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ON THE CLOSURE OF SPACES OF SUMS OF RIDGE FUNCTIONS AND THE RANGE OF THE X-RAY TRANSFORM

by Jan BOMAN

1. Introduction.

Let $\mathcal{L}$ be some set of straight lines in $\mathbf{R}^n$ and set for a function f on $\mathbf{R}^n$ with compact support

$$\widetilde{f}(\mathrm{L}) = \int_{\mathrm{L}} f\, ds \quad \text{for} \quad \mathrm{L} \in \mathcal{L}.$$

Here ds is the line element on L. When $\mathcal{L}$ is the set of *all* lines, the function $\widetilde{f}$ is usually called the X-ray transform of f. A basic problem is to describe the range of the operator $\mathrm{P} : f \longrightarrow \widetilde{f}$, the domain being specified. In this article $\mathcal{L}$ will always be the set of all lines that are parallel to one of a given finite set $\mathrm{A} = \{a^\nu\}_{\nu=1}^m$ of directions in $\mathbf{R}^n$. This choice is motivated by the situation in modern medical X-ray techniques, so-called "tomography" (see the survey articles [8] and [9]). The domain of P will typically be the set of all functions in $\mathrm{L}^p(\mathbf{R}^n)$, $1 \leqslant p \leqslant \infty$, whose support is contained in a given convex, compact set in $\mathbf{R}^n$. The main problem is to decide whether the range of P is closed.

After dualizing one is led to problems of the following kind. Let Ω be an open set in $\mathbf{R}^n$, and let $a \in \mathrm{S}^{n-1} = \{x \in \mathbf{R}^n \,;\, |x| = 1\}$. We define $\mathrm{L}^q(\Omega, a)$ to be the set of functions in $\mathrm{L}^q(\Omega)$ which are constant on almost all lines parallel to a. The question is whether the vector space

$$\mathrm{L}^q(\Omega, a^1) + \ldots + \mathrm{L}^q(\Omega, a^m) = \Sigma\{f_\nu \,;\, f_\nu \in \mathrm{L}^q(\Omega, a^\nu)\} \quad (1.1)$$

is a closed subset of $\mathrm{L}^q(\Omega)$. If Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary, the answer is yes for $1 \leqslant q \leqslant \infty$ (Corollary 1.3).

For $1 \leqslant q < \infty$ the condition on the boundary can be considerably relaxed (Theorem 1.5). In dimensions higher than two the situation is quite different. We give an example of a convex set $\Omega \subset \mathbf{R}^3$ with smooth boundary and two vectors a^1, a^2, such that (1.1) is not closed for any q. On the other hand, if $\Omega \subset \mathbf{R}^3$ is convex and the principal curvatures of the boundary are non-vanishing at every point (this condition can be relaxed), then we prove that (1.1) is closed for $1 \leqslant q \leqslant \infty$ (Theorem 1.8).

A function on $\mathbf{R}^n$, or on a subset of $\mathbf{R}^n$, that is constant on parallel lines will be called a *ridge function* ; this is an extension of a terminology introduced by Logan and Shepp [6].

As was observed by Petersen, Smith, and Solmon [7] the problem can be phrased as a question about the range for a certain matrix valued differential operator. Indeed, if Ω is convex, u belongs to (1.1) if and only if the system of partial differential equations $\sum_{\nu=1}^{m} u_\nu = u$, $D_{a^\nu} u_\nu = 0$ for $1 \leqslant \nu \leqslant m$ has a solution $(u_1, \ldots, u_m) \in L^q(\Omega)^m$; here $D_a f$ denotes the directional derivative of f with respect to the vector $a \in \mathbf{R}^n$.

The problem to decide when (1.1) is closed has been treated by a number of authors. Shepp and Logan raised the problem in [6]. Hamaker and Solmon [3] treated the case when Ω is a disk and $q = 2$. A more general case was treated by Falconer [2]. Petersen, Smith, and Solmon [7] proved that (1.1) is closed if Ω is a bounded domain in $\mathbf{R}^2$ whose boundary is a Lipschitz curve, and $1 < q < \infty$. These authors consider more general norms, Sobolev norms, instead of L^p-norms. They also treat a similar higher-dimensional problem (see Section 8 below). Svensson [10] has studied a more general problem where $L^q(\Omega, a)$ is replaced by a space of functions that are constant on a family of curves in $\mathbf{R}^2$. Our result for the three-dimensional case (Theorem 1.8) seems to have been previously known only when Ω is a ball; this special case is mentioned without proof in [11], Section 7.

We now give a precise description of our results.

Let Ω be an open subset of $\mathbf{R}^n$. For $a \in S^{n-1}$ we denote by Ω_a the set of all lines parallel to a that intersect Ω. The set

Ω_a is a subset of the factor space $\mathbf{R}^n/L_a$, $L_a = \{\lambda a ; \lambda \in \mathbf{R}\}$, but we may also think of Ω_a as a subset of some hyperplane $H_a \subset \mathbf{R}^n$ that is orthogonal to a. For $1 \leqslant p \leqslant \infty$ an operator $P_a : L^p_c(\Omega) \longrightarrow L^p_c(\Omega_a)$ (c indicates compact support) is defined as follows : $(P_a f)(L) = \int_{L\cap\Omega} f\,ds$, $L \in \Omega_a$.

The natural projection $\mathbf{R}^n \longrightarrow \mathbf{R}^n/L_a$ will be denoted p_a. The norm in $L^p_c(\Omega_a)$ is defined by integration with respect to the Lebesque measure in H_a. P_a is continuous, for if $\operatorname{supp} f$ is contained in the compact set K, then $\operatorname{supp} P_a f \subset p_a(K)$, and

$$\|P_a f\|_p \leqslant \operatorname{diam}(K)^{1-1/p} \|f\|_p \tag{1.2}$$

by Hölder's inequality.

For a given finite set $A = \{a^\nu ; \nu = 1, \dots, m\} \subset S^{n-1}$ we further define the operator $P_A : L^p_c(\Omega) \longrightarrow \prod_{\nu=1}^m L^p_c(\Omega_{a^\nu})$ by writing $P_A = (P_{a^1}, \dots, P_{a^m})$. For abbreviation we will usually write Ω_ν and P_ν instead of Ω_{a^ν} and P_{a^ν}. For given $G = (g_1, \dots, g_m)$, $g_\nu \in L^p_c(\Omega_\nu)$ we may now consider the problem to find $f \in L^p_c(\Omega)$ satisfying

$$P_A f = G. \tag{1.3}$$

To describe the necessary conditions on G for the existence of f in (1.3) it is convenient to introduce the adjoint map of P_A. Let $L^q_{\mathrm{loc}}(\Omega)$ and $L^q_{\mathrm{loc}}(\Omega_a)$ be defined in the usual way, and define the operator $Q_a : L^q_{\mathrm{loc}}(\Omega_a) \longrightarrow L^q_{\mathrm{loc}}(\Omega)$ by constant extension along lines parallel to a. The range of Q_a is the space $L^q_{\mathrm{loc}}(\Omega, a)$ of functions that are constant a.e. on almost every line parallel to a. Then P_a and Q_a are formal adjoints in the sense that $\int_{\Omega_a} (P_a f)\, u\, dy = \int_\Omega f\, Q_a u\, dx$ for $f \in L^p_c(\Omega)$, $u \in L^q_{\mathrm{loc}}(\Omega_a)$, $p^{-1} + q^{-1} = 1$. Here dy is the Lebesgue measure on Ω_a considered as a subset of the hyperplane $H_a \subset \mathbf{R}^n$. Similarly we define $Q_A : \prod_{\nu=1}^m L^q_{\mathrm{loc}}(\Omega_\nu) \longrightarrow L^q_{\mathrm{loc}}(\Omega)$ by $U = (u_1, \dots, u_m) \longrightarrow \sum_{\nu=1}^m Q_{a^\nu} u_\nu$. If we write $(f, u) = \int_\Omega f u\, dx$ for $f \in L^p_c(\Omega)$ and $u \in L^q_{\mathrm{loc}}(\Omega)$, and similarly

$$(\mathrm{F}, \mathrm{U}) = \sum_{\nu=1}^{m} \int_{\Omega_\nu} f_\nu u_\nu \, dx \quad \text{for} \quad \mathrm{F} = (f_1, \ldots, f_m), \quad f_\nu \in \mathrm{L}_c^p(\Omega_\nu),$$

and $\mathrm{U} = (u_1, \ldots, u_m)$, $u_\nu \in \mathrm{L}_{\mathrm{loc}}^q(\Omega_\nu)$, then we can state that $\mathrm{P_A}$ and $\mathrm{Q_A}$ are formal adjoints in the sense that

$$(\mathrm{P_A} f, \mathrm{U}) = (f, \mathrm{Q_A} \mathrm{U}). \tag{1.4}$$

Now, the obvious necessary condition for the existence of f in (1.3) is that G is orthogonal to ker $\mathrm{Q_A}$. Our first theorem states that this condition is also sufficient.

We will sometimes identify a function $g \in \mathrm{L}_{\mathrm{loc}}^q(\Omega_a)$ with the corresponding function $\mathrm{Q}_a g \in \mathrm{L}_{\mathrm{loc}}^q(\Omega, a)$. From this point of view Q_a is simply the imbedding operator $\mathrm{L}_{\mathrm{loc}}^q(\Omega, a) \hookrightarrow \mathrm{L}_{\mathrm{loc}}^q(\Omega)$.

We provide $\mathrm{L}_c^p(\Omega)$ with the usual topology, i.e. the inductive limit topology on $\bigcup_{j=1}^{\infty} \mathrm{L}_{\mathrm{K}_j}^p(\Omega)$, where $\mathrm{K}_j \subset \Omega$ is an increasing sequence of compact sets whose union is equal to Ω, and $\mathrm{L}_{\mathrm{K}}^p(\Omega)$ denotes the set of functions with support in K.

A continuous linear mapping T between two topological vector spaces X and Y is said to be a homorphism, if the induced mapping defined on the factor space X/ker T is a topological isomorphism between X and im T $\subset$ Y. An equivalent condition is that T is an open mapping from X onto im T with the relative topology induced from Y. If X and Y are Frechet spaces this is also equivalent to im T being closed (see Theorem 3.2).

For $a \in \mathrm{S}^{n-1}$ let $\mathcal{E}(\Omega, a)$ be the set of infinitely differentiable functions on Ω which are constant on all lines parallel to a.

THEOREM 1.1. – *Let Ω be an open convex subset of $\mathbf{R}^n$ or an open connected subset of $\mathbf{R}^2$, and let $1 \leqslant p \leqslant \infty$. Then $\mathrm{P_A}$ is a homomorphism from $\mathrm{L}_c^p(\Omega)$ into $\prod_{\nu=1}^{m} \mathrm{L}_c^p(\Omega_\nu)$, and* im $\mathrm{P_A}$ *is equal to the annihilator of* ker $\mathrm{Q_A}$. *More explicitly,* $\mathrm{G} = (g_1, \ldots, g_m)$ *belongs to* im $\mathrm{P_A}$ *if and only if*

$$\sum_{\nu=1}^{m} \int_{\Omega_\nu} g_\nu u_\nu \, dy = 0$$

for all $u_\nu \in \mathcal{E}(\Omega, a^\nu)$ such that $\sum_{\nu=1}^{m} u_\nu = 0$.

This theorem will be proved by means of an explicit construction of the solution f.

The condition on G given in the last sentence of the theorem will sometimes be abbreviated $G \in (\ker Q_A)^{\perp}$.

In the usual way we provide $L^q_{\mathrm{loc}}(\Omega)$ with the topology induced by the sequence of seminorms $q_i(f) = (\int_{K_i} |f|^q \, dx)^{1/q}$, $i = 1, 2, \ldots$, where K_i is a family of compact sets whose union is equal to Ω. In this way $L^q_{\mathrm{loc}}(\Omega)$ becomes a Frechet space.

By means of well-known functional analysis we will deduce the following statement from Theorem 1.1.

THEOREM 1.2. – *Let Ω be an open convex subset of $\mathbf{R}^n$ or an open connected subset of $\mathbf{R}^2$, and let $1 \leqslant q \leqslant \infty$. Then Q_A is a homomorphism from $\prod_{\nu=1}^{m} L^q_{\mathrm{loc}}(\Omega_\nu)$ into $L^q_{\mathrm{loc}}(\Omega)$, and hence $\sum_{\nu=1}^{m} L^q_{\mathrm{loc}}(\Omega, a^\nu)$ is closed in $L^q_{\mathrm{loc}}(\Omega)$.*

Remark. – Our proof shows in fact that the range of Q_A is equal to the annihilator in $L^q_{\mathrm{loc}}(\Omega)$ of $\ker P_A \subset \mathcal{D}(\Omega)$, the space of infinitely differentiable functions with compact support in Ω. If Ω is convex, a function $\varphi \in \ker P_A \subset \mathcal{D}(\Omega)$ must be of the form $\varphi = (\prod_{\nu=1}^{m} D_{a^\nu})\, \psi$ for some $\psi \in \mathcal{D}(\Omega)$, hence in this case $\Sigma_\nu\, L^q_{\mathrm{loc}}(\Omega, a^\nu)$ must consist precisely of those functions $f \in L^q_{\mathrm{loc}}(\Omega)$ for which $(\prod_{\nu=1}^{m} D_{a^\nu}) f = 0$ in the sense of the theory of distributions.

Let us now turn to our main problem, which is to decide whether (1.1) is closed in $L^p(\Omega)$. Let us first consider the case $n = 2$.

If $n = 2$ and the boundary is smooth one gets a positive answer as an immediate consequence of Theorem 1.2.

COROLLARY 1.3. – *Assume that Ω is an open bounded* *connected subset of* $\mathbf{R}^2$ *and that the boundary of* Ω *is of class* C^1. *Let* $1 \leqslant q \leqslant \infty$. *Then*

$$\sum_{\nu=1}^{m} L^q(\Omega, a^\nu) \quad \textit{is closed in} \quad L^q(\Omega). \tag{1.5}$$

Proof. – Assume that $u \in L^q(\Omega)$ and that u belongs to the closure of $\Sigma_\nu L^q(\Omega, a^\nu)$. By Theorem 1.2 there exist $u_\nu \in L^q_{\text{loc}}(\Omega, a^\nu)$ such that $u = \Sigma_\nu u_\nu$. It is enough to prove that each u_ν is in fact in $L^q(\Omega)$. Let B_ν be the set of points of the boundary $\partial\Omega$ which lie on some straight line parallel to a^ν not intersecting Ω (see fig. 1). Since $\partial\Omega$ is smooth, the B_ν must be pairwise disjoint. Observe first that $|u_\nu|^q$ is obviously integrable except possibly in some neighbourhood of B_ν. To prove that $u_\mu \in L^q(\Omega)$ we write $u_\mu = u - \sum_{\nu \neq \mu} u_\nu$ and observe that each term on the right hand side must belong to L^q in some neighbourhood of B_μ, since B_ν are disjoint. Hence $u_\mu \in L^q(\Omega)$, which completes the proof.

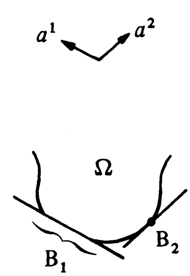

Fig. 1

In this proof the assumption that $\partial\Omega$ is smooth was used only to infer that the sets B_ν are pairwise disjoint. It follows that the same argument proves a much stronger statement, which we now formulate.

DEFINITION. – *Let* $\Omega \subset \mathbf{R}^n$ *be open and connected. A point* $x \in \partial\Omega$ *is called characteristic* (*with respect to the set* $A \subset S^{n-1}$) *if* $p_a(x)$ *belongs to the boundary of* Ω_a *for at least two different* $a \in A$.

If $\Omega \subset \mathbf{R}^2$, the condition means that there are two straight lines through x, each parallel to some $a \in A$, such that Ω is contained in one of the four angle shaped domains formed by the two lines. It is easy to see that a connected subset of $\mathbf{R}^2$ can have at most a finite number of characteristic points.

COROLLARY 1.4. – *Let* Ω *be an open bounded connected subset of* $\mathbf{R}^2$, *and assume that the boundary of* Ω *has no characteristic point. Then* (1.5) *holds for* $1 \leqslant p \leqslant \infty$.

If Ω has characteristic points, regularity assumptions on Ω are needed to guarantee (1.5). The next theorem treats this case.

An open *wedge* in the plane is an open connected set bounded by two intersecting straight lines.

DEFINITION. – *The open set* $\Omega \subset \mathbf{R}^2$ *is said to satisfy an interior wedge condition at* $x^0 \in \partial\Omega$, *if there exists a neighbourhood* V *of* x^0 *and an open wedge* Γ *with vertex at* x^0 *such that* $V \cap \Gamma \subset \Omega$.

THEOREM 1.5. – *Assume that* Ω *is an open bounded connected subset of* $\mathbf{R}^2$ *which satisfies the interior wedge condition at every characteristic point. Let* $1 \leqslant q < \infty$. *Then* (1.5) *holds.*

Note that in this theorem regularity conditions on $\partial\Omega$ are imposed only near a finite number of points, and that q is allowed to be equal to 1, whereas Petersen, Smith, and Solmon [7] require $\partial\Omega$ to be essentially Lipschitz continuous and assume $1 < q < \infty$. The method of Petersen et al. is based on the theory of so-called very strongly elliptic systems of partial differential equations. On the other hand, our proof of Theorem 1.5 depends only on Theorem 1.2 together with a very elementary integral inequality (Proposition 5.1′).

The assumptions of Theorem 1.5 are essentially the weakest possible. If $q = \infty$, Lipschitz continuity of the boundary does not suffice to imply (1.1). For $1 \leqslant q < \infty$ examples show that one cannot replace the wedge in the wedge condition by a cusp of the form $\{(x_1, x_2) ; 0 < x_2 < x_1^{1+\epsilon}\}$ for any positive ϵ. We will present these and other counterexamples in Section 7 below.

A characteristic feature of the two-dimensional case is that the kernel of Q_A is particularly simple.

PROPOSITION 1.6. – *Assume that* Ω *is an open connected subset of* $\mathbf{R}^2$ *and that* A *has* m *elements. Then* $\ker Q_A$ *consists only of polynomial functions of order at most* $m - 2$.

Proof. – Let $U = (u_1, \ldots, u_m)$, $u_\nu \in L^q(\Omega, a^\nu)$, $Q_A U = 0$, i.e. $\Sigma u_\nu = 0$. Since $D_{a^\nu} u_\nu = 0$ we have for each μ

$$(\prod_{\nu \neq \mu} D_{a^\nu}) u_\mu = 0 \qquad (1.6)$$

in the sense of the theory of distributions. But u_μ is essentially a function of one variable; thus (1.6) means that its derivative of order $m - 1$ is zero, i.e. u_μ is a polynomial of degree at most $m - 2$.

We now turn to the case of three dimensions. We begin with a counterexample, which clearly shows the radical difference between two and three dimensions.

THEOREM 1.7. – *There exists an open bounded strictly convex set* $\Omega \subset \mathbf{R}^3$ *with* C^∞ *boundary and two vectors* $a^\nu \in S^2$, $\nu = 1, 2$, *such that for each* q, $1 \leqslant q \leqslant \infty$, $L^q(\Omega, a^1) + L^q(\Omega, a^2)$ *is not closed in* $L^q(\Omega)$.

This theorem is proved in Section 7. An essential property of the set Ω constructed here is that part of $\partial\Omega$ looks like the surface $x_3 = x_1^2 + x_2^4$ near the origin, i.e. one of its principal curvatures vanishes at one point. It is therefore natural to conjecture that (1.1) must be closed if this never happens. Our next theorem states that this is indeed the case, at least if Ω is convex; it suffices even to make the curvature hypothesis at the (finitely many) characteristic points.

THEOREM 1.8. – *Let* Ω *be an open bounded convex subset of* $\mathbf{R}^3$ *whose boundary is of class* C^2, *let* $a^\nu \in S^2$, $\nu = 1, \ldots, m$, *and let* $1 \leqslant p \leqslant \infty$. *Assume that both principal curvatures of* $\partial\Omega$ *are non-zero at every characteristic point. Then*

$$\sum_{\nu=1}^{m} L^q(\Omega, a^\nu) \ \textit{is closed in} \ L^q(\Omega). \qquad (1.7)$$

The property (1.1) of Ω (relative to A) is somewhat related to P-convexity (relative to a differential operator P) studied by Hörmander in [4], chapter III. It is interesting to compare for instance the conditions on Ω in our Theorem 1.8 (Theorem 1.7) and Hörmander's Theorem 3.7.4 (Theorem 3.7.3).

The contents of this paper revolve around three central themes :

1. The constructive formula (2.1) for a solution of the equation $P_A f = G$, i.e. for a function f with prescribed projections (Theorem 1.1). The formula gives a solution whose support is slightly larger than the minimal convex set whose projections are the supports of the given data g_ν. As a consequence the dual statement (Theorem 1.2), which is deduced in Section 3, involves the space $L^q_{loc}(\Omega)$ instead of $L^q(\Omega)$. By the simple localization principle of Corollary 1.3 one obtains the corresponding result for $L^q(\Omega)$ provided $\Omega \subset \mathbf{R}^2$ and the boundary of Ω is smooth.

2. A study of the case where $\Omega \subset \mathbf{R}^2$ and the boundary is non-smooth. The main point here is Proposition 5.1; it is an elementary lemma, which generalizes the following statement for functions of one variable: if $1 \leqslant p < \infty$ and

$$\sum_{k=1}^{\infty} \int_{2^{-k-1}}^{2^{-k+1}} |f(t) - c_k|^p \, dt$$

for some constants c_k, then $f \in L^p(0,1)$.

3. The three-dimensional case. Using a localization argument (similar to the one used above in the proof of Corollary 1.3) we show that the proof of Theorem 1.8 can be reduced to the situation when all the given directions a^ν lie in one plane, say the plane $x_3 = 0$. Then we are led to study a family of two-dimensional problems on the domains $\Omega_z = \{(x_1, x_2);\ (x_1, x_2, z) \in \Omega\}$ and prove some estimates for the solutions of the equation $Q_A U = v$; the point of these estimates is that they are uniform with respect to the parameter z. These estimates are proved in Section 4. They are deduced from analogous estimates for the dual equation $P_A f = G$, which are proved in the second part of Section 2. In Section 6 we put the pieces together and complete the proof of Theorem 1.8.

2. Construction of solution in $L^p_c(\Omega)$ to $P_A f = G$.

In this section we will give a proof of Theorem 1.1 together with an estimate for the solution of the system $P_A f = G$, which is essential for the proof of our main result Theorem 1.8.

The proof of Theorem 1.1 can be outlined as follows. The first step is to study carefully the case when $m = 1$. This amounts to constructing a right inverse of the operator P_a. This is done in Lemma 2.1. By means of induction over m one reduces the proof of Theorem 1.1 to the case $g_1 = \ldots = g_{m-1} = 0$. Next one observes that $\mathrm{G} = (0, \ldots, 0, g) \in (\ker \mathrm{Q_A})^{\perp}$ implies that $g = \mathrm{S}g_0$ for some g_0 with compact support, where S is a certain differential operator (Lemma 2.2). If E is the right inverse to P_m constructed in Lemma 2.1, the solution to our problem can be written

$$f = \mathrm{D}_{a1}\, \mathrm{D}_{a2} \ldots \mathrm{D}_{am-1}\, \mathrm{E}\, g_0 \,. \tag{2.1}$$

Then it is obvious that $\mathrm{P}_\nu f = 0$ for $\nu \leqslant m-1$, and an easy computation shows that $\mathrm{P}_m f = g$.

We will now construct the right inverse of the operator P_a. We will take $\Omega = \mathbf{R}^n$, and note again that $p_a(\mathbf{R}^n) = \mathbf{R}^n/\mathrm{L}_a$ can be identified with any hyperplane H_a perpendicular to a. We may assume that $a = (0, \ldots, 0, 1)$. For $x \in \mathbf{R}^n$, write $x = (y, x_n)$, and identify H_a with $\mathbf{R}^{n-1}$. Write $\mathrm{D}_j = \partial/\partial x_j$, $\mathrm{D} = (\mathrm{D}_1 \, \ldots ,$ $\mathrm{D}_{n-1})$, $\mathrm{D}^\alpha = \mathrm{D}_1^{\alpha_1} \ldots \mathrm{D}_{n-1}^{\alpha_{n-1}}$, $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$.

LEMMA 2.1. – *Let* K *be a compact subset of* $\mathbf{R}^n$, *let* $\epsilon > 0$, *and let* $1 \leqslant p \leqslant \infty$. *Then there exists a right inverse* $\mathrm{E}_\epsilon = \mathrm{E}_{\epsilon,a}$ *of* P_a *with the following properties :*

(A) *if* $h \in \mathrm{L}^p(\mathrm{H}_a)$ *and* $\operatorname{supp} h \subset p_a(\mathrm{K})$, *then* $f = \mathrm{E}_\epsilon h \in \mathrm{L}^p(\mathbf{R}^n)$, $\mathrm{P}_a f = h$, *and* $\operatorname{supp} \mathrm{f} \subset \mathrm{K}_\epsilon = \mathrm{K} + \{x \,; |x| \leqslant \epsilon\}$;

(B) $\mathrm{E}_\epsilon h$ is C^∞ *as a function of* x_n, *and* E_ϵ *preserves regularity with respect to the y-variables in the following sense :* *if* $\mathrm{D}^\alpha h \in \mathrm{L}_c^p(\mathrm{H}_a)$, *then* $\mathrm{D}^\alpha \mathrm{D}_n^r(\mathrm{E}_\epsilon h) \in \mathrm{L}_c^p(\mathbf{R}^n)$, $r = 0, 1, \ldots$.

Proof. – Take a locally finite partition of unity $1 = \Sigma\, \phi_k$ on $\mathbf{R}^{n-1}$ such that $\phi_k \in \mathrm{C}^\infty$ and $\operatorname{diam}(\operatorname{supp} \phi_k) \leqslant \epsilon/2$. Take $\psi \in \mathrm{C}^\infty(\mathbf{R})$ such that $\int \psi\, dt = 1$ and $\operatorname{supp} \psi \subset [-\epsilon/4, \epsilon/4]$. Whenever $p_a(\mathrm{K})$ and $\operatorname{supp} \phi_k$ have some point in common, say z_k, we choose a number v_k such that $(z_k, v_k) \in \mathrm{K}$. Then we set $\mathrm{E}_\epsilon h = f$, where $f(y, x_n) = h(y) \sum\limits_k \phi_k(y)\, \psi(x_n - v_k)$. Since K is compact the sum is finite. It is obvious that all the assertions of Lemma 2.1 hold.

LEMMA 2.2. – *Let* $A = \{a^\nu\}_{\nu=1}^m$ *be a finite set of distinct elements of* S^{n-1}, *write* $H_m = H_{a^m}$, *let* $g \in L_c^p(H_m)$, *and assume that* $G = (0, \ldots, 0, g) \in (\ker Q_A)^\perp$. *Denote* $p_m(a^\nu) \in H_m$ by $b^\nu, \nu = 1, \ldots, m-1$. *Then there exists a (unique) function* $g_0 \in L_c^p(H_m)$ *such that*

$$D_{b^1} D_{b^2} \ldots D_{b^{m-1}} g_0 = g. \tag{2.2}$$

The support of g_0 *is contained in the convex hull of the support of* g.

Proof. – Note that $b^\nu \neq 0$, since $a^\nu \neq a^m$ for $\nu < m$. Again we assume that $a^m = (0, \ldots, 0, 1)$ and write $x = (y, x_n)$, where $y = p_m(x) \in H_m = \mathbf{R}^{n-1}$. In particular $a^\nu = (b^\nu, a_n^\nu)$. We note that some b^ν may be parallel although no two a^ν are parallel. After renumbering the b^ν and changing g_0 by a multiplicative constant we can write (2.2) as

$$D_{b^1}^{r_1} \ldots D_{b^k}^{r_k} g_0 = g, \tag{2.3}$$

where $\Sigma r_j = m - 1$ and $b^1, \ldots, b^k$ are pairwise non-parallel.

It is easy to see that the existence of g_0 satisfying (2.3) is equivalent to

$$\int t^s g(y + tb^\mu)\, dt = 0,\ y \in \mathbf{R}^{n-1},\ 0 \leqslant s \leqslant r_\mu - 1,\ \mu = 1, \ldots, k; \tag{2.4}$$

for if (2.4) holds, then g_0 can be constructed by successive integrations. Let us fix one μ, say $\mu = 1$. After a rotation of the coordinate system in $\mathbf{R}^{n-1} = H_m$ we may achieve that b^1 is parallel to $(1, \ldots, 0)$. Then (2.4) reads

$$\int t^s g(t, y_2, \ldots, y_{n-1})\, dt = 0,\ 0 \leqslant s \leqslant r_1 - 1. \tag{2.5}$$

The assumption that $(0, \ldots, 0, g) \in (\ker Q_A)^\perp$ implies that

$$\int_{H_m} g\, \varphi_m\, dx = 0 \tag{2.6}$$

whenever $\varphi_\nu \in C^\infty(\mathbf{R}^n)$, $D_{a^\nu} \varphi_\nu = 0$, and $\sum_{\nu=1}^m \varphi_\nu = 0$.

Our assumption that $p_m(a^\nu)$ is parallel to b^1 for r_1 values of ν, say $1 \leqslant \nu \leqslant r_1$, implies that $a^1, \ldots, a^{r_1}$ and a^m all lie in one two-dimensional plane N. Let θ^ν, $\nu = 1, \ldots, r_1$ and θ^m be non-zero vectors in N , θ^ν orthogonal to a^ν . If $s \leqslant r_1 - 1$ there are constants c_ν such that $\langle x , \theta^m \rangle^s = x_1^s = \sum_{\nu=1}^{r_1} c_\nu \langle x , \theta^\nu \rangle^s$. Here $\langle , \rangle$ denotes the inner product in $\mathbf{R}^n$. Then $D_{a^\nu} \varphi_\nu = 0$, if $\varphi_\nu(x) = \langle x , \theta^\nu \rangle^s$. The same is true if φ_ν is multiplied by an arbitrary C^∞ function of $x_2 , \ldots , x_{n-1}$, i.e.

$$\varphi_\nu(x) = \langle x , \theta^\nu \rangle^s \, \psi(x_2 , \ldots , x_{n-1}) , \; x \in \mathbf{R}^n , \; \nu = 1 , \ldots , r_1 , \; \nu = m .$$

Applying (2.6) and letting ψ vary we obtain (2.5), which completes the proof.

Proof of Theorem 1.1. – The spaces $L_c^p(\Omega)$ and $L_c^p(\Omega_\nu)$ are LF-spaces in the sense of Dieudonné and Schwarz [1], i.e. inductive limits of Frechet spaces. Theorem I page 72 in [1] implies that a continuous linear operator between LF-spaces must be a homomorphism if it has closed range. Therefore it is sufficient to prove that $\operatorname{im} P_A = (\ker Q_A)^\perp$. We identify $L_c^p(\Omega)$ with a subspace of $L_c^p(\mathbf{R}^n)$ in the obvious way. Let $G = (g_1 , \ldots , g_m) \in (\ker Q_A)^\perp$, $g_\nu \in L_c^p(\Omega_\nu)$. We use induction over m . The case $m = 1$ is trivial (c.f. Lemma 2.1.). By the induction assumption there exists $f_0 \in L_c^p(\Omega)$ such that $P_\nu f_0 = g_\nu$ for $\nu < m$. We seek f_1 so that $f = f_0 + f_1$ solves our problem, i.e. $P_\nu f_1 = 0$ for $\nu < m$ and $P_m f_1 = g_m - P_m f_0 = k_m$. It is clear that $(0 , \ldots , k_m) \in (\ker Q_A)^\perp$. Thus the problem is reduced to the special case $g_1 = \ldots = g_{m-1} = 0$.

To consider this case assume again that $a^m = (0 , \ldots , 0, 1)$ and write $g_m = g$. Let g_0 be the function constructed in Lemma 2.2. If Ω is convex, $p_m(\Omega)$ is also convex, so it is obvious that $\operatorname{supp} g_0 \subset p_m(\Omega)$. If Ω is a connected subset of $\mathbf{R}^2$, then $p_m(\Omega)$ is an interval (possibly infinite), hence convex, so again we know that $\operatorname{supp} g_0 \subset p_m(\Omega)$. Choose a compact set $K \subset \Omega$ such that $p_m(K) \supset \operatorname{supp} g_0$, and choose $\epsilon > 0$ so small that $K_\epsilon = K + \{x \in \mathbf{R}^n \, ; |x| \leqslant \epsilon\} \subset \Omega$. Let $E_\epsilon = E_{\epsilon , a^m}$ be the right inverse of P_m constructed in Lemma 2.1 and set $f = \left(\prod_{\nu=1}^{m-1} D_{a^\nu} \right) E_\epsilon g_0$. To see that $f \in L^p(\mathbf{R}^n)$ we observe that $D_{a^\nu} = D_{b^\nu} + a_n^\nu D_n$, where $a^\nu = (b^\nu , a_n^\nu)$, hence f is a linear combination of terms of the form

$$R_{m-k-1}(D) \, D_n^k (E_\epsilon g_0) , \tag{2.7}$$

where $R_{m-k-1}(D)$ denotes any product of precisely $m-k-1$ of the operators $D_{b\nu}$. But $(\Pi D_{b\nu})g_0 = g \in L^p(\mathbf{R}^{n-1})$, hence the function (2.7) belongs to $L^p(\mathbf{R}^n)$ by (B) of Lemma 2.1. This proves that $f \in L^p(\mathbf{R}^n)$. It is obvious that supp $f \subset K_\epsilon$ and that $P_\nu f = 0$ for $\nu < m$. To see that $P_m f = g$ we again write f as a linear combination of terms of the form (2.7). Then each term for which $k > 0$ is mapped to zero by P_m. Taking into account the definition of g_0 we see that the term corresponding to $k = 0$ is nothing but $E_\epsilon g$, which is clearly mapped to g by P_m. This completes the proof of Theorem 1.1.

The next theorem gives an estimate for the solution constructed in Theorem 1.1. As usual A denotes a finite set of distinct elements of S^{n-1}.

THEOREM 2.3. – *Let* K *be a compact subset of* $\mathbf{R}^n$, *let* $1 \leqslant p \leqslant \infty$, *and let* $0 < \epsilon < \operatorname{diam}(K)$. *Assume that* $g_\nu \in L^p(H_\nu)$, $\operatorname{supp} g_\nu \subset p_\nu(K)$, *and that* $G = (g_1, \dots, g_m) \in (\ker Q_A)^\perp$. *Then there exists* $f \in L^p(\mathbf{R}^n)$ *such that*

$$\operatorname{supp} f \subset K_\epsilon = K + \{x \in \mathbf{R}^n \,;\, |x| \leqslant \epsilon\}\,,\ P_A f = G\,,$$

and a constant C *depending only on* A, *such that*

$$\|f\|_p \leqslant C\,\epsilon^{-1+1/p}\left(\frac{\operatorname{diam}(K)}{\epsilon}\right)^{m^2} \sum_{\nu=1}^{m} \|g_\nu\|_p\,. \tag{2.8}$$

It is natural to begin the proof with a closer study of the inverse operator E_ϵ constructed in Lemma 2.1.

LEMMA 2.4. – *The operator* E_ϵ *of Lemma 2.1 can be constructed so that the following estimate holds. If* Q_s *is an arbitrary product of* s *directional derivatives with respect to the* y-*variables, if* $\operatorname{diam}(K) = d \geqslant \epsilon$, *and* $f(y, x_n) = f = E_\epsilon h$, *then*

$$\|Q_s D_n^r f\|_p \leqslant C_{s,r}\,\epsilon^{-r-1+1/p}\,(d/\epsilon)^s\,\|Q_s h\|_p\,.$$

For the proof we shall need a couple of lemmas.

LEMMA 2.5. – *Let* φ_k *be a collection of continuous functions on* $\mathbf{R}^{n-1}$ *such that not more than* N *of the* φ_k *are different from zero at any point of* $\mathbf{R}^{n-1}$, *and* $\sup_{y,k} |\varphi_k(y)| \leqslant B$. *Let*

$g \in L^p(\mathbf{R}^{n-1})$, $\psi \in L^p(\mathbf{R})$, and let v_k be arbitrary real numbers. Writing $x = (y, x_n)$, $y \in \mathbf{R}^{n-1}$, set

$$f(y, x_n) = g(y) \sum_{1}^{\infty} \varphi_k(y)\, \psi(x_n - v_k).$$

Then $f \in L^p(\mathbf{R}^n)$ and

$$\|f\|_p \leqslant \mathrm{B\,N}\, \|g\|_p\, \|\psi\|_p . \tag{2.9}$$

Proof. – By the assumption and Minkowski's inequality for each fixed y we have $\left(\int |f(y, x_n)|^p\, dx_n\right)^{1/p} \leqslant |g(y)|\, \mathrm{B\,N}\, \|\psi\|_p$, which immediately gives the assertion.

Proof of Lemma 2.4. – Let $Q_s = \prod_{\nu=1}^{s} D_{b^\nu}$, where $b^\nu \in \mathbf{R}^{n-1} \backslash \{0\}$ and some b^ν may be equal. We may assume that $|b^\nu| = 1$ for all ν. Define $f = E_\epsilon h$ as in the proof of Lemma 2.1. $Q_s D_n^r f$ can be written as a sum of 2^s terms of the form

$$Q_{s-t}\, h(y) \sum_k Q_t\, \phi_k(y)\, D_n^r\, \psi(x_n - v_k), \tag{2.10}$$

where Q_t is a product of t of the D_{b^ν}. It is possible to choose the partition of unity $\{\phi_k\}$ in the proof of Lemma 2.1 so that $\operatorname{diam}(\operatorname{supp} \phi_k) \leqslant \epsilon/2$, and $\sup |Q_t\, \phi_k| \leqslant C_t\, \epsilon^{-t}$, $t = 1, 2, \ldots$, where C_t is independent of ϵ and k. Similarly, $\psi \in C^\infty(\mathbf{R})$ can be chosen so that $\int \psi\, dt = 1$, $\operatorname{supp} \psi \subset [-\epsilon/4, \epsilon/4]$, and $\sup |D^r \psi| \leqslant C_r\, \epsilon^{-r-1}$. By means of Lemma 2.5 the L^p-norm of the function (2.10) can be estimated by a constant times

$$C_t\, \epsilon^{-t}\, N\, \|Q_{s-t}\, h\|_p\, C_r\, \epsilon^{-r-1+1/p}\, \|\psi\|_p . \tag{2.11}$$

Since $\operatorname{supp} h$ is contained in a ball of radius $\leqslant d$ we have by well-known estimates $\|Q_{s-t}\, h\|_p \leqslant d^t\, \|Q_s\, h\|_p$.

Inserting this inequality in (2.11) and summing over t, assuming $d \geqslant \epsilon$, we obtain the estimate of Lemma 2.4.

Proof of Theorem 2.3. – As in the proof of Theorem 1.1 we consider first the case $g_1 = \ldots = g_{m-1} = 0$. Set $g_m = g$. It is enough to estimate the L^p-norm of the expression (2.7). By Lemma 2.4 this quantity can be estimated by

$$c_m\, \epsilon^{-k-1+1/p}\, (d/\epsilon)^{m-k-1}\, \|R_{m-k-1}\, g_0\|_p\,. \tag{2.12}$$

Since R_{m-k-1} is a product of $m-k-1$ of the factors D_{b^ν}, $1 \leqslant \nu \leqslant m-1$, we have $g = S_k R_{m-k-1}\, g_0$, if S_k denotes the product of the remaining k factors. Furthermore we use the estimate $\|R_{m-k-1}\, g_0\|_p \leqslant C\, d^k\, \|g\|_p$, where C depends only on the length of the b^ν, hence only on A. Inserting this estimate in (2.12) and summing over k gives with a new C

$$\|f\|_p \leqslant C\, \epsilon^{-1+1/p}\, (d/\epsilon)^{m-1}\, \|g\|_p\,.$$

In order to prove Theorem 2.3 in the general case we use induction over m. The case $m = 1$ is already settled. By the induction assumption we can choose f_1 such that $\operatorname{supp} f_1 \subset K_\epsilon$, $P_\nu f_1 = g_\nu$ for $\nu \leqslant m-1$, and

$$\|f_1\|_p \leqslant C\, \epsilon^{-1+1/p}\, (d/\epsilon)^{(m-1)^2} \sum_{\nu=1}^{p-1} \|g_\nu\|_p\,. \tag{2.13}$$

Here and in what follows we denote by C different constants depending only on A. By the special case already treated we can then choose f_2 such that $P_\nu f_2 = 0$ for $\nu \leqslant m-1$, $P_m f_2 = g_m - P_m f_1 = k$, $\operatorname{supp} f_2 \subset K_{2\epsilon}$, and

$$\|f_2\|_p \leqslant C\, \epsilon^{-1+1/p}\, (d/\epsilon)^{m-1}\, \|k\|_p\,. \tag{2.14}$$

By (1.2) we have

$$\|P_m f_1\|_p \leqslant (\operatorname{diam} K_\epsilon)^{1-1/p}\, \|f_1\|_p \leqslant (2d)^{1-1/p}\, \|f_1\|_p\,,$$

hence

$$\|k\|_p \leqslant \|g_m\|_p + \|P_m f_1\|_p \leqslant \|g_m\|_p + (2d)^{1-1/p}\, \|f_1\|_p\,. \tag{2.15}$$

Set $f = f_1 + f_2$. Clearly $P_\nu f = g_\nu$ for all ν. Combining (2.13), (2.14), and (2.15) we obtain the desired result.

3. The range of Q_A in $L^q_{\mathrm{loc}}(\Omega)$.

We will now study the map

$$Q_A : \sum_{\nu=1}^{m} L^q_{\mathrm{loc}}(\Omega_\nu) \longrightarrow L^q_{\mathrm{loc}}(\Omega)\,, \tag{3.1}$$

$(u_1, \ldots, u_m) \longrightarrow \Sigma\, u_\nu$, which is the adjoint of the map P_A studied in the previous section. By means of functional analytic arguments we will deduce Theorem 1.2 from Theorem 1.1.

Denoting by X^* the dual space of X we have the isomorphisms $(p^{-1} + q^{-1} = 1)$

$$L_c^p(\Omega) \cong L_{\text{loc}}^q(\Omega)^* \quad \text{for } 1 \leqslant q < \infty,$$

$$L_{\text{loc}}^q(\Omega) = L_c^q(\Omega)^* \quad \text{for } 1 \leqslant p < \infty.$$

If the dual spaces are provided with the strong dual topology, these isomorphisms are topological. Since P_a obviously is a homomorphism from $L_c^p(\Omega)$ onto $L_c^p(\Omega_a)$ we have the topological isomorphism $L_c^p(\Omega_a) \cong L_c^p(\Omega)/\ker P_a$.

LEMMA 3.1. – *For arbitrary open* $\Omega \subset \mathbf{R}^n$ *and* $1 \leqslant p \leqslant \infty$ *the annihilator in* $L_{\text{loc}}^q(\Omega)$ *of* $\ker P_a \subset L_c^p(\Omega), p^{-1} + q^{-1} = 1$, *is* $L_{\text{loc}}^q(\Omega, a)$.

Proof. – It is obvious that the annihilator in question contains $L_{\text{loc}}^q(\Omega, a)$. To prove the converse inclusion assume $f \in L_{\text{loc}}^q(\Omega)$ and that $\int f \varphi \, dx = 0$ for every $\varphi \in \ker P_a \subset \mathcal{D}(\Omega)$; here $\mathcal{D}(\Omega)$ denotes the space of infinitely differentiable functions with compact support in Ω. If Ω is convex every such φ is equal to $D_a \psi$ for some $\psi \in \mathcal{D}(\Omega)$. Thus $\int f D_a \psi \, dx = 0$ for all $\psi \in \mathcal{D}(\Omega)$.

But this means that $D_a f = 0$ in the sense of the theory of distributions, hence $f \in L_{\text{loc}}^q(\Omega, a)$ if Ω is convex. For a general Ω this argument shows that f is constant a.e. on each component of $\Omega \cap L$ for almost every line L parallel to a. Applying the hypothesis to suitably chosen $\varphi \in \ker P_a \subset \mathcal{D}(\Omega)$ one easily proves that f must actually be globally constant on $\Omega \cap L$. We leave out the details.

By virtue of Lemma 3.1 we have the following natural topological isomorphisms for $p^{-1} + q^{-1} = 1$

$$L_c^p(\Omega_a) \cong L_{\text{loc}}^q(\Omega, a)^* \quad \text{for } 1 \leqslant q < \infty,$$

$$L_{\text{loc}}^q(\Omega, a) \cong L_c^p(\Omega_a)^* \quad \text{for } 1 \leqslant p < \infty.$$

Proof of Theorem 1.2 for $1 \leqslant q < \infty$. – Let X and Y be locally convex topological vector spaces. Assume that $T : X \longrightarrow Y$ is a homomorphism, and let T^* be the adjoint map from Y^* into X^*.

Then it is known that $\operatorname{im} T^*$ must equal $(\ker T)^\perp$, the annihilator of $\ker T \subset X$. In fact this follows directly from the Hahn-Banach Theorem. Now take for $1 \leqslant p < \infty$

$$X = L_c^p(\Omega), \quad Y = \prod_{\nu=1}^{m} L_c^p(\Omega_\nu),$$

and $T = P_A$. Then $T^* = Q_A$ as in (3.1), $p^{-1} + q^{-1} = 1$. Since P_A is known to be a homomorphism by Theorem 1.1, we conclude that $\operatorname{im} Q_A = (\ker P_A)^\perp$, which implies that $\operatorname{im} Q_A$ is closed. By the open mapping theorem Q_A must be a homomorphism.

To treat the case $q = 1$ in Theorem 1.2 we must deduce information about T from assumptions on T^*, which is somewhat harder than the other way round. But in this case we can rely on a theorem about mappings between Frechet spaces (Theorem 21.9 in [5]).

THEOREM 3.2. – *Assume that* X *and* Y *are Frechet spaces and* T *is a continuous linear mapping from* X *into* Y. *Then the following conditions are equivalent :*

(i) $\operatorname{im} T$ *is closed in* Y

(ii) T *is a homomorphism*

(iii) $\operatorname{im} T^*$ *is weak* closed in* X^*.

End of proof of Theorem 1.2. – In Theorem 3.2 we take $X = \prod_{\nu=1}^{m} L_{\mathrm{loc}}^1(\Omega, a^\nu)$, $Y = L_{\mathrm{loc}}^1(\Omega)$, and $T = Q_A$. Then $T^* = P_A$, so we know by Theorem 1.1 that $\operatorname{im} T^*$ is weak* closed in X^*. This proves that $T = Q_A$ has the properties (i) and (ii) as claimed.

4. Estimates for solutions in $L^q(\Omega)^m$ of $Q_A U = v$.

The estimates given in Section 2 for solutions of the equation $P_A f = G$ will now be translated into analogous estimates for the adjoint equation $Q_A U = v$. These estimates are all local in the sense that the solution U is estimated on a certain set in terms of v on a somewhat larger set. However, for convex $\Omega \subset \mathbf{R}^2$ with smooth boundary it is possible to deduce global estimates in $L^q(\Omega)$-norms (Proposition 4.3) by means of the simple argument of Corollary 1.3. These estimates are crucial for the study of the three-dimensional problem.

Let Ω be a bounded open subset of $\mathbf{R}^n$. We will work with the Banach spaces $L^p(\Omega)^m$, $1 \leqslant p \leqslant \infty$. The norm in $L^p(\Omega)^m$ is $\|F\|_p = \sum_{\nu=1}^{m} \left(\int_\Omega |f_\nu|^p \, dx\right)^{1/p}$ for $F = (f_1, \ldots, f_m)$, $f_\nu \in L^p(\Omega)$. For $G = (g_1, \ldots, g_m) \in L^q(\Omega)^m$, $p^{-1} + q^{-1} = 1$, and F as above we set $(F, G) = \sum_{\nu=1}^{m} \int_\Omega f_\nu g_\nu \, dx$.

In this section we will always assume that $1 \leqslant p \leqslant \infty$ and that $p^{-1} + q^{-1} = 1$.

Let E be a subspace of $L^p(\Omega)^m$. We denote by $E^\perp$ the space of all $G \in L^q(\Omega)^m$ such that $(F, G) = 0$ for all $F \in E$.

LEMMA 4.1. – *Assume that* E *is a closed subspace of* $L^q(\Omega)^m$, *and let* $V \in L^q(\Omega)^m$. *Then*

$$\begin{aligned}\inf\{\|V - H\|_q \,; H \in E\} \\ = \sup\{|(V, \phi)| \,; \phi \in E^\perp \subset L^p(\Omega)^m, \|\phi\|_p = 1\}.\end{aligned}$$

Proof. – Let X be a Banach space, X^* its dual, E a closed subspace of X, $E^\perp$ its annihilator in X. Let $x \in X$ and $\xi \in X^*$. Then it is well-known that

$$\sup\{|\eta(x)| \,; \eta \in E^\perp, \|\eta\| \leqslant 1\} = \inf\{\|x - y\| \,; y \in E\},$$

and

$$\sup\{|\xi(y)| \,; y \in E, \|y\| \leqslant 1\} = \inf\{\|\xi - \eta\| \,; \eta \in E^\perp\}.$$

Moreover we note that the dual of $L^p(\Omega)^m$ is isometrically isomorphic to $L^q(\Omega)^m$ for $1 \leqslant p < \infty$. These facts imply the assertion of the lemma.

The next proposition is a dual version of Theorem 2.3. Recall that $\operatorname{im} Q_A$ is equal to the space of solutions of the equation $\left(\prod_{\nu=1}^{m} D_{a^\nu}\right) v = 0$ (see the remark after Theorem 1.2). For $\epsilon > 0$ we write $\Omega^\epsilon = \{x \in \Omega \,; d(x, \partial\Omega) > \epsilon\}$; here $d(x, \Sigma)$ is the distance from x to the set Σ.

PROPOSITION 4.2. – *Let* Ω *be a bounded convex subset of* $\mathbf{R}^n$, *let* $1 \leqslant q \leqslant \infty$ *and* $0 < \epsilon < \operatorname{diam}(\Omega)$. *Let* $v \in L^q(\Omega)$ *satisfy* $\left(\prod_{\nu=1}^{m} D_{a^\nu}\right) v = 0$. *Then there exist a constant* C *depending only* *on* $A = \{a^\nu\}_{\nu=1}^m$, *and functions* u_ν *such that* $\Sigma u_\nu = v$ *and*

$$\Big(\sum_{\nu=1}^m \int_{\Omega^\epsilon} |u_\nu|^q\, dx\Big)^{1/q} \leqslant C\Big(\frac{\operatorname{diam}(\Omega)}{\epsilon}\Big)^{m^2+1} \Big(\int_\Omega |v|^q\, dx\Big)^{1/q}.$$

Proof. – By Theorem 1.2 and the remark following it there exists $V = (v_1, \ldots, v_m)$, $v_\nu \in L^q_{\mathrm{loc}}(\Omega, a^\nu)$, such that $\Sigma v_\nu = v$ in Ω. Then $v_\nu \in L^q(\Omega^\epsilon)$ and hence $V \in L^q(\Omega^\epsilon)^m$. Considering Q_A as an operator from $\prod_{\nu=1}^m L^q(\Omega^\epsilon, a^\nu)$ to $L^q(\Omega^\epsilon)$ and identifying the domain of Q_A with a subspace of $L^q(\Omega^\epsilon)^m$ we may consider $\ker Q_A$ as a subspace of $L^q(\Omega^\epsilon)^m$. Set $E = \ker Q_A$. We are going to estimate $\inf\{\|V - H\|_q\,;\, H \in E\}$ by means of Lemma 4.1 ; here the norm refers of course to the space $L^q(\Omega^\epsilon)$. Take an arbitrary $\phi = (\varphi_1, \ldots, \varphi_m) \in E^\perp \subset L^p(\Omega^\epsilon)^m$. Extending ϕ by zero outside Ω^ϵ we get an element of $L^p_c(\mathbf{R}^n)^m$, which is also denoted ϕ, such that $\operatorname{supp}\phi \subset \overline{\Omega}^\epsilon$. It is easily seen that the element $\widetilde{\phi} = (P_1\varphi_1, \ldots, P_m\varphi_m)$ belongs to $(\ker Q_A)^\perp$ in the sense of Theorem 1.1 and Theorem 2.3. Thus by Theorem 2.3 there exists $\psi \in L^p_c(\mathbf{R}^n)$ such that $\operatorname{supp}\psi \subset \Omega$, $P_A\,\psi = \widetilde{\phi}$, and

$$\|\psi\|_p \leqslant C\Big(\frac{\operatorname{diam}(\Omega)}{\epsilon}\Big)^{m^2} \epsilon^{-1+1/p} \sum_{\nu=1}^m \|P_\nu\varphi_\nu\|_p\,. \tag{4.1}$$

Since $\|P_\nu\varphi_\nu\|_p \leqslant \operatorname{diam}(\Omega)^{1/q}\|\varphi_\nu\|_p$, we deduce from (4.1) that $\|\psi\|_p \leqslant C\Big(\frac{\operatorname{diam}(\Omega)}{\epsilon}\Big)^{m^2+1}\|\phi\|_p$. Identifying $v_\nu \in L^q(\Omega^\epsilon, a^\nu)$ with the corresponding function $v_\nu \in L^q(\Omega^\epsilon_\nu)$ we now get

$$(V, \phi) = \sum_{\nu=1}^m \int_{\Omega^\epsilon} v_\nu\varphi_\nu\, dx = \sum_{\nu=1}^m \int_{\Omega^\epsilon_\nu} v_\nu P_\nu\varphi_\nu\, dy$$

$$= \sum_{\nu=1}^m \int_{\Omega^\epsilon_\nu} v_\nu P_\nu\,\psi\, dy = \int_\Omega \Big(\sum_{\nu=1}^m v_\nu\Big)\psi\, dx = \int_\Omega v\,\psi\, dx = (v, \psi).$$

Taking account of (4.1) we finally obtain

$$|(V, \phi)| = |(v, \psi)| \leqslant \Big(\int_\Omega |v|^q\, dx\Big)^{1/q} C\Big(\frac{\operatorname{diam}(\Omega)}{\epsilon}\Big)^{m^2+1}\|\phi\|_p$$

for $\phi \in E^\perp \subset L^p(\Omega^\epsilon)^m$. The assertion of the proposition now follows from Lemma 4.1.

For a convex domain $\Omega \subset \mathbf{R}^2$ with smooth boundary we can now easily deduce an estimate on the entire domain Ω of solutions to $Q_A U = v$. The argument needed is only a quantitative version of the proof of Corollary 1.4.

PROPOSITION 4.3. – *Assume that $\Omega \subset \mathbf{R}^2$ is convex and bounded, that the boundary $\partial\Omega$ of Ω is of class C^2, and let κ be the maximal curvature of $\partial\Omega$. Let $1 \leqslant q \leqslant \infty$, let $v \in L^q(\Omega)$, and $(\Pi D_{a^\nu}) v = 0$ in Ω. Then there exist $u_\nu \in L^q(\Omega, a^\nu)$ such that $\Sigma u_\nu = v$, and a constant C depending only on $A = \{a^\nu\}_{\nu=1}^m$, such that*

$$\Big(\sum_{\nu=1}^m \int_\Omega |u_\nu|^q \, dx\Big)^{1/q} \leqslant C(\kappa \operatorname{diam}(\Omega))^{m^2+1} \Big(\int_\Omega |v|^q \, dx\Big)^{1/q}.$$

Proof. – For given $\epsilon > 0$ and $a \in \mathbf{R}^2 \backslash \{0\}$ let $V(\epsilon, a)$ be the set of all $x \in \Omega$ such that the line through x parallel to a intersects Ω^ϵ. Thus $V(\epsilon, a)$ is that part of Ω where a function in $L^q(\Omega, a)$ is determined by its values on Ω^ϵ. An easy geometrical argument shows that $|\Omega \cap L| \leqslant 2|\Omega^\epsilon \cap L|$ for any line L intersecting $\Omega^{2\epsilon}$; here $|\ |$ denotes length. Hence we have the estimate

$$\int_{V(2\epsilon, a)} |f|^q \, dx \leqslant 2 \int_{\Omega^\epsilon} |f|^q \, dx \tag{4.2}$$

for $f \in L^q(\Omega, a)$. Now choose ϵ so small that all the sets $\Omega \backslash V(2\epsilon, a^\nu)$ are disjoint. Another elementary geometrical consideration shows that it suffices to take ϵ so that $\cos(\gamma/3) \geqslant 1 - \kappa\, 2\epsilon$, where γ is the minimum angle between different a^ν. In figure 2 we have drawn the "extremal" case, in which $\partial\Omega$ is a circular arc with radius $1/\kappa$. By Proposition 4.2 and (4.2) we infer that u_ν can be chosen so that (set $d = \operatorname{diam}(\Omega)$)

$$\Big(\int_{V(2\epsilon, a^\nu)} |u_\nu|^q \, dx\Big)^{1/q} \leqslant C(d\kappa)^{m^2+1} \Big(\int_\Omega |v|^q \, dx\Big)^{1/q}. \tag{4.3}$$

The rest of the argument is parallel to the proof of Corollary 1.3. Set $F_\nu = \Omega \backslash V(2\epsilon, a^\nu)$. Since the F_ν are disjoint, we have $F_\mu \subset \Omega \backslash F_\nu = V(2\epsilon, a^\nu)$ for $\nu \neq \mu$. Using the identity $u_\mu = v - \sum_{\nu \neq \mu} u_\nu$ we get with a new C

$$\left(\int_{F_\mu} |u_\mu|^q \, dx\right)^{1/q} \leqslant \left(\int_\Omega |v|^q \, dx\right)^{1/q} + \sum_{\nu \neq \mu} \left(\int_{V(2\epsilon, a^\nu)} |u_\nu|^q \, dx\right)^{1/q}$$

$$\leqslant C(d\kappa)^{m^2+1} \left(\int_\Omega |v|^q \, dx\right)^{1/q}. \qquad (4.4)$$

Combination of (4.3) and (4.4) gives the desired estimate.

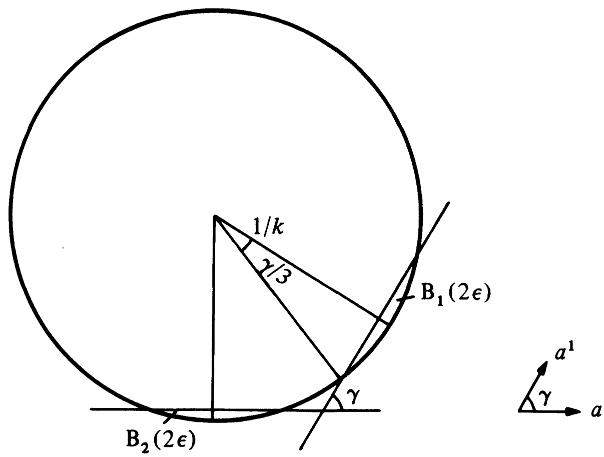

Fig. 2

5. The range of Q_A in $L^q(\Omega)$, $\Omega \subset \mathbf{R}^2$.

In this section we will complete the proof of Theorem 1.5. It remains to consider the case when $\partial\Omega$ has characteristic points. The main step is to treat the case when Ω has one "corner", i.e. Ω looks like a wedge near one point. Let Γ be a bounded or unbounded wedge. A basic idea is to introduce a new norm on $L^q(\Gamma)$ by factoring out polynomials on a sequence of overlapping dyadic parts of the wedge. Since functions in $L^q(\Gamma, a)$ actually depend only on one variable, we can formulate the lemma that we need in terms of functions on $\mathbf{R}_+ = \{t\,; t > 0\}$ rather than Γ.

Let $1 \leqslant q < \infty$, let P_r be the set of polynomials in one variable of degree at most r, and set

$$I_k = \{t\,; 2^{k-1} \leqslant t \leqslant 2^{k+1}\}, \quad k = 0, \pm 1, \ldots.$$

We will consider the factor space $L^q(I_k)/P_r$ for each k. Moreover for σ real > -1 we introduce the Banach space $S_r^{q,0}$ of sequences $V = \{v_k\}_{k=-\infty}^{\infty}$, $v_k \in L^q(I_k)/P_r$, such that $v_k - v_{k+1} = 0$ in $L^q(I_k \cap I_{k+1})/P_r$, and

$$\|V\|_{q,r} = \|V\|_{q,r,\sigma} = \left\{ \sum_{k=-\infty}^{\infty} 2^{k\sigma} \|v_k\|^q_{L^q(I_k)/P_r} \right\}^{1/q} < \infty .$$

If we think of v_k as *functions* in $L^q(I_k)$ rather than elements of the factor spaces $L^q(I_k)/P_r$ these conditions mean that $v_k - v_{k+1} \in P_r$ for all k and that

$$\sum_{k=-\infty}^{\infty} 2^{k\sigma} \inf_{h \in P_r} \int_{I_k} |v_k - h|^q \, dt < \infty. \tag{5.1}$$

Let $L^{q,\sigma}(\mathbf{R}_+)$ be the Banach space of functions v on $\mathbf{R}_+$ such that $\|v\|_{q,\sigma} = \left\{ \int_0^\infty t^\sigma |v(t)|^q \, dt \right\}^{1/q} < \infty$. Then there is a natural continuous linear map $\eta : L^{q,\sigma}(\mathbf{R}_+) \longrightarrow S_r^{q,\sigma}$ defined by the sequence of restrictions and projections

$$L^{q,\sigma}(\mathbf{R}_+) \ni u \longrightarrow u_k \in L^q(I_k)/P_r \,, \; k = 0, \pm 1, \ldots .$$

PROPOSITION 5.1. – *The map η is surjective, if $1 \leqslant q < \infty$ and $\sigma > -1$.*

If $u \in L^{q,\sigma}(\mathbf{R}_+)$ and $\eta(u) = 0$, then u must be a polynomial, hence $u = 0$. Thus η is injective. By the Closed Graph Theorem η must be an isomorphism of Banach spaces. In other words, the expression (5.1) is a norm on $L^{q,\sigma}(\mathbf{R}_+)$, equivalent to the usual one.

Denote by $L^q_{\mathrm{loc}}(\mathbf{R}_+)$ the set of functions u such that $u(t)$ belongs to L^q over every compact subset of $\mathbf{R}_+$. An arbitrary element of $S_r^{q,\sigma}$ can by its definition obviously be represented by a function in $L^q_{\mathrm{loc}}(\mathbf{R}_+)$. Thus Proposition 5.1 is equivalent to the following statement.

PROPOSITION 5.1′ – *Let $1 \leqslant q \leqslant \infty$ and $\sigma > -1$. There exists a constant C depending only on r, q, and σ with the following property. If $u \in L^q_{\mathrm{loc}}(\mathbf{R}_+)$ and*

$$\sum_{k=-\infty}^{\infty} 2^{k\sigma} \inf_{h \in P_r} \int_{I_k} |u - h|^q \, dt \leqslant C_1 < \infty , \tag{5.2}$$

then there exists $h \in P_r$ *such that* $u - h \in L^q(\mathbf{R}_+)$ *and*

$$\int_0^\infty t^\sigma \, |u(t) - h(t)|^q \, dt \leqslant C C_1 .$$

Remark. – The spaces $L^{q,\sigma}(\mathbf{R}_+)$ and $S_r^{q,\sigma}$ make of course good sense for $q = \infty$; in this case the parameter σ is immaterial and we may choose $\sigma = 0$. However, Proposition 5.1 is not true in this case. To see this take $r = 0$ and $u(t) = \log t$. Then $u \in L^\infty(\mathbf{R}_+)$, but $(k-1)\log 2 \leqslant u(t) \leqslant (k+1)\log 2$ on I_k, hence $\inf_c \sup_{t \in I_k} |u(t) - c| = \log 2$ for all k.

Proof of Proposition 5.1 – Choose $h_k \in P_r$ such that $\sum_k b_k^q < \infty$, where $b_k = \left\{ 2^{k\sigma} \int_{I_k} |u - h_k|^q \, dt \right\}^{1/q}$.

We will choose $h = \lim_{k \to \infty} h_k$. To prove that this limit exists, observe that

$$\left\{ \int_{I_k \cap I_{k-1}} |h_k - h_{k-1}|^q \, dt \right\}^{1/q}$$

$$\leqslant \left\{ \int_{I_k \cap I_{k-1}} |h_k - u|^q \, dt \right\}^{1/q} + \left\{ \int_{I_k \cap I_{k-1}} |h_{k-1} - u|^q \, dt \right\}^{1/q}$$

$$\leqslant b_k \, 2^{-k\sigma/q} + b_{k-1} \, 2^{-(k-1)\sigma/q} .$$

Since $h_k - h_{k-1} \in P_r$, there is a constant C_r depending only on r, such that

$$\sup_{I_k \cap I_{k-1}} |h_k - h_{k-1}| \leqslant C_r \, 2^{-k/q} \left(b_k \, 2^{-k\sigma/q} + b_{k-1} \, 2^{-(k-1)\sigma/q}\right). \tag{5.3}$$

Extend h_k to $\mathbf{R}_+$. Then the estimate (5.3) is valid on $(0,2^k)$ with a new C_r. Thus

$$|h_k - h| \leqslant \sum_{k+1}^{\infty} |h_j - h_{j-1}| \leqslant C_r \sum_k^{\infty} 2^{-j(\sigma+1)/q} \, b_j \tag{5.4}$$

on $(0,2^k)$. Now

$$\int_0^\infty t^\sigma \, |u - h|^q \, dt \leqslant 2^\sigma \sum_{-\infty}^{\infty} 2^{k\sigma} \int_{I_k} |u - h|^q \, dt.$$

Replacing $u - h$ by $u - h_k + h_k - h$ in the k^{th} term in the sum and using Minkowski's inequality for sequences we get

$$\left\{ \int_0^\infty t^\sigma \, |u - h|^q \, dt \right\}^{1/q} \leq 2^{\sigma/q} \left(\sum_{-\infty}^{\infty} b_k^q \right)^{1/q}$$

$$+ \; 2^{\sigma/q} \left\{ \sum_{-\infty}^{\infty} 2^{k\sigma} \int_{I_k} |h_k - h|^q \, dt \right\}^{1/q} .$$

To estimate the last term we use (5.4) and obtain

$$\left(2^{k\sigma} \int_{I_k} |h_k - h|^q \, dt \right)^{1/q} \leq 2^{k\sigma/q} \, 2^{k/q} \, C_r \sum_k^\infty 2^{-j(\sigma+1)/q} \, b_j$$

$$= C_r \sum_{j=0}^{\infty} 2^{-j(\sigma+1)/q} \, b_{k+j} \, .$$

Denote the expression on the left hand side by λ_k. Using Minkowski's inequality for sequences we get

$$\sum_{k=-\infty}^{\infty} (\lambda_k^q)^{1/q} \leq C_r \sum_{j=0}^{\infty} 2^{-j(\sigma+1)/q} \left(\sum_{k=-\infty}^{\infty} b_{k+j}^q \right)^{1/q}$$

$$= \frac{C_r}{1 - 2^{-(\sigma+1)/q}} \left(\sum_{-\infty}^{\infty} b_k^q \right)^{1/q} .$$

This completes the proof.

The link between Proposition 5.1 and Theorem 1.5 is provided by the next lemma. Its proof depends only on Theorem 1.2 together with the fact that the wedge is invariant under dilation.

LEMMA 5.2. – *Let $a^1 = (0,1)$, let Γ be a wedge with vertex at the origin, whose closure except the origin is contained in the half-plane $x_1 > 0$, and let $1 \leq q \leq \infty$. Assume $u = \Sigma\, v_\nu \in L^q(\Gamma)$, $v_\nu \in L^q_{\mathrm{loc}}(\Gamma, a^\nu)$, $\nu = 1, \ldots, m$. Then $v_1(x_1, x_2) = w(x_1)$ satisfies (5.2) for $\sigma = 1$ and $r = m - 2$*, i.e.

$$\sum_{k=-\infty}^{\infty} 2^k \inf_{h \in P_{m-2}} \int_{I_k} |w(t) - h(t)|^q \, dt < \infty. \tag{5.5}$$

Proof. – Let K_0 and K_1 be compact subsets of Γ such that $K_0 \subset\subset K_1$. From Theorem 1.2 we know that Q_A is a homomorphism in the topology of $L^q_{\mathrm{loc}}(\Gamma)$. This implies that

$$\inf_{h \in P_{m-2}} \int_{K_0} |v - h|^q \, dx \leq C \int_{K_1} |u|^q \, dx .$$

By homogeneity this inequality holds with the same C if K_0 and K_1 are replaced by λK_0 and λK_1, $\lambda > 0$. Let Γ_0 and Γ_1 be two smaller wedges with vertex at the origin such that $\Gamma_0 \subset\subset \Gamma_1 \subset\subset \Gamma$ (in an obvious sense), choose

$$K_0 = K_0(k) = \Gamma_0 \cap \{(x_1, x_2) ; 2^{k-1} < x_1 < 2^{k+1}\},$$
$$K_1 = K_1(k) = \Gamma_1 \cap \{(x_1, x_2) ; 2^{k-2} < x_1 < 2^{k+2}\},$$

and sum over k from $-\infty$ to ∞. The sum on the right hand side can be estimated by a constant times $\int_\Gamma |u|^q \, dx$, hence is finite. For $\nu = 1$ the sum on the left is larger than a constant times the expression in (5.5), since v_1 is independent of x_2. Thus the lemma is proved for $1 \leqslant q < \infty$. The argument is easily seen to be valid for $q = \infty$ as well; in this case (5.5) should be interpreted as

$$\sup_k \inf_{h \in P_{m-2}} \sup_{I_k} |w - h| < \infty .$$

Remark. – The statement of the lemma is valid with obvious modifications if Γ is replaced by a bounded wedge, say $\{x \in \Gamma ; |x| < \delta\}$; we simply restrict the summation in (5.3) to $k \leqslant k_0$ for some suitable k_0.

Proof of Theorem 1.5. – Assume that $u \in L^q(\Omega)$ belongs to the closure of $\Sigma\, L^q(\Omega, a^\nu)$. By Theorem 1.2 $u = \Sigma\, v_\nu$, where $v_\nu \in L^q_{\mathrm{loc}}(\Omega, a^\nu)$. We have to prove that each $v_\nu \in L^q(\Omega)$. It is enough to prove that $v_\nu \in L^q$ in a neighbourhood of an arbitrary point of the boundary of Ω. By the proof of Corollary 1.3 it remains only to consider a neighbourhood of a characteristic point, which we may assume to be the origin. Let V_δ be an open disk with radius δ and center at the origin. By the wedge condition there exist an open wedge Γ with vertex at the origin and a $\delta > 0$ such that $\Gamma \cap V_\delta \subset \Omega$. Fix an arbitrary μ, $1 \leqslant \mu \leqslant m$. We need to prove that

$$v_\mu \in L^q(\Omega \cap V_\delta) . \tag{5.6}$$

We have to distinguish between different cases with regard to the position of Ω relative to the direction a^μ. Let π_ν be the line through the origin with direction a^ν, $\nu = 1, \ldots, m$. If π_μ intersects Ω, (5.6) follows immediately from $v_\mu \in L^q_{\mathrm{loc}}(\Omega, a^\mu)$. Thus we assume that π_μ does not intersect Ω. If a^μ is parallel to one of the edges of Γ we choose Γ somewhat smaller, so that $\pi_\mu \cap \bar{\Gamma} = \{0\}$. Then we combine Proposition 5.1 with Lemma 5.2 and the remark following it to conclude that $v_\mu \in L^q(\bar{\Gamma} \cap V_\delta)$. At this point we have used the assumption $q < \infty$. If $\pi_\mu \cap \bar{\Omega} = \{0\}$, this immediately implies (5.6).

There remains the case when π_μ intersects $\bar{\Omega} \cap V_\delta$ at points other than the origin, but π_μ is disjoint from Ω (see fig. 4). By the argument given so far $|v_\mu|^q$ must be integrable over $(\Omega \cap V_\delta) \backslash \Sigma$, where Σ is any open wedge containing a^μ. But if Σ is sufficiently small we know already that $v_\nu \in L^q(\Omega \cap \Sigma \cap V_\delta)$ for all $\nu \neq \mu$. Hence the identity $\Sigma\, v_\nu = u \in L^q(\Omega)$ implies (5.6), and the proof is complete.

In the proof of Theorem 1.5 we used only the "negative half" $k < 0$ of the mapping η in Proposition 5.1. Using the full strength of that proposition we can prove results for certain unbounded regions. However, we have not investigated unbounded regions systematically, so we will give only one result of that kind here.

PROPOSITION 5.3. – *Let* Γ *be an (unbounded) open wedge in* $\mathbf{R}^2$, *and let* $1 \leqslant q < \infty$. *Then* $\sum_{\nu=1}^{m} L^q(\Gamma, a^\nu)$ *is closed in* $L^q(\Gamma)$.

Proof. – We may assume that the vertex of Γ is the origin. Then we observe that $L^q(\Gamma, a) = \{0\}$, if $a \in \bar{\Gamma}$. Thus we may assume $a^\nu \notin \bar{\Gamma}$ for all ν. Denote by $P_r(a)$ the set of polynomial functions in $\mathbf{R}^2$ of degree at most r which are constant on all lines parallel to a. Arguing as in the proof of Theorem 1.5 using Proposition 5.1 and Lemma 5.2 we conclude that there exist $h_\nu \in P_{m-2}(a^\nu)$ so that $v_\nu - h_\nu \in L^q(\Gamma)$ and $\Sigma\, v_\nu = u$. Then $\Sigma(v_\nu - h_\nu) = \Sigma\, v_\nu - \Sigma\, h_\nu = u - \Sigma\, h_\nu$, so that $\Sigma\, h_\nu \in L^q(\Gamma)$. But $\Sigma\, h_\nu$ is a polynomial, hence $\Sigma\, h_\nu = 0$. This proves that $u = \Sigma(v_\nu - h_\nu)$ and thus completes the proof of the proposition.

6. The range of Q_A in $L^q(\Omega)$, $\Omega \subset \mathbf{R}^3$.

This section is devoted to a proof of Theorem 1.8. We will actually prove the following statement, which implies Theorem 1.8.

PROPOSITION 6.1. – *Assume that* Ω *satisfies the assumptions of Theorem 1.8 and let* $1 \leqslant q \leqslant \infty$. *Assume that* $u = \Sigma\, u_\nu \in L^q(\Omega)$, $u_\nu \in L^q_{\text{loc}}(\Omega, a^\nu)$, $\nu = 1, \dots, m$. *Then there exist* $v_\nu \in L^q(\Omega, a^\nu)$, *such that* $u = \Sigma\, v_\nu$.

To prove that this proposition implies Theorem 1.8 we argue as in the proof of Corollary 1.3. Assume that $u \in L^q(\Omega)$ belongs to the closure of $\Sigma\, L^q(\Omega, a^\nu)$. By Theorem 1.2 u must lie in $\Sigma\, L^q_{\text{loc}}(\Omega, a^\nu)$. But this means that the assumptions of Proposition 6.1 are fulfilled, and the conlusion of that proposition is our assertion.

An important difference between the two-dimensional and three-dimensional cases should be noted here. In the proof of Corollary 1.3 we could assert that the functions u_ν, a priori assumed to belong to $L^q_{\text{loc}}(\Omega, a^\nu)$, were actually in $L^q(\Omega)$. In Proposition 6.1, however, we cannot assert this, but must choose new functions v_ν, which belong to $L^q(\Omega, a^\nu)$. The reason is of course that in the two-dimensional case all solutions $u_\nu \in L^q_{\text{loc}}(\Omega, a^\nu)$ of $\Sigma\, u_\nu = 0$ are polynomials (Proposition 1.6), which is not the case in higher dimensions.

Our first step is to solve *locally* the problem of constructing the functions v_ν in Proposition 6.1. We formulate this step as a lemma. Recall that the u_ν are automatically in L^q near non-characteristic points of the boundary, so we need only consider a neighbourhood of a characteristic point.

LEMMA 6.2. – *Assume that* Ω *satisfies the hypothesis of Theorem 1.8 and let* $1 \leqslant q \leqslant \infty$. *Assume that* $u \in L^q(\Omega)$, $u = \Sigma\, u_\nu, u_\nu \in L^q_{\text{loc}}(\Omega, a^\nu)$, *and let* $x^0 \in \partial\Omega$ *be a characteristic point. Then there exist a neighbourhood (relative to* Ω*)* ω *of* x^0 *and functions* $z_\nu \in L^q_{\text{loc}}(\Omega, a^\nu)$ *such that* $z_\nu = u_\nu$ *in* $\Omega\backslash\omega$, $\Sigma\, z_\nu = u$ *in* Ω, *and the restriction of* z_ν *to* ω *belongs to* $L^q(\omega)$ *for each* ν.

Proof. – Let A_0 be the set of all $a^\nu \in A = \{a^\nu\}_{\nu=1}^m$ that are parallel to the tangent plane to $\partial\omega$ at x^0. We may assume that A_0 is the set of the k first elements of A. Then there exists a neighbourhood ω_0 of x_0 such that $u_\nu \in L^q(\omega_0)$ for $\nu > k$, i.e.

$$\sum_{\nu=1}^{k} u_\nu = u - \sum_{k+1}^{m} u_\nu \in L^q(\omega_0).$$

We may assume that $x^0 = (0, \dots, 0)$, the tangent plane at x^0 is $x_3 = 0$, and that $\Omega \subset \{x ; x_3 > 0\}$. We are going to choose $\omega = \{x \in \Omega ; x_3 < \delta\}$, where $\delta > 0$ is so small that $\omega \subset \omega_0$. For fixed $t \in (0, \delta)$ we consider the region

$$\Omega_t = \{(x_1, x_2) ; (x_1, x_2, t) \in \Omega\}.$$

Set $u - \sum_{k+1}^{m} u_\nu = w$. By Proposition 4.3 applied to $\Omega_t \subset \mathbf{R}^2$ and A_0 we can find $v_\nu^t \in L^q(\Omega_t, a^\nu)$ so that $\Sigma_1^k v_\nu^t = w^t$, where $w^t(x_1, x_2) = w(x_1, x_2, t)$, and (assume $q < \infty$, the case $q = \infty$ is similar)

$$\int_{\Omega_t} |v_\nu^t|^q \, dx \leqslant C \int_{\Omega_t} |w^t|^q \, dx, \quad \nu = 1, \dots, k. \tag{6.1}$$

We claim that C may be chosen independent of t. This follows from Proposition 4.3 if we show that $\kappa_t \operatorname{diam}(\Omega_t)$ is bounded for small t; here κ_t is the maximal curvature of the boundary of Ω_t. But this is an easy consequence of the hypothesis about the principal curvatures of $\partial\Omega$ at x^0. To complete the proof we set $z_\nu = u_\nu$ in all of Ω for $\nu > k$, $z_\nu = u_\nu$ in $\Omega \backslash \omega$ for $\nu \leqslant k$, and $z_\nu(x_1, x_2, t) = v_\nu^t(x_1, x_2)$ for $(x_1, x_2, t) \in \omega$ for $\nu \leqslant k$. Then $z_\nu \in L^q(\omega)$ by virtue of (6.1), and the lemma is proved.

End of proof of Proposition 6.1. – Choose, for every characteristic point $x^i, i = 1, \dots, N$, functions z_ν^i according to Lemma 6.2. Set $w_\nu^i = z_\nu^i - u_\nu$ and $v_\nu = u_\nu + \sum_{i=1}^{N} w_\nu^i$. Then $v_\nu \in L^q(\Omega, a^\nu)$ and $\Sigma\, v_\nu = u$, which proves the theorem.

7. Counterexamples.

Our first example shows that the statement of Theorem 1.5 is not valid for $q = \infty$, not even for convex Ω. The second example treats the case $1 \leqslant q < \infty$ and shows that one cannot weaken the assumptions of Theorem 1.5 by replacing the wedge condition by a Lip(α)-condition for any $\alpha < 1$. Finally we construct the example announced in Theorem 1.7, which shows that the curvature hypothesis on $\partial\Omega$ in Theorem 1.8 cannot be omitted.

In the first two examples we shall take $A = \{a^1, a^2\}$, $a^1 = (1,0)$, $a^2 = (0,1)$. For abbreviation we shall write

$$E_q = L^q(\Omega, a^1) + L^q(\Omega, a^2).$$

The closure of E_q in $L^q(\Omega)$ will be denoted $\bar{E}_q$. In the first example we take

$$\Omega = \{(x_1, x_2); x_1 < x_2 < 2x_1, 0 < x_1 < 1/4\}, \text{ and}$$

$$f(x_1, x_2) = \log|\log x_2| - \log|\log x_1|, x \in \Omega.$$

PROPOSITION 7.1. – *$f \in \bar{E}_\infty$, but $f \notin E_\infty$.*

Proof. – Since any representation $f = f_1 + f_2$, $f_\nu \in L^\infty(\Omega, a^\nu)$, differs from (7.1) by an additive constant, we see at once that $f \notin E_\infty$. Next observe that $f(x) \longrightarrow 0$ as $x \longrightarrow (0,0)$ in Ω, hence $f \in L^\infty(\Omega)$. To see that $f \in \bar{E}_\infty$ define f_ϵ in Ω by $f_\epsilon(x) = f(x)$ for $x_1 > \epsilon$, $f_\epsilon(x) = 0$ for $x_1 < \epsilon$. Then $f_\epsilon(x_1, x_2) = u_\epsilon(x_1) - u_\epsilon(x_2) + v_\epsilon(x_1, x_2)$, where $u_\epsilon(t) = \log|\log t|$ for $t \geqslant \epsilon$, $u_\epsilon(t) = \log|\log \epsilon|$ for $t < \epsilon$, and $v_\epsilon(x_1, x_2) = \log|\log x_2| - \log|\log \epsilon|$ in the region $\{x \in \Omega; x_1 < \epsilon < x_2\}$, and $v_\epsilon = 0$ elsewhere in Ω. Then $v_\epsilon \longrightarrow 0$ and $f_\epsilon \longrightarrow f$ in $L^\infty(\Omega)$, which proves the statement.

Remark. – In [7], p. 154, the function $f(x_1, x_2) = \log(x_2/x_1)$ is claimed to have the properties of Proposition 7.1. However, it is not true that $f \in \bar{E}_\infty$. To see this, assume $g = h(x_1) + k(x_2) \in E_\infty$, and $\|f - g\|_\infty < (\log 2)/4$. Then

$$k(2^{-k+1}) - k(2^{-k}) = g(2^{-k}, 2^{-k+1}) - g(2^{-k}, 2^{-k})$$
$$\geqslant f(2^{-k}, 2^{-k+1}) - f(2^{-k}, 2^{-k}) - (\log 2)/2 = (\log 2)/2 .$$

But this implies that $k(x_2) \longrightarrow \infty$ as $x_2 \longrightarrow 0$, which is a contradiction.

We now turn to the case $1 \leqslant q < \infty$. Let β be any real number > 1 and choose

$$\Omega = \{(x_1, x_2); x_1 < x_2 < x_1 + x_1^\beta, 0 < x_1 < 1\},$$

and $f(x_1, x_2) = x_1^{-\alpha} - x_2^{-\alpha}, x \in \Omega$, where $\alpha = (1 + 3\beta)/2q$.

PROPOSITION 7.2. – *For $1 \leqslant q < \infty$ we have $f \in \bar{E}_q$, but $f \notin E_q$.*

Proof. – The functions $f_j(x) = x_j^{-\alpha}$ do not belong to $L^q(\Omega)$, hence $f \notin E_q$. Next we claim that $f \in L^q(\Omega)$. By a straightforward computation

$$\int_\Omega |f(x)|^q\, dx = \int_0^1 x_1^{1-\alpha q} \int_0^{x_1^{\beta-1}} (1 - (1+t)^{-\alpha})^q\, dt\, dx_1 .$$

The inner integral has the order of magnitude $(x_1^{\beta-1})^{q+1}$ as $x_1 \longrightarrow 0$. Hence the condition for convergence of the outer integral becomes $1 - \alpha q + (\beta - 1)(q+1) > -1$, which is satisfied for the value of α that was chosen. It still remains to show that $f \in \overline{E}_q$. Set $f_\epsilon(x) = f(x)$ for $x_1 > \epsilon$, $f_\epsilon(x) = 0$ for $x_1 < \epsilon$ and write $f_\epsilon(x) = u_\epsilon(x_1) - u_\epsilon(x_2) + v_\epsilon(x_1, x_2)$, where $u_\epsilon(t) = t^{-\alpha}$ for $t > \epsilon$, $u_\epsilon(t) = 0$ for $t < \epsilon$. Then $v_\epsilon = x_2^{-\alpha}$ on the region $D_\epsilon = \{x \in \Omega ; x_1 < \epsilon < x_2\}$ and $v_\epsilon = 0$ elsewhere. It suffices to show that $v_\epsilon \longrightarrow 0$ in $L^q(\Omega)$ as $\epsilon \longrightarrow 0$. The area of D_ϵ is $\leqslant \epsilon^{2\beta}$ and $|v_\epsilon| \leqslant \epsilon^{-\alpha}$, hence $\|v_\epsilon\|_q \leqslant \epsilon^{-\alpha + 2\beta/q}$, which tends to zero as $\epsilon \longrightarrow 0$ for our value of α.

We will now go back to the three-dimensional case and prove Theorem 1.7. The idea of the construction is that the two-dimensional sections Ω_t of Ω (see below) are very narrow and close to the line $x_1 = x_2$ for small t, hence the function (7.2) is integrable to q^{th} power although the g_j are not.

Proof of Theorem 1.7. – Take $a^1 = (1,0,0)$, $a^2 = (0,1,0)$. Set $\psi(x_1, x_2) = (x_1 - x_2)^2 + (x_1 + x_2)^4$. Choose Ω bounded and strictly convex with C^∞ boundary so that $\partial\Omega$ near the origin has the form $x_3 = \psi(x_1, x_2)$ and Ω lies in the region $x_3 > \psi(x_1, x_2)$. For $z > 0$ let Ω_z be the two-dimensional region $\Omega \cap \{x_3 = z\}$. For small z, Ω_z is an ellips-like domain with length and width proportional to $\sqrt[4]{z}$ and $\sqrt{z}$, respectively. Assume first that $1 \leqslant q < \infty$. Set $f(x) = x_3^{-\alpha}(x_2 - x_1)$, $x \in \Omega$, where $\alpha = (3 + 14q^{-1})/8$. Then $f \in L^q(\Omega)$, since

$$\begin{aligned}\int_\Omega |f|^q\, dx &\leqslant 4 \int_0^1 z^{-\alpha q} z^{1/4} \int_0^{\sqrt{z}} t^q\, dt dz \\ &= \frac{4}{q+1} \int_0^1 z^{1/4 + (q+1)/2 - \alpha q}\, dz \\ &= \frac{4}{q+1} \int_0^1 z^{-1+q/8}\, dz < \infty.\end{aligned}$$

Setting $f_\epsilon = f$ for $x_3 > \epsilon$, $f_\epsilon = 0$ for $x_3 < \epsilon$, we have $f_\epsilon \in L^q(\Omega, a^1) + L^q(\Omega, a^2)$ and $f_\epsilon \longrightarrow f$ in $L^q(\Omega)$ as $\epsilon \longrightarrow 0$, hence f belongs to the closure of $L^q(\Omega, a^1) + L^q(\Omega, a^2)$. On the other hand, we claim that $f \notin L^q(\Omega, a^1) + L^q(\Omega, a^2)$. We have the decomposition

$$f(x) = x_3^{-\alpha} x_2 - x_3^{-\alpha} x_1 = g_1(x_2, x_3) - g_2(x_1, x_3), \tag{7.2}$$

where g_ν are constant in the a^ν-direction. Since any decomposition of this kind differs from this one by a function of x_3 only (see Proposition 1.6), it is enough to prove that $g_1 \notin L^q(\Omega)$. In fact, for some $c > 0$, $\delta > 0$,

$$\begin{aligned}\int_\Omega |g_1|^q\, dx &\geqslant c \int_0^\delta z^{-\alpha q}\, z^{1/2} \int_0^{z^{1/4}} t^q\, dt\, dz \\ &= \frac{c}{q+1} \int_0^\delta z^{-\alpha q + 1/2 + (q+1)/4}\, dz \\ &= \frac{c}{q+1} \int_0^\delta z^{-1-q/8}\, dz = \infty.\end{aligned}$$

Finally, if $q = \infty$, we take $\alpha = 3/8$ and define f in the same way. The proof for this case is obvious.

8. Generalizations.

It seems very plausible that Theorem 1.8 can be generalized to the case $\Omega \subset \mathbf{R}^n$, $n > 3$. This would mean that (1.1) is closed for arbitrary bounded convex $\Omega \subset \mathbf{R}^n$ with boundary of class C^2, whose principal curvatures are all different from zero at every characteristic point. If this condition is satisfied, the set of characteristic points is a finite union of $n-3$-dimensional submanifolds of $\partial\Omega$. In fact, if B_ν is the set of points of $\partial\Omega$ where the tangent plane is parallel to a^ν, then the characteristic set is equal to the union of all $B_\nu \cap B_\mu$, $\nu \neq \mu$.

In the special case when Ω is a ball in $\mathbf{R}^n$ we can prove that (1.1) is closed by means of a reasoning similar to the proof of Theorem 6.1 together with an induction over n; what makes this case simple is that the intersection of a ball and a plane is always a ball, and hence estimates like those of Proposition 4.3 automatically hold for families of such intersections.

One can generalize the problem by replacing the directions a^ν by subspaces H_ν of arbitrary dimension k, $1 \leqslant k \leqslant n-1$. Let us write $u \in L^q(\Omega, H)$, if $u \in L^q(\Omega)$ and u is constant almost everywhere on $\Omega \cap E$ for almost every translate $E = x + H$ of H. Let $\{H^\nu\}_{\nu=1}^m$ be a finite set of such subspaces. The question is whether

$$\sum_{\nu=1}^{m} L^q(\Omega, H_\nu) \text{ is closed in } L^q(\Omega). \tag{8.1}$$

The case when

$$H_\nu + H_\mu \overset{\text{def}}{=} \{x + y\,;\, x \in H_\nu, y \in H_\mu\} = \mathbf{R}^n \text{ whenever } \nu \neq \mu \tag{8.2}$$

has been treated by a number of authors. Petersen et al. [7] show that (8.1) holds if $1 < p < \infty$, (8.2) is satisfied, and $\partial\Omega$ is Lipschitz continuous (their result is in fact more general than this). Lars Svensson [10] proves a similar result for the more general problem where the family of translates of a subspace is replaced by the family of level sets of a smooth mapping from $\mathbf{R}^n$ to $\mathbf{R}^k$, $1 \leqslant k \leqslant n-1$. If (8.2) is not satisfied one can construct an example based on the idea of Theorem 1.7, where Ω is strictly convex, $\partial\Omega$ is infinitely differentiable, and (8.1) does not hold. On the other hand, it is natural to expect that the non-vanishing of the principal curvatures again guarantees (8.1).

BIBLIOGRAPHY

[1] J. Dieudonne, L. Schwartz, La dualité dans les espaces (F) et (LF), *Ann. Inst. Fourier,* 1 (1949), 61-101.

[2] K.J. Falconer, Consistency conditions for a finite set of projections of a function, *Math. Proc. Cambridge Philos. Soc.,* 85 (1979), 61-68.

[3] C. Hamaker, D.C. Solmon, The angles between the null spaces of X-rays, *J Math. Anal. Appl.,* 62 (1978), 1-23.

[4] L. Hormander, *Linear partial differential operators,* Springer-Verlag, Berlin, 1963.

[5] J.L. KELLEY, I. NAMIOKA, *Linear topological spaces,* Van Nostrand, 1963.

[6] B.F. LOGAN, L.A. SHEPP, Optimal reconstruction of a function from its projections, *Duke Math. J.,* 42 (1975), 645-659. 659.

[7] B.E. PETERSEN, K.T. SMITH, D.C. SOLMON, Sums of plane waves and the range of the Radon transform, *Math. Ann.,* 243 (1979), 153-161.

[8] L.A. SHEPP, J.B. KRUSKAL, Computerized tomography : the new medical X-ray technology, *Amer. Math. Monthly,* 85 (1978), 420-438.

[9] K.T. SMITH, D.C. SOLMON, S.L. WAGNER, Practical and mathematical aspects of the problem of reconstructing objects from radiographs, *Bull. A.M.S.,* 83 (1977), 1227-1270.

[10] L. SVENSSON, When is the sum of closed subspaces closed ? An example arising in computerized tomography, *Research Report, Royal Inst. Technology (Stockholm),* 1980.

[11] J.V. LEAHY, K.T. SMITH, D.C. SOLMON, Uniqueness, nonuniqueness and inversion in the X-ray and Radon problems, to appear in *Proc. Internat. Symp. on Ill-posed Problems,* Univ. of Delaware, Newark, Delaware, 1979.

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