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Joseph F. Plante

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MEASURE PRESERVING PSEUDOGROUPS AND A THEOREM OF SACKSTEDER

by J.F. PLANTE

Introduction.

A classical theorem of Denjoy says that a minimal set of a C^2 diffeomorphism of the circle is either a periodic orbit or the entire circle. In [7, 8] Sacksteder proved a generalization of Denjoy's result which has been extensively exploited by authors studying codimension one foliations. For example, in [3] and [6] it is shown that C^2 codimension one foliations whose leaves have non exponential growth cannot have exceptional minimal sets. The proof uses Sacksteder's theorem as well as the generalization for codimension one foliations of the Poincaré-Bendixson theorem. In the present note we will obtain this as well as other applications of Sacksteder's result directly, that is, without using topological results such as the Poincaré-Bendixson theorem. For example, we show that a finitely generated solvable group of C^2 diffeomorphisms of the circle cannot have an exceptional minimal set. Our approach is based on the problem of finding invariant measures for a pseudogroup of homeomorphisms defined on a compact Hausdorff space.

1. Definitions and Sacksteder's theorem.

Let X be a compact Hausdorff space and let Γ be a collection of homeomorphisms for which the domain and range are open subsets of X. If $\gamma \in \Gamma$ then the domain and range of γ will be denoted by $D(\gamma)$ and $R(\gamma)$, respectively.

DEFINITION. – Γ *is said to be a pseudogroup of (local) homeomorphisms of* X *if the following conditions are satisfied*

1) *If* $\gamma \in \Gamma$ *then* $\gamma^{-1} \in \Gamma$ *and* $D(\gamma^{-1}) = R(\gamma)$, $R(\gamma^{-1}) = D(\gamma)$.

2) *If* $\gamma_1, \gamma_2 \in \Gamma$ *and* $\gamma : D(\gamma_1) \cup D(\gamma_2) \to R(\gamma_1) \cup R(\gamma_2)$ *is a homeomorphism such that* $\gamma(x) = \gamma_i(x)$ $(i = 1, 2)$ *whenever* $x \in D(\gamma_i)$ *then* $\gamma \in \Gamma$.

3) *If* $\gamma_1, \gamma_2 \in \Gamma$ *then* $\gamma_1 \circ \gamma_2 \in \Gamma$ *where*

$$D(\gamma_1 \circ \gamma_2) = \gamma_1^{-1}(R(\gamma_2) \cap D(\gamma_1)) .$$

4) *If* $\gamma \in \Gamma$ *and* $U \subset D(\gamma)$ *is an open subset then* $\gamma \mid U \in \Gamma$.

5) *The identity map of* X *is an element of* Γ.

The *orbit* $\Gamma(x)$ of a point $x \in X$ is defined by

$$\Gamma(x) = \{\gamma(x) \in X \mid \gamma \in \Gamma\} .$$

A set $S \subset X$ is *invariant* if it is a union of orbits. A *minimal set* for Γ is a non-empty closed invariant subset of X which is minimal with respect to these properties. A minimal set is said to be *exceptional* if it is nowhere dense but is not a finite orbit. Γ is *finitely generated with respect to a compact subset* $K \subset X$ if there exist $\gamma_1, \ldots, \gamma_k \in \Gamma$ which generate every element of Γ via the operations 1), 2), 3), 4) above and such that each of the subsets $D(\gamma_i) \cap K$ is compact $(i = 1, \ldots, k)$.

If X is a differentiable manifold and Γ consists of C^k diffeomorphisms, then Γ is called a pseudogroup of C^k diffeomorphisms.

DEFINITION. – *Let* Γ *be a pseudogroup of homeomorphisms of* X. $\gamma \in \Gamma$ *has a contracting fixed point at* $x \in X$ *if there is neighborhood* U *of* x *such that* $U \subset D(\gamma)$ *and*

i) $\gamma(U) \subset U$

ii) $\bigcap_{n \geq 0} \gamma^n(U) = \{x\}$.

The following is a result of Sacksteder [7, 8] which generalizes earlier results of Denjoy and Schwartz.

1.1. THEOREM. – *Let* Γ *be a pseudogroup of* C^2 *diffeomorphisms of* S^1 *which is finitely generated with respect to an exceptional mi-*

nimal set $\mathfrak{M}$. *Then there exists* $\gamma \in \Gamma$, $x \in \mathfrak{M}$ *such that* $\gamma(x) = x$ *and* $|\gamma'(x)| < 1$. *In particular,* γ *has a contracting fixed point at* x.

2. Invariant measures.

Let Γ be a pseudogroup of homeomorphisms of a compact Hausdorff space X and let μ be a Borel measure on X.

DEFINITIONS. – μ *is said to be* Γ*-invariant if for every* $\gamma \in \Gamma$ *and every Borel set* $B \subset D(\gamma)$, $\mu(\gamma(B)) = \mu(B)$. μ *is said to be normalized if* $\mu(X) = 1$. *The support of* μ *(denoted* supp (μ)*) is the complement in* X *of the union of all open sets having* μ*-measure zero.*

2.1. LEMMA. – *If* μ *is a normalized* Γ*-invariant Borel measure and* $\gamma \in \Gamma$ *has a contracting fixed point at* $x \in$ supp (μ), *then* $\Gamma(x)$ *is finite.*

Proof. – Let U be a neighborhood of x such that $\gamma(U) \subset U$ and $\bigcap_{n \geq 0} \gamma^n(U) = \{x\}$. Since $x \in$ supp (μ) we have $\mu(U) > 0$. By Γ-invariance of μ,

$$\mu\left(\bigcap_{n \geq 0}^{\infty} \gamma^n(U)\right) = \mu(\gamma^n(U)) = \mu(U)$$

so $\mu(\{x\}) > 0$. Since μ is a finite measure this implies that $\Gamma(x)$ is finite.

2.2. PROPOSITION. – *Let* Γ *be a pseudogroup of* C^2 *diffeomorphisms of* S^1 *which is finitely generated with respect to every nowhere dense minimal set. If* μ *is a normalized* Γ*-invariant Borel measure on* S^1 *then either*

1) supp $(\mu) = S^1$

or 2) supp (μ) *contains a finite orbit.*

Proof. – Suppose that neither of 1), 2) holds, i.e., suppose that supp (μ) is not all of S^1 and contains no finite orbits. Since μ is Γ-invariant, supp (μ) is a closed, non-empty, Γ-invariant set which must contain a minimal set $\mathfrak{M}$ which is an exceptional minimal set since it is not finite. By (1.1) $\mathfrak{M}$ contains a point x at which $\gamma \in \Gamma$ has a contracting fixed point. This, however, contradicts the conclusion of (2.1) since $\Gamma(x)$ is not finite and, thus, completes the proof of (2.2).

If Γ is a pseudogroup of homeomorphisms of a compact Hausdorff space X we say that Γ is *topologically conjugate to a pseudogroup of isometries* if there exists a compact metric space M and a homeomorphism $h : X \to M$ such that the map $h \circ \gamma \circ h^{-1}$ is an isometry for every $\gamma \in \Gamma$.

2.3. PROPOSITION. – *Let* Γ *be a pseudogroup of* C^2 *diffeomorphisms of* S^1 *which is finitely generated with respect to nowhere dense minimal sets and suppose that* μ *is a normalized* Γ*-invariant Borel measure on* S^1. *If* Γ *has no finite orbits then it is topologically conjugate to a pseudogroup of isometries. If, in addition, every element of* Γ *is orientation preserving then* Γ *is conjugate to a pseudogroup of rotations.*

Proof. – If Γ has no finite orbits, then by (2.2) supp $(\mu) = S^1$. Now let M denote the real numbers mod 1.

We will now define a homeomorphism $h : S^1 \to M$. Fix $x_0 \in S^1$ and an orientation of S^1, and for $x \in S^1 (x \neq x_0)$ let $[x_0 , x]$ denote the closed interval from x to x_0 in the positive direction. h is now defined by

$$h(x) = \begin{cases} 0 & \text{if} \quad x = x_0 \\ \mu([x_0 , x]) & \text{if} \quad x \neq x_0 \, . \end{cases}$$

It is easily checked that h is a homeomorphism and, clearly, the induced pseudogroup on M consists of isometries. Since the only orientation preserving isometries of M are translations (mod 1) the proof of (2.3) is complete.

3. Some sufficient conditions for the existence of an invariant measure.

Let Γ be a pseudogroup of homeomorphisms of a compact Hausdorff space X. In this section we give examples of conditions on

Γ which guarantee the existence of a normalized Γ-invariant Borel measure on X.

For the moment we assume that X is a metrice space.

DEFINITION. – *Γ is said to be equicontinuous at $x \in X$ if given $\epsilon > 0$ there exists $\delta > 0$ such that $d(\gamma(x), \gamma(y)) < \epsilon$ whenever $d(x, y) < \delta$ where $x, y \in D(\gamma)$, $\gamma \in \Gamma$.*

The following is a result of Sacksteder which is proved in [7].

3.1. THEOREM. – *Let Γ be a pseudogroup of homeomorphisms of X which is equicontinuous at $x \in X$. Then there exists a Γ-invariant normalized Borel measure μ on X such that* $\text{supp}(\mu) \subset \overline{\Gamma(x)}$.

Now let us consider the case where X is merely a compact Hausdorff space and the pseudogroup Γ is finitely generated by a finite subset Γ_0. Furthermore, assume that Γ_0 is symmetric in the sense that $\gamma \in \Gamma_0$ implies $\gamma^{-1} \in \Gamma_0$. We now define analogues of the growth functions considered in [5].

DEFINITION. – *For $x \in X$, n a positive integer we define $\Gamma_n(x)$ as follows.*

$$\Gamma_n(x) = \{ y \in X \mid y = \gamma_{i_k} \circ \ldots \circ \gamma_{i_1}(x) \quad \textit{for some} \quad k \leqslant n \quad \textit{and some} \quad \gamma_{i_1}, \ldots, \gamma_{i_k} \in \Gamma_0 \}.$$

Letting $|\Gamma_n(x)|$ denote the cardinality of $\Gamma_n(x)$ we say that Γ *has exponential growth at x* if there exist positive numbers A, α such that $|\Gamma_n(x)| \geqslant A \exp(\alpha n)$. Otherwise, we say that Γ has *non-exponential growth at x*. If Γ has exponential (non-exponential) growth at every $x \in X$, then Γ is said to have exponential (non-exponential) growth.

3.2. THEOREM. – *Let Γ be a finitely generated pseudogroup of homeomorphisms of a compact Hausdorff space X. If Γ has non-exponential growth at $x \in X$ with respect to some symmetric finite generating set Γ_0 then there exists a normalized Γ-invariant Borel measure μ on X such that* $\text{supp}(\mu) \subset \overline{\Gamma(x)}$.

Proof. – By the Riesz representation theorem it suffices to define an invariant functional on the space of continuous real valued functions defined on X. Let $n_k \to \infty$ be a sequence of positive integers such that

$$\lim_{k \to \infty} \frac{|\Gamma_{n_k+1}(x) - \Gamma_{n_k-1}(x)|}{|\Gamma_{n_k}(x)|} = 0 , \qquad (*)$$

where $|\ |$ denotes cardinality. This can be done since Γ has non-exponential growth at x. Now let $f : X \to \mathbf{R}$ be continuous and define the functional I_k by

$$I_k(f) = \frac{1}{|\Gamma_{n_k}(x)|} \sum_{y \in \Gamma_{n_k}(x)} f(y) .$$

Clearly the measures corresponding to the I_k are normalized and by compactness of the set of normalized measures in the weak* topology there exists $k_j \to \infty$ such that I_k converges to a normalized functional I as $j \to \infty$. Clearly, if μ is the Borel measure corresponding to I, supp $(\mu) \subset \overline{\Gamma(x)}$. It remains to show that μ is Γ-invariant. To do this it suffices to show that if $\gamma \in \Gamma_0$ and $f : X \to \mathbf{R}$ is continuous such that $\{x \in X \mid f(x) \neq 0\} \subset R(\gamma)$ then $I(f) = I(f_\gamma)$ where f_γ is defined by

$$f_\gamma(z) = \begin{cases} f(\gamma(z)) & z \in D(\gamma) \\ 0 & \text{otherwise} \end{cases}$$

Now, letting n_j denote the sequence n_{k_j} ,

$$I(f) - I(f_\gamma) = \lim_{j \to \infty} [I_{k_j}(f) - I_{k_j}(f_\gamma)]$$

$$= \lim_{j \to \infty} \frac{1}{|\Gamma_{n_j}(x)|} \sum_{y \in \Gamma_{n_j}(x)} [f(y) - f(\gamma(y))]$$

$$|I(f) - I(f_\gamma)| \leq \lim_{j \to \infty} \frac{1}{|\Gamma_{n_j}(x)|} \sum_{y \in \Gamma_{n_j}(x) \Delta \gamma \Gamma_{n_j}(x)} |f(y)|$$

where Δ denotes symmetric difference. Since f is bounded, in order to show that $I(f) - I(f_\gamma) = 0$, it is sufficient to show that

$$\lim_{j \to \infty} \frac{|\Gamma_{n_j}(x) \, \Delta \gamma \Gamma_{n_j}(x)|}{|\Gamma_{n_j}(x)|} = 0 \, .$$

This last equation, however, follows from condition (*) above and the inequality

$$|\Gamma_{n_j}(x) \, \Delta \gamma \Gamma_{n_j}(x)| \leqslant |\Gamma_{n_j+1}(x) - \Gamma_{n_j-1}(x)| \, .$$

This completes the proof of (3.2).

4. Groups of diffeomorphisms.

Let $\mathrm{Diff}^2(S^1)$ denote the group of C^2 diffeomorphisms of the circle and let $\Gamma \subset \mathrm{Diff}^2(S^1)$ be a subgroup. In particular, Γ is a pseudogroup, and if Γ is finitely generated as a group then it is automatically finitely generated with respect to every compact subset of S^1. For the record, we recall the following result of Sacksteder which is proved in [8]. Γ is said to be *fixed point free* if $\gamma \in \Gamma, x \in S^1, \gamma(x) = x$ implies that γ = identity.

4.1. THEOREM. – *If $\Gamma \subset \mathrm{Diff}^2(S^1)$ is fixed point free group then Γ is conjugate to a group of rotations.*

Now suppose that Γ is generated by a finite symmetric generating set Γ_0. Define $\Gamma_n \subset \Gamma$ by

$$\Gamma_n = \{\gamma \in \Gamma \mid \gamma = \gamma_{i_k} \circ \ldots \circ \gamma_{i_1} \quad \text{for some} \quad k \leqslant n$$
$$\text{and} \quad \gamma_{i_1}, \ldots, \gamma_{i_k} \in \Gamma_0\} \, .$$

If for some positive A, α we have $|\Gamma_n| \geqslant A \exp(\alpha n)$ then Γ is said to have *exponential growth.* Otherwise Γ has *non-exponential growth.* These notions are independent of the generating set Γ_0. The following result follows immediately from (2.2), (2.3) and (3.2).

4.2. THEOREM. – *If $\Gamma \subset \mathrm{Diff}^2(S^1)$ is a finitely generated group which has non-exponential growth then*

1) *Γ has no exceptional minimal sets.*

2) *If Γ has no finite orbits then Γ is conjugate to a group of isometries.*

Actually, (4.2) can be improved substantially. First we need to recall some concepts and facts which may be found, for example, in [1].

DEFINITION. – *A discrete group* Γ *is said to be amenable if there is a left invariant linear functional* Φ *on the space of bounded maps* $\Gamma \to \mathbf{R}$ *such that*

1) $f \geqslant 0$ *implies* $\Phi(f) \geqslant 0$, *and*

2) $\Phi(1) = 1$.

Remarks. – 1) Solvable groups and groups with non-exponential growth [2] are amenable. On the other hand, free groups on more than one generator are not amenable.

2) If Γ is an amenable group of homeomorphisms of a compact Hausdorff space X then there exists a normalized Γ-invariant Borel measure on X. In particular, if $\mathcal{M}$ is a minimal set of Γ then such a measure can be found with support equal to $\mathcal{M}$.

The above facts together with (2.2) and (2.3) imply the following.

4.3. THEOREM. – *If* $\Gamma \subset \mathrm{Diff}^2(S^1)$ *is a finitely generated amenable group then*

1) Γ *has no exceptional minimal sets.*

2) *If* Γ *has no finite orbits then* Γ *is topologically conjugate to a group of isometries. In particular, if every element of* Γ *is orientation preserving then* Γ *is conjugate to a group of rotations.*

Remarks. – In [6], (4.2) is proved only for finitely presented groups. Furthermore, (4.3) is a significant generalization of (4.2) since any solvable group which does not have a nilpotent subgroup of finite index must have exponential growth [9].

Several of the results on groups of diffeomorphisms of the circle may be summarized by the following.

4.4. THEOREM. – *Let* $\Gamma \subset \mathrm{Diff}^2(S^1)$ *be a finitely generated group of orientation preserving diffeomorphisms without finite orbits. For such* Γ *the following are equivalent.*

1) Γ *is topologically conjugate to a group of rotations.*

2) Γ *is fixed point free.*

3) Γ *has non-exponential growth.*

4) Γ *is amenable.*

5. Some remarks on rotation numbers.

Let $\mathcal{H}(S^1)$ denote the group of homeomorphisms of S^1 and let $\mathcal{H}^+(S^1)$ be the subgroup of orientation preserving homeomorphisms. In most treatments of the classical Denjoy theorem (see for example [4]) one considers rotation numbers of elements of $\mathcal{H}^+(S^1)$ which are defined as follows. Let $\gamma \in \mathcal{H}^+(S^1)$ and let $\overline{\gamma} : \mathbf{R} \to \mathbf{R}$ be a lifting of γ with respect to the covering projection $\mathbf{R} \to S^1$ given by $t \to e^{2\pi i t}$ (where we think of S^1 as being the unit circle in the complex plane).

The *rotation number* of γ, $\rho(\gamma)$ is defined by

$$\rho(\gamma) = \lim_{n \to \infty} \frac{\overline{\gamma}^n(t)}{n} \pmod 1$$

where the definition is independent of $t \in \mathbf{R}$ and the lifting chosen. Assume that S^1 has a fixed orientation and for $x, y \in S^1$ let $[x, y)$ denote the half open interval going from x to y in the positive direction.

The following result indicates the relationship between invariant measures and rotation numbers.

5.1. Proposition. – *Let* $\Gamma \subset \mathcal{H}^+(S^1)$ *be a subgroup and let* μ *be a normalized* Γ*-invariant Borel measure on* S^1. *Then for* $\gamma \in \Gamma$, $x \in S^1$ *we have* $\rho(\gamma) = \mu([x, \gamma(x)))$. *In particular,* $\rho(\gamma)$ *is independent of* x *and* μ.

Proof. – By choosing $\overline{\gamma}$ so that $0 \leqslant \overline{\gamma}(t) - t < 1$ for all $t \in \mathbf{R}$ we observe that

$$\rho(\gamma) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \mu([\gamma^j(x), \gamma^{j+1}(x))) .$$

By Γ-invariance of μ the right hand side is just $\mu([x\,,\gamma(x)))$ which proves (5.1).

5.2. Corollary. – *If* Γ *and* μ *are as in* (5.1) *then the map* $\rho : \Gamma \to \mathbf{R}/\mathbf{Z}$ *is a homomorphism. In particular, the restriction of* ρ *to any amenable subgroup* $\Gamma \subset \mathcal{H}^+(S^1)$ *is a homomorphism.*

Proof. – Let α, $\beta \in \Gamma$, $x \in S^1$. By (5.1) we have

$$\begin{aligned}\rho(\alpha\beta) &= \mu([x\,,\alpha\beta(x)))\\ &= \mu([x\,,\beta(x))) + \mu([\beta(x)\,,\alpha\beta(x)))\\ &= \rho(\beta) + \rho(\alpha)\end{aligned}$$

which proves (5.2).

6. Codimension one foliations.

In this section we indicate how previous results may be applied in the study of codimension one foliations. Let $\mathfrak{F}$ be a codimension one foliation of class C^2 of a compact manifold M. If τ is a smoothly embedded closed curve transverse to the leaves of $\mathfrak{F}$ then it is well-known that there is a pseudogroup $\Gamma(\tau)$ of C^2 diffeomorphisms of τ which is determined by translating along the leaves of $\mathfrak{F}$ (see for example [7]). If $\mathfrak{M}$ is a minimal set of $\mathfrak{F}$ with $\mathfrak{M} \cap \tau \neq \phi$ then $\mathfrak{M} \cap \tau$ is a minimal set of $\Gamma(\tau)$. Furthermore, there is a finitely generated subpseudogroup Γ of $\Gamma(\tau)$ such that Γ contains the restriction of $\Gamma(\tau)$ to $\mathfrak{M} \cap \tau$. This follows from the compactness of $\mathfrak{M}$ as follows. Assume that M has a Riemannian metric and given $x \in M$, $R > 0$ let $D_R(x)$ be the set of points in the same leaf as x which can be connected to x by a path in the leaf of length $< R$. By compactness of $\mathfrak{M}$ there exists a sufficiently large $\rho > 0$ such that every point of $\mathfrak{M}$ is contained in $D_\rho(x)$ for some $x \in \mathfrak{M} \cap \tau$. Each $x \in \tau$ has a sufficiently small neighborhood $U_x \subset \tau$ such that the set

$$W_x = \bigcup_{y \in U_x} D_{3\rho}(y)$$

determines a finite subset $\Gamma_x \subset \Gamma(\tau)$ each of which has domain U_x. By compactness of $\mathfrak{M} \cap \tau$ there exist $x_1, \ldots, x_k \in \mathfrak{M} \cap \tau$ such that the open sets $\bigcup_{y \in U_{x_i}} D_\rho(x_i)$; $i = 1, \ldots, k$ cover $\mathfrak{M}$ which means that the subpseudogroup Γ generated by $\Gamma_0 = \Gamma_{x_1} \cup \ldots \cup \Gamma_{x_k}$ has the desired property.

DEFINITION. – *$\mathcal{F}$ is said to have exponential growth at $x \in M$ if* vol $D_R(x) \geqslant A \exp(\alpha R)$ *for some $A > 0$, $\alpha > 0$ (where* vol $D_R(x)$ *denotes the volume of* $D_R(x)$ *as a Riemannian manifold). Otherwise, $\mathcal{F}$ has non-exponential growth at x.*

6.1. LEMMA. – *Let Γ be the finitely generated pseudogroup described above and let $x \in \mathfrak{M} \cap \tau$. If $\mathcal{F}$ has non-exponential growth at x then Γ has non-exponential growth at x.*

Proof. – Let $\delta > 0$ be sufficiently small so that $y, z \in \tau$, $y \neq z$ imply that $D_\delta(y) \cap D_\delta(z) = \phi$. Also let

$$\sigma = \inf_{y \in \tau} \operatorname{vol} D_\delta(y) .$$

Thus, for $R > 0$ we have

$$\operatorname{vol} D_{R+\delta}(x) \geqslant \gamma(x, R) \cdot \sigma$$

where $\gamma(x, R)$ denotes the number of elements in $D_R(x) \cap \tau$. It is clear from the definition of Γ_0 that

$$\gamma(x, 3n\rho) \geqslant |\Gamma_n(x)| .$$

Therefore,

$$\begin{aligned} \operatorname{vol} D_{3n\rho+\delta}(x) &\geqslant \gamma(x, 3n\rho) \cdot \sigma \\ &\geqslant |\Gamma_n(x)| \cdot \sigma \end{aligned}$$

If Γ has exponential growth at x then so does $\mathcal{F}$ by the last inequality. This proves (6.1).

Remark. – The converse of (6.1) is also true but since we will not use it we omit its proof.

Since any non-compact leaf of $\mathcal{F}$ intersects some closed transversal τ, the following result follows from (1.1), (2.1), (3.2), and (6.1).

6.2. THEOREM. – *If $\mathfrak{M}$ is an exceptional minimal set of a C^2 codimension one foliation then every leaf in $\mathfrak{M}$ has exponential growth.*

Remark. – (6.2) is an improvement over the corresponding results in [3] and [6] in that it is not necessary to assume that the individual leaves of $\mathfrak{F}$ are immersed submanifolds of class C^3.

6.3. THEOREM. – *Let $\mathfrak{F}$ be a transversely oriented* C^2 *codimension one foliation of* M *such that $\mathfrak{F}$ has non-exponential growth at every* $x \in M$. *If $\mathfrak{F}$ has no compact leaves then there is a continuous flow* $\phi_t : M \to M$ *with the following properties.*

1) The orbits of ϕ_t define a smooth one dimensional foliation transverse to $\mathfrak{F}$.

2) If L is a leaf of $\mathfrak{F}$ then $\phi_t(L)$ is also a leaf of $\mathfrak{F}$ for every $t \in \mathbf{R}$. Furthermore, $\phi_t \mid L$ is a C^2 diffeomorphism of L onto $\phi_t(L)$.

Proof. – By (6.2) M must be a minimal set of $\mathfrak{F}$ so any closed transverse curve τ intersects every leaf of $\mathfrak{F}$. Let X be a smooth vector field transverse to $\mathfrak{F}$ which has τ as a closed orbit. Using (2.3), (3.2), and the arguments used in the proof of Theorem 6 of [7] we obtain ϕ_t as a reparametrization of the X-flow.

Remark. – Note that in the present proof of (6.3) it is not necessary to show that the holonomy of $\mathfrak{F}$ is trivial. Of course, the conclusion of (6.3) implies that $\mathfrak{F}$ has trivial holonomy.

In the case that M is a bundle with fiber S^1 and the leaves of $\mathfrak{F}$ are transverse to the fibers (4.3) yields the following result.

6.4. THEOREM. – Assume that M is a bundle with fiber S^1 over base manifold B (with smooth projection map) and that $\mathfrak{F}$ is a C^2 codimension one foliation of M which is everywhere transverse to the fibers. If $\pi_1(B)$ is an amenable group then $\mathfrak{F}$ does not have any exceptional minimal sets.

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J.F. PLANTE,
Department of Mathematics
University of North Carolina
Chapel Hill N.C. 27514 (USA).