ANNALES DE L'INSTITUT FOURIER

ALEXANDRU BUIUM

Degree of the fibres of an elliptic fibration

Annales de l'institut Fourier, tome 33, n° 1 (1983), p. 269-276

<http://www.numdam.org/item?id=AIF_1983__33_1_269_0>

© Annales de l'institut Fourier, 1983, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (http://annalif.ujf-grenoble.fr/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Fourier, Grenoble
33, 1 (1983), 269-276

DEGREE OF THE FIBRES OF AN ELLIPTIC FIBRATION

by Alexandru BUIUM

1. Statement of the results.

Let $f: X \longrightarrow B$ be an elliptic fibration over the complex field i.e. a morphism from a smooth complex projective surface X to a smooth curve B such that the general fibre F of f is a smooth elliptic curve and no fibre contains exceptional curves of the first kind. Consider the following subsets of Pic(X):

$$N_e = \{\mathcal{L} \in \mathrm{Pic}(X),\ \mathcal{L} = \mathcal{O}_X(D) \text{ for some effective } D\}$$

$$N_s = \{\mathcal{L} \in \mathrm{Pic}(X),\ \mathcal{L} \text{ is spanned by global sections}\}$$

$$N_a = \{\mathcal{L} \in \mathrm{Pic}(X),\ \mathcal{L} \text{ is ample}\}$$

$$N_v = \{\mathcal{L} \in \mathrm{Pic}(X),\ \mathcal{L} \text{ is very ample}\}$$

and let n_e, n_s, n_a, n_v be the minima of the non-zero intersection numbers $(\mathcal{L}.F)$ when $\mathcal{L}$ runs through N_e, N_s, N_a and N_v respectively. In [3] p. 259, Enriques investigates the possibility of finding a birational model of X in the projective space $\mathbf{P}^3$ such that the fibres of f have degree n_e. His analysis suggests the following problem: find the minimum possible degree of the fibres of f in an embedding of X in a projective space. In other words: find n_v. There obviously exist inequalities: $n_e \leqslant n_s \leqslant n_v$ and $n_a \leqslant n_v$.

Let m denote the maximum of the multiplicities of the fibres of f. The aim of this paper is to prove the following propositions:

PROPOSITION 1. – *Equality* $n_e = n_s$ *holds if and only if* $n_e \geqslant 2m$.

PROPOSITION 2. – *Equality* $n_a = n_v$ *holds if and only if* $n_a \geqslant 3m$.

The statements above are consequences of the following more precise results:

THEOREM 1. – *There exists a constant* C_1 *depending only of the fibration such that for any effective divisor* D *on* X *which does not contain in its support any component of any reducible fibre and such that* D *is either reduced dominating* B, *or ample, the following conditions are equivalent:*

1) $(D.F) \geqslant 2m$.

2) $\mathcal{O}_X(D) \otimes f^*L$ *is spanned by global sections for any* $L \in \mathrm{Pic}(B)$ *with* $\deg(L) \geqslant C_1$.

3) $\mathcal{O}_X(D) \otimes f^*L$ *is spanned by global sections for some* $L \in \mathrm{Pic}(B)$.

THEOREM 2. – *There exists a constant* C_2 *depending only on the fibration such that for any ample sheaf* $\mathcal{L} \in \mathrm{Pic}(X)$ *the following conditions are equivalent:*

1) $(\mathcal{L}.F) \geqslant 3m$.

2) $\mathcal{L} \otimes f^*L$ *is very ample for any* $L \in \mathrm{Pic}(B)$ *with* $\deg(L) \geqslant C_2$.

3) $\mathcal{L} \otimes f^*L$ *is very ample for some* $L \in \mathrm{Pic}(B)$.

Our proofs are based on Bombieri's technique from [2]. Therefore the main point will be to prove that certain divisors on X are numerically connected.

2. Two lemmas.

LEMMA 1. – *Let* D *be an effective divisor on* X *which does not contain in its support any component of any reducible fibre. Suppose* D *is either reduced or ample and put* $T = D + a_1F_1 + \ldots + a_pF_p$ *where* F_i *are distinct fibres and* $a_i \in \mathbf{Q}$, $a_i > 0$ *for* $1 \leqslant i \leqslant p$. *Suppose furthermore that* $a_1 + \ldots + a_p \geqslant 2$. *Then we have:*

1) *If* $(D.F) \geqslant 2m$ *then* T *is 2-connected.*

2) *If* $(D.F) \geqslant 3m$ *and* D *is integral and ample then* T *is 3-connected.*

Proof. – Suppose $T = T_1 + T_2$ where $T_k > 0$ and

$$T_k = D_k + A_k$$
$$D_1 + D_2 = D$$
$$A_1 + A_2 = A = a_1 F_1 + \ldots + a_p F_p .$$

We get

$$(T_1 . T_2) = (D_1 . D_2) + (D_1 . A_2) + (D_2 . A_1) + (A_1 . A_2).$$

If in addition D is integral we may suppose $D_2 = 0$. Since by [6] ample divisors are 1-connected it follows that in any case $(D_1 . D_2) \geqslant 0$. On the other hand we have $(D_1 . A_2) \geqslant 0$ and $(D_2 . A_1) \geqslant 0$ because any common component of D and A must be a rational multiple of a fibre. We may write $A_2 = Z_1 + \ldots + Z_p$ where $Z_i \leqslant a_i F_i$ for $1 \leqslant i \leqslant p$. We get

$$(A_1 . A_2) = (A - A_2 . A_2) = -(A_2^2) = -(Z_1^2) - \ldots - (Z_p^2).$$

By [1] p. 123 we have $(Z_i^2) \leqslant 0$ for any i. Suppose first that there exists an index i such that $(Z_i^2) < 0$. By [5], $(Z_i^2) = -2$, consequently $(T_1 . T_2) \geqslant 2$. If an addition D is integral and ample then $A_2 \neq 0$ (because otherwise $T_2 = 0$) hence $(D_1 . A_2) \geqslant 1$ and we get $(T_1 . T_2) \geqslant 3$.

Now suppose $(Z_i^2) = 0$ for any i. Then by [1] p.123, we must have $Z_i = c_{i2} F_i$ where $c_{i2} \in \mathbf{Q}$, $0 \leqslant c_{i2} \leqslant a_i$, hence

$$A_1 = c_{11} F_1 + \ldots + c_{p1} F_p$$

where $c_{i1} + c_{i2} = a_i$. If both D_1 and D_2 dominate B we get $(D_k . F) \geqslant 1$ for $k = 1, 2$ hence

$$(T_1 . T_2) \geqslant (D_1 . A_2) + (D_2 . A_1) \geqslant c_{12} + \ldots + c_{p2} + c_{11} + \ldots + c_{p1}$$
$$= a_1 + \ldots + a_p \geqslant 2$$

and we are done. If $D_k = 0$ for $k = 1$ or $k = 2$ then $A_k \neq 0$ hence there exists an index i_0 such that $c_{i_0 k} > 0$. Now if m_0 denotes the multiplicity of F_{i_0} we have $c_{i_0 k} \geqslant 1/m_0 \geqslant 1/m$. Consequently we get $(T_1 . T_2) = (A_k . D) \geqslant c_{i_0 k}(D . F) \geqslant (D . F)/m$ and we are done again. Finally if $D_k \neq 0$ and D_k does not dominate B we get $(T_1 . T_2) \geqslant (D_1 . D_2) = (D . D_k) \geqslant (D . F)/m$ and the lemma is proved.

LEMMA 2. – *Let* $m_1, \ldots, m_r$ *denote the multiplicities of the multiple fibres of* f. *Then for any reduced effective divisor* D *not*

containing in its support any component of any reducible fibre we have $(D^2) \geqslant -(D.F)\,(\chi(\mathcal{O}_X) + \sum_{j=1}^{r} (m_j - 1)/m_j)$.

Proof. – We may suppose $D = D_1 + \ldots + D_t$ where D_i are integral, distinct, dominating B. For any $i = 1, \ldots, t$ let E_i be the normalization of D_i. By adjuction formula and by Hurwitz formula we get:

$$(D_i^2) + (D_i.K) = 2p_a(D_i) - 2 \geqslant 2p_a(E_i) - 2 \geqslant [E_i : B]\,(2p_a(B) - 2).$$

Consequently:

$$(D^2) \geqslant \sum_{i=1}^{t} (D_i^2) \geqslant \Big(\sum_{i=1}^{t} [E_i : B]\Big)(2p_a(B) - 2) - (D.K)$$

$$= (D.F)\,(2p_a(B) - 2) - (D.F)\,(2p_a(B) - 2 + \chi(\mathcal{O}_X) + \sum_{j=1}^{r} (m_j - 1)/m_j)$$

because of the formula for the canonical divisor K (see [4] p. 572) and we are done.

3. Proofs of Theorems 1 and 2.

Suppose $m_1 Y_1, \ldots, m_r Y_r$ are all the multiple fibres of f each having multiplicity m_j, $1 \leqslant j \leqslant r$ and take $b_j \in B$ such that $m_j Y_j = f^*(b_j)$. By the formula for the canonical divisor K we may write

$$\mathcal{O}_X(K) = f^*M \otimes \mathcal{O}_X\Big(\sum_{j=1}^{r} (m_j - 1)\, Y_j\Big)$$

where $M \in \mathrm{Pic}(B)$, $\deg(M) = 2p_a(B) - 2 + \chi(\mathcal{O}_X)$.

Furthermore for any points x, x_1, x_2 on X denote by $p : \tilde{X} \longrightarrow X$ and $q : \hat{X} \longrightarrow X$ the blowing ups of X at x and $\{x_1, x_2\}$ respectively and let W, W_1, W_2 be the corresponding exceptional curves. Put $y = f(x)$, $y_1 = f(x_1)$, $y_2 = f(x_2)$.

Proof of Theorem 1. – To prove 1) $\Longrightarrow$ 2) it is sufficient by [2] to prove that $H^1(\tilde{X}, p^*\mathcal{O}_X(D) \otimes p^*f^*L \otimes \mathcal{O}_{\tilde{X}}(-W)) = 0$ for any $x \in X$ hence by Bombieri-Ramanujam vanishing theorem [2] to prove that the linear system

$$\Lambda = |p^* \mathcal{O}_X(D - K) \otimes p^* f^* L \otimes \mathcal{O}_{\tilde{X}}(-2W)|$$

contains an 1-connected divisor with selfintersection > 0. Now by Lemma 2 the selfintersection of Λ is

$$(D^2) - 2(D.K) + 2(D.F)\deg(L) - 4 > 0$$

provided $\deg(L) \geqslant \alpha_1$ where α_1 is a constant depending only on the fibration. Now by Riemann-Roch on B we get that

$$|L \otimes M^{-1} \otimes \mathcal{O}_B(-b_1 - \ldots - b_r - 2y)| \neq \phi$$

provided $\deg(L) - \deg(M) - r - 2 \geqslant p_a(B)$. Hence there exists a constant α_2 depending only on f such that for $\deg(L) \geqslant \alpha_2$ we may find a divisor $\underline{b} \in |L \otimes M^{-1}|$ with $b_1 + \ldots + b_r + 2y \leqslant \underline{b}$. It follows that

$$G = p^*(D + f^*\underline{b} - \sum_{j=1}^{r} (m_j - 1)Y_j) - 2W \in \Lambda.$$

Now for $\deg(L) - \deg(M) - \sum_{j=1}^{r} (m_j - 1)/m_j \geqslant 2$ the divisor $D + f^*\underline{b} - \sum_{j=1}^{r} (m_j - 1)Y_j$ must be 2-connected by Lemma 1. It follows by a standard computation that in this case G is 1-connected. Hence we may choose $C_1 = \max\{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_3 = \deg(M) + \sum_{j=1}^{r} (m_j - 1)/m_j + 2$ and we are done.

2) $\Longrightarrow$ 3) is obvious.

To prove 3) $\Longrightarrow$ 1) we may suppose that L is trivial and that D has no common components with Y, where mY is some fibre of multiplicity m. We only have to prove that $(D.Y) \geqslant 2$. Suppose $(D.Y) = 1$. By Riemann-Roch on the (possibly singular) curve Y we get

$$h^0(\mathcal{O}_Y(D)) = h^0(\omega_Y(-D)) + \deg(\mathcal{O}_Y(D)) + \chi(\mathcal{O}_Y)$$
$$= h^0(\mathcal{O}_Y(-D)) + 1$$

because the dualizing sheaf ω_Y is trivial. Now since $\mathcal{O}_Y(-D) \subset \mathcal{O}_Y$ we get $H^0(\mathcal{O}_Y(-D)) \subset H^0(\mathcal{O}_Y)$. Since by [5], $H^0(\mathcal{O}_Y)$ consists only of constants and since $\mathcal{O}_Y(-D)$ is not trivial we get $h^0(\mathcal{O}_Y(-D)) = 0$ hence $h^0(\mathcal{O}_Y(D)) = 1$. Since $\mathcal{O}_Y(D)$ is not trivial, it follows that $\mathcal{O}_Y(D)$ cannot be spanned by global sections, contradiction.

Proof of Theorem 2. – Note that 2) $\Longrightarrow$ 3) is obvious and that 3) $\Longrightarrow$ 1) follows easily considering as above a multiple fibre of the form mY and noting that Y must have degree at least 3 with respect to any very ample divisor because $p_a(Y) = 1$.

Let us prove 1) $\Longrightarrow$ 2). Start with an ample $\mathcal{L} \in \mathrm{Pic}(X)$ with $(\mathcal{L}.F) \geqslant 3m$, put $\mathcal{M} = \mathcal{L} \otimes f^*L$ for $L \in \mathrm{Pic}(B)$ and let us prove first that $|\mathcal{M}|$ has no fixed components among the components of the reducible fibres of f provided $\deg(L) \geqslant \beta_1$ for some constant β_1. Let Z_1 be a component of a reducible fibre F and look for a divisor in $|\mathcal{M}|$ not containing Z_1 in its support. Note that by [5], Z_1 is smooth rational with selfintersection $(Z_1^2) = -2$. According to [5] there are two cases which may occur: either $(Z_1 . Z_2) \leqslant 1$ for any other component Z_2 of F, or $F = b(Z_1 + Z_2)$ for some natural b where Z_2 is smooth rational with $(Z_2^2) = -2$ and $(Z_1 . Z_2) = 2$. In the first case put $Z = Z_1$ and choose a point $p \in Z$. In the second case, since $b(\mathcal{L} . Z_1) + b(\mathcal{L} . Z_2) = (\mathcal{L} . F) \geqslant 3m \geqslant 3b$ we must have $(\mathcal{L} . Z_k) \geqslant 2$ for $k = 0$ or $k = 1$. Put in this case $Z = Z_1 + Z_2 - Z_k$ and take $p \in Z_1 \cap Z_2$. It will be sufficient to find a divisor in $|\mathcal{M}|$ not passing through p. We have the following exact sequence:

$$0 \longrightarrow H^0(\mathcal{M}(-Z)) \longrightarrow H^0(\mathcal{M}) \longrightarrow H^0(\mathcal{O}_{p1}(c)) \longrightarrow H^1(\mathcal{M}(-Z))$$

where $c = (\mathcal{L} . Z) \geqslant 1$. It is sufficient to prove that $H^1(\mathcal{M}(-Z)) = 0$. We use Ramanujam's vanishing theorem [6]. By Serre duality it is sufficient to prove that

$$(\mathcal{M}(-Z-K)^2) > 0 \quad \text{and} \quad (\mathcal{M}(-Z-K) . R) \geqslant 0$$

for any integral curve R. Now

$$(\mathcal{M}(-Z-K)^2) = (\mathcal{L}^2) + 2(\mathcal{L} . F)\deg(L) - 2 - 2(\mathcal{L} . Z) - 2(\mathcal{L} . K)$$
$$> 2(\mathcal{L} . F)(\deg(L) - 1 - d) - 2$$

where $d \in \mathbf{Q}$, $K \equiv dF$. Consequently the selfintersection is > 0 for $\deg(L) \geqslant d + 2$.

To check the second inequality suppose first that R is contained in a fibre F. We get $(\mathcal{M}(-Z-K) . R) = (\mathcal{L} . R) - (Z . R) \geqslant 0$ because the only case when $(Z . R) = 2$ is $F = b(Z_1 + Z_2)$ and $R = Z_k$. Now if R dominates B we get

$$(\mathcal{M}(-Z-K) . R) = (\mathcal{L} . R) + (F . R)\deg(L) - (Z . R) - (K . R)$$
$$> (F . R)\deg(L) - (F . R) - d(F . R) \geqslant 0$$

for $\deg(L) \geqslant d + 1$, and we are done. Now if β_1 is chosen also such that $\beta_1 \geqslant 2p_a(B)$ it follows that $\mathfrak{M}$ is still ample hence by Theorem 1 the linear system $|\mathcal{L} \otimes f^*L|$ is ample and base point free provided $\deg(L) \geqslant \beta_2 = \beta_1 + C_1$. By Bertini's theorem the above system contains an integral member D. To prove 1) $\Longrightarrow$ 2) it is sufficient by [2] to prove that

$$H^1(\tilde{X}, p^*\mathcal{O}_X(D) \otimes p^*f^*L \otimes \mathcal{O}_{\tilde{X}}(-2W)) = 0$$
$$H^1(\hat{X}, q^*\mathcal{O}_X(D) \otimes q^*f^*L \otimes \mathcal{O}_{\hat{X}}(-W_1 - W_2)) = 0$$

for any $x, x_1, x_2 \in X$, provided $\deg(L) \geqslant \beta_3$ for some constant β_3; in this case the constant $C_2 = \beta_2 + \beta_3$ will be convenient for our purpose.

Now exactly as in the proof of the Theorem 1 we may find a constant β_3 such that for $\deg(L) \geqslant \beta_3$ the linear systems

$$|p^*\mathcal{O}_X(D - K) \otimes p^*f^*L \otimes \mathcal{O}_{\tilde{X}}(-3W)|$$

and

$$|q^*\mathcal{O}_X(D - K) \otimes q^*f^*L \otimes \mathcal{O}_{\hat{X}}(-2W_1 - 2W_2)|$$

have strictly positive selfintersections and contain divisors of the form

$$G_1 = p^*\Big(D + \sum_i a_i F_i\Big) - 3W$$

and

$$G_2 = q^*\Big(D + \sum_i b_i F_i\Big) - 2W_1 - 2W_2$$

with $a_i, b_i \in \mathbf{Q}$, $a_i \geqslant 0$, $b_i \geqslant 0$, $\sum_i a_i \geqslant 2$, $\sum_i b_i \geqslant 2$ and where F_i are fibres. Then by Lemma 1 the divisors $D + \sum_i a_i F_i$ and $D + \sum_i b_i F_i$ are 3-connected hence by a standard computation, G_1 and G_2 are 1-connected and the Theorem is proved.

BIBLIOGRAPHY

[1] A. BEAUVILLE, Surfaces algébriques complexes, *Astérisque*, 54 (1978).

[2] E. BOMBIERI, Canonical models of surfaces of general type, *Publ. Math. IHES*, 42 (1972), 171-220.

[3] F. ENRIQUES, *Le superficie algebriche,* Zanichelli, 1949.

[4] P. GRIFFITHS and J. HARRIS, *Principles of algebraic geometry,* John Wiley & Sons, New York, 1978.

[5] K. KODAIRA, On compact complex analytic surfaces II, *Ann. of Math.,* 77 (1963).

[6] C.P. RAMANUJAM, Remarks on the Kodaira vanishing theorem, *J. of the Indian Math. Soc.,* 36 (1972), 41-51; Supplement to the article "Remarks on the Kodaira vanishing theorem", *J. of the Indian Math. Soc.,* 38 (1974), 121-124.

Manuscrit reçu le 20 avril 1982.

Alexandru BUIUM,
Department of Mathematics
National Institute for Scientific
and Technical Creation
Bd. Pacii 220
79622 Bucarest (Romania).