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## A NOTE ON REARRANGEMENTS OF FOURIER COEFFICIENTS

by **Hugh L. MONTGOMERY**

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Let  $\{\varphi_k\}$  be a sequence of functions on  $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ , with the property that they are uniformly bounded,

$$(1) \quad \|\varphi_k\|_\infty \leq M,$$

and satisfy a Bessels inequality

$$(2) \quad \sum_k \left| \int_0^1 f \varphi_k \right|^2 \leq M^2 \int_0^1 |f|^2.$$

For the sake of simplicity we suppose that  $M$  has the same value in (1) and (2); this does not occasion any loss of generality. Suppose that  $\sum_k |a_k|^2 < \infty$ . Then

$$(3) \quad f(x) = \sum_k a_k \varphi_k(x)$$

is a member of  $L^2(\mathbf{T})$ , since the dual of (2) asserts that

$$(4) \quad \int_0^1 \left| \sum_k a_k \varphi_k \right|^2 \leq M^2 \sum_k |a_k|^2.$$

In this note we obtain bounds for  $\int_{\mathbf{E}} |f|^2$  in terms of the measure of the set  $\mathbf{E}$  and the numbers  $|a_k|$ . Following Hardy and Littlewood, we let the numbers  $a_0^*, a_1^*, \dots$  be the numbers  $|a_k|$ , permuted so that  $a_n^* \searrow$ . Then we set

$$(5) \quad f^*(x) = \sum_{n=0}^{\infty} a_n^* \cos 2\pi n x.$$

**THEOREM 1.** — *Let  $\{\varphi_k\}$  be a sequence of functions satisfying (1) and (2), let  $f$  and  $f^*$  be defined by (3) and (5). Then*

for any measurable set  $E \subseteq T$ , with measure  $|E| = 2\theta$ , we have

$$(6) \quad \int_E |f|^2 \leq 20M^2 \int_{-\theta}^{\theta} |f^*|^2.$$

If  $C \in L^2(T)$ ,  $C(x) \sim \sum_{n=0}^{\infty} C_n \cos 2\pi nx$ , and if  $C_n \searrow$ , then  $C = C^*$ , so (6) implies that

$$\int_E |C|^2 \leq 20 \int_{-\theta}^{\theta} |C|^2,$$

where  $E \subseteq T$ ,  $|E| = 2\theta$ . Thus, although it is not necessarily true that  $C(x)$  is decreasing on  $\left[0, \frac{1}{2}\right)$ , in a certain sense it is still the case that  $C$  is largest near 0.

Using a simple inequality of A. Baernstein [2], we shall derive from Theorem 1 the following.

**THEOREM 2.** — *Let  $\psi$  be a convex increasing function from  $[0, \infty)$  to  $\mathbf{R}$ . Then, in the above notation,*

$$\int_E \psi(|f|^2) \leq \int_{-\theta}^{\theta} \psi(20 M^2 |f^*|^2).$$

Taking  $\psi(t) = t^{q/2}$ , we see from the above that

$$(7) \quad \|f\|_q \leq 5M \|f^*\|_q \quad (q \geq 2).$$

Inequalities of this type have a long history. Hardy and Littlewood [3, 4] proved that

$$(8) \quad \|f\|_q \leq c_q \|f^*\|_q \quad (q \geq 2)$$

in the case  $\varphi_k(x) = e^{2\pi i k x}$ ,  $-\infty < k < +\infty$ . Littlewood [6] has shown that  $c_q$  is bounded in this case, and F. R. Keogh [5] has shown that  $c_p \rightarrow 1$  as  $q \rightarrow \infty$ . In the opposite direction, Littlewood [7] showed that  $c_q > 1$  except when  $q$  is an even integer. Consequently, the constant 20 in Theorems 1 and 2 can not be replaced by 1. R. E. A. C. Paley [9] extended (8) to the case of arbitrary uniformly bounded orthonormal  $\varphi_k$  (see Zygmund [11, XII § 5] for a simple proof). Theorem 2 does not seem to follow from the special case (7), since in general a convex increasing function  $\psi(t)$  is not comparable to a sum  $\sum_r c_r t^{a_r}$ ,  $c_r \geq 0$ ,  $a_r \geq 1$ .

If one were to consider, in place of  $f^*$ , a function

$$f^-(x) = \sum_{n=0}^{\infty} a_n * \varphi_n(x),$$

then one does not in general expect the inequality

$$\|f\|_q \leq c_q \|f^-\|_q \quad (q \geq 2)$$

to be valid, even when the  $\varphi_n$  are given in some natural order. (See G. A. Bachelis [1], and H. S. Shapiro [10]). However, in the special case of ordinary Dirichlet series, there are good reasons to believe that something positive may be said. For example, we can formulate a

*Conjecture.* — Let  $\varepsilon > 0$ , and  $2 \leq q \leq 4$ . Then for  $T \geq 2$ ,  $N > N_0(\varepsilon, q)$ , we have

$$\int_{-T}^T \left| \sum_{n=1}^N a_n n^{-it} \right|^q dt \leq (T + N^{q/2}) N^{q/2+\varepsilon},$$

for arbitrary coefficients  $a_n$  satisfying  $|a_n| \leq 1$ .

The above is known to be true when  $q = 2$ ,  $q = 4$ ; thus by Hölder's inequality it suffices to consider the case  $T = N^{q/2}$ . The Conjecture is of special interest in multiplicative number theory, since from it one can deduce (see Montgomery [8, Theorem 12.6]) that the interval  $(x, x + x^{\frac{1}{2}+\varepsilon})$  contains a prime number, for all  $x > x_0(\varepsilon)$ .

We now prove Theorem 1. We have only countably many functions  $\varphi_k$ , so without loss of generality we may suppose that  $0 \leq k < \infty$ . Let  $\pi$  be the permutation such that  $a_n^* = |a_{\pi(n)}|$ . Put  $N = [(2\theta)^{-1}]$ , and set

$$\mathcal{N} = \{\pi(n) : 0 \leq n \leq N\}.$$

Thus  $\mathcal{N}$  is the set of indices of the  $N + 1$  coefficients of largest absolute value. Break the sum (3) into two parts,

$$f = \sum_{n \in \mathcal{N}} + \sum_{n \notin \mathcal{N}} = f_1 + f_2,$$

say. On one hand,

$$\int_{\mathbb{E}} |f_1|^2 \leq \|f_1\|_{\infty}^2 \int_{\mathbb{E}} 1 \leq 2\theta \left( M \sum_{n=0}^N a_n^* \right)^2,$$

in view of (1). On the other hand, from (4) we see that

$$\int_{\mathbb{E}} |f_2|^2 \leq \int_0^1 |f_2|^2 \leq M^2 \sum_{n>N} a_n^{*2}.$$

For each  $x$ ,  $|f|^2 \leq 2|f_1|^2 + 2|f_2|^2$ , so on combining the above we find that

$$(9) \quad \int_{\mathbb{E}} |f|^2 \leq 4\theta \left( M \sum_{n=0}^N a_n^* \right)^2 + 2M^2 \sum_{n>N} a_n^{*2}.$$

It now remains to relate the right hand side above to  $\int_{-1}^1 |f^*|^2$ .

Let  $K(x) = \max(0, 1 - |x|\theta^{-1})$  for  $|x| \leq \frac{1}{2}$ . Then

$$\int_{-1}^1 |f^*|^2 \geq \int_{-\frac{1}{2}}^{\frac{1}{2}} K|f^*|^2 = \frac{1}{2} \sum_{m,n=0}^{\infty} a_m^* a_n^* (\hat{K}(m+n) + \hat{K}(m-n)).$$

Now  $\hat{K}(m) = \theta \left( \frac{\sin \pi m \theta}{\pi m \theta} \right)^2 \geq 0$ , so

$$(10) \quad \frac{1}{2} \sum_{m,n=0}^{\infty} a_m^* a_n^* \hat{K}(m-n) \leq \int_{-1}^1 |f^*|^2.$$

If  $|m-n| \leq N$  then

$$(11) \quad \hat{K}(m-n) \geq \theta \left( \frac{\sin \pi N \theta}{\pi N \theta} \right)^2 \geq \theta \left( \frac{\sin \frac{1}{2} \pi}{\frac{1}{2} \pi} \right)^2 = 4\pi^{-2}\theta,$$

since  $N \leq (2\theta)^{-1}$ . But  $a_n^* \geq 0$ , so

$$(12) \quad \theta \left( \sum_{0 \leq n \leq N} a_n^* \right)^2 \leq \frac{1}{4} \pi^2 \sum_{0 \leq m, n \leq N} \hat{K}(m-n) a_m^* a_n^*.$$

If  $0 < n - N \leq m \leq n$  then  $a_m^* \geq a_n^*$ , so from (11) we find that

$$\sum_{n-N \leq m \leq n} a_m^* \hat{K}(m-n) \geq 4\pi^{-2}\theta(N+1)a_n^* \geq 2\pi^{-2}a_n^*,$$

since  $N+1 > (2\theta)^{-1}$ . Hence

$$(13) \quad \sum_{n>N} a_n^{*2} \leq \frac{1}{2} \pi^2 \sum_{\substack{n>N \\ n-N \leq m \leq n}} \hat{K}(m-n) a_m^* a_n^*.$$

Combining (9) with (12), (13), we find that

$$\int_E |f|^2 \leq \pi^2 \theta M^2 \sum_{m, n=0}^{\infty} a_m^* a_n^* \hat{K}(m - n).$$

But  $2\pi^2 < 20$ , so by (10) our proof is complete.

We note that once (9) is established, the remainder of the proof can be effected in several ways. In proving (8), Hardy and Littlewood [3] established that

$$\int_0^1 |f^*|^q \approx_q \sum_{n=0}^{\infty} a_n^{*q} (n+1)^{q-2}.$$

One can modify their proof of this (see also Keogh [5]) to show that

$$\sum_{n > 6^{-1}} n^{-2} \left( \sum_{0 \leq m \leq n} a_m^* \right)^2 < c \int_{-1}^0 |f^*|^2.$$

Theorem 1 follows easily from the above and (9), apart from the values of constants.

To prove Theorem 2 we require the following result of A. Baernstein [2].

LEMMA. — For  $f \in L^1(\mathbf{T})$ ,  $0 \leq \theta \leq \frac{1}{2}$ , let  $f^+(\theta) = \sup_E \int_E |f|$ , where the supremum is taken over all measurable sets  $E \subseteq [0, 1)$  such that  $|E| = 2\theta$ . For two functions  $r, s \in L^1(\mathbf{T})$ , the following are equivalent:

(a) For all  $\theta \in \left[0, \frac{1}{2}\right)$ ,  $r^+(\theta) \leq s^+(\theta)$ ;

(b) For any  $\psi(t)$ , convex and increasing on  $[0, \infty)$ , we have

$$\int_0^1 \psi(|r|) \leq \int_0^1 \psi(|s|).$$

In the language of this lemma, we find from Theorem 1 that  $(|f|^2)^+(\theta) \leq (20M|f^*|^2)^+(\theta)$ . Hence

$$\|\psi(|f|^2)\|_1 \leq \|\psi(20M|f^*|^2)\|_1.$$

However, with a little more care we obtain the full strength of Theorem 2. Let  $E \subseteq [0, 1)$  be a set with  $|E| = 2\theta$ . Put  $r = |f|^2 \chi_E$ ,  $s = 20M^2 |f^*|^2 \chi_{(-\theta, \theta)}$ . Then by Theorem 1,  $r^+ \leq s^+$ ,

so by the Lemma above,  $\|\psi(r)\|_1 \leq \|\psi(s)\|_1$ . If  $\psi(0) = 0$ , then this asserts that

$$\int_E \psi(|f|^2) \leq \int_{-1}^0 \psi(20M^2|f^*|^2).$$

To obtain this for general  $\psi$  we have only to add a constant to both sides of the inequality. This completes the proof of Theorem 2.

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