i. We define the *up* and *down operators* $U, D \in End(KP)$ associated with graph G by

$$Ux = \sum_y a_1(x, y)y$$

and

$$Dy = \sum_x a_2(x, y)x.$$

Graded graphs G_1 and G_2 with a common vertex set and rank function are said to be *dual* if

$$DU - UD = I,$$

where I is an identity operator acting on KP. Thus, we see that KP is a representation of the Weyl algebra. Furthermore, it is a representation of a very special kind, where U and D act in a particularly nice combinatorial way on a fixed basis.

One would then expect that variations of the Weyl algebra would correspond to some variations of the theory of dual graded graphs. One such variation is the *q-Weyl algebra* defined by the relation

$$DU - qUD = 1.$$

The corresponding theory of *quantum dual graded graphs* was pursued by Lam in [14].

In this paper, we shall study the theory arising from an analogue W of the Weyl algebra with defining relation

$$DU - UD = D + 1.$$

We shall refer to it as the Ore algebra [20], and we shall see that it is a very natural object. In particular, the corresponding theory of *dual filtered graphs* fits naturally with the existing body of work on the K-theory of Grassmannians.

REMARK 1.1. It is easy to see that by rescaling U and D, we can pass from arbitrary $DU - UD = \alpha D + \beta$ to $DU - UD = D + I$. We thus suffer no loss in generality by writing down the relation in the latter form.

1.2. DIFFERENTIAL VS DIFFERENCE OPERATORS. The following provides some intuitive explanation for the relation between representations of Weyl and Ore algebras. The simplest representation of the Weyl algebra is that acting on a polynomial ring. Indeed, consider polynomial ring $R = \mathbb{R}[x]$, and for $f \in R$ let

$$U(f) = xf, \quad D(f) = \frac{\partial f}{\partial x}.$$

It is an easy exercise to verify that this indeed produces a representation of the Weyl algebra. In fact, this is how it is often defined.

Let us now redefine the down operator to be the *difference operator*:

$$U(f) = xf, \quad D(f) = f(x+1) - f(x).$$

LEMMA 1.2. *This gives a representation of the Ore algebra W. In other words, those operators satisfy*

$$DU - UD = D + I,$$

where I is the identity map on R.

Proof. We have

$$\begin{aligned}(DU - UD)(f) &= ((x+1)f(x+1) - xf(x)) - x(f(x+1) - f(x)) \\ &= f(x+1) = f(x) + (f(x+1) - f(x)) = (I + D)(f). \end{aligned}$$ □

It can also be noted that differential and difference operators can be related as follows:

$$D = e^{\frac{\partial}{\partial x}} - 1.$$

We omit the easy proof for brevity. As we shall see in Example 7.10, this representation of W corresponds to a very basic dual filtered graph.

1.3. PIERI AND MÖBIUS CONSTRUCTIONS. We make use of two conceptual ways to build examples of dual filtered graphs. The first construction starts with an algebra A, a derivation d on A, and an element $f \in A$ such that $d(f) = f + 1$. We often build the desired derivation d using a *bialgebra* structure on A. This construction is very close to an existing one in the literature [1, 16, 19], where dual graded graphs are constructed from graded Hopf algebras. In fact, if we were to require $d(f) = 1$, we would get dual graded graphs instead of dual filtered graphs in our construction. We refer to this method as the *Pieri construction* or sometimes as the *Pieri deformation*.

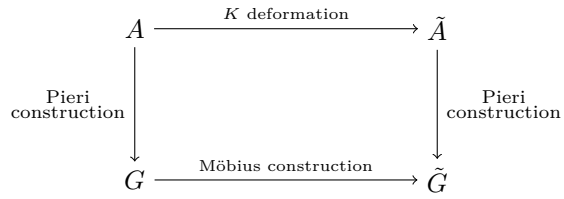
Instead, we can also start with an existing dual graded graph $G = (P, \rho, E_1, E_2)$ (see definition below), composed of graphs $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$, and adjust E_1 and E_2 in the following manner. To obtain G_1', we add $\#\{x \mid y \text{ covers } x \text{ in } G_1\}$ loops at each vertex $y \in P$ to E_1. As for G_2', we create a new edge set E_2' by forming

$$a_2'(x, y) = |\mu(x, y)|$$

edges between vertices x and y, where μ denotes the Möbius function in $G_2 = (P, \rho, E_2)$. We refer to this construction as the *Möbius construction* or *Möbius deformation*. Note that this does not always produce a pair of dual filtered graphs, and it is mysterious to determine when it does. In some major examples, however, it is

the result of this construction that relates to Robinson–Schensted-like algorithms, for example to *Hecke insertion* of [6].

The following observation seems remarkable, and we call it the *Möbius via Pieri phenomenon*. Let A be a graded Hopf algebra, and let bialgebra $\tilde{A}$ be its *K-theoretic deformation* in some appropriate sense, which we do not know how to formalize. Let G be a natural dual graded graph associated with A. What we observe is that applying the Möbius construction to G yields *the same result* as a natural Pieri construction applied to $\tilde{A}$. In other words, the following diagram commutes.



This happens for Young's lattice, see Section 4.5, and for the binary tree dual graded graph, see Section 7.1. We expect this phenomenon to also occur for the shifted Young's lattice.

The crucial condition necessary for this phenomenon to be observed is for A to be the associated graded algebra of $\tilde{A}$, but this is not always the case. For example, this is not the case for K-theoretic analogues of the Poirier–Reutenauer and Malvenuto–Reutenauer Hopf algebras, as described in Sections 7.2, 7.3. To put it simply, the numbers of basis elements for the filtered components of A and $\tilde{A}$ are distinct in those cases, thus there is no hope to obtain corresponding graphs one from the other via Möbius construction. Interestingly, those are exactly the examples we found where the Möbius construction fails to produce a dual filtered graph.

1.4. Synopsis of the paper. In Section 2, we recall the definition of dual graded graphs from [10] as well as three major examples: Young's lattice, Young–Fibonacci lattice, and shifted Young's lattice. We then remind the reader of the definition of the Robinson–Schensted algorithm and how one can obtain it locally via the machinery of *growth diagrams*, also introduced by Fomin in [10, 11].

In Section 3, we formulate our definition of dual filtered graphs. We then introduce the trivial, Pieri, and Möbius constructions.

In Section 4, we build the Pieri and Möbius deformations of Young's lattice. We recall the definition of Hecke insertion and observe it is a map into a pair of paths in the Möbius deformation of Young's lattice. We provide growth rules that realize Hecke insertion. We also show how the Pieri construction applied to the ring generated by *Grothendieck polynomials* yields the same result as the Möbius construction applied to Young's lattice. Thus we demonstrate an instance of the Möbius via Pieri phenomenon.

In Section 5, we build Pieri and Möbius deformations of the shifted Young's lattice. We introduce *shifted Hecke insertion* and remark that its result always coincides with that of K-theoretic jeu de taquin rectification as defined in [8]. We also provide the corresponding growth rules.

In Section 6, we build Pieri and Möbius deformations of the Young–Fibonacci lattice. We define K-Young–Fibonacci tableaux and suggest the corresponding insertion algorithm and growth rules.

In Section 7, we consider some other examples of dual filtered graphs. Of special interest are the Pieri constructions associated to the quasisymmetric functions, to the

Poirier–Reutenauer and Malvenuto–Reutenauer Hopf algebras, as well as to their K-theoretic analogues, which we draw from [15, 21]. The Hopf algebra of quasisymmetric functions and its K-theoretic analogue provide another instance of the Möbius via Pieri phenomenon.

In Section 8, we use the calculus of up and down operators to prove some identities similar to the ones known for dual graded graphs. In particular, we formulate and prove a K-theoretic analogue of the Frobenius–Young identity.

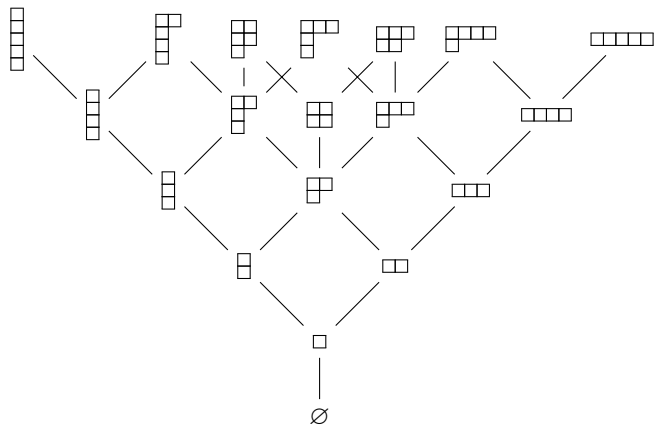
2. Dual graded graphs

2.1. Dual graded graphs. This section follows [10], and we refer the reader to this source for further reading on dual graded graphs.

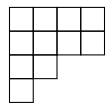
A graded graph is a triple $G = (P, \rho, E)$, where P is a discrete set of vertices, $\rho : P \to \mathbb{Z}$ is a rank function, and E is a multiset of edges/arcs (x, y), where $\rho(y) = \rho(x)+1$. In other words, each vertex is assigned a rank, and edges can only join vertices in successive ranks. For the set of vertices P, let P_n denote the subset of vertices of rank n. For any field K of characteristic zero, the formal linear combinations of vertices of G form the vector space KP.

In future examples, the idea of *inner corners* and *outer corners* of certain configurations of boxes or cells will be necessary. We will call a cell an inner corner if it can be removed from the configuration and the resulting configuration is still valid. A cell is an outer corner of a configuration if it can be added to the configuration, and the resulting configuration is still valid.

Example 2.1. Young's lattice is an example of a graded graph where for any partition λ, $\rho(\lambda) = |\lambda|$, and there is an edge from λ to μ if μ can be obtained from λ by adding one box. Ranks zero through five are shown below. We see that $(3, 31) \in E$ and $\rho(3) + 1 = 3 + 1 = \rho(31) = 4$.



If we look at the partition (4,4,2,1), visually represented below, we see that the inner corners are in positions (2,4), (3,2), and (4,1), and the outer corners are in positions (1,5), (3,3), (4,2), and (5,1).



Let $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ be a pair of graded graphs with a common vertex set and rank function. From this pair, define an *oriented graded graph* $G =$

(P, ρ, E_1, E_2) by orienting the edges of G_1 in the direction of increasing rank and the edges of G_2 in the direction of decreasing rank. Let $a_i(x, y)$ denote the number of edges of E_i joining x and y or the multiplicity of edge (x, y) in E_i. We define the *up* and *down operators* $U, D \in End(KP)$ associated with graph G by

$$Ux = \sum_y a_1(x, y)y$$

and

$$Dy = \sum_x a_2(x, y)x.$$

For example, in Young's lattice shown above, $U(21) = 31{+}22{+}211$ and $D(21) = 2{+}11$.

The restrictions of U and D to "homogeneous" subspaces KP_n are denoted by U_n and D_n, respectively. Graded graphs G_1 and G_2 with a common vertex set and rank function are said to be *r-dual* if

$$D_{n+1}U_n = U_{n-1}D_n + rI_n$$

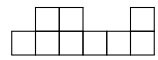
for some $r \in \mathbb{R}$ and simply *dual* if

$$D_{n+1}U_n = U_{n-1}D_n + I_n.$$

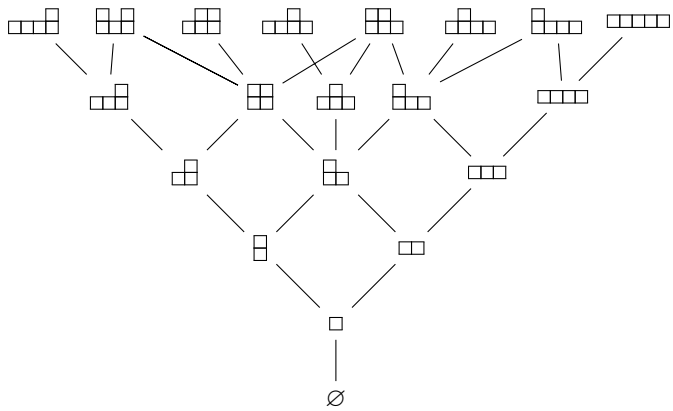
We focus on the latter.

Example 2.2. Young's lattice is an example of a self-dual graded graph.

Example 2.3. Another well-known example of a self-dual graded graph is the *Young–Fibonacci lattice*, $\mathbb{YF}$. The first six ranks are shown below. The vertices of the graph are all finite words in the alphabet $\{1, 2\}$, where the rank of a word is the sum of its entries. The covering relations are as follows: a word w' covers w if and only if either $w' = 1w$ or $w' = 2v$ for some v covered by w. For example, the word 121 covers the word 21 since 121 is obtained by concatenating 1 and 21, and 121 is covered by 221 because 121 covers 21. The words in the alphabet $\{1, 2\}$ are sometimes called snakes and can be represented as a collection of boxes whose heights correspond to the entries of the word called snakeshapes. For example, the word 122112 can be pictured as

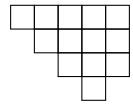


Thought of this way, the rank of a word is the number of boxes in its corresponding snakeshape.



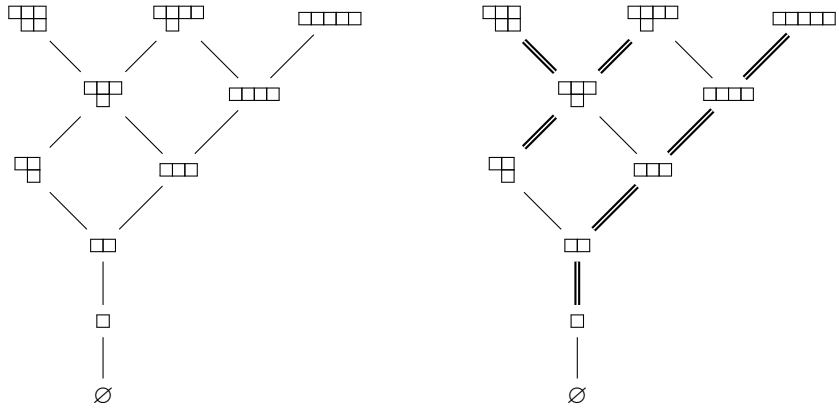
Young's lattice and the Young–Fibonacci lattice discussed above are the most interesting examples of self-dual graded graphs. We close this section by describing the graph of shifted shapes, $\mathbb{SY}$, and its dual, which together form a dual graded graph.

Example 2.4. Given a strict partition λ (i.e. $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_k)$), the corresponding shifted shape λ is an arrangement of cells in k rows where row i contains λ_i cells and is indented $i-1$ spaces. For example, the strict partition $(5,4,3,1)$ corresponds to the shifted shape shown below.



We say that cells in row i and column i are *diagonal* cells and all other cells are *off-diagonal*.

The graph of shifted shapes, $\mathbb{SY}$, has shifted shapes as vertices, the rank function counts the number of cells in a shape, and shifted shape λ covers μ if λ is obtained from μ by adding one cell. The first six ranks of the graph $\mathbb{SY}$ are shown below on the left. The graph shown on the right is its dual. In the dual graph, there is one edge between λ and μ if λ is obtained from μ by adding a diagonal cell, and there are two edges between λ and μ if λ is obtained from μ by adding an off-diagonal cell. To form the dual graded graph, the edges of $\mathbb{SY}$ are oriented upward and those of its dual are oriented downward.



2.2. Growth rules as generalized RSK. There is a well-known combinatorial correspondence between words in the alphabet of positive integers and pairs consisting of a semistandard Young tableau and a standard Young tableau of the same shape called the Robinson–Schensted–Knuth (RSK) correspondence. We briefly review the correspondence here and refer the reader to [27] for more information.

Given a word w, the RSK correspondence maps w to a pair of tableaux via a row insertion algorithm consisting of inserting a positive integer into a tableau. The algorithm for inserting positive integer k into a row of a semistandard tableau is as follows. If k is greater than or equal to all entries in the row, add a box labeled k to the end of the row. Otherwise, find the first y in the row with $y > k$. Replace y with k in this box, and proceed to insert y into the next row. To insert k into semistandard tableau P, we start by inserting k into the first row of P. To create the *insertion tableau* of a word $w = w_1 w_2 \dots w_r$, we first insert w_1 into the empty tableau, insert w_2 into the result of the previous insertion, insert w_3 into the result of the previous insertion, and so on until we have inserted all letters of w. We denote the resulting insertion tableau by $P(w)$. The insertion tableau will always be a semistandard tableau.

To obtain a standard Young tableau from w, we define the *recording tableau*, $Q(w)$, of w by labeling the box of $P(w_1 \dots w_s)/P(w_1 \dots w_{s-1})$ by s. For example, $w = 14252$

has insertion and recording tableau shown below.

$$P(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 2 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \qquad\qquad Q(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$$

The RSK correspondence described above is a bijection between words consisting of positive integers and pairs (P, Q), where P is a semistandard Young tableau and Q is a standard Young tableau of the same shape.

If $w = w_1 w_2 \dots w_k$ is a permutation, we can also obtain $(P(w), Q(w))$ from w by using growth diagrams. First, we create a $k \times k$ array with an X in the w_i^{th} square from the bottom of column i. For example, if $w = 14253$, we have

$$\begin{array}{|c|c|c|c|c|}\hline & & & X & \\ \hline & X & & & \\ \hline & & & & X \\ \hline & & X & & \\ \hline X & & & & \\ \hline\end{array}$$

We will label the corners of each square with a partition. We begin by labeling all corners along the bottom row and left side of the diagram with the empty shape, $\varnothing$.

To complete the labeling of the corners, suppose the corners μ, λ, and ν are labeled, where μ, λ, and ν are as in the picture below. We label γ according to the following rules.

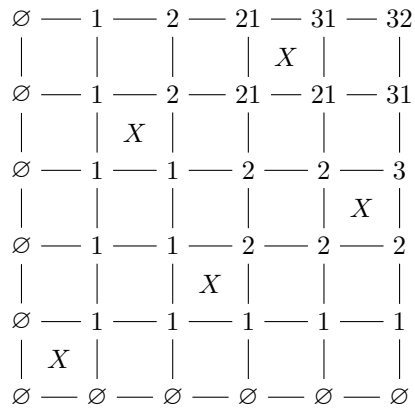
$$\begin{array}{ccc} \mu & \longrightarrow & \gamma \\ \uparrow & & \uparrow \\ \lambda & \longrightarrow & \nu \end{array}$$

(L1) If the square does not contain an X and $\lambda = \mu = \nu$, then $\gamma = \lambda$.
(L2) If the square does not contain an X and $\lambda \subset \mu = \nu$, then μ/λ is one box in some row i. In this case, γ is obtained from μ by adding one box in row $i+1$.
(L3) If the square does not contain an X and $\mu \neq \nu$, define $\gamma = \mu \cup \nu$.
(L4) If the square contains an X, then $\lambda = \nu = \mu$ and γ is obtained from μ by adding one box in the first row.

Following these rules, there is a unique way to label the corners of the diagram. The resulting array is called the *growth diagram* of w, and rules (L1)–(L4) are called *growth rules*. For the remainder of this paper, we will use the word "square" when referring to a square in the growth diagram and the word "box" or "cell" when referring to a box in a partition, shifted partition, or snakeshape.

The chains of partitions read across the top of the diagram and up the right side of the diagram determine $Q(w)$ and $P(w)$, respectively. To obtain a tableau of shape λ from a chain of partitions $\varnothing \subset \lambda^1 \subset \lambda^2 \subset \cdots \subset \lambda^k = \lambda$, label the box of λ^i/λ^{i-1} by i.

Example 2.5. The growth diagram for the word 14253 is shown below.



From the chains of partitions on the rightmost edge and uppermost edge of the diagram, we can read the insertion tableau and recording tableau, respectively.

$$P(14253) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array} \qquad\qquad Q(14253) = \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}$$

We next draw a connection between the growth rules described and an explicit cancellation in Young's lattice that shows that $DU - UD = I$. We shall start by giving explicit cancellation rules to pair all down-up paths from μ to ν with up-down paths from μ to ν, which will leave one up-down path unpaired with $\mu = \nu$. (Note that if $\rho(\mu) \neq \rho(\nu)$, then there are no up-down or down-up paths from μ to ν.)

(C1) If $\mu \neq \nu$, there is only one down-up path from μ to ν, which passes through $\mu \cap \nu$. Pair this with the only up-down path from μ to ν, which passes through $\mu \cup \nu$.

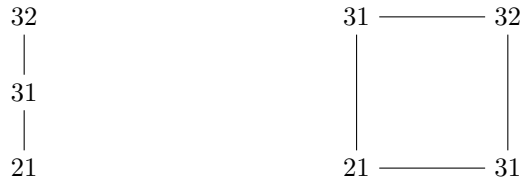
(C2) If $\mu = \nu$, then any down-up path can be represented by the inner corner of μ that is deleted in the downward step, and any up-down path can be represented by the outer corner of μ that is added in the upward step. Pair each down-up path with the up-down path that adds the first outer corner of μ strictly below the inner corner of the down-up path.

Now, the only up-down path that has not been paired with a down-up path is the path corresponding to adding the outer corner of μ in the first row. This gives $(DU - UD)\mu = \mu$.

It is easy to see that this cancellation yields exactly the growth rules described above. Cancellation rule (C1) determines growth rule (L3), cancellation rule (C2) determines growth rule (L2), and the up-down path left unmatched determines growth rule (L4). If we started with a different explicit cancellation, we could modify (L2), (L3), and (L4) accordingly to obtain new growth rules. Note that growth rule (L1) is simply a way to transfer information and will remain the same for any explicit cancellation.

Example 2.6. Starting with $\mu = (3, 1)$, cancellation rule (C2) says that the down-up path from μ to itself passing through $(2, 1)$ is paired with the up-down pathpassing

through $(3,2)$. The corresponding growth rule in this situation, (L4), is illustrated on the right.



3. Dual filtered graphs

3.1. The definition. A weak filtered graph is a triple $G = (P, \rho, E)$, where P is a discrete set of vertices, $\rho : P \to \mathbb{Z}$ is a rank function, and E is a multiset of edges/arcs (x, y), where $\rho(y) \geqslant \rho(x)$. A strict filtered graph is a triple $G = (P, \rho, E)$, where P is a discrete set of vertices, $\rho : P \to \mathbb{Z}$ is a rank function, and E is a multiset of edges/arcs (x, y), where $\rho(y) > \rho(x)$. Let P_n as before denote the subset of vertices of rank n.

Let $G_1 = (P, \rho, E_1)$ and $G_2 = (P, \rho, E_2)$ be a pair of filtered graphs with a common vertex set, G_1 weak while G_2 strict. From this pair, define an *oriented filtered graph* $G = (P, \rho, E_1, E_2)$ by orienting the edges of G_1 in the positive filtration direction and the edges of G_2 in the negative filtration direction. Recall that $a_i(x, y)$ denotes the number of edges of E_i joining x and y or the multiplicity of edge (x, y) in E_i. Using the *up* and *down operators* associated with graph G as previously defined, we say that G_1 and G_2 are *dual filtered graphs* if for any $x \in G$,

$$(DU - UD)x = \alpha x + \beta Dx$$

for some $\alpha, \beta \in \mathbb{R}$. We focus on examples where $DU - UD = D + I$, see Remark 1.1.

In the next sections, we describe three constructions of dual filtered graphs.

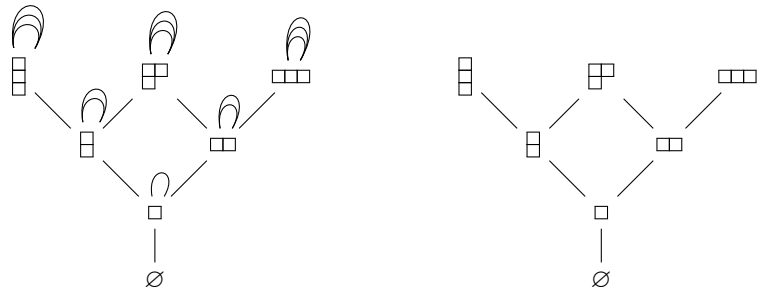
3.2. Trivial construction. Every dual graded graph has an easy deformation that makes it a dual filtered graph. To construct it, simply add $\rho(x)$ "upward" loops to each element x of the dual graded graph, where ρ is the rank function. We will call this the *trivial construction*.

Theorem 3.1. *For any dual graded graph G, the trivial construction gives a dual filtered graph G'.*

Proof. It suffices to show that if q covers p in G_2, then $[p](DU - UD)(q) = [p]D(q)$, where $[p](DU - UD)(q)$ and $[p]D(q)$ denote the coefficient of p in the linear combination $(DU - UD)(q)$ and $(D)(q)$, respectively. If we restrict to all paths that do not contain a loop, the coefficient of p in $(DU - UD)(q)$ is 0 since without loops, we have a dual graded graph. Hence we may restrict our attention to up-down paths and down-up paths that contain a loop. The up-down paths beginning at q and ending at p are exactly the paths consisting of a loop at q followed by an edge from q to p. There are $\rho(q)e(q \to p)$ such paths, where $e(q \to p)$ denotes the number of directed edges from q to p in G. The down-up paths containing a loop are the paths consisting of an edge from q to p followed by a loop at p. There are $e(q \to p)\rho(p)$ such paths. Thus

$$[p](DU - UD)(q) = e(q \to p)(\rho(q) - \rho(p)) = e(q \to p) = [p]D(q). \qquad \square$$

Example 3.2. The trivial construction for the first four ranks of Young's lattice is shown below. The edges in the graph on the left are oriented upward, and the edges in the graph on the right are oriented downward.



3.3. Pieri construction. The following construction is close to the one for dual graded graphs described in the works of Bergeron–Lam–Li [1], Nzeutchap [19], and Lam–Shimozono [16].

Let

$$A = \bigoplus_{i \geqslant 0} A_i$$

be a filtered, commutative algebra over a field $\mathbb{F}$ with $A_0 = \mathbb{F}$ and a linear basis $a_\lambda \in A_{|\lambda|}$. Here, the λ's belong to some infinite indexing set P, each graded component of which

$$P_i = \{\lambda \mid |\lambda| = i\}$$

is finite. The property of being filtered means that $a_\lambda a_\mu = \sum c_\nu a_\nu$, where $|\lambda| + |\mu| \leqslant |\nu|$.

Let $f = f_1 + f_2 + \cdots \in \hat{A}$ be an element of the completion $\hat{A}$ of A such that $f_i \in A_i$. Assume $d \in End(A)$ is a derivation on A satisfying $d(f) = f + 1$ and $d(a_\lambda) \in \bigoplus_{0 \leqslant i \leqslant |\lambda|} A_i$.

To form a graph, we define E_1 by defining $a_1(\lambda, \mu)$ to be the coefficient of a_λ in $d(a_\mu)$. We also define E_2 by defining $a_2(\lambda, \mu)$ to be the coefficient of a_μ in $a_\lambda f$. We assume here that $a_i(\lambda, \mu)$ are non-negative integers.

Theorem 3.3. *The resulting graph is a dual filtered graph.*

Proof. It follows from

$$d(f a_\lambda) = d(f) a_\lambda + f d(a_\lambda),$$

or

$$d(f a_\lambda) - f d(a_\lambda) = a_\lambda + d(a_\lambda). \qquad \square$$

Assume A_1 is one-dimensional, and let g be its generator. We shall often find a derivation d from a *bialgebra* structure on A. Indeed, assume Δ is a coproduct on A such that $\Delta(a) = 1 \otimes a + a \otimes 1 + \cdots$, where the rest of the terms lie in $(A_1 \oplus A_2 \oplus \cdots) \otimes (A_1 \oplus A_2 \oplus \cdots)$. Assume also ξ is a map $A \otimes A \longrightarrow A$ such that $\xi(p \otimes g) = p$ and for any element $q \in A_i$, $i \neq 1$, we have $\xi(p \otimes q) = 0$.

Lemma 3.4. *The map $D(a) = \xi(\Delta(a))$ is a derivation.*

Proof. We have $\xi(\Delta(ab)) = \xi(\Delta(a)\Delta(b)))$. Pick a linear basis for A that contains 1 and g. Express $\Delta(a)$ and $\Delta(b)$ so that the right side of tensors consists of the basis elements. The only terms that are not killed by ξ are the ones of the form $p \otimes g$. The only way we can get such terms is from $(p_1 \otimes 1) \cdot (p_2 \otimes g)$ or $(p_1 \otimes g) \cdot (p_2 \otimes 1)$. Those are exactly the terms that contribute to

$$\xi(\Delta(a)) \cdot b + a \cdot \xi(\Delta(b)). \qquad \square$$

3.4. Möbius construction. Let $G = (P, \rho, E_1, E_2)$ be a dual graded graph. From G, form a pair of filtered graphs as follows. Let $G_1' = (P, \rho, E_1')$ have the set of edges E_1' consisting of the same edges as E_1 plus $\#\{x \mid y \text{ covers } x \text{ in } G_1\}$ loops at each vertex $y \in P$. Let $G_2' = (P, \rho, E_2')$ have the set of edges E_2' consisting of

$$a_2'(x, y) = |\mu(x, y)|$$

edges between vertices x and y, where μ denotes the Möbius function in (P, ρ, E_2). Compose G_1' and G_2' into an oriented filtered graph $\tilde{G}$.

As we shall see, in all three of our major examples, the Möbius deformation forms a dual filtered graph. Unlike with the Pieri construction, we do not have the algebraic machinery explaining why the construction yields dual filtered graphs. Instead, we provide the proofs on a case-by-case basis. Nevertheless, the Möbius construction is the most important one, as it is this construction that relates to insertion algorithms and thus to K-theory of Grassmannians.

It is not the case that every dual graded graph's Möbius deformation makes it a dual filtered graph. For example, it is easy to see that the Möbius deformation of the graphs in Example 7.7 and Example 7.4 are not dual filtered graphs.

Problem 3.5. *Determine for which dual graded graphs G the Möbius construction $\tilde{G}$ yields a dual filtered graph.*

4. Major examples: Young's lattice

4.1. Pieri deformations of Young's lattice. Let $A = \Lambda$ be the ring of symmetric functions. Let s_λ, p_λ, and h_λ be its bases of Schur functions, power sum symmetric functions, and complete homogeneous symmetric functions. Let

$$f = h_1 + h_2 + \cdots .$$

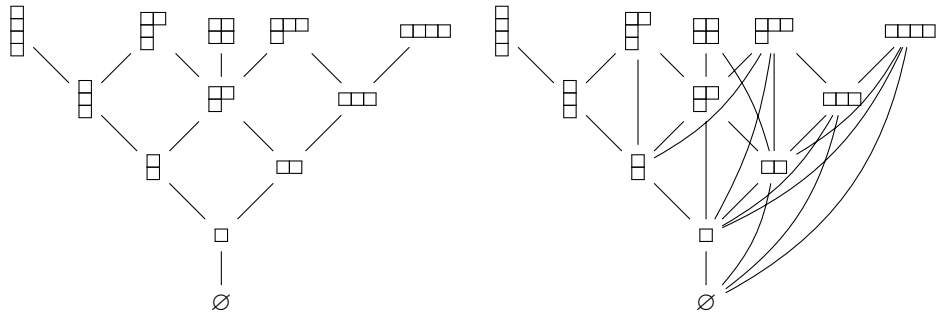
Define up and down edges of a filtered graph $G = (P, \rho, E_1, E_2)$ by letting $a_1(\mu, \nu)$ be the coefficient of s_ν in $p_1 s_\mu$, and $a_2(\mu, \nu)$ be the coefficient of s_ν in $f s_\mu$.

Recall that given two partitions, μ and ν, μ/ν forms a *horizontal strip* if no two boxes of μ/ν lie in the same column. For example, $(4, 2, 1)/(2, 1)$ forms a horizontal strip while $(4, 3, 1)/(2, 1)$ does not.

Lemma 4.1. *The up edges E_1 coincide with those of Young's lattice, while the down edges E_2 connect shapes that differ by a horizontal strip.*

Proof. The statement follows from the Pieri rule and the fact that $p_1 = h_1$, see [27]. □

In the figure below, the first six ranks are shown. The edges in the graph on the left are upward-oriented, and the edges on the graph on the right are downward-oriented.



THEOREM 4.2. *The Pieri deformation of Young's lattice is a dual filtered graph with*

$$DU - UD = D + I.$$

Recall the following facts regarding the ring of symmetric functions.

THEOREM 4.3 ([27, 18]).

(a) Λ *is a free polynomial ring in* $p_1, p_2, \ldots$.
(b) Λ *can be endowed with a standard bilinear inner product determined by* $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda,\mu}$.

Because of the first property, we can differentiate elements of Λ with respect to p_1 by expressing them first as a polynomial in the p_i's. We shall need the following two properties of f and p_1.

LEMMA 4.4.

(a) *For any* $h, g \in \Lambda$ *we have*

$$\langle g, p_1 h \rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}p_1} g, h \right\rangle.$$

(b) *We have*

$$\frac{\mathrm{d}}{\mathrm{d}p_1} f = f + 1.$$

Proof. By bilinearity, it suffices to prove the first claim for g, h in some basis of Λ. Let us choose the power sum basis. Then we want to argue that

$$\langle p_\lambda, p_1 p_\mu \rangle = \left\langle \frac{\mathrm{d}}{\mathrm{d}p_1} p_\lambda, p_\mu \right\rangle.$$

This follows from the fact that p_λ's form an orthogonal basis and [27, Proposition 7.9.3].

For the second claim, using [27, Proposition 7.7.4] we have

$$1 + f = e^{p_1 + \frac{p_2}{2} + \cdots},$$

which of course implies $\frac{\mathrm{d}}{\mathrm{d}p_1}(1+f) = 1 + f$. □

Now we are ready for the proof of Theorem 4.2.

Proof. Applying Lemma 4.4 we see that $A = \Lambda$, $a_\lambda = s_\lambda$, $d = \frac{\mathrm{d}}{\mathrm{d}p_1}$ and $f = h_1 + h_2 + \cdots$ satisfy the conditions of Theorem 3.3. The claim follows. □

Remark 4.5. We can make a similar argument for the analogous construction where the edges in E_1 are again those of Young's lattice and the edges in E_2 connect shapes that differ by a vertical strip. In this case we use $f = e_1 + e_2 + \cdots$, where the e_i are the elementary symmetric functions.

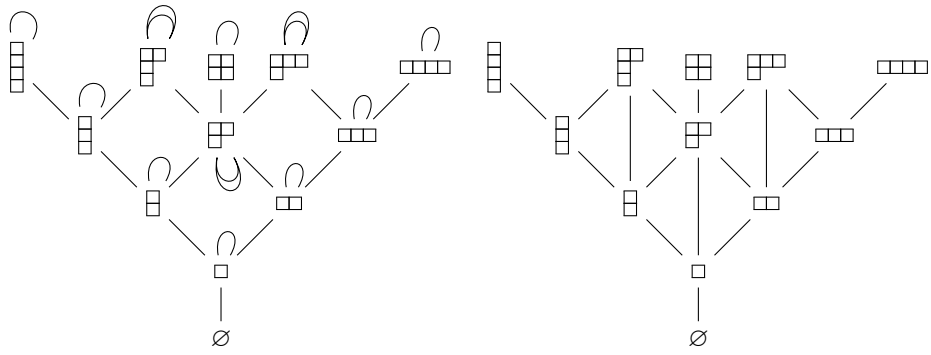
4.2. Möbius deformation of Young's lattice. Given two partitions, λ and ν, λ/ν forms a *rook strip* if no two boxes of λ/ν lie in the same row and no two boxes of λ/ν lie in the same column. In other words, λ/ν is a rook strip if it is a disconnected union of boxes. We state the following well-known result about the Möbius function on Young's lattice.

Proposition 4.6. *We have*

$$|\mu(\lambda,\nu)| = \begin{cases} 1 & \text{if } \lambda/\nu \text{ is a rook strip;} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The statement follows from the fact that Young's lattice is a distributive lattice, and the interval $[\lambda,\mu]$ is isomorphic to a Boolean algebra if and only if μ/λ is a rook strip. See [28, Example 3.9.6]. □

The Möbius deformation of Young's lattice is shown below with upward-oriented edges shown on the left and downward-oriented edges shown on the right. Notice that loops at a partition λ may be indexed by inner corners of λ.



Theorem 4.7. *The Möbius deformation of Young's lattice forms a dual filtered graph with*

$$DU - UD = D + I.$$

Proof. It suffices to show that $[\lambda](DU - UD)\mu = [\lambda]D\mu$ when μ covers λ in G_2 since we started with a dual graded graph.

Suppose shape λ is covered by shape μ. Each inner corner of μ contributes one to the coefficient of λ in $DU\mu$ via taking the loop corresponding to that inner corner of μ as an upward move and then going down to λ.

Next, consider shape μ and mark the outer corners of λ contained in μ. The marked boxes will be inner corners of μ. The shapes with an upward arrow from μ and a downward arrow to λ are exactly those obtained by adding one box to an outer corner of μ which is not adjacent to one of the marked boxes. There are $\#\{\text{outer corners of } \lambda\} - |\mu/\lambda|$ possible places to add such a box. Thus the coefficient of λ in $DU\mu$ is

$$\#\{\text{inner corners of } \mu\} + \#\{\text{outer corners of } \lambda\} - |\mu/\lambda|.$$

For down-up paths that involve three shapes, consider the inner corners of μ that are also inner corners of λ. Every shape obtained by removing one of these inner corners determines one down-up path from μ to λ, and there are $\#\{\text{inner corners of } \mu\} - |\mu/\lambda|$ such shapes. The remaining down-up paths are given by going from μ to λ and then taking a loop at λ. Hence the coefficient of λ in $UD\mu$ is

$$\#\{\text{inner corners of } \mu\} - |\mu/\lambda| + \#\{\text{inner corners of } \lambda\}.$$

Since all shapes have one more outer corner than inner corner, the coefficient of λ in $(DU - UD)\mu$ is 1. □

4.3. Hecke insertion. We next describe an insertion and reverse insertion procedure that correspond to the Möbius deformation of Young's lattice in the same way that RSK corresponds to Young's lattice, which is called *Hecke insertion* [6]. In other words, if we insert a word w of length n using Hecke insertion, the construction of the insertion tableau will be represented as a path of length n downward that ends at $\varnothing$ in the Möbius deformation of Young's lattice, and the construction of the recording tableau is represented as a path of length n upward starting at $\varnothing$. We illustrate this in Example 4.15 after introducing the necessary definitions.

An *increasing tableau* is a filling of a Young diagram with positive integers such that the entries in rows are strictly increasing from left to right and the entries in columns are strictly increasing from top to bottom.

Example 4.8. The tableau shown on the left is an increasing tableau. The tableau on the right is not an increasing tableau because the entries in the first row are not strictly increasing.

1	2	4	5
2	3	5	7
6	7		
8			

1	2	2	4
3	4		
5			

We follow [6] to give a description of Hecke (row) insertion of a positive integer x into an increasing tableau Y resulting in an increasing tableau Z. The shape of Z is obtained from the shape of Y by adding at most one box. If a box is added in position (i, j), then we set $c = (i, j)$. In the case where no box is added, then $c = (i, j)$, where (i, j) is a special corner indicating where the insertion process terminated. We will use a parameter $\alpha \in \{0, 1\}$ to keep track of whether or not a box is added to Y after inserting x by setting $\alpha = 0$ if $c \in Y$ and $\alpha = 1$ if $c \notin Y$. We use the notation $Z = (Y \xleftarrow{H} x)$ to denote the resulting tableau, and we denote the outcome of the insertion by (Z, c, α).

We now describe how to insert x into increasing tableau Y by describing how to insert x into R, a row of Y. This insertion may modify the row and may produce an output integer, which we will insert into the next row. To begin the insertion process, insert x into the first row of Y. The process stops when there is no output integer. The rules for insertion of x into R are as follows:

(H1) If x is weakly larger than all integers in R and adjoining x to the end of row R results in an increasing tableau, then Z is the resulting tableau and c is the new corner where x was added.

(H2) If x is weakly larger than all integers in R and adjoining x to the end of row R does not result in an increasing tableau, then $Z = Y$, and c is the box at the bottom of the column of Z containing the rightmost box of the row R.

For the next two rules, assume R contains boxes strictly larger than x, and let y be the smallest such box.

(H3) If replacing y with x results in an increasing tableau, then replace y with x. In this case, y is the output integer to be inserted into the next row

(H4) If replacing y with x does not result in an increasing tableau, then do not change row R. In this case, y is the output integer to be inserted into the next row.

Example 4.9.

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 4 & 6 \\ \hline 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array} \xleftarrow{H} 3 \;=\; \begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 4 & 6 \\ \hline 6 \\ \cline{1-1} 7 \\ \cline{1-1} \end{array}$$

We use rule (H4) in the first row to obtain output integer 5. Notice that the 5 cannot replace the 6 in the second row since it would be directly below the 5 in the first row. Thus we use (H4) again and get output integer 6. Since we cannot add this 6 to the end of the third row, we use (H2) and get $c = (4, 1)$. Notice that the shape did not change in this insertion, so $\alpha = 0$.

Example 4.10.

$$\begin{array}{|c|c|c|} \hline 2 & 4 & 6 \\ \hline 3 & 6 & 8 \\ \hline 7 \\ \cline{1-1} \end{array} \xleftarrow{H} 5 \;=\; \begin{array}{|c|c|c|} \hline 2 & 4 & 5 \\ \hline 3 & 6 & 8 \\ \hline 7 & 8 \\ \cline{1-2} \end{array}$$

The integer 5 bumps the 6 from the first row using (H3). The 6 is inserted into the second row, which already contains a 6. Using (H4), the second row remains unchanged and we insert 8 into the third row. Since 8 is larger than everything in the third row, we use (H1) to adjoin it to the end of the row. Thus $\alpha = 1$.

Using this insertion algorithm, we define the *Hecke insertion tableau* of a word $w = w_1 w_2 \dots w_n$ to be

$$P_H(w) = (\dots((\varnothing \xleftarrow{H} w_1) \xleftarrow{H} w_2)\dots) \xleftarrow{H} w_n.$$

In [6], Buch, Kresch, Shimozono, Tamvakis, and Yong give the following algorithm for reverse Hecke insertion starting with the triple (Z, c, α) as described above and ending with a pair (Y, x) consisting of an increasing tableau and a positive integer.

(rH1) If y is the cell in square c of Z and $\alpha = 1$, then remove y and reverse insert y into the row above.

(rH2) If $\alpha = 0$, do not remove y, but still reverse insert it into the row above.

Let x be the largest integer in the row above square c (if such a row exists) such that $x < y$.

(rH3) If replacing x with y results in an increasing tableau, then we replace x with y and reverse insert x into the row above.

(rH4) If replacing x with y does not result in an increasing tableau, leave the row unchanged and reverse insert x into the row above.

(rH5) If there is no row above in (rH4), the final output consists of x and the modified tableau.

Theorem 4.11 ([6, Theorem 4]). *Hecke insertion $(Y, x) \mapsto (Z, c, \alpha)$ and reverse Hecke insertion $(Z, c, \alpha) \mapsto (Y, x)$ define mutually inverse bijections between the set of pairs consisting of an increasing tableau and a positive integer and the set of triples consisting of an increasing tableau, a corner cell of the increasing tableau, and $\alpha \in \{0, 1\}$.*

We next describe the tableaux that will act as recording tableau for Hecke insertion.

A *set-valued tableau* T of shape λ is a filling of the boxes with finite, non-empty subsets of positive integers so that

(1) the smallest number in each box is greater than or equal to the largest number in the box directly to the left of it (if that box is present), and
(2) the smallest number in each box is strictly greater than the largest number in the box directly above it (if that box is present).

We call a set-valued tableau *standard* if it contains exactly the integers $[n]$ for some n.

Given a word $w = w_1 w_2 \dots w_l$, we can associate a pair of tableaux $(P_H(w), Q_H(w))$, where $P_H(w)$ is the Hecke insertion tableau described previously and $Q_H(w)$ is a standard set-valued tableau called the *Hecke recording tableau* obtained as follows. Start with $Q_H(\varnothing) = \varnothing$. At each step of the insertion of w, let $Q_H(w_1 \dots w_k)$ be obtained from $Q_H(w_1 \dots w_{k-1})$ by labeling the special corner, c, in the insertion of w_k into $P_H(w_1 \dots w_{k-1})$ with the positive integer k. Then $Q_H(w) = Q_H(w_1 w_2 \dots w_l)$ is the resulting standard set-valued tableau.

Example 4.12. Let w be 15133. We obtain $(P_H(w), Q_H(w))$ with the following sequence, where in column k, $Q_H(w_1 \dots w_k)$ is shown below $P_H(w_1 \dots w_k)$.

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 5 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 5 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 5 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 5 \\ \cline{1-1} \end{array} = P_H(w)$$

$$\begin{array}{|c|}\hline 1 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 34 \\ \cline{1-1} \end{array} \quad \begin{array}{|c|c|}\hline 1 & 25 \\ \hline 34 \\ \cline{1-1} \end{array} = Q_H(w)$$

Call a word w *initial* if the letters appearing in it are exactly the numbers in $[k]$ for some positive integer k. For example, the word 13422 is initial since the letters appearing in it are the numbers from 1 to 4. On the other hand, the word 1422 is not initial because the letters appearing in it are 1, 2, and 4 and do not form a set $[k]$ for any k. Similarly, call an increasing tableau *initial* if the entries appearing in it are exactly the numbers in $[k]$ for some positive integer k.

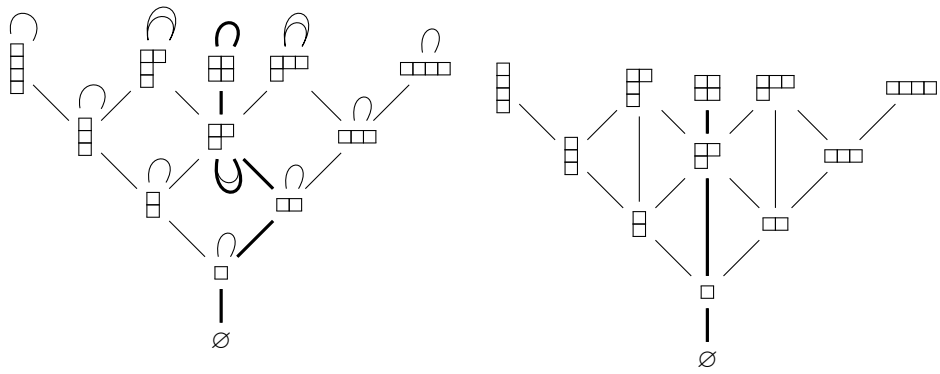
As with RSK, Hecke insertion gives a useful bijection.

Theorem 4.13 ([21]). *The map sending $w = w_1 w_2 \dots w_n$ to $(P_H(w), Q_H(w))$ is a bijection between words and ordered pairs of tableaux of the same shape (P, Q), where P is an increasing tableau and Q is a set-valued tableau with entries $\{1, 2, \dots, n\}$. It is also a bijection if there is an extra condition of being initial imposed both on w and P.*

Remark 4.14. Hecke insertion is closely related to the K-theoretic jeu de taquin algorithm introduced in [29]. This relationship is explored in [7, 31]

Example 4.15. We may now see the correspondence between initial increasing tableaux and walks in G_1 and between standard set-valued tableaux and walks in G_2 in an example. The thickened walks in the graphs correspond to the tableaux below, where we assign the lower loop on partition 21 to the corner in row 2. Suppose the walk in G_2 has includes n edges. The ith step upward from $\varnothing$ in G_1 tells us the location of i in the set-valued tableau, and the jth step downward from the top shape of the walk in G_2 tells us the location(s) of $n - j + 1$ in the increasing tableau.

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 34 & 56 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & 3 \\ \hline \end{array}$$

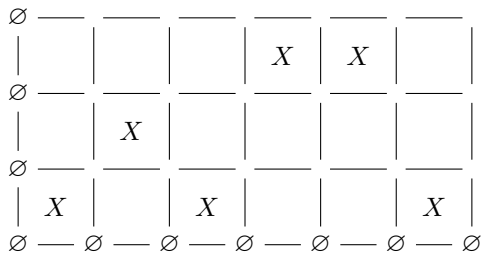


Notice that if the path on the left contains no loops and the path on the right contains to edges between shapes differing in size by more than one box, i.e., both are paths in Young's lattice, the correspondence yields a pair of standard Young tableaux.

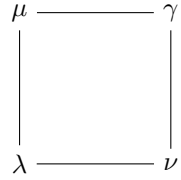
4.4. Hecke growth and decay. Given any word, $h = h_1 h_2 \dots h_k$ containing $n \leqslant k$ distinct numbers, we can create an $n \times k$ array with an X in the h_i^{th} square from the bottom of column i. Note that there can be multiple X's in the same row but is at most one X per column. For example, if $h = 121331$, we have the array below.

			X	X	
	X				
X		X			X

We will label each of the four corners of each square with a partition and some of the horizontal edges of the squares with a specific inner corner of the partition on the left vertex of the edge, which will indicate the box where the insertion terminates. We do this by recording the row of the inner corner at which the insertion terminates. We begin by labeling all corners along the bottom row and left side of the diagram with the partition $\varnothing$.



To complete the labeling of the corners, suppose the corners μ, λ, and ν are labeled, where μ, λ, and ν are as in the picture below. We label γ according to the following rules.



- *If the square contains an X:*
  (1) If $\mu_1 = \nu_1$, then γ/μ consists of one box in row 1.
  (2) If $\mu_1 \neq \nu_1$, then $\gamma = \mu$ and the edge between them is labeled by the row of the highest inner corner of μ.
- *If the square does not contain an X and if either $\mu = \lambda$ or $\nu = \lambda$ with no label between λ and ν:*
  (3) If $\mu = \lambda$, then set $\gamma = \nu$. If $\nu = \lambda$, then $\gamma = \mu$.
- *If the square does not contain an X, does not satisfy the conditions of previous bullet point, and if $\nu \not\subseteq \mu$:*
  (4) In the case where $\nu \not\subseteq \mu$, $\gamma = \nu \cup \mu$.
- *If the square does not contain an X, does not satisfy the conditions of second bullet point, and if $\nu \subseteq \mu$:*
  (5) If ν/λ is one box in row i and μ/ν has no boxes in row $i+1$, then γ/μ is one box in row $i+1$.
  (6) If ν/λ is one box in row i and μ/ν has a box in row $i+1$, then $\gamma = \mu$ and the edge between then is labeled $i+1$.
  (7) If $\nu = \lambda$, the edge between them is labeled i, and there are no boxes of μ/ν immediately to the right or immediately below the inner corner of ν in row i, then $\gamma = \mu$ with the edge between them labeled i.
  (8) If $\nu = \lambda$, the edge between them is labeled i, and there is a box of μ/ν directly below the inner corner of ν in row i, then $\gamma = \mu$ with the edge between them labeled $i+1$.
  (9) If $\nu = \lambda$, the edge between them is labeled i, and there is a box of μ/ν immediately to the right of the inner corner of ν in row i but no box of μ/ν in row $i+1$, then γ/μ is one box in row $i+1$.
  (10) If $\nu = \lambda$, the edge between them is labeled i, and there is a box of μ/ν immediately to the right of the inner corner of ν in row i and a box of μ/ν in row $i+1$, then $\gamma = \mu$ with the edge between them labeled $i+1$.

The four bullet points clearly cover all possibilities, and it is easy to check that the options below each cover all subcases of that bullet point.

We call the resulting array the *Hecke growth diagram* of h. In our previous example with $h = 121331$, we would have the diagram below.

$$\begin{array}{ccccccccccccc}
\varnothing & - & 1 & - & 2 & - & 21 & - & 31 & \overset{1}{-} & 31 & \overset{2}{-} & 31 \\
| & & | & & | & & | & X & | & X & | & & | \\
\varnothing & - & 1 & - & 2 & - & 21 & - & 21 & - & 21 & \overset{2}{-} & 21 \\
| & & | & X & | & & | & & | & & | & & | \\
\varnothing & - & 1 & - & 1 & \overset{1}{-} & 1 & - & 1 & - & 1 & \overset{1}{-} & 1 \\
| & X & | & & | & X & | & & | & & | & X & | \\
\varnothing & - & \varnothing & - & \varnothing & - & \varnothing & - & \varnothing & - & \varnothing & - & \varnothing
\end{array}$$

Let $\mu_0 = \varnothing \subseteq \mu_1 \subseteq \ldots \mu_k$ be the sequence of partitions across the top of the Hecke growth diagram, and let $\nu_0 = \varnothing \subseteq \nu_1 \subseteq \ldots \nu_n$ be the sequence of partitions on the right side of the Hecke growth diagram. The sequence across the top corresponds to a standard set-valued tableau $Q(h)$ and the sequence on the right corresponds to an increasing tableau $P(h)$. If the edge between μ_i and μ_{i+1} is labeled j, then $\mu_i = \mu_{i+1}$, and the label $i+1$ of $Q(h_1 \ldots h_{i+1})$ is placed in the box at the end of row j of $Q(h_1 \ldots h_i)$. Also note that if $\mu_i = \mu_{i+1}$, the edge between them will have a label. This is because there is an X somewhere in the corresponding column, and it follows that growth rule (3) will not be applied to obtain μ_{i+1}. For the array above, we have

the following $P(h)$ and $Q(h)$.

$$P(h) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad\qquad Q(h) = \begin{array}{|c|c|c|}\hline 1 & 2 & 45 \\ \hline 36 \\ \cline{1-1}\end{array}$$

Theorem 4.16. *For any word h, the increasing tableau $P(h)$ and set-valued tableau $Q(h)$ obtained from the sequence of partitions across the right side of the Hecke growth diagram for h and across the top of the Hecke growth diagram for h, respectively, are $P_H(h)$ and $Q_H(h)$, the Hecke insertion and recording tableau for h.*

Proof. We first prove two necessary facts.

Claim 4.17. *Oriented as in the square above, $|\nu/\lambda| \leqslant 1$.*

Proof of Claim 4.17. By the induction hypothesis, λ is the recording tableau after inserting the numbers in columns 1 through $s-1$, and ν is the insertion tableau after inserting the number in column s. Since we are only inserting one number, $|\nu/\lambda|$ is at most 1. □

Claim 4.18. *Oriented as in the square above, μ/λ is a rook strip, that is, no two boxes in μ/λ are in the same row or column.*

Proof of Claim 4.18. By induction, μ/λ represents the positions of boxes filled with t at some point in the insertion. Since insertion results in an increasing tableau, μ/λ must be a rook strip. □

We use the notation of Theorem 7.13.5 of [27]. Let $\gamma(i,j)$ denote the partition in row i and column j, where rows are numbered from bottom to top and columns from left to right. For fixed j, we have

$$\varnothing = \gamma(0,j) \subseteq \gamma(1,j) \subseteq \cdots \subseteq \gamma(n,j),$$

where $|\gamma(i,j)/\gamma(i-1,j)| \geqslant 0$. Let $T(i,j)$ be the tableau of shape $\gamma(i,j)$ obtained by inserting k into $\gamma(k,j)/\gamma(k-1,j)$ when $0 \leqslant k \leqslant i$ and $|\gamma(k,j)/\gamma(k-1,j)| > 0$.

Similarly, for fixed i, we have

$$\varnothing = \gamma(i,0) \subseteq \gamma(i,1) \subseteq \cdots \subseteq \gamma(i,n),$$

where $|\gamma(i,j)/\gamma(i,j-1)| = 0$ or 1. Let $R(i,j)$ be the set-valued tableau of shape $\gamma(i,j)$ obtained by inserting k into $\gamma(i,k)/\gamma(i,k-1)$ when $0 \leqslant k \leqslant j$ and $|\gamma(k,j)/\gamma(k-1,j)| = 1$ and by inserting k into the rightmost cell in row i of $\gamma(i,k)$ when $0 \leqslant k \leqslant j$ and $|\gamma(k,j)/\gamma(k-1,j)| = 0$ with the edge between them having label i. For example, in the growth diagram for $h = 121331$ shown above, we have the tableaux below.

$$T(1,3) = \begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad T(2,3) = T(3,3) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 \\ \cline{1-1}\end{array}$$

$$R(2,1) = \begin{array}{|c|}\hline 1 \\ \hline\end{array} \qquad R(2,2) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} \qquad R(2,3) = R(2,4) = R(2,5) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$$

$$R(2,6) = \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 36 \\ \cline{1-1}\end{array}$$

We will prove that $T(i,j)$ and $R(i,j)$ can also be defined as follows. Suppose the X's to the left of and below $T(i,j)$ are in positions $(i_1,j_1),\ldots,(i_k,j_k)$ with $j_1 < \cdots < j_k$. Then

$$T(i,j) = ((\varnothing \xleftarrow{H} i_1) \xleftarrow{H} i_2) \xleftarrow{H} \cdots \xleftarrow{H} i_k$$

and $R(i,j)$ is a recording tableau for this insertion using entries $j_1, \ldots, j_k$. We prove this fact by induction on $i+j$. We do it simultaneously for both $T(i,j)$ and $R(i,j)$ but focus on $T(i,j)$. The claim is clearly true when $i=0$ or $j=0$.

Consider now some $T(t,s)$ for $t,s>0$. By the induction hypothesis, $T(t-1,s)$, $T(t,s-1)$, and $T(t-1,s-1)$ satisfy the desired conditions. We next check that $T(t,s)$ satisfies the conditions.We will denote the shape of $T(t-1,s)$ by ν,the shape of $T(t,s-1)$ by μ, the shape of $T(t-1,s-1)$ by λ, and the shape of $T(t,s)$ by γ to match the description of the growth rules.

(1) Since our square contains an X, there is no other X in the same column. It follows that $\lambda=\nu$. Also, the X in the square we are considering corresponds to inserting a t into $T(t,s-1)$. Since, by the induction hypothesis, shapes along columns represent the insertion tableaux, $\mu_1=\nu_1=\lambda_1$ means that before adding the t, there are no t's in the first row of $T(t,s-1)$. Since t is weakly the largest number inserted to this point, inserting the t will result in a t being added to the first row of $T(t,s-1)$. Thus $\gamma=(\mu_1+1,\mu_2,\mu_3,\ldots)$.

(2) If $\mu_1\neq\nu_1=\lambda_1$, then there is a t at the end of the first row of $T(t,s-1)$. It follows that inserting another t will result in this t merging with the t at the end of the first row, and the special corner in this case becomes the bottom box of the last column, or in other words, the highest inner corner of μ.

(3) If $\mu=\lambda$ and there is no X in our square, then there is no X in row t in columns 1 to s. Thus there is no t to add to $T(t-1,s)$ to obtain $T(t,s)$, and so $\nu=\gamma$. Similarly for the case where $\nu=\lambda$.

(4) Suppose that ν/λ is one box in row i and μ/λ is a rook strip containing boxes in rows $j_1,j_2,\ldots,j_k$. Then $\nu\not\subseteq\mu$ implies that $i\notin\{j_1,j_2,\ldots,j_k\}$. Since ν/λ is one box, the last action in the insertion sequence of r into $T(t-1,s-1)$ is a box being added in row i, where r is the row index of the X in column s. (We will continue to denote this row index by r for the rest of the proof.) Since there is not a t in row i of $T(t,s-1)$, the bumping sequence when inserting r into $T(t,s-1)$ will not disturb the t's.

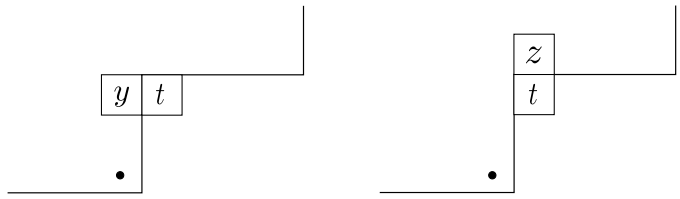
(5) Suppose ν is λ plus one box in position (i,j). Then since $\nu\subseteq\mu$, μ/λ contains the box at (i,j), and so there is a t in position (i,j) of $T(t,s-1)$. It follows that inserting r into $T(t,s-1)$ will result in bumping the t in position (i,j) since the bumping path of r inserted into $T(t-1,s-1)$ ends by adding a box at (i,j). Since there is no box in row $i+1$ of μ/λ, everything in row $i+1$ of $T(t,s-1)$ is strictly less than t. It follows that the t bumped from (i,j) can be added to the end of row $i+1$.

(6) This is almost identical to the proof of Rule (4). Since there is now a box in row $i+1$ of μ/λ, there is a t at the end of row $i+1$ of $T(t,s-1)$. Inserting r will bump the t in row i as above, but now the t will merge with the t in row $i+1$. Thus the special corner is now at the inner corner of μ in row $i+1$.

(7) There are two ways that inserting r into $T(t,s-1)$ will involve the t's. Assume the edge between ν and λ is labeled i, and the inner corner of ν in row i is in column j. Assume rows k through i all have length j in ν for some $k\leqslant i$.

Case 1. A box containing some $y\geqslant r$ was bumped from row $k-1$ of $T(t,s-1)$ or inserted into $T(t,s-1)$ if $k=1$, merged with itself in row k, and there was a t to the right of the y in row k. Note that $k\neq i$. Then the y duplicates and bumps the t in row k of $T(t,s-1)$, but this t cannot be added to row $k+1$ as it would be directly below the original t. Thus, by the rules of Hecke insertion, the special corner is the box at the bottom of column j, box (i,j). See the left-hand figure below where the dot indicates the box (i,j).

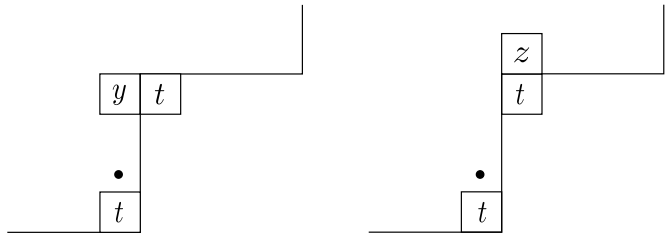
Case 2. Consider the situation in the right-hand figure below where the dot indicates the box (i, j). Assume that during the insertion of r into $T(t, s-1)$, some number merged with itself to the left of z in row $k-1$ and z is duplicated bumped to the next row. Following the rules of Hecke insertion, z bumps the t in row k, but as it cannot replace the t, the t remains at the end of row k. The t that was bumped cannot be added to end of row $k+1$, so no new boxes are created in the insertion and the special corner is the box at the bottom of column j, box (i, j).



(8) There are three ways that inserting r into $T(t, s-1)$ will involve the t's. Assume the edge between ν and λ is labeled i, and the inner corner of ν in row i is in column j. Assume rows k through i all have length j in ν.

Case 1. If there is a t in square $(i, j+1)$, we are in the situation described by Rule (9).

Cases 2 and 3 are described in the proof of Rule (7). The difference is that the bottom of column j after we insert r is now the box $(i+1, j)$, so the special corner in each case will be $(i+1, j)$, as desired.



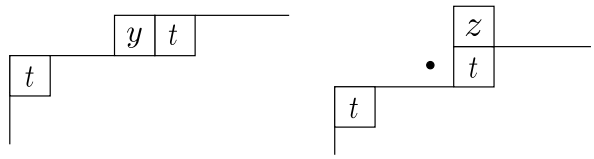
(9) Note that an edge label of i between λ and ν with a box of μ/ν in row i implies that $\nu_{i-1} > \nu_i > \nu_{i+1}$. There are two ways the special corner of the insertion of r into the $T(t-1, s-1)$ could have been (i, j).

Case 1. Some y was bumped from row $i-1$ and merged with itself in box (i, j). Since there is a t to the right of box (i, j) in $T(t, s-1)$, this will result in duplicating and bumping t. Since there is nothing in row $i+1$ of μ/ν, everything in row $i+1$ of $T(t, s-1)$ is strictly less than t. Hence the t bumped from row i of $T(t, s-1)$ can be added to the end of row $i+1$.

Case 2. As in the figure below, z was duplicated and bumped from box $(i-1, j+1)$. This z cannot replace the t in box $(i, j+1)$ of $T(t, s-1)$ since it would be directly below the original z, but it bumps the t to row $i+1$. Since there is nothing in row $i+1$ of μ/ν, everything in row $i+1$ of $T(t, s-1)$ is strictly less than t. Hence the t bumped from row i can be added to the end

of row $i+1$ of $T(t, s-1)$ to obtain $T(t, s)$.

(10) This is similar to the situation described in the proof of Rule 9. The difference is that a box in row $i+1$ of μ/ν means there is already a t at the end of row $i+1$ of $T(t, s-1)$, so each insertion will end with the bumped t merging with itself at the end of row $i+1$.



□

4.5. Möbius via Pieri. The following shows that we have an instance of the Möbius via Pieri phenomenon in the case of Young's lattice. Namely, the Möbius deformation of Young's lattice is obtained by the Pieri construction from an appropriate K-theoretic deformation of the underlying Hopf algebra.

Let A be the subring of the completion $\hat{\Lambda}$ of the ring of symmetric functions with basis $\{G_\lambda\}$, the stable Grothendieck polynomials. For details, see [5, 15]. Define the *signless stable Grothendieck polynomials* $\tilde{G}_\lambda$ by omitting the sign. Note that these coincide with the $\tilde{K}_\lambda$ defined in [15]. The structure constants will coincide with those of the stable Grothendieck polynomials up to sign. We recall these structure constants here.

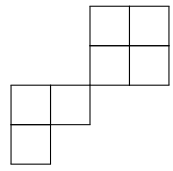
Let S_μ be the superstandard tableau of shape μ, that is, the first row of S is filled with $1, 2, \ldots, \mu_1$, the second row with $\mu_1+1, m_1+2, \ldots, \mu_1+\mu_2$, etc. Given an increasing tableau R, let $\mathrm{row}(R)$ denote the *row word* of R: the word obtained by reading the entries of R left to right across rows from the bottom row to the top row. This space has a bialgebra product structure given by

$$\tilde{G}_\lambda \tilde{G}_\mu = \sum c^\nu_{\lambda,\mu} \tilde{G}_\nu,$$

where $c^\nu_{\lambda,\mu}$ is the number of increasing tableaux R of skew shape ν/λ such that $P_H(\mathrm{row}(R)) = S_\mu$. The coproduct structure is given by

$$\Delta(\tilde{G}_\nu) = \sum_{\lambda,\mu} d^\nu_{\lambda,\mu} \tilde{G}_\lambda \otimes \tilde{G}_\mu,$$

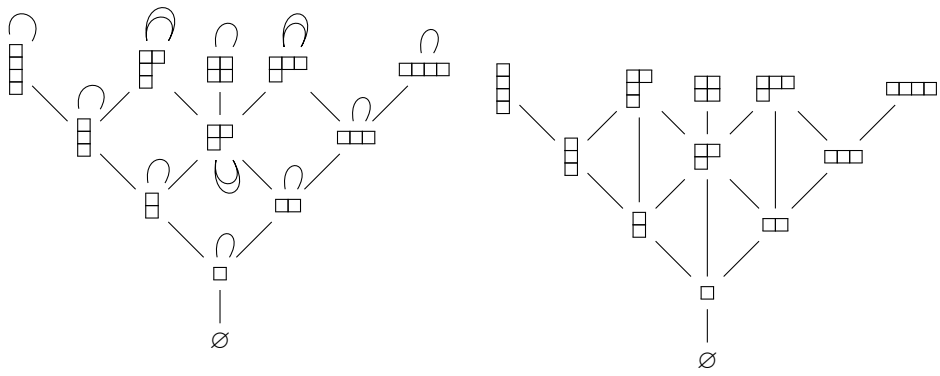
where $d^\nu_{\lambda,\mu}$ is the number of increasing tableaux R of skew shape $\lambda \oplus \mu$ such that $P_H(\mathrm{row}(R)) = S_\nu$, see [7, 21, 30]. Here $\lambda \oplus \mu$ denotes the skew shape obtained by joining the rightmost top corner of λ to the leftmost bottom corner of μ. For example, $(2,1) \oplus (2,2) = (4,4,2,1)/(2,2)$ is shown below.



We consider the following Pieri construction in A. Let $g = \tilde{G}_1$, and define an operator d by $d(\tilde{G}_\nu) = \xi(\Delta(\tilde{G}_\nu))$. We define a graph G, where elements are partitions,

$a_1(\mu,\lambda)$ is the coefficient of $\tilde{G}_\mu$ in $d(\tilde{G}_\lambda)$, and $a_2(\mu,\lambda)$ is the coefficient of $\tilde{G}_\lambda$ in $\tilde{G}_\mu\tilde{G}_1$. Here, our element $f = f_1 + f_2 + \cdots \in \hat{A}$ is $\tilde{G}_1$.

Proposition 4.19. *The resulting graph coincides with the one obtained via Möbius construction in Section* 4.2.



Proof. To verify this claim, we first show why the coefficient of $\tilde{G}_\nu$ in $\tilde{G}_\lambda\tilde{G}_1$ is 1 if ν/λ is a rook strip and is 0 otherwise. This follows from that fact that the only words that Hecke insert into S_1 are words consisting only of the letter 1. Thus, $\mathrm{row}(R) = 11\ldots1$ for any R contributing to $c^\nu_{\lambda,1}$. Increasing tableaux of shape ν/λ with row word $11\ldots1$ are precisely rook strips.

Secondly, it must be true that $d^\lambda_{\lambda,1}$ is the number of inner corners of λ and if $\nu \neq \lambda$, $d^\nu_{\lambda,1}$ is 1 if ν is obtained from λ by adding one box and 0 otherwise. Suppose $\nu \neq \lambda$ and consider all skew tableaux R of shape $\lambda \oplus 1$ such that $P_H(\mathrm{row}(R)) = S_\nu$. To obtain such λ, simply reverse insert an entry of S_ν with $\alpha = 1$ and use the output integer to fill the single box in $\lambda \oplus 1$. The resulting shapes are the shapes obtained from ν by deleting one inner corner of ν. Similarly, if $\nu = \lambda$, we reverse insert each inner corner of S_λ using $\alpha = 0$ and then let R be $S_\lambda \oplus x$, where x is the result of the reverse insertion. □

5. Major examples: shifted Young's lattice

5.1. Pieri deformation of shifted Young's lattice. Let $A = \Lambda'$ be the subring of the ring of symmetric functions generated by the odd power sums $p_1, p_3, \ldots$. Let Q_λ and P_λ be the bases of Schur Q and Schur P functions for this ring. Here λ varies over the set of partitions with distinct parts. We refer the reader to [18] for background. Let q_i be defined by

$$q_i = \sum_{0 \leqslant j \leqslant i} e_j h_{i-j},$$

and let $f = q_1 + q_2 + \cdots$.

Define up and down edges of the Pieri deformation of shifted Young's lattice $G = (P, \rho, E_1, E_2)$ by letting $a_2(\mu,\nu)$ be the coefficient of Q_μ in fQ_ν and $a_1(\mu,\nu)$ be the coefficient of P_μ in p_1P_ν.

We will fill a shifted partition λ with ordered alphabet $1' < 1 < 2' < 2 < 3' < 3\ldots$ according to the usual rules:

- the filling must be weakly increasing in rows and columns;
- there can be at most one instance of k' in any row;
- there can be at most one instance of k in any column;
- there can be no primed entries on the main diagonal.

For example, the shifted partition below is filled according to the rules stated above.

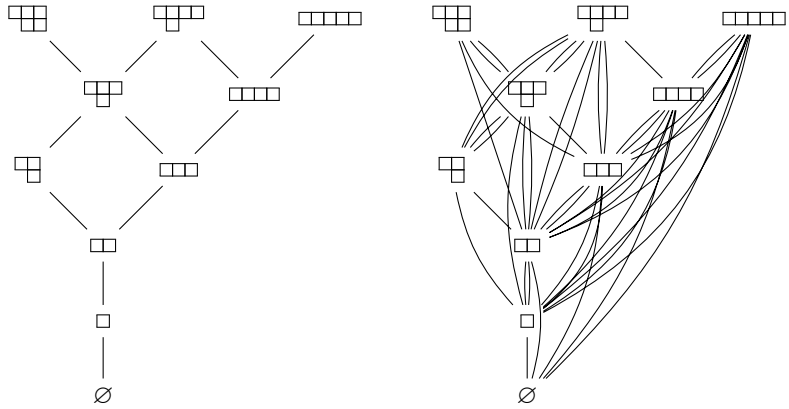
1	$2'$	2	2	$4'$
	2	$3'$	$5'$	5
		4	$5'$	
			6	

Given two shifted shapes, μ and ν, we say that μ/ν forms a *border strip* if it contains no 2-by-2 square.

LEMMA 5.1. *The Pieri deformation of the shifted Young's lattice is formed by adding downward-oriented edges from μ to ν whenever μ/ν forms a border strip. The number of such edges added is the same as the number of ways to fill the boxes of the border strip with k and k' according to the usual rules.*

Proof. The proof follows from Formula (8.15) in Chapter III of [18], which is an analogue of the Pieri rule for Schur P functions, and the fact that $q_1 = 2p_1$. □

The first six ranks of the Pieri deformation are shown below.



Notice that there are two edges from 41 to 2 corresponding to the following fillings of the border strip $(4,1)/(2)$.

THEOREM 5.2. *The Pieri deformation of the dual graded graph of shifted shapes satisfies*

$$DU - UD = D + I.$$

We shall need the following two properties of Λ'.

THEOREM 5.3 ([18]).

- *Λ' is a free polynomial ring in odd power sums $p_1, p_3, \ldots$.*
- *Λ' inherits the standard bilinear inner product from Λ, which satisfies $\langle Q_\lambda, P_\mu \rangle = 2^{l(\mu)}\delta_{\lambda,\mu}$.*

Because of the first property, we can again differentiate elements of Λ' with respect to p_1 by expressing them first as a polynomial in the p_i's. We shall need the following property of f.

LEMMA 5.4. *We have*

$$\frac{\mathrm{d}}{\mathrm{d}p_1} f = 2f + 2.$$

Proof. From [18, Ex. 6 (a), III, 8] we have

$$1 + f = e^{2p_1 + \frac{2p_3}{3} + \cdots},$$

which of course implies $\frac{\mathrm{d}}{\mathrm{d}p_1}(1+f) = 2 + 2f$. □

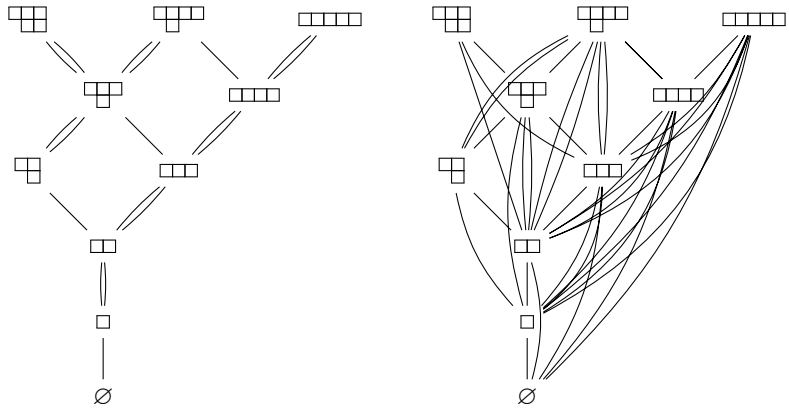
Now we are ready for the proof of Theorem 5.2.

Proof. Applying Lemma 4.4 we see that $A = \Lambda'$, $a_\lambda = Q_\lambda$, $d = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}p_1}$ and $f = q_1 + q_2 + \cdots$ satisfy the conditions of Theorem 3.3. The claim follows. □

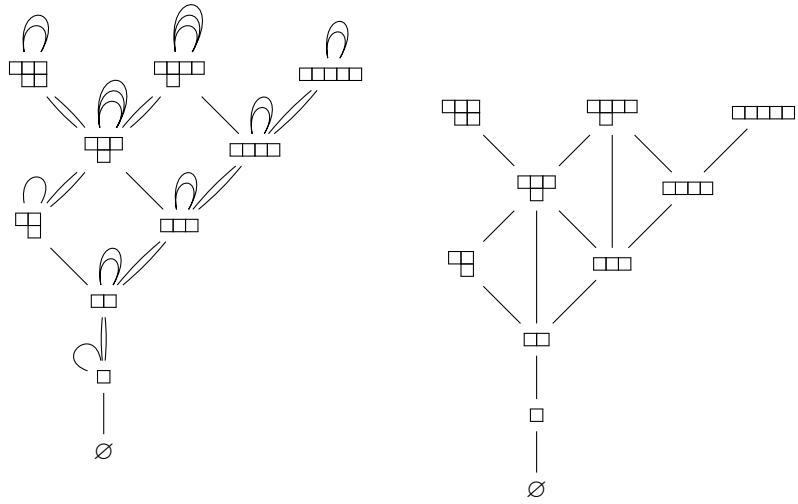
Remark 5.5. If we instead take $a_1(\mu, \nu)$ to be 2 times the coefficient of Q_ν in $p_1 Q_\mu$, and $a_2(\mu, \nu)$ to be $\frac{1}{2}$ times the coefficient of P_μ in fP_ν, we get a dual filtered graph with

$$DU - UD = 2D + I.$$

This choice corresponds to first swapping E_1 and E_2 in the original dual graded graph of $\mathbb{SY}$ and then adding downward edges to E_2 as described above.



5.2. Möbius deformation of shifted Young's lattice. The Möbius deformation of the shifted Young's lattice is shown below with upward-oriented edges shown on the left and downward-oriented edges on the right. Note that we swapped the E_1 and E_2 of Example 2.4.



LEMMA 5.6. *For the lattice of shifted shapes, the Möbius function is given by the following formula:*

$$\mu(p,q) = \begin{cases} (-1)^{|p|-|q|} & \text{if } q/p \text{ is a disjoint union of boxes} \\ 0 & \text{otherwise} \end{cases}$$

Proof. It is known that $\mathbb{SY}$ is a distributive lattice [10, Example 2.2.8], and so it follows that $\mu(p,q)$ is nonzero if and only if the interval $[p,q]$ is a Boolean lattice, in which case $\mu(p,q)$ is as described above, [27, Example 3.9.6]. And $[p,q]$ is a Boolean lattice exactly when q/p is a disjoint union of boxes, i.e. no two boxes of q/p share an edge. □

THEOREM 5.7. *The Möbius deformation of the shifted Young's lattice forms a dual filtered graph with*

$$DU - UD = D + I.$$

Proof. Given that the non-deformed pair form a pair of dual graded graphs, it suffices to show that

$$[\lambda](DU - UD)(\mu) = [\lambda](D)(\mu)$$

when μ covers λ in G_2.

For a shifted shape λ, let $I_{diag}(\lambda)$ denote the size of the set of inner corners of λ that are on the diagonal, $I_{odiag}(\lambda)$ denote the size of the set of inner corners of λ that are not on the diagonal, $O_{diag}(\lambda)$ denote the size of the set of outer corners of λ that are on the diagonal, and $O_{odiag}(\lambda)$ denote the size of the set of outer corners of λ that are not on the diagonal.

LEMMA 5.8. *For any shifted shape λ,*

$$O_{diag}(\lambda) + 2O_{odiag}(\lambda) - I_{diag}(\lambda) - 2I_{odiag}(\lambda) = 1.$$

Proof. Consider some element $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$. Suppose $\lambda_k = 1$. We can add a box at some subset of rows $\{i_1 = 1, i_2, \ldots, i_t\}$ and fill each with either s or s'. Thus $O_{diag}(\lambda) + 2O_{odiag}(\lambda) = 2t$. We can delete a box in rows $\{i_2 - 1, i_3 - 1, \ldots, i_t - 1, k\}$, where each can be filled with s or s' except the box in row k, which is on the diagonal. Thus $I_{diag}(\lambda) + 2I_{odiag}(\lambda) = 2(t-1) + 1$.

Now suppose $\lambda_k \geqslant 2$. We can add a box at some subset of rows $\{i_1 = 1, i_2, \ldots, i_t, k+1\}$ and fill each with either s or s' except for the box in row $k+1$, which is on the diagonal. This gives $O_{diag}(\lambda) + 2O_{odiag}(\lambda) = 2t + 1$. We can delete boxes in rows $\{i_2 - 1, \ldots, i_t - 1, k\}$, where each can be filled with s or s', giving $I_{diag}(\lambda) + 2I_{odiag}(\lambda) = 2t$. □

First, consider all up-down paths from μ to λ that begin with a loop at μ. There are

$$2I_{odiag}(\mu) + I_{diag}(\mu)$$

such paths since there are the same number of loops at μ. Up-down paths that do not start with a loop can be counted by the number of outer corners of λ that are not inner corners of q counted with multiplicity two if they are off the main diagonal since each of these corners may be added to q for the step up and then removed in addition to the boxes of μ/λ for the step down. There are

$$2(O_{odiag}(\lambda) - I_{odiag}(\mu/\lambda)) + (O_{diag}(\lambda) - I_{diag}(\mu/\lambda))$$

such corners. Thus there are

$$2I_{odiag}(\mu) + I_{diag}(\mu) + 2(O_{odiag}(\lambda) - I_{odiag}(\mu/\lambda)) + (O_{diag}(\lambda) - I_{diag}(\mu/\lambda))$$

up-down paths from μ to λ.

Next, consider all down-up paths from μ to λ that end with a loop at λ. There are exactly

$$2I_{odiag}(\lambda) + I_{diag}(\lambda)$$

such paths. Down-up paths that do not end with a loop can be counted by the inner corners of q that are also inner corners of λ counted twice if they are off the main diagonal since removing this type of inner corner in addition to the boxes of λ/μ will result in a shifted shape covered by λ. There are

$$2(I_{odiag}(\mu) - I_{odiag}(\mu/\lambda)) + (I_{diag}(\mu) - I_{diag}(\mu/\lambda))$$

such corners. Thus there are

$$2I_{odiag}(\lambda) + I_{diag}(\lambda) + 2(I_{odiag}(\mu) - I_{odiag}(\mu/\lambda)) + (I_{diag}(\mu) - I_{diag}(\mu/\lambda))$$

down-up paths from μ to λ.

Hence

$$\begin{aligned}[\lambda](DU - UD)(\mu) &= 2I_{odiag}(\mu) + I_{diag}(\mu) + 2(O_{odiag}(\lambda) - I_{odiag}(\mu/\lambda)) \\ &\quad + (O_{diag}(\lambda) - I_{diag}(\mu/\lambda)) - (2I_{odiag}(\lambda) + I_{diag}(\lambda) \\ &\quad + 2(I_{odiag}(\mu) - I_{odiag}(\mu/\lambda)) + (I_{diag}(\mu) - I_{diag}(\mu/\lambda)))) \\ &= 2O_{odiag}(\lambda) + O_{diag}(\lambda) - 2I_{odiag}(\lambda) - I_{diag}(\lambda) \\ &= 1\end{aligned}$$

by Lemma 5.8. □

5.3. Shifted Hecke insertion.

Definition 5.9. *An* increasing shifted tableau *is a filling of a shifted shape with positive integers such that entries are strictly increasing across rows and down columns.*

The following are examples of increasing shifted tableaux.

1	2	4
	4	

2	3	5	7	8
	5	8		
		9		

We now describe an insertion procedure for increasing shifted shapes that is similar to Hecke insertion, which we call *shifted Hecke insertion*. This procedure is a natural analogue of that of Sagan and Worley, [24, 32]. As before, any insertion tableau will correspond to a walk downward ending at $\varnothing$ in the dual filtered graph of shifted shapes from the Möbius deformation and any recording tableau corresponds to a walk upward starting at $\varnothing$. The correspondence is completely analogous to that in Example 4.15. In this case, whenever there are two edges connecting λ and μ, we assign one of the two edges to be the "primed edge" and the other to be the "unprimed edge".

We start by describing how to insert x into an increasing shifted tableau Y to obtain increasing shifted tableau Z. We begin by inserting $x = y_1$ into the first row of Y. This insertion may modify the first row of Y and may produce an output integer y_2. Following the rules below, we then insert y_2 into the second row of Y and continue in this manner until one of two things happens. Either at some stage the insertion will not create an output integer, in which case the insertion of x into Y terminates, or y_k will replace some diagonal element of Y. In the latter case, we continue the insertion process along the columns: y_{k+1} is inserted into the column to the right of its original position and so on until some step in this process does not create an output integer.

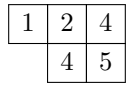
For each insertion, we designate a specific box of the resulting tableau, Z, as being the box where the insertion terminated. We will later use this notion to define recording tableaux.

The rules for inserting any positive integer x into a row or column are as follows:

(S1) If x is weakly larger than all integers in the row (resp. column) and adjoining x to the end of the row (resp. column) results in an increasing tableau, then Z is the resulting tableau. We say that the insertion terminated at this new box.

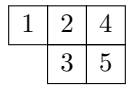
Example 5.10. Inserting 4 into the first row of the tableau below on the left gives the tableau on the right. This insertion terminated in position $(1, 3)$.

Inserting 5 into the third column of the resulting tableau gives the tableau shown below. This insertion terminated in position $(2, 3)$.

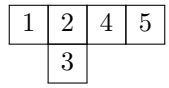


(S2) If x is weakly larger than all integers in the row (resp. column) and adjoining x to the end of the row (resp. column) does not result in an increasing tableau, then $Z = Y$. If x was row inserted into a nonempty row, we say that the insertion terminated at the box at the bottom of the column containing the rightmost box of this row. If x was row inserted into an empty row, we say the insertion terminated at the rightmost box of the previous row. If x was column inserted, we say the insertion terminated at the rightmost box of the row containing the bottom box of the column x could not be added to.

Example 5.11. Inserting 4 into the first row of the tableau below does not change the row and does not give an output integer, and so the insertion does not change the tableau. The insertion terminated in position $(2, 3)$.



Inserting 4 into the third column of the tableau below does not change the column and does not produce an output integer, and so the insertion does not change the tableau. The insertion terminated in position $(1, 4)$.



Inserting 2 into the (empty) second row of the tableau below does not change the row. The insertion terminated in position (1,2)

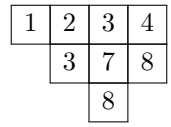
For the last two rules, suppose the row (resp. column) contains a box with label strictly larger than x, and let y be the smallest such box.

(S3) If replacing y with x results in an increasing tableau, then replace y with x. In this case, y is the output integer. If x was inserted into a column or if y was on the main diagonal, proceed to insert all future output integers into the next column to the right. If x was inserted into a row and y was not on the main diagonal, then insert y into the row below.

Example 5.12. Inserting 6 into the second row of the tableau on the left results in the tableau on the right with output integer 7 to be inserted into the third row.

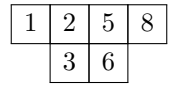
Inserting 6 into the third column of the tableau on the left also results in the tableau on the right with output integer 7, but this time, 7 is to be inserted into the fourth column.

Inserting 3 into the second row of the tableau on the left results in the tableau shown below with output integer 4 to be inserted into the third column.



(S4) If replacing y with x does not result in an increasing tableau, then do not change the row (resp. column). In this case, y is the output integer. If x was inserted into a column or if y was on the main diagonal, proceed to insert all future output integers into the next column to the right. If x was inserted into a row, then insert y into the row below.

Example 5.13. Inserting 2 into the first row of the tableau below does not change the tableau and produces output integer 5 to be inserted into the second row.



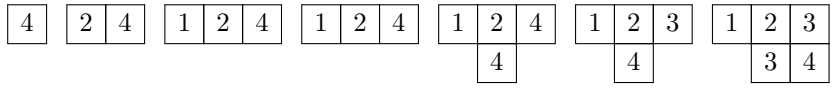
Inserting 5 into the third column does not change the tableau and gives output integer 6 to be inserted into the fourth column.

Inserting 2 into the second row does not change the tableau. In this case, the output integer is 3 and is to be inserted into the third column.

Using this insertion algorithm, we define the *shifted Hecke insertion tableau* of a word $w = w_1 w_2 \dots w_n$ to be

$$P_S(w) = (\dots((\varnothing \leftarrow w_1) \leftarrow w_2)\dots) \leftarrow w_n.$$

Example 5.14. We show the sequence of shifted tableaux obtained while computing $P_S(4211232)$. We start by inserting 4 into the first row.



Remark 5.15. The result of such insertion agrees with that of K-theoretic jeu de taquin rectification described in [8]. See [12].

In this setting, recording tableaux will be set-valued shifted tableaux.

Definition 5.16. *A* set-valued shifted tableau T *of shifted shape* λ *is a filling of the boxes with finite, nonempty subsets of primed and unprimed positive integers so that*

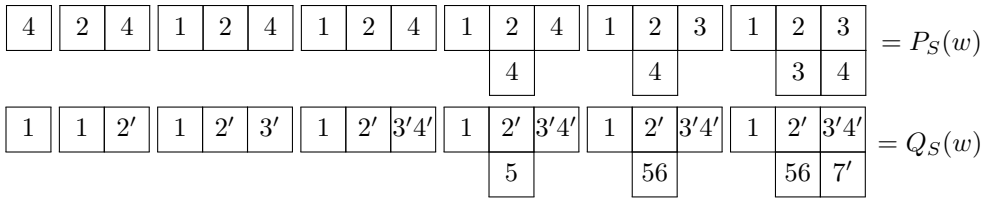
(1) *the smallest number in each box is greater than or equal to the largest number in the box directly to the left of it (if that box is present),*
(2) *the smallest number in each box is greater than or equal to the largest number in the box directly to the above it (if that box is present),*

(3) *any positive integer appears at most once, either primed or unprimed, but not both, and*
(4) *there are no primed entries on the main diagonal.*

A set-valued shifted tableau is called *standard* if the set of labels consists of $1, 2, \ldots, n$, each appearing either primed or unprimed exactly once, for some n.

A recording tableau for a word $w = w_1 w_2 \ldots w_n$ is a standard shifted set-valued tableau and is obtained as follows. Begin with $Q_S(\varnothing) = \varnothing$. If the insertion of w_k into $P_S(w_1 \ldots w_{k-1})$ resulted in adding a new box to $P_S(w_1 \ldots w_{k-1})$, add this same box with label k if the box was added via row insertion and k' if the box was added via column insertion to $Q_S(w_1 \ldots w_{k-1})$ to obtain $Q_S(w_1 \ldots w_k)$. If the insertion of w_k into $P_S(w_1 \ldots w_{k-1})$ did not change the shape of $P_S(w_1 \ldots w_{k-1})$, obtain $Q_S(w_1 \ldots w_k)$ from $Q_S(w_1 \ldots w_{k-1})$ by adding the label k to the box where the insertion terminated if the last move was a row insertion into a nonempty row and k' if the last move was a column insertion. If the last move was row insertion into an empty row, label the box where the insertion terminated k'.

Example 5.17. The top row of tableaux shows the sequence of tableaux obtained from inserting $w = 4211232$ as in the previous example, and the bottom row shows the corresponding steps to form $Q_S(w)$.



We next define a reverse insertion procedure so that given a pair $(P_S(w), Q_S(w))$, we can recover w.

First locate the box containing the largest label of $Q_S(w)$, call the label n and the position of the box (i_n, j_n), and find the corresponding box in $P_S(w)$. Say the integer in position (i_n, j_n) of $P_S(w)$ is y_n. We then perform reverse insertion on cell (i_n, j_n) of $P_S(w)$ by following the rules below.

(rS1) If n is the only label in cell (i_n, j_n) of $Q(w)$, remove box (i_n, j_n) from $P_S(w)$ and reverse insert y_n into the row above if n is unprimed and into the column to the left if n is primed.
(rS2) If n is not the only label in cell (i_n, j_n) of $Q_S(w)$, do not remove box (i_n, j_n) from $P_S(w)$, but still reverse insert y_n into the row above if n is unprimed and into the column to the left if n is primed.

If y_n is reverse inserted into row $i_n - 1$, let x be the largest label in row $i_n - 1$ with $x < y_n$. If y_n is reverse inserted into column $j_n - 1$, let x be the largest label in column $j_n - 1$ with $x < y_n$.

(rS3) If replacing x with y_n results in an increasing shifted tableau, replace x with y_n. If y_n was reverse column inserted and x was not on the main diagonal, reverse insert x into the column to the left. Otherwise, reverse insert x into the row above.
(rS4) If replacing x with y_n does not result in an increasing shifted tableau, leave the row or column unchanged. If y_n was reverse column inserted and x was not on the main diagonal, reverse insert x into the column to the left. Otherwise, reverse insert x into the row above.

If we are in the first row of the tableau and the last step was a reverse row insertion or we are in the first column and the last step was a reverse column insertion, then

x and the modified tableau, which we will call $P_{S,1}(w)$, are the final output value. Define $x_1 = x$.

Repeat this process using the pair $(P_{S,1}(w), Q_{S,n-1}(w))$, where $Q_{S,n-1}(w)$ is the result of removing the entry n or n' from $Q_S(w)$. Define x_2 to be the output value and $P_{S,2}(w)$ to be the modified tableau. Continue this process with pairs $(P_{S,2}(w), Q_{S,n-2}(w)), \ldots, (P_{S,n-1}(w), Q_{S,1}(w))$. Then we claim $w = x_1 x_2 \ldots x_n$.

THEOREM 5.18. *There is a bijection between pairs consisting of an increasing shifted tableau and a standard set-valued shifted tableau of the same shape, (P, Q), and words, where the word w corresponds to the pair $(P_S(w), Q_S(w))$.*

Proof. We show that given a pair of tableaux of the same shape, (P, Q), where P is an increasing shifted tableau on some $[n]$ and Q is standard shifted set-valued tableau on $1 < 2' < 2 < 3' < \cdots < n' < n$, we can recover w so that $P = P_S(w)$ and $Q = Q_S(w)$. Assume first that (P, Q) has been obtained by inserting some integer h into a pair (Y, Z). We must show that the reverse insertion procedure defined above recovers h and (Y, Z) from (P, Q).

It is clear that if the insertion of h ended by adding a new box to Y and Z using (S1), then reverse insertion steps (rS1), (rS2), (rS3), and (rS4) undo insertion steps (S1), (S2), (S3), and (S4), respectively. Therefore, it suffices to show that if the insertion of h does not result in adding a new box to Y and Z using (S2), then the reverse insertion procedure can recover h from (Y, Z).

Suppose the last step of the insertion of h into Y is inserting some x into nonempty row i using (S2) and the insertion terminates in row $i + j$. We first use (rS2) and reverse insert the entry at the end of row $i + j$ into row $i + j - 1$, which will not change row $i + j - 1$ because the entry being reverse inserted is strictly larger than all entries in row $i + j - 1$ but cannot replace the entry at the end of row $i + j - 1$ since this would put it directly above the occurence of the entry in row $i + j$. Next, we reverse insert the entry at the end of row $i + j - 1$ into $i + j - 2$, and in the same manner, it will not change row $i + j - 2$. This process continues until we reverse insert the entry at the end of row $i + 1$ into i and obtain output integer x. From this point onward, the reverse insertion rules will exactly undo the corresponding insertion rules to recover h. We can use the transpose of this argument if x is column inserted into column j using (S2) and the insertion terminates in column $j + i$.

For example, inserting 4 into the first row of the tableau below does not change the tableau, and the insertion terminates at $(3, 4)$. Starting reverse insertion at $(3, 4)$, we reverse insert 6 into the second row. It cannot replace the 5, so the third row doesn't change, and 5 is reverse inserted into the first row. The 5 cannot replace the 4, so the first row is unchanged, and we end with the original tableau and the integer 4.

1	2	3	4
	3	4	5
		5	6

If x is inserted into empty row i using (S2), then our original tableau looked like the one shown below, where the bottom row shown is row $i - 1$ and the dot indicates where the insertion terminated. In this case, x must have been bumped by a row insertion of a into row $i - 1$. We show reverse insertion starting from the box where the insertion terminated recovers the step of inserting a into row $i - 1$.

	a	x				$\cdot$	

Using (rS4), we reverse insert the entry at the end of row $i-1$ into the column to the left. The column will not change since the entry being reverse inserted is strictly larger than all entries in the column and cannot replace the entry at the end of the column. We then reverse insert the second rightmost entry of row $i-1$ into the column to its left. Continuing in this manner, we eventually reverse column insert x into the main diagonal, which results in reverse row inserting a into row $i-2$. From this point on, the reverse insertion rules will exactly undo the corresponding insertion rules to recover h.

For example, suppose some insertion ends with inserting 2 into the empty second row of the tableau below. The insertion terminates in position $(1,4)$, and we see that the 2 must have been bumped by a row insertion of 1 into the first row. We start reverse column insertion in position $(1,4)$ and reverse column insert 4 into the third column. The integer 4 cannot replace the 3 in the third column since it would be directly to the left of the 4 in the fourth column, so the third column is left unchanged. We reverse column insert 3 into the second column. Again, the column remains unchanged, and we reverse insert 2 into the first column. The 2 cannot replace the 1, so the column is unchanged. Since the 1 was on the main diagonal, we next reverse row insert 1 into the row above.

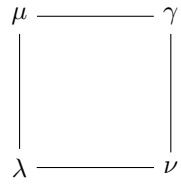
1	2	3	4

□

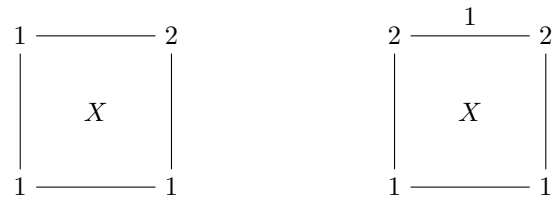
5.4. Shifted Hecke growth and decay. As before, given any word $w = w_1 w_2 \dots w_k$, containing $n \leqslant k$ distinct numbers, we can create an $n \times k$ array with an X in the w_i^{th} square from the bottom of column i. Note that there can be multiple X's in the same row but is at most one X per column.

We will label the corners of each square with a shifted partition and label some of the horizontal edges of the squares with a specific inner corner where the insertion terminated and/or a 'c' to designate column insertion. To denote a specific inner corner, we will give either its row or column. For example, the edge label 2 denotes the inner corner at the end of the second row of the shifted diagram, and an edge labeled 2c denotes the inner corner at the end of the second column of the shifted diagram. We begin by labeling all corners along the bottom row and left side of the diagram with the empty shape, $\varnothing$, as before.

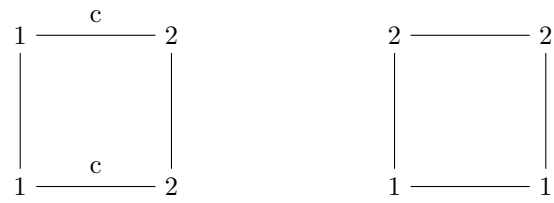
To complete the labeling of the corners, suppose the corners μ, λ, and ν are labeled, where μ, λ, and ν are as in the picture below. We label γ according to the following rules.



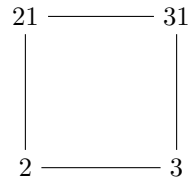
- *If the square contains an X:*
  (1) If $\lambda_1 = \mu_1$, then γ/μ consists of one box in the first row.
  (2) If $\lambda_1 + 1 = \mu_1$, then $\gamma = \mu$ and the edge between them is labeled with a 1 to signify the inner corner in the first row of γ.



- *If the square does not contain an X and $\mu = \lambda$ or if the square does not contain an X and $\nu = \lambda$ with no edge label on the bottom edge*
  (3) If $\mu = \lambda$, then set $\gamma = \nu$ and label the top edge label as the bottom edge if one exists. If $\nu = \lambda$, then $\gamma = \mu$.



- *If $\nu \not\subseteq \mu$, the square does not contain an X, and the square does not satisfy the conditions of the previous bullet point:*
  (4) In the case where $\nu \not\subseteq \mu$, $\gamma = \nu \cup \mu$, and the top edge label is the same as the bottom edge label.



- *If $\nu \subseteq \mu$, the square does not contain an X, and the square does not satisfy the conditions of the second bullet point:*

  Suppose the square of ν/λ is one box in position (i, j). If (i, j) is not on the diagonal and the edge between λ and ν is not labeled c:
  (5) If there is no box of μ/λ in row $i+1$ of μ, then γ is obtained from μ by adding a box to the end of row $i+1$.
  (6) If there is a box of μ/λ in row $i+1$ of μ, then $\gamma = \mu$ and the top edge is labeled $i+1$.

  If (i, j) is on the main diagonal or the bottom edge is labeled c:
  (7) If there is no box of μ/λ in column $j+1$ of μ, then γ is obtained from μ by adding a box to the end of column $j+1$. The top edge is labeled c.
  (8) If there is a box of μ/λ in column $j+1$ of μ, then $\gamma = \mu$. The top edge is labeled with c and $j+1$.

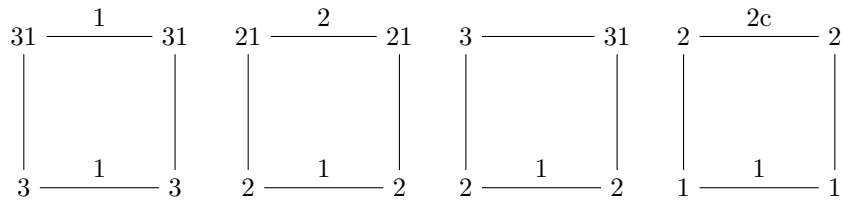
If $\lambda = \nu$ and the bottom edge is labeled with i but not with c:

(9) If there is no box of μ/λ immediately to the right of or immediately below the box at the end of row i of ν, then $\gamma = \mu$ and the top edge is labeled by i.

(10) If there is a box of μ/λ directly below the box at the end of row i of ν, then $\gamma = \mu$ and the edge between them is labeled by $i+1$.

(11) If there is a box of μ/λ immediately to the right of the box at the end of row i of ν, say in position $(i, j+1)$, no box of μ/λ in row $i+1$, and the outer corner of row $i+1$ is not in column j, then γ/μ is one box in row $i+1$.

(12) If there is a box of μ/λ immediately to the right of the box at the end of row i of ν, say in position $(i, j+1)$, no box of μ/λ in row $i+1$, and the outer corner of row $i+1$ is in column $j+1$, then the outer corner of row $i+1$ is on the main diagonal. In this case, $\gamma = \mu$, and the top edge is labeled with $j+1$ and c.

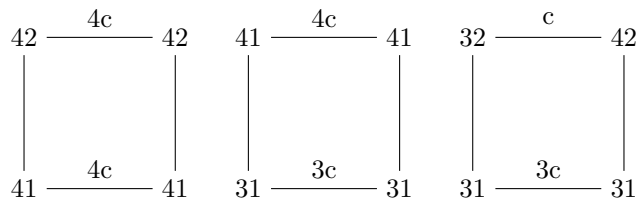


If $\lambda = \nu$ and the bottom edge is labeled with j and with c:

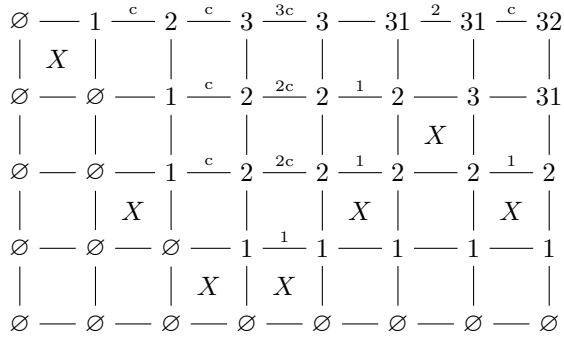
(13) If there is no box of μ/λ immediately to the right of or immediately below the box at the end of column j of ν, then $\gamma = \mu$ and the edge between them is labeled by j and c.

(14) If there is a box of μ/λ immediately to the right of the box at the end of column j of ν, then $\gamma = \mu$ and the top edge is labeled by $j+1$ and c.

(15) If there is a box of μ/λ directly below the box at the end of column j of ν but no box of μ/λ in column $j+1$, then γ/μ is one box in column $j+1$. The top edge is labeled c.



We call the resulting array the *shifted Hecke growth diagram* of w. In our previous example with $w = 4211232$, we would have the diagram below.



Let $\mu_0 = \varnothing \subseteq \mu_1 \subseteq \dots \mu_k$ be the sequence of shifted partitions across the top of the growth diagram, and let $\nu_0 = \varnothing \subseteq \nu_1 \subseteq \dots \nu_n$ be the sequence of shifted partitions on the right side of the growth diagram. These sequences correspond to increasing shifted tableaux $Q(w)$ and $P(w)$, respectively. If the edge between μ_i and μ_{i+1} is labeled j, then $\mu_i = \mu_{i+1}$, and the label $i+1$ of $Q(w_1 \dots w_{i+1})$ is placed in the box at the end of row j of $Q(w_1 \dots w_i)$. If the edge between μ_i and μ_{i+1} is labeled jc, then $\mu_i = \mu_{i+1}$, and the label $i+1'$ of $Q(w_1 \dots w_{i+1})$ is placed in the box at the end of column j of $Q(w_1 \dots w_i)$. Since there is an X in each column of a shifted Hecke growth diagram, there must be an edge label when $\mu_i = \mu_{i+1}$. For the shifted Hecke growth diagram above, we have:

$$P(w) = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline \multicolumn{1}{c}{} & 3 & 4 \\ \cline{2-3}\end{array} \qquad Q(w) = \begin{array}{|c|c|c|}\hline 1 & 2' & 3'4' \\ \hline \multicolumn{1}{c}{} & 56 & 7' \\ \cline{2-3}\end{array}$$

We see that this P and Q agree with $P_S(w)$ and $Q_S(w)$, which is not a coincidence.

Theorem 5.19. *For any word w, the increasing shifted tableaux P and Q obtained from the sequence of shifted partitions across the right side of the shifted Hecke growth diagram for w and across the top of the shifted Hecke growth diagram for w, respectively, are $P_S(w)$ and $Q_S(w)$, the shifted Hecke insertion and recording tableau for w.*

Proof. Recall the definitions of $T(i,j)$ and $R(i,j)$ from the proof of Theorem 4.16. We define them analogously in this context and prove the analogous result by induction on $i+j$. The desired statement is clearly true when $i = 0$ or $j = 0$. Suppose we are considering some $T(s,t)$ with $s,t > 0$.

We will use the same two claims from the Hecke growth diagram proof, which have analogous proofs.

Claim 5.20. *Oriented as in the square above, $|\nu/\lambda| \leqslant 1$.*

Claim 5.21. *Oriented as in the square above, μ/λ is a rook strip, that is, no two boxes in μ/λ are in the same row or column.*

(1-2) Note that if the square contains an X, then $\lambda = \nu$ because there are no other X's in column t. If $\lambda_1 = \mu_1$, then there are no s's in the first row of $T(s,t-1)$. Thus the inserted s will be added to the end of the first row, creating γ. If $\lambda_1 = \mu_1$, then there is already an s at the end of the first row of $T(s,t-1)$. The inserted s cannot be added to the first row, so the $\mu = \gamma$. In this case, the insertion terminated at the end of the first row. We know this is an inner corner because s is the largest number inserted so far, so there can not be anything directly below the box labeled s in the first row.

(3) If $\mu = \lambda$, there is no X to the left of square (s,t) in row s. Thus nothing changes between ν and γ as we consider occurrences of entry s. If $\lambda = \nu$, there is no X below square (s,t) in column t and so no new insertion.Thus nothing changes between μ and γ.

(4) Suppose ν/λ is one box in row n and μ/λ contains boxes in rows exactly $\{j_1, j_2, \ldots, j_k\}$. Suppose the box in ν/λ corresponds to inserting $r < s$. Then $\nu \not\subseteq \mu$ implies that $n \notin \{j_1, j_2, \ldots, j_k\}$. Since ν/λ is one box, the last action in the insertion sequence of r into the $T(s-1, t-1)$ is a box being inserted in row n. Since there is no s in row n, the bumping sequence when inserting r into the $T(s, t-1)$ will not disturb the s's. The edge between λ and ν is either labeled c or not, corresponding to whether or not inserting the number in column t involves column insertions. This remains unchanged.

(5-6) Since ν/λ is one box in position (i,j), there is some X in position (r,t) for $r < s$. It follows that inserting r into $T(s, t-1)$ will result in bumping the s in position (i,j) of $T(s, t-1)$. If there is no box in row $i+1$ of μ/λ, there is not an s in row $i+1$ of $T(s, t-1)$. Thus the s in position (i,j) is bumped from row i by r and added to the end of row $i+1$. If there is a box in row $i+1$ of μ/λ, there is already an s in row $i+1$ of $T(s, t-1)$. It follows that the s bumped from row i cannot be added to row $i+1$, and so $\gamma = \mu$. The insertion terminates at the end of row $i+1$ since there cannot be any boxes directly below the s in row $i+1$.

(7-8) If (i,j) is on the main diagonal or the edge between λ and ν is labeled c, any bumped entries will be column inserted into the column j. The rules are the transpose of rules 5 and 6, and the new edge is labeled c since all subsequent bumping will be column inserted after the first column insertion.

(9) Since there is no box directly to the right of (i,j), nothing is bumped when inserting $r < s$ into $T(s, t-1)$. Since there is no box directly below (i,j), the insertion terminates at inner corner (i,j).

(10) In the situation described above, if there is a box directly below (i,j), the insertion now terminates at the inner corner in row $i+1$.

(11) When inserting r into $T(s, t-1)$, the s in position $(i, j+1)$ is bumped but not replaced and inserted into row $i+1$. Since there is no s in row $i+1$, this s can be added to the row as long as it is not directly below the s in position (i,j).

(12) When inserting r into $T(s, t-1)$, the s in position $(i, j+1)$ is bumped but not replaced and inserted into row $i+1$. Since there is no s in row $i+1$, we attempt to add this s to the end of row $i+1$. However, this s would end directly below the s in row i, and so cannot be added. If the outer corner of row $i+1$ is not on the main diagonal, the insertion terminates at the end of row $i+1$. If it is on the diagonal, the insertion terminates at the end of row i, and we know the last box in row i must be in column j since there are no entries larger than s. This edge is labeled with c since this action may bump an entry from the main diagonal in column j in future steps.

(13-15) Rules 13–15 are the transpose of rules 10–12, and may be explained in an analogous way. □

We can also formulate the rules for the reverse insertion, as follows. The proof is omitted for brevity.

Reverse Rules.

(R1) If γ/μ is one box in the first row, then the square has a X and $\lambda = \nu$.

(R2) If $\gamma = \mu$ and the edge between them is labeled 1, then the square has an X and $\lambda = \nu$.
(R3) If $\gamma = \mu$ with no edge label, then $\lambda = \mu$. If $\gamma = \nu$, then $\lambda = \mu$ and the edge label between λ and ν is the same as the label between γ and μ.
(R4) If $\nu \not\subseteq \mu$, then $\lambda = \mu \cap \nu$.
(R5) If γ/μ is one box in row $i+1$ for some $i > 0$, the edge between γ and μ has no label, and μ/ν has no box in row i, then ν/λ is one box in row i.
(R11) If γ/μ is one box in row $i+1$ for some $i > 0$, the edge between γ and μ has no label, and μ/ν has a box in row i, then $\lambda = \nu$ and the edge between them is labeled i.
(R7) If γ/μ is one box in column $j+1$ for some $j > 0$, the edge between γ and μ is labeled c, and μ/ν has no box in column j, then ν/λ is one box in column j and the edge between ν and λ is labeled c.
(R15) If γ/μ is one box in column $j+1$ for some $j > 0$, the edge between γ and μ is labeled c, and μ/ν has a box in column j, then $\nu = \lambda$ and the edge between them is labeled j and c.
(R6) If $\gamma = \mu$, the edge between them is labeled $i+1$ for some $i > 0$, μ/ν has a box in row $i+1$, and $\mu_i \neq \mu_{i+1} + 1$, then ν/λ is one box in row i.
(R9) If $\gamma = \mu$, the edge between them is labeled $i+1$ for some $i > 0$, μ/ν has no box in row $i+1$, then $\lambda = \nu$ and the edge between them is labeled $i+1$.
(R10) If $\gamma = \mu$, the edge between them is labeled $i+1$ for some $i > 0$, μ/ν has a box in row $i+1$, and $\mu_i = \mu_{i+1} + 1$, then $\lambda = \nu$ and the edge between them is labeled i.
(R8) If $\gamma = \mu$, the edge between them is labeled $j+1$ and c for some $j > 0$, μ/ν has a box in column $j+1$, and the j^{th} and $j+1^{th}$ columns of μ are not the same size, then ν/λ is one box in column j.
(R12) If $\gamma = \mu$, the edge between them is labeled $j+1$ and c for some $j > 0$, and the bottom box of column $j+1$ is in the last row of μ, then $\lambda = \nu$ and the edge between them is labeled with the bottom row of ν.
(R13) If $\gamma = \mu$, the edge between them is labeled $j+1$ and c for some $j > 0$, μ/ν has no box in column $j+1$, then $\lambda = \nu$ and the edge between them is labeled $j+1$ and c.
(R14) If $\gamma = \mu$, the edge between them is labeled $j+1$ and c for some $j > 0$, μ/ν has a box in column $j+1$, and the j^{th} and $j+1^{th}$ columns of μ are the same size, then $\lambda = \nu$ and the edge between them is labeled j and c.

6. Major examples: Young–Fibonacci lattice

6.1. Möbius deformation of the Young–Fibonacci lattice. In order to describe the Möbius deformation of the Young–Fibonacci lattice, we first need to determine its Möbius function.

Theorem 6.1. *Let $X > Y$ be elements in the Young–Fibonacci lattice and suppose X has n edges below it. Then*

$$\mu(Y,X) = \begin{cases} n-1 & \text{if } 2Y = X \\ -1 & \text{if } X \text{ covers } Y \\ 0 & \text{otherwise} \end{cases}$$

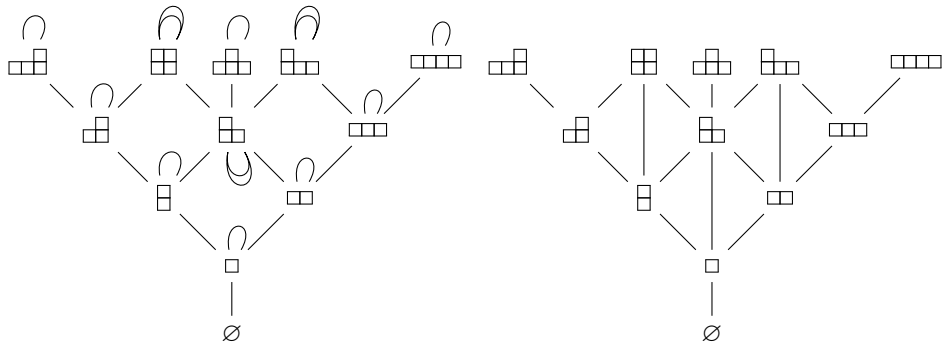
Proof. Suppose $X > Y$. If $\rho(X) = \rho(Y) + 1$, then clearly, $\mu(Y,X) = -1$.

If $\rho(X) = \rho(Y) + 2$ and $X = 2Y$, then $\mu(Y,X) = n-1$ since $2Y$ covers exactly the elements that cover Y. If $X \neq 2Y$, then it can be checked that X covers exactly one element that covers Y. Therefore $\mu(Y,X) = 0$.

Assume now $\rho(X) \geqslant \rho(Y) + 2$. We will argue that $\mu(Y, X) = 0$ by induction on $\rho(X)$. The base case $\rho(X) = \rho(Y) + 2$ was just proved. Let us argue the step of induction. There are two cases.

- We have $2Y < X$. In this case, we argue that $\sum_{Y \leqslant W < X} \mu(Y, W) = 0$. Indeed, $\sum_{Y \leqslant W \leqslant 2Y} \mu(Y, W) = 0$. If W is in the interval $[Y, X]$ but not in $[Y, 2Y]$, then W does not cover Y and $W \neq 2Y$. Thus by the inductive hypothesis $\mu(Y, W) = 0$.
- We have $2Y \not< X$. We claim there is only one Z that covers Y such that $Z < X$. Indeed, since the Young–Fibonacci lattice is a lattice, and since the join of any two distinct such Z's is $2Y$, we would have a contradiction. Then $\mu(Y, Y) + \mu(Y, Z) = 1 + (-1) = 0$. For the rest of $Y < W < X$ we have $\mu(Y, W) = 0$ by the induction assumption. □

By the previous result, for each element X in the Young–Fibonacci lattice, the Möbius deformation is formed by adding an upward-oriented loop for each of the n elements X covers and adding $n-1$ downward-oriented edges from X to Y if $X = 2Y$. For example, there are 2 edges from 221 to 21. The first few ranks are shown below.



THEOREM 6.2. *The Möbius deformation of the Young–Fibonacci lattice forms a dual filtered graph with*

$$DU - UD = D + I.$$

Proof. Suppose that shape X covers shape Y, and first assume that $\rho(Y) = \rho(X) - 1$. The up-down paths from X to Y that consist of a loop at X followed by following the edge from X to Y can be counted by the number of loops at X, i.e. the number of edges below X in the Young–Fibonacci lattice. In the second type of up-down path, we start at X, move up to a distinct shape W, and then move down to Y. The only W with $\rho(W) = \rho(Y) + 2$ that cover Y is the W defined by $W = 2Y$. This W covers X by definition of the Young–Fibonacci lattice. To count the number of paths through W, we need to count the number of edges from W to Y. This number is the same as one fewer than the number edges below W in $\mathbb{YF}$, which is the same as one fewer than the number of edges above Y in $\mathbb{YF}$.

To count down-up paths, we first count the paths that consist of the edge from X to Y followed by a loop at Y. These can be counted by the number of loops at Y or, in other words, the one fewer than the number of edges above Y in $\mathbb{YF}$. The second type of down-up paths involve traveling from X down to some Z with $\rho(Z) + 2 = \rho(X)$ and then going up to Y. There is a unique such Z defined by $2Z = X$, and Y covers this Z. We thus need to count the number of ways to get from X to Z, which is the same as one fewer than the number of edges below X in $\mathbb{YF}$.

Putting the previous two arguments together, $[Y](DU-UD)(X)=1$ in this case, as desired.

Now suppose $\rho(Y)=\rho(X)-2$. The only up-down paths from X to Y consist of a loop at X followed by an edge from X to Y. There are

$$(\#\{\text{edges below } X \text{ in } \mathbb{YF}\})(\#\{\text{edges below } X \text{ in } \mathbb{YF}\}-1)$$

such paths.

To count down-up paths, we must count the number of ways to choose an edge from X to Y and then choose a loop of Y. There are

$$(\#\{\text{edges below } X \text{ in } \mathbb{YF}\}-1)(\#\{\text{edges above } Y \text{ in } \mathbb{YF}\}-1)$$

such paths. Since

$$\#\{\text{edges down from } X \text{ in } \mathbb{YF}\}=\#\{\text{edges above } Y \text{ in } \mathbb{YF}\},$$

we have that

$$[Y](DU-UD)(X)=\#\{\text{edges below } X \text{ in } \mathbb{YF}\}-1=[Y]D(X). \qquad \square$$

6.2. K-Young–Fibonacci insertion. As with the previous examples of Möbius construction, we describe an insertion procedure that corresponds to the Möbius deformation of the Young–Fibonacci lattice, which is an analogue of the Young–Fibonacci insertion of Fomin. We recommend [10] for details about Young–Fibonacci insertion. In this analogue, any walk upward from $\varnothing$ will correspond to a recording tableau and any walk downward to $\varnothing$ will correspond to an insertion tableau. (See Example 6.10.) We begin by describing the insertion tableaux in this setting.

Definition 6.3. *A K-*Young–Fibonacci (KYF) tableau *is a filling of a snakeshape with positive integers such that*

(i) *for any pair* $\begin{array}{|c|}\hline B\\\hline A\\\hline\end{array}$ *the inequality $A \leqslant B$ holds;*

(ii) *to the right of any* $\begin{array}{|c|}\hline B\\\hline A\\\hline\end{array}$*, there are no numbers from the interval $[A,B]$ in either the upper or lower rows;*

(iii) *if the position above* $\boxed{A}$ *is not occupied yet, then to the right of* $\boxed{A}$ *there are no numbers greater than or equal to A in either the lower or upper rows;*

(iv) *any pair* $\begin{array}{|c|}\hline A\\\hline A\\\hline\end{array}$ *must come before any single box* $\boxed{B}$ *and must not be in rightmost column of the snakeshape. In addition, A may not be the smallest entry in the snakeshape.*

Example 6.4. The three snakeshapes below are valid KYF tableaux.

$$\begin{array}{|c|c|c|c|c|c|}\hline \multicolumn{1}{c}{} & 3 & 4 & \multicolumn{1}{c}{} & 7 & \multicolumn{1}{c}{} \\\hline 9 & 2 & 1 & 8 & 6 & 5 \\\hline\end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 2 & 3 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & 5 \\\hline 2 & 1 & 7 & 6 & 4 \\\hline\end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 4 & 5 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & 2 \\\hline 4 & 5 & 6 & 3 & 1 \\\hline\end{array}$$

The snakeshapes below are not valid KYF tableaux. The first violates condition (iii), the second violates condition (ii), and the third, fourth, and fifth violate condition (iv).

$$\begin{array}{|c|c|c|}\hline \multicolumn{1}{c}{} & 3 & 4 \\\hline 4 & 2 & 1 \\\hline\end{array} \qquad \begin{array}{|c|c|c|c|}\hline 3 & 4 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\\hline 2 & 1 & 4 & \multicolumn{1}{c}{} \\\hline\end{array} \qquad \begin{array}{|c|c|c|c|c|}\hline 4 & 6 & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} & 2 \\\hline 4 & 5 & 7 & 3 & 2 \\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 3 & 2 \\\hline 1 & 2 \\\hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\\hline 1 & 2 \\\hline\end{array}$$

Let τ be a valid KYF tableau. We show how to insert positive integer x into τ to obtain a new valid KYF tableau that may or may not be different than τ. Notice that this insertion procedure is different than Hecke insertion and shifted Hecke insertion in that the algorithm does not proceed row-by-row or column-by-column. As before, we designate a specific box of the resulting tableau, τ', as being the box where the insertion terminated. We will later use this notion to define recording tableaux.

The rules for inserting any positive integer x into τ are as follows:

(YF1) If x is equal to the smallest entry in τ, then $\tau = \tau'$. In this case, we say the insertion terminated in the top cell of the first column of $\tau = \tau'$. If this is not the case, continue to step 2.

EXAMPLE 6.5. Inserting 1 into the tableau on the left or 2 into the tableau on the right will not change the tableaux.

	2	
4	1	3

3		
2	5	4

(YF2) Attach a new box $\boxed{x}$ just to the left of τ in the lower row.

(YF3) Find all the entries of τ that are greater than or equal to x and sort them:

$$x \leqslant a_1 \leqslant a_2 \leqslant \cdots \leqslant a_k.$$

If in τ we have $\begin{array}{|c|}\hline A \\ \hline A \\ \hline \end{array}$, then we consider the A in the upper row to be the larger of the two.

(YF4) If $a_i = a_{i+1}$, then replace a_{i+1} with $*$.

(YF5) Now put a box $\boxed{\bullet}$ just above $\boxed{x}$ and move the a_i chainwise according to the following rule: a_1 moves into the new box, a_2 moves to a_1's original position, a_3 moves to a_2's original position, etc. The box that was occupied by a_k disappears. If it was located in the lower row, then the left and right parts of the snake are concatenated.

(YF6) If there is a box with $*$ and no box directly above it, delete this box and concatenate the left and right parts of the snake. If this happens, we say the insertion terminated at the box in the upper row of the column directly to the right of this concatenation. Otherwise, a box was added in the insertion process, and we say the insertion terminated at this new box.

(YF7) Replace any pair $\begin{array}{|c|}\hline A \\ \hline * \\ \hline \end{array}$ with $\begin{array}{|c|}\hline A \\ \hline A \\ \hline \end{array}$.

EXAMPLE 6.6. Let's insert 3 into $\tau = \begin{array}{|c|c|c|}\hline 3 & 4 & \\ \hline 2 & 4 & 1 \\ \hline \end{array}$. We first attach $\boxed{3}$ as shown below.

	3	4	
3	2	4	1

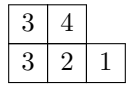
Next, we locate and order

$$3 \leqslant (a_1 = 3) \leqslant (a_2 = 4) \leqslant (a_3 = 4),$$

and we replace a_3 with $*$. After shifting the boxes as in (YF5), we have

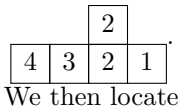
3	4		
3	2	*	1

We then delete the box with $*$ to obtain the final KYF tableau below.



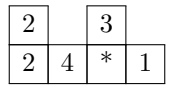
This insertion terminates at the box at the top of the third column.

Notice also that inserting 1 into τ does not change the tableau, and this insertion terminates at the box at the top of the first column.

Example 6.7. We insert 2 into KYF tableau  . We first attach a box containing 2 to the left of the original tableau. We then locate and order

$$2 \leqslant (a_1 = 2) \leqslant (a_2 = 2) \leqslant (a_3 = 3) \leqslant (a_4 = 4).$$

We replace a_2 with $*$ and shift the boxes to obtain



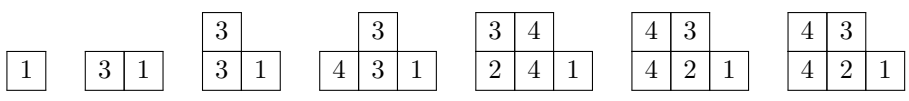
After performing the last step of the insertion procedure, we end with the tableau shown below.

This insertion terminated at the box at the top of the first column of the resulting tableau.

As usual, for a word $w = w_1 w_2 \dots w_n$, we define $P_{YF}(w)$ by setting

$$P_{YF}(w_1 \dots w_k) = P_{YF}(w_1 \dots w_{k-1}) \leftarrow w_k.$$

Example 6.8. The sequence of KYF tableaux below shows the intermediate tableaux obtained in computing $P_{YF}(1334241)$, which is shown on the right.



We need the next definition to define the recording tableaux for KYF insertion.

DEFINITION 6.9. *A* standard set-valued KYF tableau *is a snakeshape whose boxes are filled with finite, nonempty subsets of positive integers that satisfy the following conditions, where* $\bar{A}, \bar{B}, \bar{C}$ *are subsets,* $\bar{A} < \bar{B}$ *when* $\max(\bar{A}) < \min(\bar{B})$*, and the letters* $[n]$ *are used exactly once for some* n*:*

(i) *for any pair* $\begin{array}{|c|}\hline \bar{B} \\ \hline \bar{A} \\ \hline \end{array}$ *the inequality* $\bar{A} < \bar{B}$ *holds;*

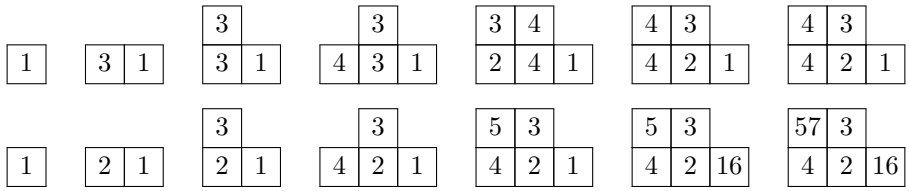
(ii) *to the right of any* $\begin{array}{|c|}\hline \bar{B} \\ \hline \bar{A} \\ \hline \end{array}$*, there are no numbers from the interval* $[\max(\bar{A}), \min(\bar{B})]$ *in either the upper or lower rows;*

(iii) *if the position above* $\boxed{\bar{A}}$ *is not occupied yet, then to the right of* $\boxed{\bar{A}}$ *there are no entries greater than* $\min(\bar{A})$ *in either the lower or upper rows;*

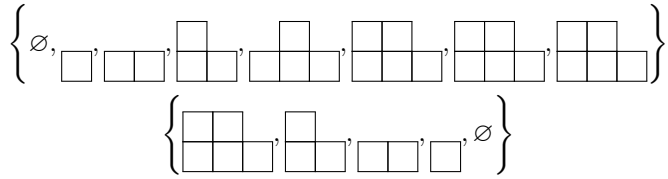
A recording tableau for a word $w = w_1 w_2 \dots w_n$ is a standard set-valued KYF tableau and is obtained as follows. Begin with $Q_{YF}(\varnothing) = \varnothing$. If the insertion of w_k into $P_{YF}(w_1 \dots w_{k-1})$ resulted in adding a new box to $P_{YF}(w_1 \dots w_{k-1})$, add this same box with label k to $Q_{YF}(w_1 \dots w_{k-1})$ to obtain $Q_{YF}(w_1 \dots w_k)$.

If the insertion of w_k into $P_{YF}(w_1 \dots w_{k-1})$ did not change the shape of $P_{YF}(w_1 \dots w_{k-1})$, obtain $Q_{YF}(w_1 \dots w_k)$ from $Q_{YF}(w_1 \dots w_{k-1})$ by adding the label k to the box where the insertion terminated.

EXAMPLE 6.10. In Example 6.8, we computed $P_{YF}(1334241)$. We repeat this computation on the top row and show the corresponding steps of building $Q_{YF}(1334241)$ on the bottom row.



The top row below shows the walk in G_1 corresponding to the recording tableau. Here, we must pair the loops on each shape to its inner corners. In the walk shown, the last two steps use distinct loops at the shape 221. The second row below shows the walk in G_2 corresponding to the insertion tableau.



We next define a reverse insertion procedure so that given a pair $(P_{YF}(w), Q_{YF}(w))$, we can recover $w = w_1 \dots w_n$.

First locate the box containing the largest label of $Q_{YF}(w)$, call the label n and its column c, and find the corresponding box in $P_{YF}(w)$. Let x denote the label in the leftmost box of the bottom row of $P_{YF}(w)$.

(rYF1) If the label n of Q was not the only label in its box and was located in the upper row of the first column, then $P_{YF}(w) = P_{YF}(w_1 \dots w_{n-1}) \leftarrow s$, where s is the smallest entry in $P_{YF}(w_1 \dots w_{n-1})$. Finally, $Q_{YF}(w_1 \dots w_{n-1})$ is obtained from $Q_{YF}(w)$ by removing the label n.

In all remaining cases, $P_{YF}(w) = P_{YF}(w_1 \dots w_{n-1}) \leftarrow x$, and we describe how to construct $P_{YF}(w_1 \dots w_{n-1})$. In each case, $Q_{YF}(w_1 \dots w_{n-1})$ is obtained from $Q_{YF}(w)$ by removing the label n. If n is the only label in its box, the box is removed.

(rYF2) If the label n of Q was the only label in its box:
- (a) Delete the leftmost square in the bottom row.
- (b) Let k denote the largest entry in $P_{YF}(w)$, and sort the entries of $P_{YF}(w)$ as shown below:
$$x \leqslant b_1 \leqslant b_2 \leqslant \cdots \leqslant b_t = k.$$
If we have $\begin{array}{|c|}\hline A \\ \hline A \\ \hline \end{array}$, then we consider the A in the upper row to be the larger of the two.
- (c) If $b_i = b_{i+1}$, replace b_i with $*$.
- (d) Move the b_i chainwise according the following rule: b_1 moves into b_2's position, b_2 moves into b_3's position, etc until b_{t-1} moves into b_t's position. The box that was occupied by b_1 disappears.
- (e) Place b_t in the position determined by the shape of $Q_{S,n-1}$.
- (f) Replace any pair $\begin{array}{|c|}\hline * \\ \hline A \\ \hline \end{array}$ with $\begin{array}{|c|}\hline A \\ \hline A \\ \hline \end{array}$.

(rYF3) If the label n of Q was not the only label in its box and the largest entry in $P_{YF}(w)$ is k:
- (a) Add the column $\begin{array}{|c|}\hline \bullet \\ \hline k \\ \hline \end{array}$ directly to the left of column c to $P_{YF}(w)$.
- (b) Sort the entries of $P_{YF}(w)$ as shown below:
$$k = b_1 \geqslant b_2 \geqslant b_2 \cdots \geqslant x = b_t.$$
If we have $\begin{array}{|c|}\hline A \\ \hline A \\ \hline \end{array}$, then we consider the A in the upper row to be the larger of the two.
- (c) If $b_i = b_{i+1}$, replace b_{i+1} with $*$.
- (d) Move the b_i chainwise according to the following rule: b_1 moves into the new box $\begin{array}{|c|}\hline \bullet \\ \hline \end{array}$, b_2 moves to b_1's original position, b_3 moves to b_2's original position, etc. The box that was occupied by $b_t = x$ disappears.
- (e) Replace any pair $\begin{array}{|c|}\hline * \\ \hline A \\ \hline \end{array}$ with $\begin{array}{|c|}\hline A \\ \hline A \\ \hline \end{array}$.

The steps above clearly reverse the KYF insertion steps, giving us the result below.

Theorem 6.11. *KYF insertion and reverse KYF insertion define mutually inverse bijections between the set of words on $\mathbb{N}$ and the set of pairs (P_{YF}, Q_{YF}) consisting of a KYF tableau and a set-valued KYF tableau of the same shape.*

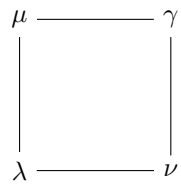
Proof. Omitted for brevity. □

6.3. KYF growth and decay. As before, given any word $w = w_1 w_2 \dots w_k$, containing $n \leqslant k$ distinct numbers, we can create an $n \times k$ array with an X in the w_i^{th} square from the bottom of column i. Note that there can be multiple X's in the same row but is at most one X per column.

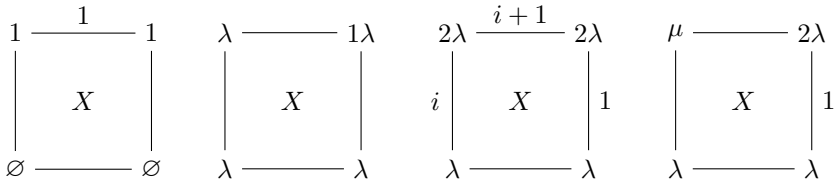
We will label the corners of each square with a snakeshape, label some of the horizontal edges of the squares with a specific inner corner by writing the column of the inner corner, and label some vertical edges corresponding to where a 2 was

added to the bottom shape to obtain the top shape. For example, if the corner at the bottom of the vertical edge is labeled with 221, the corner at the top is labeled 2221, and the vertical edge has label 3, this means that the third column of 2's in 2221 is the column that was added when going from 221 to 2221. We begin by labeling all corners in the bottom row and left side of the diagram with the empty shape, $\varnothing$.

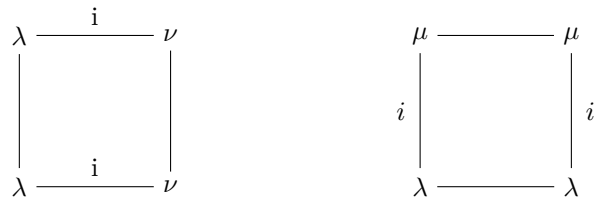
To complete the labeling of the corners, suppose the corners μ, λ, and ν are labeled, where μ, λ, and ν are as in the picture below. We label γ according to the following rules.



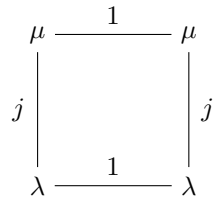
- *If the square contains an X:*
  (1) If $\lambda = \nu = \varnothing$, then $\gamma = \mu = 1$ and the top edge is labeled 1.
  (2) If $\lambda = \mu = \nu$, then $\gamma = 1\lambda$.
  (3) If $\mu = 2\lambda$ and the left edge is labeled i, then $\gamma = 2\lambda$, the top edge is labeled $i + 1$, and the right edge is labeled 1.
  (4) If $\mu \neq \lambda$, $\mu \neq 2\lambda$, and $\mu \neq 1$, then $\gamma = 2\lambda$ and the right edge is labeled 1.



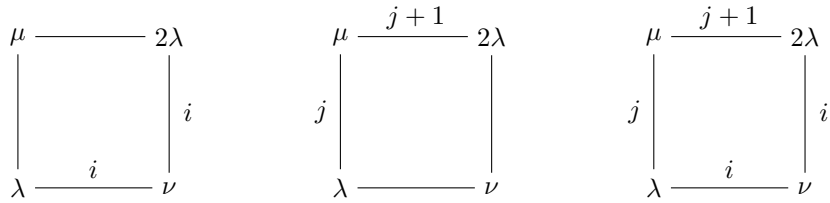
- *If the square does not contain an X and $\mu = \lambda$ or if the square does not contain an X and $\nu = \lambda$ with no edge label between λ and ν:*
  (5) If $\mu = \lambda$, then set $\gamma = \nu$ and label the top edge with the same label as the bottom edge if one exists. If $\nu = \lambda$, then $\gamma = \mu$ with the right edge labeled with the same label as the left edge if such a label exists.



- *If the square does not contain an X, $\nu = \lambda$, and the bottom edge is labeled 1:*
  (6) If $\nu = \lambda$ and the bottom edge is labeled 1, then $\gamma = \mu$, the top edge is labeled 1, and the label on the left edge (if one exists) is the same as the label on the right edge.

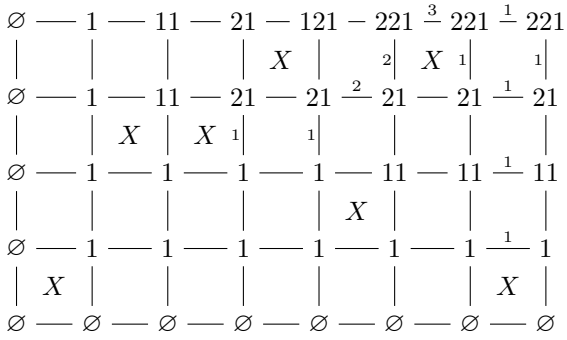


- *If no previous cases apply, then set $\gamma = 2\lambda$ and follow the rule below:*
  (7) If the bottom edge is labled i, then label the right edge i. If the left edge is labled j, then label the top edge $j+1$.



We call the resulting array the KYF growth diagram of w. For example, continuing with the word from Example 6.8, $w = 1334241$, we would have the diagram below.

Example 6.12. Below is the KYF growth diagram for the word 1334241.



Let $\mu_0 = \varnothing \subseteq \mu_1 \subseteq \dots \mu_k$ be the sequence of snakeshapes across the top of the growth diagram, and let $\nu_0 = \varnothing \subseteq \nu_1 \subseteq \dots \nu_n$ be the sequence of snakeshapes on the right side of the KYF growth diagram. These sequences correspond to KYF tableaux $Q(w)$ and $P(w)$, respectively.

If the edge between ν_i and ν_{i+1} is labeled j, then $2\nu_i = \nu_{i+1}$, and a column of i's is added in the j^{th} column of ν_i to obtain ν_{i+1}. If the edge between μ_i and μ_{i+1} is labeled j, then $\mu_i = \mu_{i+1}$ and the label $i+1$ of $Q(w_1 \dots w_{i+1})$ is placed in the box at the top of the j^{th} column of $Q(w_1 \dots w_i)$.

In the example above, we have

$$P = \begin{array}{|c|c|c|}\hline 4 & 3 & \multicolumn{1}{c}{} \\ \hline 4 & 2 & 1 \\ \hline \end{array} \qquad Q = \begin{array}{|c|c|c|}\hline 57 & 3 & \multicolumn{1}{c}{} \\ \hline 4 & 2 & 16 \\ \hline \end{array}$$

which agrees with the insertion and recording tableau from Example 6.8. As for Hecke insertion and shifted Hecke insertion, this is not a coincidence.

Theorem 6.13. *For any word w, the KYF tableau and set-valued KYF tableau P and Q obtained from the sequence of snakeshapes across the right side of the KYF growth diagram for w and across the top of the KYF growth diagram for w, respectively, are $P_{YF}(w)$ and $Q_{YF}(w)$, the KYF insertion and recording tableau for w.*

Proof. Recall the definitions of $T(i,j)$ and $R(i,j)$ from the proof of Theorem 4.16. We define them analogously in this context and prove the analogous result by induction on $i+j$. The desired statement is clearly true when $i = 0$ or $j = 0$.

Suppose we are considering some $T(t,s)$ with $s,t > 0$. Assume also the X in column s is in row r. We may or may not have $r = t$. As in the previous growth diagram proofs, we have the following claims.

Claim 6.14. *Oriented as in the square above, $|\nu/\lambda| \leqslant 1$.*

Claim 6.15. *Oriented as in the square above, to obtain μ from λ, we add either one box or a column of 2 boxes.*

Rule 1 follows from the fact that inserting t into the tableau $\boxed{t}$ does not change the tableau. Rules 2 applies when inserting positive integer t that is strictly larger than all other entries of the KYF tableau. Thus a box with t will be added directly to the left of the tableau. In Rule 3, we are adding integer t to a tableau that already has 2 boxes labeled t in the i^{th} column, and t is the largest integer in the tableau. Following the insertion rules, we end with a column of t's in the first column, hence the right edge is labeled 1. The special corner of this insertion is in the $(i+1)^{th}$ column, the column where the t on the top row of the original KYF tableau was deleted in the insertion procedure.

In Rule 4, there is exactly one t in $T(t, s-1)$. Thus, inserting a t will result in a tableau where the first column has two boxes filled with a t.

If $\lambda = \nu$ as in Rule 5, then $r > t$. Thus no integer is being inserted as we move from μ to γ, so $\mu = \gamma$ and the edge labels are unchanged. Similarly, if $\lambda = \mu$, then there is no X to the left of the square we are considering, and so no boxes labeled t in the $T(t, s)$. Thus there is no change in the insertion and recording tableaux.

Rule 6 follows directly from insertion rule YF1.

In considering Rule 7, note that $r < t$ because if not, we would be in the situation described in Rule 5. In Rule 7, $\gamma = 2\lambda$ because no matter what r is inserted, since $\mu \neq \lambda$, there is at least one entry of $T(t, s-1)$ that is larger than r. This means that when we insert r into $T(t, s-1)$, a box with r will be added to the first column, and after the shifting of YF5, there will be a box directly above this new box with r. An entry t will shift into the position of any box that was "lost" when shifting during the insertion of r into $T(t-1, s-1)$, and if there were two boxes labeled t in $T(t, s-1)$, the second will be deleted in YF6. It follows that $\gamma = 2\lambda$.

If the bottom edge is labeled i, then insertion of r into $T(t-1, s-1)$ included the situation described by YF6 in column i of λ. When we insert r into $T(t, s-1)$, column i now becomes $\begin{array}{|c|}\hline t \\ \hline * \\ \hline \end{array}$ and thus $\begin{array}{|c|}\hline t \\ \hline t \\ \hline \end{array}$ in $T(t-1, s-1)$. This means that the edge between ν and γ should be labeled i, indicating where the t's are in $T(t, s)$.

If the left edge is labeled j, then the difference between $T(t-1, s-1)$ and $T(t, s-1)$ is a column $\begin{array}{|c|}\hline t \\ \hline t \\ \hline \end{array}$ in column j of $T(t, s-1)$. The process of inserting r into $T(t, s-1)$ will match that of inserting r into $T(t-1, s-1)$ until we get to the point where we are shifting the entries t and $*$. Since t is the largest entry in $T(t, s-1)$, the insertion of r into $T(t, s-1)$ will end with deleting the unmatched $*$ from column $j+1$ of $T(t, s)$, and therefore the top edge is labeled $j+1$. □

We can also formulate the rules for the reverse insertion as follows. The proof is omitted for brevity.

Reverse Rules.

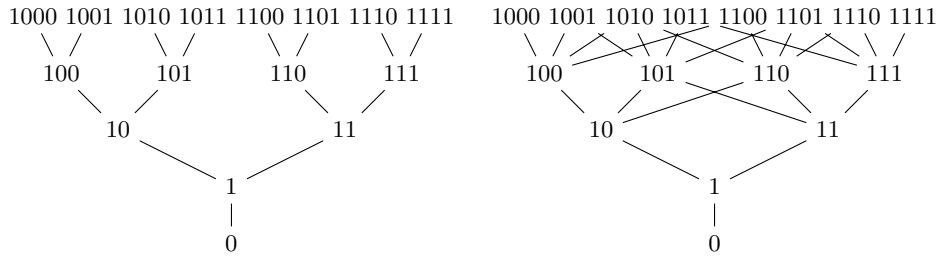
- (R1) If $\mu = \gamma = 1$, $\nu = \varnothing$, at the top edge is labeled 1, then the square contains an X and $\lambda = \varnothing$.
- (R2) If $\mu = \nu$ and $\gamma = 1\mu = 1\nu$, then the square contains an X and $\lambda = \mu = \nu$.
- (R3) $\mu = \gamma = 2\nu$, the top edge is labeled i and the right edge is labeled 1, then there is an X in the square, $\lambda = \nu$ and the left edge is labeled $i-1$.
- (R4) If $\gamma = 2\nu$ and the right edge is labeled 1, then the square contains an X and $\lambda = \nu$.

(R5) If $\gamma = \nu$, then set $\lambda = \mu$ and label the bottom edge with the same label as the top edge if one exists. If $\gamma = \mu$, then $\lambda = \nu$ and the left edge has the same label as the right edge if such a label exists.
(R6) If $\gamma = \mu$ and the top edge is labeled 1, then $\lambda = \nu$, the bottom edge is labeled 1, and the left edge has the same label as the right edge.
(R7) If no previous cases apply, then λ is obtained from γ by removing the first 2. If the right edge is labeled i, then the bottom edge is labeled i. If the top edge is labeled j, then the left edge is labeled $j - 1$.

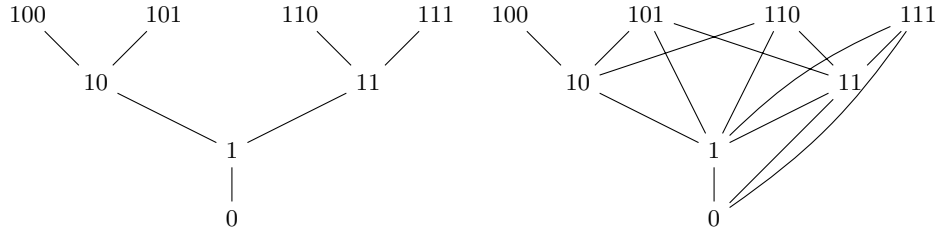
7. Other examples

7.1. Binary tree deformations.

Example 7.1. The lifted binary tree is shown on the left, where vertices can be naturally labeled by bit strings: 0, 1, 10, 11, 100, 101,.... The graph *BinWord* with the same set of vertices and rank function is shown on the right. In *BinWord*, an element x covers y if y can be obtained from x by deleting a single digit from x, and in addition, 1 covers 0. The lifted binary tree and *BinWord* form a dual graded graph, see [10, Example 2.4.1].



We form a Pieri deformation by interpreting the graph in the context of the ring of quasisymmetric functions, QSym (see [27]). If we interpret "1" as L_1, "10" as L_2, "100" as L_3, 11001 as L_{131}, and so on, we see that x covers y in the graph on the right exactly when x appears in the product $L_y \cdot L_1$, where L_α is the fundamental quasisymmetric function with the usual product. Let d be the operator that subtracts 1 from the rightmost number in the subscript or deletes the rightmost number if it is 1. For example, $d(L_{14}) = L_{13}$ and $d(L_{1421}) = L_{142}$. Then x covers y in the graph on the left exactly when $y = d(x)$. To form the Pieri deformation, we can let $f = L_1 + L_{11} + L_{111} + \cdots$. In other words, the multiplicity of the edge from x down to y is the coefficient of x in $y \cdot (L_1 + L_{11} + \cdots)$.



Using $A = \mathrm{QSym}$ and d and f as defined above, it is easy to see that $d(f) = L_\varnothing + f = id + f$. Using the Hopf algebra structure of QSym, it is clear that d is a derivation by Lemma 3.4. Thus the resulting graph is a dual filtered graph by Theorem 3.3.

EXAMPLE 7.2. Consider the Hopf algebra of multi-quasisymmetric functions, $\mathfrak{m}$QSym, defined in [15]. This Hopf algebra has a basis of multi-fundamental quasisymmetric functions indexed by compositions, $\{\tilde{L}_\alpha\}$. Notice that a graph with vertices indexed by $\{\tilde{L}_\alpha\}$ has the same vertices as in Example 7.1. To define the product, we first define the concept of a multiword. Let $u = u_1u_2\dots u_k$ be a word. We call $w = w_1w_2\dots w_m$ a *multiword* of u if there is a surjective and non-decreasing map $t : [m] \to [k]$ so that $w_j = u_{t(j)}$. For example, 11335662 and 133335562 are multiwords of 13562. Let $u = u_1u_2\dots u_k$ and $v = v_1v_2\dots v_l$ be two words, and assume that all letters v_i and u_j are distinct. A word $w = w_1w_2\dots w_m$ is a *multishuffle* of u and v if

(1) for any $i \in [1, m-1]$, $w_i \neq w_{i+1}$, and
(2) when restricted to alphabets v_i and u_j, w becomes a multiword of v and u, respectively.

For example, 1818373567627 is a multishuffle of 13562 and 87. If u or v contain repeated letters, obtain the multishuffle by first replacing u and v with words with distinct letters, taking the multishuffle product, and then replacing the letters in the result with the original letters. For example, if multishuffling 121 with 1, we can instead multishuffle 132 with 4 to get $1324 + 13242 + 41342 + \cdots$, and then substitute the original letters to get $1214 + 12111 + 11211 + \cdots$. Note that multiplicities may occur in this case.

To multiply $\tilde{L}_\alpha$ and $\tilde{L}_\beta$, first choose words w_α and w_β such that the composition of the descent set of u is α and of v is β. For example, if $\alpha = (2,1)$ and β is (1), we can choose $w_\alpha = 231$ and $w_\beta = 1$. Say that w_α is on $[n]$ for some $n \in \mathbb{N}$, and let $w_\beta[n]$ be the word obtained by adding n to each letter of w_β. In the previous example, $w_\beta[3] = 4$. Now

$$\tilde{L}_\alpha\tilde{L}_\beta = \sum_w \tilde{L}_{\mathcal{C}(w)},$$

where we sum over multishuffles of w_α and $w_\beta[n]$ and $\mathcal{C}(w)$ is the composition associated to the descent set of word w. For example, $\tilde{L}_{(2,1)}\tilde{L}_{(1)} = \tilde{L}_{(2,2)} + \tilde{L}_{(2,1,1)} + \tilde{L}_{(3,1)} + \tilde{L}_{(12121)} + \cdots$, where the terms shown correspond to multishuffles 2314, 2431, 2341, 4243141.

Define the coproduct using the *cuut coproduct* of a word w: $\blacktriangle(w_1w_2\dots w_k) = \varnothing \otimes w_1\dots w_k + w_1 \otimes w_1\dots w_k + w_1 \otimes w_2\dots w_k + w_1w_2 \otimes w_2\dots w_k + \cdots + w_1\dots w_k \otimes w_k + w_1\dots w_k \otimes \varnothing$. Then let

$$\Delta(\tilde{L}_\alpha) = \sum_{u\otimes u' \text{ in } cuut(w_\alpha)} \tilde{L}_{\mathcal{C}(u)} \otimes \tilde{L}_{\mathcal{C}(u')}.$$

For example,

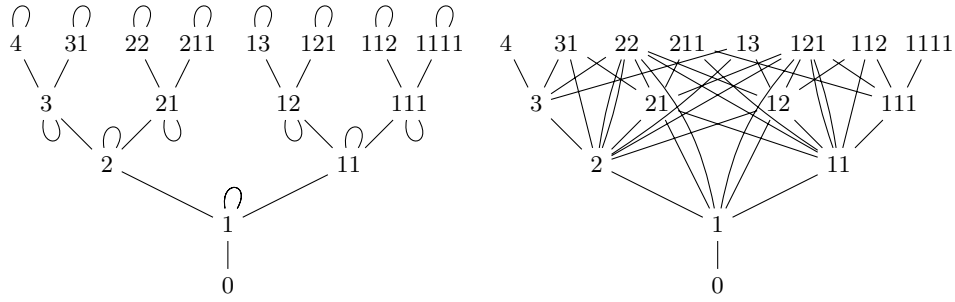
$$\Delta(\tilde{L}_{(2,1)}) = \varnothing \otimes \tilde{L}_{(2,1)} + \tilde{L}_{(1)} \otimes \tilde{L}_{(2,1)} + \tilde{L}_{(1)} \otimes \tilde{L}_{(1,1)} + \tilde{L}_{(2)} \otimes \tilde{L}_{(1,1)} + \tilde{L}_{(2)} \otimes \tilde{L}_{(1)} + \tilde{L}_{(2,1)} \otimes \tilde{L}_{(1)} + \tilde{L}_{(2,1)} \otimes \varnothing$$

since

$$\blacktriangle(231) = \varnothing \otimes 231 + 2 \otimes 231 + 2 \otimes 31 + 23 \otimes 31 + 23 \otimes 1 + 231 \otimes 1 + 231 \otimes \varnothing.$$

Taking A to be $\mathfrak{m}$QSym with the basis $\{\tilde{L}_\alpha\}$ indexed by compositions, we create a dual filtered graph by taking $f = \tilde{L}_{(1)}$ and $d = \xi \circ \Delta$ with $g = \tilde{L}_{(1)}$. The first five

ranks are shown below.

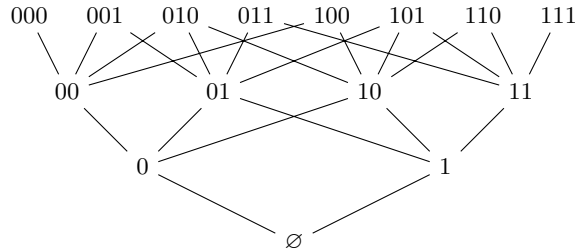


This is a dual filtered graph by Theorem 3.3.

The following proposition shows that we have another instance of Möbius via Pieri phenomenon, just like in the case of Young's lattice.

Proposition 7.3. *The Pieri construction above is a Möbius deformation of the dual graded graph in Example* 7.1*.*

Proof. We can transform the poset *BinWord* from Example 7.1 into the poset of subwords shown below by ignoring the first 1 in every word. In doing this, the element $\varnothing$ in *BinWord* disappears, 1 becomes $\varnothing$ in the subword poset, 10 becomes 0, 11001 becomes 1001 and so on. With this change, $x < y$ in *BinWord* exactly when x can be found as a subword of y, ignoring the initial 1's in each.



Given a word $y = y_1 y_2 \dots y_n$, define its *repetition set* $\mathcal{R}(y) = \{i : y_{i-1} = y_i\}$. An embedding of x in y is a sequence $1 \leqslant i_1 < i_2 < \cdots < i_k \leqslant n$ such that $x = y_{i_1} y_{i_2} \dots y_{i_k}$. It is a *normal embedding* if $\mathcal{R}(y) \subseteq \{i_1, i_2, \dots, i_k\}$. For two words, x and y, let $\binom{y}{x}_n$ denote the number of normal embeddings of x in y. For example, $\binom{10110}{10}_n = 1$ and $\binom{1010}{10}_n = 3$. Now, from Theorem 1 in [3],

$$\mu(x, y) = (-1)^{|y|-|x|} \binom{y}{x}_n.$$

We show that the coefficient of $\tilde{L}_\beta$ in $\tilde{L}_\alpha \tilde{L}_{(1)}$ is the number of normal embeddings of α into β after viewing α and β as sequences of 0's and 1's and deleting the first 1 in each by showing the unique way to insert strings of alternating 1's and 0's into α at a specific location to obtain β in the multishuffle. So, for example, the coefficient of $\tilde{L}_{(1111)}$ in $\tilde{L}_{(11)} \tilde{L}_{(1)}$ is zero since $\binom{111}{1}_n = 0$ and the coefficient of $\tilde{L}_{(122)}$ in $\tilde{L}_{(12)} \tilde{L}_{(1)}$ is 3 since $\binom{1010}{10}_n = 3$.

Choose word w_α on $[n]$ and multishuffle it with $n+1$. We note that adding two 0's to α that are not separated by a 1 in α in the multishuffle means increasing the length of an increasing segment of w_α by two, which is impossible as adding $n+1$ to the end of an increasing segment is the only way to increase its length, and the letters of w_α must stay in the same relative order in the multishuffle. Similarly for adding two 1's

to w_α that are not separated by a 0 in α. This agrees with the Möbius function since there are no normal embeddings in this case, e.g. $\binom{1010001}{101}_n = 0$.

Suppose we want to insert a segment of alternating 1's and 0's in α in a place with a 0 (if anything) on the left and a 1 (if anything) on the right. Let w_α be $w_1 w_2 \dots w_{k-1} w_k w_{k+1} \dots w_n$, where the w_k and w_{k+1} correspond to the 1 and 0 of α mentioned above. The only way to add an alternating sequence beginning with 0 is to insert the string $(n+1)w_k(n+1)w_k(n+1)\dots$ between w_k and w_{k+1}. The resulting word,

$$w_1 \dots w_{k-1} w_k (n+1) w_k (n+1) w_k \dots w_{k+1} \dots w_n$$

is clearly a multishuffle of w_α and $n+1$. Similarly, to add an alternating sequence beginning with 1, "delete" the w_k from w and replace it with $(n+1)w_k(n+1)w_k$ $(n+1)\dots$. The resulting word,

$$w_1 \dots w_{k-1} (n+1) w_k (n+1) w_k \dots w_{k+1} \dots w_n$$

is a multishuffle of w_α and $n+1$.

If instead we insert the alternating string in a place of α with a 1 on the left and a 0 on the right, we only need to consider strings beginning with 0 as the 1 at the beginning of the string can be added as in the previous case. To insert an alternating string beginning with 0, we must insert the segment $(n+1)w_k(n+1)w_k\dots$ between w_k and w_{k+1} of w_α.

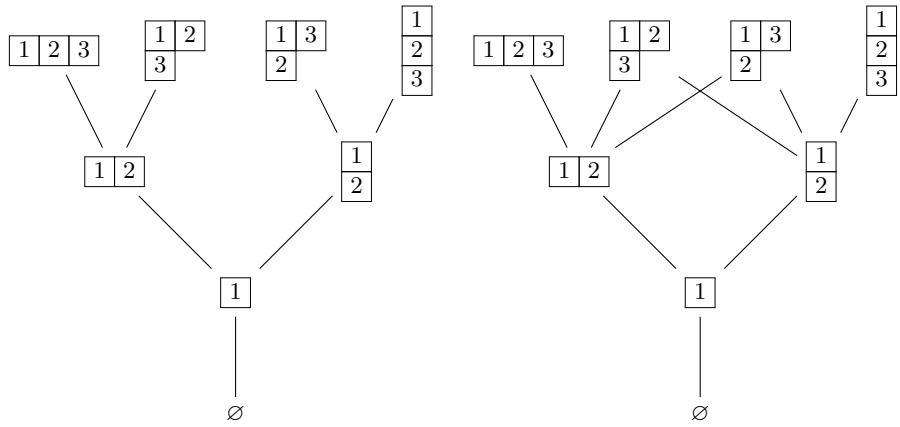
If we insert the alternating string in place of α with 0's to the left and right, we only need to consider strings that begin with 1. To do this, delete the w_k from w and insert the segment $(n+1)w_k(n+1)w_k\dots$ between w_{k-1} and w_{k+1}.

If we insert the alternating string in place of α with 1's to the left and right, we only need to consider strings that begin with 0. To do this, insert the segment $(n+1)w_k(n+1)w_k\dots$ between w_k and w_{k+1}. □

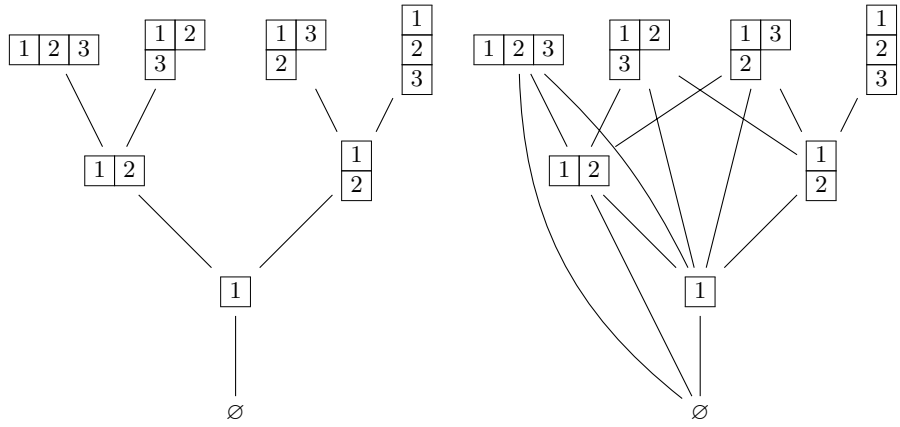
7.2. Poirier–Reutenauer deformations.

Example 7.4. The *SYT-tree* has as vertices standard Young tableaux, and standard Young tableau T_1 covers standard Young tableau T_2 if T_2 is obtained from T_1 by deleting the box with largest entry. It is shown below on the left. It is dual to the *Schensted graph*, which is shown on the right.

The *Schensted graph* is constructed using RSK insertion. A standard Young tableau T_1 covers T_2 if T_1 appears in the product of T_2 with the single-box standard tableau, where multiplication of standard Young tableaux is as follows. Suppose T_1 and T_2 are standard Young tableaux with row words $\mathrm{row}(T_1)$ on $[n]$ and $\mathrm{row}(T_2)$ on $[m]$, respectively. Then let $T_1 \cdot T_2$ be the sum of tableaux obtained by first lifting $\mathrm{row}(T_1)$ to $[n+m]$ in all ways that preserve relative order and then RSK inserting the corresponding lifting of $\mathrm{row}(T_2)$. For example, in multiplying $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}$ by itself, we would get the sum of $(P(12) \leftarrow 3) \leftarrow 4$, $(P(13) \leftarrow 2) \leftarrow 4$, $P((14) \leftarrow 2) \leftarrow 3$, $(P(23) \leftarrow 1) \leftarrow 4$, $(P(24) \leftarrow 1) \leftarrow 3$, and $(P(34) \leftarrow 1) \leftarrow 2$.



Let $f = T_1 + T_2 + T_3 + \cdots$, where T_i is the one-row tableau with row reading word $123\ldots i$, and create a Pieri deformation by adding edges to the *Schensted graph* so that $a_2(T_i, T_j)$ is the coefficient of T_j in $T_i \cdot f$.



To see that this gives a dual filtered graph, take A to be the algebra of standard Young tableaux with multiplication as defined above. Note that A is dual to the Poirier–Reutenauer Hopf algebra [22]. Take d to be the operator that deletes the box with the largest entry, and notice that T_1 covers T_2 in *SYT-tree* exactly when $d(T_1) = T_2$. The operator d is a derivation by Lemma 3.4 using the fact that $\Delta(T) = \sum(u \otimes v)$, where we sum over words u and v such that $\mathrm{row}(T)$ is a shuffle of u and v. Lastly, it is easy to see that $d(f) = \varnothing + f$. Thus by Theorem 3.3, we have a dual filtered graph.

Example 7.5. Take A to be the K-theoretic Poirier–Reutenauer bialgebra (KPR) defined in [21] with a basis of K-Knuth classes of initial words (i.e. words containing only $1, 2, \ldots, n$ for some n) on $\mathbb{N}$. The K-Knuth relations are as follows:

$$\begin{aligned} pp &\equiv p && \text{for all } p \\ pqp &\equiv qpq && \text{for all } q, p \\ pqs \equiv qps \text{ and } spq &\equiv sqp && \text{for } p < s < q \end{aligned}$$

See [7] for details on this relation.

Multiplication of two classes $[w_1]$ and $[w_2]$ is done by shuffling each pair of elements, one from $[w_1]$ and one from $[w_2[n]]$, where w_1 is a word on n and $w_2[n]$ is obtained by adding n to each letter in w_2, and writing the result as a sum of classes. For example,

$$[\![12]\!] \cdot [\![312]\!] = [\![53124]\!] + [\![51234]\!] + [\![35124]\!] + [\![351234]\!] + [\![53412]\!] + [\![5351234]\!].$$

The coproduct is similarly defined by taking the sum of the coproduct, $\Delta(w_1 \dots w_k) = \varnothing \otimes w_1 \dots w_k + std(w_1) \otimes std(w_2 \dots w_k) + \cdots + std(w_1 \dots w_{k-1} w_k) + std(w_1 \dots w_k) \otimes \varnothing$, of each element in the class and writing the result as a sum of classes. Here, $std(w)$ sends w to the unique word with the same relative order using all letters $\{1, 2, \dots, n\}$ for some n. For example, $std(13375) = 12243$. Using this, we compute,

$$\Delta([\![12]\!]) = [\![\varnothing]\!] \otimes [\![12]\!] + [\![1]\!] \otimes [\![1]\!] + [\![1]\!] \otimes [\![12]\!] + [\![12]\!] \otimes [\![1]\!] + [\![12]\!] \otimes [\![\varnothing]\!].$$

Letting $g = [1]$ and $f = [1]$, we create a Pieri construction using Theorem 3.3. The partial dual filtered graph is shown below. The first five ranks are shown completely, and there are a few additional elements shown in the graph on the right to illustrate all elements that cover [212]. To see that there is an upward edge from [3412] to [3124], we first notice that $34124 \in [3124]$ and that $3412 \otimes 1$ appears in the coproduct of 34124.

In Figures 7.1 and 7.2, K-Knuth equivalence classes of words are represented by an increasing tableau. Note that there may be more than one increasing tableau in any given class.

7.3. Malvenuto–Reutenauer deformations.

Example 7.6. Figure 7.3 shows dual graphs where vertices are permutations. On the left, a permutation σ covers π if π is obtained from σ by deleting the rightmost number (in terms of position in the permutation) and standardizing. On the right, a permutation σ covers π if π is obtained from σ by deleting the largest number in the permutation. There are other similar constructions, which can be found in [10, Example 2.6.8].

Consider the Hopf algebra of permutations with the shuffle product and coproduct defined by $\Delta(w) = \sum std(u) \otimes std(v)$, where the concatenation of u and v is w and $std(w)$ sends w to the unique permutation with the same relative order. For example, $std(1375) = 1243$. Take d to be the operator that deletes the rightmost letter of the permutation. Then we see that $a_1(v, w)$ is the coefficient of v in $d(w)$. We create a Pieri deformation by letting $f = 1 + 12 + 123 + \cdots$ and $a_2(v, w)$ be the coefficient of w in $f \cdot v$. Clearly $d(f) = \varnothing + f$, and we use Lemma 3.4 to see that d is a derivation. It then follows from Theorem 3.3 that the resulting graph is a dual filtered graph. This dual filtered graph is shown in Figure 7.3.

Example 7.7. We next describe a K-theoretic analogue of the Malvenuto–Reutenauer Hopf algebra. For details, see [15]. A small multi-permutation or $\mathfrak{m}$-permutation of $[n]$ is a word w in the alphabet $1, 2, \dots, n$ such that no two consecutive letters in w are equal. Now let the small multi-Malvenuto–Reutenauer Hopf algebra, $\mathfrak{m}$MR be the free $\mathbb{Z}$-module of arbitrary $\mathbb{Z}$-linear combinations of multi-permutations. Recall the definition of the multishuffle product from Example 7.2. Given two $\mathfrak{m}$-permutations $w = w_1 \dots w_k$ and $u = u_1 \dots u_l$, define their product to be the multishuffle product of w with $u[n]$, where w contains exactly the numbers $1, 2, \dots, n$ and $u[n] = (u_1 + n)(u_2 + n) \dots (u_l + n)$.

Let $st(w)$ send a word w to the unique $\mathfrak{m}$-permutation u of the same length such that $w_i \leqslant w_j$ if and only if $u_i \leqslant u_j$ for each $1 \leqslant i, j \leqslant l(w)$. Define the coproduct in $\mathfrak{m}$MR by $\Delta(w) = st(\blacktriangle(w))$, where $\blacktriangle(w)$ is the cuut coproduct defined in Example 7.2.

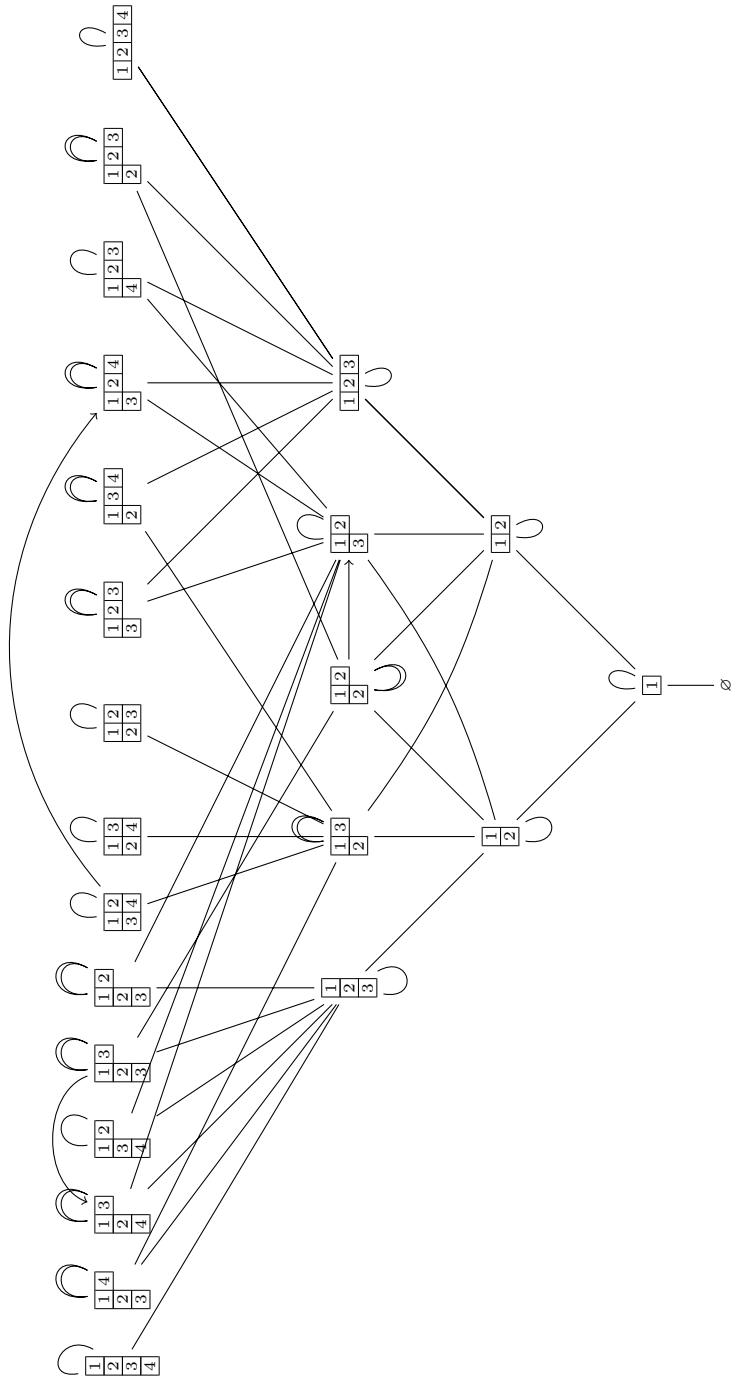


FIGURE 7.1. The upward-oriented edges of the Pieri construction of KPR as described in Example 7.5.

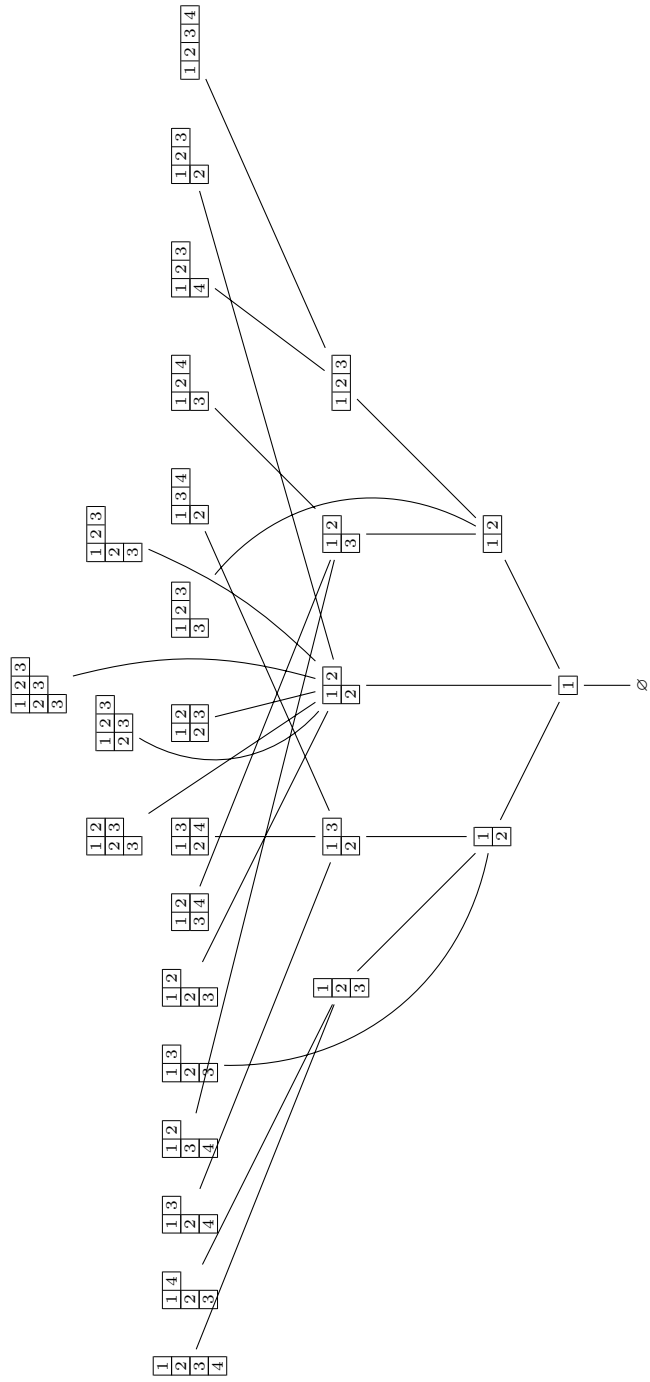


FIGURE 7.2. The downward-oriented edges of the Pieri construction of KPR as described in Example 7.5.

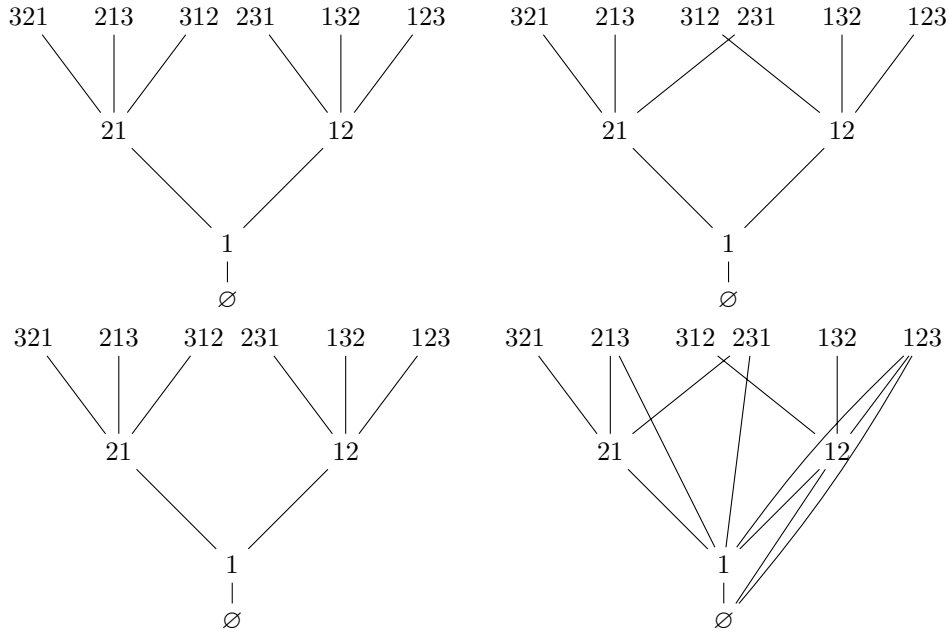
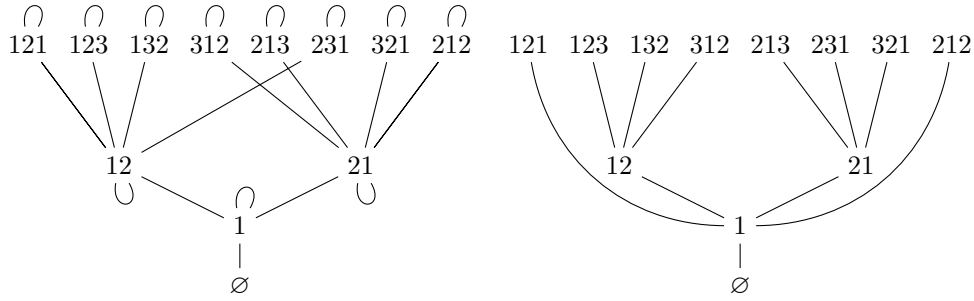


Figure 7.3. The top row shows the dual graded graph with permutations for vertices as described in Example 7.6. The bottom row is the corresponding Pieri deformation.

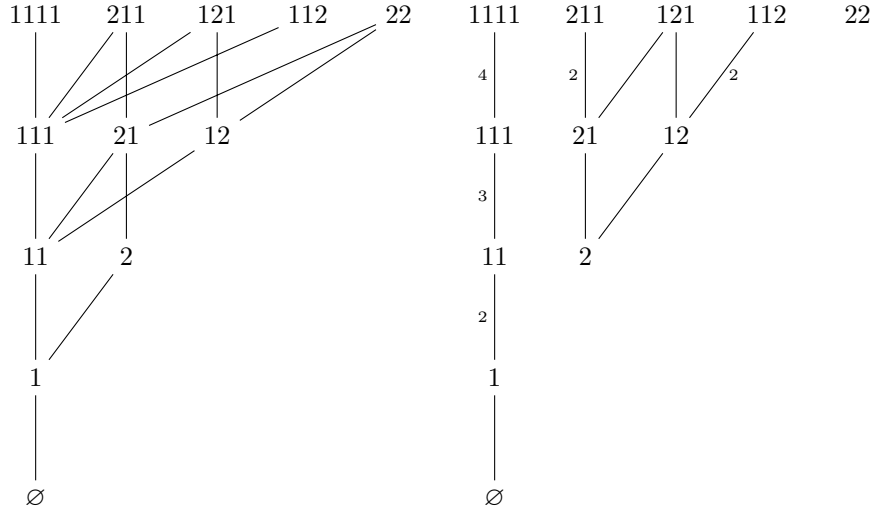
Using $g = 1$, $d = \xi \circ \Delta$, and $f = 1$, we form the dual filtered graph partially shown below. Notice that $d(w_1 \dots w_k) = w_1 \dots w_k + w_1 \dots w_{k-1}$, so $d(f) = \varnothing + f$.



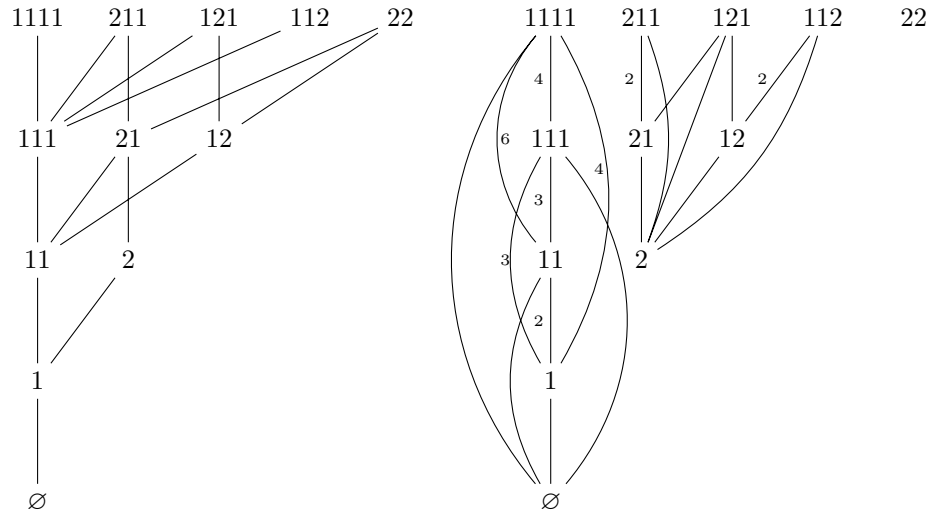
7.4. Stand-alone examples.

Example 7.8. There is another graph with the same set of vertices as the Young–Fibonacci lattice called the Fibonacci graph. The vertices of the Fibonacci graph are words on the alphabet $\{1, 2\}$, or snakeshapes, and a word w is covered by w' if w' is obtained from w by adding a 1 at the end or by changing any 1 into a 2. In the graph that is dual to the Fibonacci graph, w is covered by w' if w is obtained from w' by deleting a 1, and the multiplicity of the edge between w and w' is the number of ways to delete a 1 from w' to get w. The pair form a dual graded graph shown below,

see [10, Example 2.3.7], where the numbers next to the edges denote multiplicity.



We construct a Pieri deformation of this dual graded graph by adding downward-oriented edges so that in the resulting set of downward-oriented edges, there is an edge from w' to w for every way w can be obtained from w' by deleting at least one 1. For example, there are six edges from 1111 to 11 since there are six ways to delete two 1's from 1111 to obtain 11.

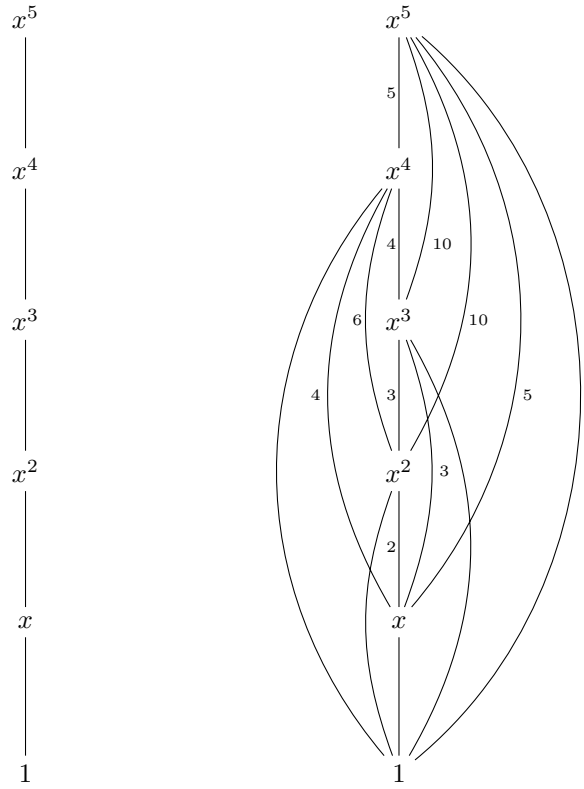


THEOREM 7.9. *The resulting graph is a dual filtered graph.*

Proof. Consider an algebra structure on words in alphabet $\{1, 2\}$ with shuffle product, as in [10]. Consider d to be the operator that changes any 2 into a 1 or deletes the last digit if it is 1. According to [10, Lemma 2.3.9], d is a derivation such that the coefficient of v in $d(w)$ is the multiplicity $a_1(v, w)$ for the Fibonacci graph. Take $f = 1 + 11 + 111 + \cdots$. It is easy to see that $d(f) = id + f$ and that the coefficient of w in fv is the multiplicity $a_2(v, w)$. Thus, Theorem 3.3 applies, and the statement follows. □

EXAMPLE 7.10. The dual filtered graph on the polynomial ring shown below is that mentioned in Section 1.2. We take U to be multiplication by x and $D = e^{\frac{\partial}{\partial x}} - 1$. It

is easy to check that $DU - UD = D + I$, so the result is a dual filtered graph.



8. Enumerative theorems via up-down calculus

Let $\varnothing$ be the minimal element of a dual filtered graph satisfying $DU - UD = 1 + D$. Let $T(n, k)$ be the number of ways n labeled objects can be distributed into k nonempty parcels. We have

$$T(n, k) = k! \cdot S(n, k),$$

where $S(n, k)$ is the Stirling number of the second kind.

Theorem 8.1. *For any dual filtered graph, the coefficient of $\varnothing$ in $D^k U^n(\varnothing)$ is $T(n, k)$.*

Proof. We think of replacing the fragments DU inside the word by either UD, D, or 1. This way the initial word gets rewritten until there is no DU in any of the terms:

$$D^k U^n = D^{k-1} U D U^{n-1} + D^k U^{n-1} + D^{k-1} U^{n-1} = \cdots .$$

Only the terms of the form U^t that appear at the end can contribute to the coefficient we are looking for, since $D(\varnothing) = 0$. Among those, only the terms $U^0 = 1$ can contribute. Thus, we are looking for all the terms where D's and U's eliminated each other.

It is easy to see that each D eliminates at least one U, and each U must be eliminated by some D. The number of ways to match D's with U's in this way is exactly $T(n, k)$. □

Let f^λ be the number of increasing tableaux of shape λ. Let g_n^λ be the number of set-valued tableaux of shape λ and content $1, \ldots, n$. Let $F(n)$ denote the *Fubini number*, or *ordered Bell number* - the number of ordered set partitions of $[n]$.

COROLLARY 8.2. *We have*

$$\sum_{|\lambda| \leqslant n} f^\lambda g_n^\lambda = F(n).$$

Proof. The left side is clearly the coefficient of $\varnothing$ in $(D + D^2 + \cdots + D^n)U^n(\varnothing)$. It remains to note that $\sum_{k=1}^n T(n,k) = F(n)$. □

This is the analogue of the famous Frobenius–Young identity. Of course, this also follows from bijectivity of Hecke insertion. The advantage of our proof is that a similar result exists for any dual filtered graph.

The following result is analogous to counting oscillating tableaux.

THEOREM 8.3. *For any dual filtered graph the coefficient of $\varnothing$ in $(D+U)^n(\varnothing)$ is equal to the number of set partitions of $[n]$ with parts of size at least 2.*

Proof. As before, the desired coefficient is equal to the number of ways for all D's to eliminate all U's via the commutation relation. Each factor in the product

$$(D + U)(D + U) \cdots (D + U)$$

thus either eliminates one of the factors to the right of it or is eliminated by a factor to the left. Grouping together such factors, we get a set partition with parts of size at least 2. On the other hand, any such set partition corresponds to a choice of D in the first factor of each part and of U's in the rest. Thus, it corresponds to term 1 after such D's eliminate such U's. The statement follows. □

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