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HOLOMORPHIC MAPS BETWEEN MODULI SPACES

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Abstract. — We prove that every non-constant holomorphic map $\mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ between moduli spaces of Riemann surfaces is a forgetful map, provided that $g \geqslant 6$ and $g' \leqslant 2g-2$.

Résumé. — Nous démontrons que toute application non-constante et holomorphe $\mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ entre deux espaces de modules est une application d'oubli, à condition que $g \geqslant 6$ et $g' \leqslant 2g-2$.

1.

Let $\mathcal{M}_{g,p}$ denote the moduli space of Riemann surfaces of genus g with p labelled marked points. Moduli space has a natural structure as a complex orbifold. In this paper we study holomorphic maps, in the category of orbifolds, between distinct moduli spaces. Examples of such maps are the so-called *f*orgetful maps [7, 11]: given $(i_1, \ldots, i_{p'})$ with $i_j \in \{1, \ldots, p\}$ and $i_j \neq i_k$ for $j \neq k$, set

$$\mathcal{M}_{g,p} \to \mathcal{M}_{g,p'}, \quad (X, x_1, \ldots, x_p) \mapsto (X, x_{i_1}, \ldots, x_{i_{p'}}). \tag{1.1}$$

We prove that under suitable genus bounds, there are no other non-constant holomorphic maps:

Theorem 1.1. — *Suppose that $g \geqslant 6$ and $g' \leqslant 2g-2$. Every non-constant holomorphic map $\mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ is a forgetful map.*

As a direct consequence of Theorem 1.1 we obtain:

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Corollary 1.2. — *Suppose that $g \geqslant 6$ and $g' \leqslant 2g-2$. If there is a non-constant holomorphic map $\mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$, then $g = g'$ and $p \geqslant p'$.* □

Theorem 1.1 remains true under slightly more generous conditions, compare with the discussion at the end of section 2. We get for instance that, as long as $g \geqslant 4$, every non-constant holomorphic map $\mathcal{M}_{g,p} \to \mathcal{M}_{g,p}$ is induced by a permutation of marked points, and is hence a biholomorphism (see Corollary 3.3). On the other hand, some genus bounds are necessary for Theorem 1.1 to hold. For instance, it follows from [2] that for all $g \geqslant 2$ there are $g' > g$ and a holomorphic embedding $\mathcal{M}_{g,0} \hookrightarrow \mathcal{M}_{g',0}$.

Remark. — Moduli space is not compact, but a natural compactification $\widehat{\mathcal{M}}_{g,p}$, a projective variety, was constructed in [10] by Deligne and Mumford. Morphisms between Deligne–Mumford compactifications have been studied by several authors (see for example [9, 15]). Notice that in Theorem 1.1 we assume neither that the holomorphic maps in question extend to the Deligne–Mumford compactification, nor that they are surjective or have connected fibers.

We sketch briefly the proof of Theorem 1.1. Since we are working in the category of orbifolds, every continuous map $f : \mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ induces a homomorphism $f_* : \mathrm{Map}_{g,p} \to \mathrm{Map}_{g',p'}$ between the associated mapping class groups. In [3], the two last authors classified all homomorphisms between mapping class groups under the genus bounds in Theorem 1.1; it follows easily from this classification that f is either homotopically trivial or homotopic to a forgetful map h (Proposition 2.3). The claim then follows easily from a result asserting that if M is smooth and quasi-projective variety, then any two non-constant homotopic holomorphic maps $M \to \mathcal{M}_{g,p}$ are identical (Proposition 3.2).

In an earlier version of this note, the last two authors had used an argument due to Eells–Sampson [13] to prove Proposition 3.2 when M is moduli space. Later on, the first author and at the time referee of the paper, pointed out that Proposition 3.2 is in fact a consequence of the particular case when M is a Riemann surface. This was proved by Imayoshi and Shiga [17] using tools of classical complex analysis together with some properties of the asymptotic behaviour of the Teichmüller metric near generic points in the Bers compactification of Teichmüller space. In the last section of this paper we give a proof of the Imayoshi–Shiga theorem using the Eells–Sampson technique.

We would like to end this introduction with the following question [4, §5.3]. We believe that the methods discussed in this paper offer a good starting point for addressing this question and suggest a positive answer.

QUESTION. — *Let* $\phi : \mathrm{Map}_{g,p} \to \mathrm{Map}_{g',p'}$ *be an irreducible*(1) *homomorphism and* $g \geqslant 3$*. Is there a* ϕ*-equivariant holomorphic map* $f : \mathcal{T}_{g,p} \to \mathcal{T}_{g',p'}$*?*

It would also be interesting to investigate whether or not Theorem 1.1 remains true for small values of $g \leqslant 6$. Finally, we would like to refer to the discussion in [21] for a different perspective on this topic and related ideas.

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2.

Throughout this paper we make use of standard facts about Teichmüller spaces and mapping class groups. We refer to [14, 16, 18, 23] for an extensive treatment of these subjects, and to [8] for a nice survey.

Let $S_{g,p}$ be the closed surface of genus g with p distinct labeled marked points. The mapping class group $\mathrm{Map}_{g,p}$ is the group of isotopy classes of orientation preserving self-homeomorphisms of $S_{g,p}$ fixing each individual marked point. It acts discretely on Teichmüller space and this action preserves the standard complex structure on $\mathcal{T}_{g,p}$ (see [1]). Moduli space is the complex orbifold

$$\mathcal{M}_{g,p} = \mathcal{T}_{g,p} / \mathrm{Map}_{g,p} .$$

In particular, maps and covers are taken in the category of orbifolds. For instance, for every continuous map $f : \mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ there are a homomorphism $f_* : \mathrm{Map}_{g,p} \to \mathrm{Map}_{g',p'}$ and a continuous f_*-equivariant map $\tilde{f} : \mathcal{T}_{g,p} \to \mathcal{T}_{g',p'}$ such that the following diagram commutes:

$$\begin{tikzcd} \mathcal{T}_{g,p} \arrow[r, "\tilde{f}"] \arrow[d] & \mathcal{T}_{g',p'} \arrow[d] \\ \mathcal{M}_{g,p} \arrow[r, "f"] & \mathcal{M}_{g',p'} . \end{tikzcd}$$

The homomorphism f_* and the lift $\tilde{f}$ are unique up to simultaneous conjugation by a mapping class. All of this amounts to saying that Teichmüller space is the (orbifold) universal cover of moduli space. We discuss briefly the example of forgetful maps:

(1) The homomorphism ϕ is irreducible if its image in $\mathrm{Map}_{g',p'}$ fixes no multicurves.

Example 2.1. — Let $f : \mathcal{M}_{g,p} \to \mathcal{M}_{g,p'}$ be the forgetful map defined in (1.1). The lift $\tilde{f} : \mathcal{T}_{g,p} \to \mathcal{T}_{g,p'}$ is given by the same formula, and the homomorphism $f_* : \mathrm{Map}_{g,p} \to \mathrm{Map}_{g,p'}$ is the one given by forgetting the marked points $\{x_1, \ldots, x_p\} \setminus \{x_{i_1}, \ldots, x_{i_{p'}}\}$ [3, 14]. In fact, both f and $\tilde{f}$ are holomorphic fiber bundles, and the long exact sequence of homotopy groups corresponding to the fiber bundle f yields the Birman exact sequence for f_*.

Returning to the general setting, note that the homotopy class of the map $f : \mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ (in the sense of orbifolds) is determined by the homomorphism f_*. This is so because Teichmüller space is a classifying space for proper actions $\underline{E}\,\mathrm{Map}_{g,p}$ of the mapping class group [19]. In other words, $\mathcal{T}_{g,p}$ is contractible, and fixed-point sets of infinite subgroups of $\mathrm{Map}_{g,p}$ are empty, while those of finite subgroups are non-empty and contractible. To see the latter property, one may use that any two points in Teichmüller space $\mathcal{T}_{g,p}$ are connected by a unique Weil–Petersson geodesic segment, plus the fact that the Weil–Petersson metric on Teichmüller space is induced by a negatively curved and geodesically convex Riemannian metric. See [16, 18] for background on the Weil–Petersson metric d_{WP}.

Anyways, the importance of the fact that the homotopy class of f be determined by f_* arises from the fact that there is a number of rigidity results for homomorphisms between mapping class groups. More concretely, we will need such a rigidity theorem taken from [3]. To be able to state it precisely, we need to recall some terminology from that paper. Let S and S' be compact surfaces, possibly with boundary, and P and P' finite sets of marked points in the interior of S and S' respectively. By an *embedding* $\iota : (S, P) \to (S', P')$ we understand a continuous injective map $\iota_{top} : S \to S'$ with the property that $\iota_{top}^{-1}(P') \subset P$. Every embedding $\iota : (S, P) \to (S', P')$ induces a (continuous) homomorphism $\mathrm{Homeo}(S, P) \to \mathrm{Homeo}(S', P')$ between the corresponding groups of self-homeomorphisms fixing pointwise the boundary and the set of marked points. In particular, ι induces a homomorphism

$$\iota_\# : \mathrm{Map}(S, P) \to \mathrm{Map}(S', P')$$

between the associated mapping class groups. The main result proved in [3] asserts that, subject to suitable genus bounds, every non-trivial homomorphism between mapping class groups is in fact induced by an embedding.

Theorem 2.2 (Aramayona–Souto). — *Suppose that S and S' are compact surfaces, possibly with boundary, and that P and P' are finite sets of marked points in the interior of S and S' respectively. If S has genus $g \geqslant 6$*

and S' has genus $g' \leqslant 2g-2$, then every nontrivial homomorphism

$$\mathrm{Map}(S,P) \to \mathrm{Map}(S',P')$$

is induced by an embedding $(S,P) \to (S',P')$.

Armed with Theorem 2.2, we prove:

PROPOSITION 2.3. — *If $g \geqslant 6$ and $g' \leqslant 2g-2$, then every map $f : \mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$ is either homotopically trivial or homotopic to a forgetful map.*

Proof. — Let $f_* : \mathrm{Map}_{g,p} \to \mathrm{Map}_{g',p'}$ be the homomorphism associated to f, let $\tilde{f} : \mathcal{T}_{g,p} \to \mathcal{T}_{g',p'}$ be an f_*-equivariant lift of f, and recall once again that the homotopy type of f is determined by f_*. In particular, if f_* is trivial, then f is homotopically trivial and we have nothing to prove. Suppose from now on that this is not the case.

Let (S,P) and (S',P') be, respectively, closed surfaces of genus g and g', with p and p' marked points. Identifying $\mathrm{Map}(S,P) = \mathrm{Map}_{g,p}$ and $\mathrm{Map}(S',P') = \mathrm{Map}_{g',p'}$, we obtain from Theorem 2.2 that the homomorphism f_* is induced by an embedding

$$\iota : (S,P) \to (S',P').$$

Since S and S' are closed, the underlying injective map $\iota_{top} : S \to S'$ is a homeomorphism and $\iota_{top}(P) \supset P'$. In other words, the embedding ι is obtained by forgetting marked points.

In the same way that we have identified mapping class groups, we also identify Teichmüller spaces $\mathcal{T}_{g,p} = \mathcal{T}(S,P)$ and $\mathcal{T}_{g',p'} = \mathcal{T}(S',P')$. The embedding ι induces an f_*-equivariant map

$$\tilde{h} : \mathcal{T}_{g,p} \to \mathcal{T}_{g',p'}$$

obtained again by forgetting marked points. By construction, $\tilde{h}$ descends to a forgetful map

$$h : \mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}.$$

Both $\tilde{f}$ and $\tilde{h}$ are homotopic because both of them are f_*-equivariant. □

3.

In this section we prove Theorem 1.1 from the introduction. To do so we will rely on Proposition 2.3 together with the following results:

Lemma 3.1. — *Let M be a quasi-projective variety and let $f: M \to \mathcal{M}_{g,q}$ be a holomorphic map. If f is homotopically trivial, then it is constant.*

Proof. — If f is homotopically trivial, then we can lift it to a holomorphic map $\tilde{f}: M \to \mathcal{T}_{g,q}$. The claim follows since Teichmüller space $\mathcal{T}_{g,p}$ is a bounded domain and any bounded holomorphic function on a quasi-projective variety is constant. □

Proposition 3.2. — *Let M be a quasi-projective variety and $f_1, f_2: M \to \mathcal{M}_{g,q}$ non-constant holomorphic maps. If f_1 and f_2 are homotopic (in the category of orbifolds) then $f_1 = f_2$.*

If M is a Riemann surface, then Proposition 3.2 is due to Imayoshi and Shiga [17]:

Imayoshi–Shiga theorem. — *Let Σ be a Riemann surface of finite type and $f_1, f_2 : \Sigma \to \mathcal{M}_{g,q}$ non-constant holomorphic maps. If f_1 and f_2 are homotopic (in the category of orbifolds) then $f_1 = f_2$.*

As we just pointed out, the Imayoshi–Shiga theorem is a special case of Proposition 3.2. Conversely, Proposition 3.2 follows easily from the Imayoshi–Shiga's theorem. We will give a logically independent proof of the Imayoshi–Shiga's theorem in the next section.

Proof of Proposition 3.2. — We first note that the Imayoshi–Shiga theorem holds for arbitrary algebraic curves $\widetilde{\Sigma}$. Indeed, we can apply it to the Riemann surface Σ obtain from $\widetilde{\Sigma}$ by removing its finite set of singularities, and use the fact that two holomorphic maps on $\widetilde{\Sigma}$ that agree on a Zariski dense subset are equal.

It follows that if f_1 and f_2 are homotopic maps on M, then f_1 and f_2 agree along every curve $\widetilde{\Sigma} \subset M$, unless they are both constant along $\widetilde{\Sigma}$. In particular, f_1 and f_2 agree on a generic curve in M, and hence $f_1 = f_2$. □

We note that the moduli space $\mathcal{M}_{g,p}$ admits a finite (orbifold) cover

$$\pi : \mathcal{N} \to \mathcal{M}_{g,q} \tag{3.1}$$

which is a smooth quasi-projective variety. See [5] for algebraic geometric properties of the moduli space $\mathcal{M}_{g,p}$ and for example [20] for the construction of a cover which is not only smooth, but whose Deligne–Mumford compactification is smooth as well. In order to avoid any unnecessary technical difficulties in the proof given below, we will apply Proposition 3.2 to the smooth quasi-projective variety $\mathcal{N}$ from (3.1). We are now ready to prove:

Proof of Theorem 1.1. — Suppose that $g \geqslant 6$, $g' \leqslant 2g-2$ and let

$$f : \mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$$

be a non-constant holomorphic map. With $\pi : \mathcal{N} \to \mathcal{M}_{g,p}$ is as in (3.1), consider the composition $f \circ \pi : \mathcal{N} \to \mathcal{M}_{g',p'}$. The map $f \circ \pi$ is not constant, and hence Lemma 3.1 yields that it cannot be homotopic to a constant map. It follows thus from Proposition 2.3 that f is homotopic to a forgetful map $h : \mathcal{M}_{g,p} \to \mathcal{M}_{g',p'}$. Now, since both $f \circ \pi$ and $h \circ \pi$ are holomorphic and non-constant we get that $f \circ \pi = h \circ \pi$ from Proposition 3.2. Since π is a covering, and thus surjective, it follows that $f = h$, as we needed to prove. □

Besides Theorem 2.2 cited above, there are many other rigidity results for homomorphisms between mapping class groups; see for example [4] for a survey of results in this direction. Combining any such theorem with Proposition 3.2 one obtains a rigidity result for holomorphic maps between the corresponding moduli spaces. For instance, the version of Theorem 2.2 proved in [3] covers a few more cases than the ones stated here. From this more general version, it follows that Theorem 1.1 also holds for maps $\mathcal{M}_{g,p} \to \mathcal{M}_{2g-1,p'}$ with $p' \geqslant 1$, and for maps $\mathcal{M}_{g,p} \to \mathcal{M}_{g,p'}$ as long as $g \geqslant 4$. In particular, we have:

Corollary 3.3. — *Suppose that $g \geqslant 4$. Every non-constant holomorphic map $\mathcal{M}_{g,p} \to \mathcal{M}_{g,p}$ is induced by a permutation of marked points, and is hence a biholomorphism.* □

Note that the isomorphism, for $g \geqslant 2$, between the group of biholomorphisms of $\mathcal{M}_{g,p}$ and the symmetric group Σ_p follows also from Royden's characterization of the biholomorphism group of Teichmüller space [24, 12].

4.

In this section we give a proof of the Imayoshi–Shiga theorem. Suppose from now on that $f_1, f_2 : \Sigma \to \mathcal{M}$ are homotopic holomorphic maps from a Riemann surface of finite type to moduli space $\mathcal{M} = \mathcal{M}_{g,p}$. Suppose that f_1 is not constant. To begin with recall that by a theorem of Royden [24] the Teichmüller metric d_{T} is equal to the Kobayashi metric of hyperbolic space. In particular, since f_1 is assumed to be non-constant, it follows that Σ admits a conformal hyperbolic metric d_Σ. The maps f_1, f_2 are then 1-Lipschitz with respect to the hyperbolic metric in the domain and the Teichmüller metric in the target. We will however consider $\mathcal{M}$ to be endowed with the Weil–Petersson metric d_{WP}, but this does not change things much. Indeed, the identity map $\mathrm{Id} : (\mathcal{T}_{g,p}, \mathrm{d_T}) \to (\mathcal{T}_{g,p}, \mathrm{d_{WP}})$ is Lipschitz [22, Proposition 2.4]. In particular, when endowing the domain with the hyperbolic metric and the target with the Weil–Petersson metric we get that the maps

$$f_1, f_2 : (\Sigma, \mathrm{d}_\Sigma) \to (\mathcal{M}, \mathrm{d_{WP}})$$

are L-Lipschitz for some suitable L. Since we are working in the category of orbifolds we should possibly be a bit more careful: what we mean when we say that f_1, f_2 are L-Lipschitz is that the induced maps between the universal covers are L-Lipschitz. Since the maps in question are smooth this just means that the norm of their first derivatives is bounded by L at every point.

Let now $\hat{F} : [1,2] \times \Sigma \to \mathcal{M}$ be a homotopy between f_1, f_2. Since the Weil–Petersson metric is negatively curved and geodesically convex, we can replace $\hat{F}$ with the straight homotopy

$$F : [1,2] \times \Sigma \to \mathcal{M}, \quad F(t,x) = f_t(x)$$

determined by the fact that $t \mapsto f_t(x)$ is the d_{WP}-geodesic segment joining $f_1(x)$ and $f_2(x)$ in the homotopy class of $\hat{F}([1,2] \times \{x\})$.

Note that f_t is L-Lipschitz for all t because f_1 and f_2 are. Indeed, for $v \in T_x\Sigma$ the vector field $t \mapsto \mathrm{d}(f_t)_X v$ is a Jacobi field along $t \mapsto f_t(x)$. Since the Weil–Peterson metric is negatively curved, the length of Jacobi fields is a convex function, and hence attains its maximum at $t \in \{1,2\}$.

A priori, the map F itself need not be Lipschitz: the norm of $\mathrm{d}\,F_{(t,x)} \frac{\partial}{\partial t}$ is equal to the length of the geodesic arc $t \mapsto F(t,x)$, and when Σ is not compact there is no reason for this to be bounded. However, fixing $x_0 \in \Sigma$ there is a constant k, independent of x, such that the segment $t \mapsto F(t,x)$ has length at most $k + 2L\,\mathrm{d}_\Sigma(x, x_0)$ because f_1, f_2 are L-Lipschitz. It follows that there are constants A, B with

$$\|\,\mathrm{d}\,F_{(t,x)}\|^2 \leqslant A \cdot \mathrm{d}_\Sigma(x_0, x)^2 + B$$

for all $(t,x) \in [1,2] \times \Sigma$. Here, $\|\,\mathrm{d}\,F_{(t,x)}\|$ is the operator norm of $\mathrm{d}\,F_{(t,x)}$.

The convexity of Jacobi fields also implies the convexity of the energy density

$$t \mapsto E_x(f_t) \stackrel{\mathrm{def}}{=} \frac{1}{2}(\|\,\mathrm{d}\,f_t|_x v_1\|^2_{\mathrm{WP}} + \|\,\mathrm{d}\,f_t|_x v_2\|^2_{\mathrm{WP}})$$

where v_1, v_2 is an arbitrary orthonormal basis of $T_x\Sigma$. This function is strictly convex if one of $df_1|_x$ or $df_2|_x$ has rank at least 2. In particular, if the holomorphic maps f_1, f_2 are distinct and one of them is non-constant, then the energy

$$t \mapsto E(f_t) \overset{\text{def}}{=} \int_\Sigma E_x(f)\omega_\Sigma$$

is strictly convex. Here ω_Σ is the Riemannian volume form of Σ, and $E(f_t) < \infty$ because f_t is L-Lipschitz.

We summarize this discussion in the following lemma:

Lemma 4.1. — *Let $(\Sigma, \mathrm{d}_\Sigma)$ be a hyperbolic surface of finite type and*

$$f_1, f_2 : (\Sigma, \mathrm{d}_\Sigma) \to (\mathcal{M}, \mathrm{d}_{\mathrm{WP}})$$

homotopic holomorphic maps. Consider the straight homotopy

$$F : [1,2] \times \Sigma \to \mathcal{M}, \quad F(t,x) = f_t(x)$$

between them. Then:

(1) *There is $L > 0$ such that $f_t : \Sigma \to \mathcal{M}$ is L-Lipschitz and has finite energy $E(f_t) < \infty$ for all t.*

(2) *For $x_0 \in \Sigma$, there are $A, B > 0$ with*

$$\| \mathrm{d}\, F_{(t,x)} \|^2 \leqslant A \cdot \mathrm{d}_\Sigma(x_0, x)^2 + B$$

for all $(t,x) \in [1,2] \times \Sigma$.

(3) *The energy function $t \mapsto E(f_t)$ is convex. Moreover, it is strictly convex unless either $f_1 = f_2$ or both are constant.* □

At this point we will use that both the hyperbolic metric on Σ and the Weil–Petersson metric on $\mathcal{M}$ are Kähler. This means that the 2-form $\omega = \langle \cdot\,, J\cdot \rangle$, is closed, where $\langle \cdot\,,\cdot \rangle$ is the relevant Riemannian metric and J is the endomorphism of the tangent bundle given by complex multiplication. Note that in the case of Σ, the Kähler form is nothing other than the volume form ω_Σ. See [6] for facts on Kähler manifolds and [13] for a proof of the following key proposition:

Proposition 4.2. — *Let $f : \mathbb{H}^2 \to (\mathcal{T}_{g',p'}, d_{\mathrm{WP}})$ be a smooth map and $x \in \mathbb{H}^2$. We have $E_x(f)\omega_{\mathbb{H}^2} \geqslant f^*(\omega_{\mathrm{WP}})$ with equality if and only if f is holomorphic at x.*

We are now ready to prove the Imayoshi–Shiga theorem:

Proof of the Imayoshi–Shiga theorem. — Suppose that f_1 is not constant and $f_1 \neq f_2$, and let

$$F : [1,2] \times \Sigma \to \mathcal{M}, \quad F(t,x) = f_t(x)$$

be the straight homotopy between them. From Lemma 4.1 we know that the function $t \mapsto E(f_t)$ is strictly convex; we may hence assume that

$$E(f_t) < E(f_1)$$

for all $t \in (1,2)$. We are going to contradict this assertion.

Choose now a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact subsets of Σ with the following properties:

(1) $\Sigma = \bigcup_{n\in\mathbb{N}} K_n$, and $K_n \subset K_{n+1}$ for all n,
(2) $\ell_\Sigma(\partial K_n) \leqslant ce^{-n}$ for all n, and
(3) K_n is contained within distance $1+n$ of K_0 for all n.

Since Kähler forms are closed, we deduce from Stokes theorem that

$$\begin{aligned}
0 &= \int_{[1,t]\times K_n} \mathrm{d}\,(F^*\omega_{\mathrm{WP}}) = \int_{\partial([1,t]\times K_n)} (F^*\omega_{\mathrm{WP}}) \\
&= \int_{\{t\}\times K_n} (F^*\omega_{\mathrm{WP}}) - \int_{\{1\}\times K_n} (F^*\omega_{\mathrm{WP}}) + \int_{[1,t]\times\partial K_n} (F^*\omega_{\mathrm{WP}}) \\
&= \int_{K_n} (f_t^*\omega_{\mathrm{WP}}) - \int_{K_n} (f_1^*\omega_{\mathrm{WP}}) + \int_{[1,t]\times\partial K_n} (F^*\omega_{\mathrm{WP}}).
\end{aligned}$$

Below we will prove:

$$\lim_{n\to\infty} \int_{[1,t]\times\partial K_n} (F^*\omega_{\mathrm{WP}}) = 0. \tag{4.1}$$

Assuming (4.1), we obtain from the computation above that

$$\lim_{n\to\infty} \left(\int_{K_n} (f_t^*\omega_{\mathrm{WP}}) - \int_{K_n} (f_1^*\omega_{\mathrm{WP}}) \right) = 0.$$

Taking into account that f_t and f_1 are Lipschitz and that Σ has finite volume, we deduce that

$$\int_\Sigma (f_t^*\omega_{\mathrm{WP}}) = \int_\Sigma (f_1^*\omega_{\mathrm{WP}}).$$

Now, Proposition 4.2 yields

$$E(f_t) \geqslant \int_\Sigma (f_t^*\omega_{\mathrm{WP}}) = \int_\Sigma (f_1^*\omega_{\mathrm{WP}}) = E(f_1)$$

where the last equality holds because f_1 is holomorphic. This gives the desired contradiction to the fact that $E(f_t) < E(f_1)$. Therefore, it remains to prove (4.1).

Fix $(t,x) \in [0,1] \times \partial K_n$ and let v_1, v_2 be an orthonormal basis of $T_{(t,x)}([0,1]\times\partial K_n)$. We have

$$\left|(F^*\omega_{\mathrm{WP}})(v_1,v_2)\right| = \left|\langle \mathrm{d}\,F_{(t,x)}v_1, J\,\mathrm{d}\,F_{(t,x)}v_2\rangle_{\mathrm{WP}}\right| \leqslant \|\,\mathrm{d}\,F_{(t,x)}\|^2$$

where $\|\,\mathrm{d}\,F_{(t,x)}\|$ is the operator norm of $\mathrm{d}\,F_{(t,x)}$. Fixing $x_0 \in \Sigma$ we get from Lemma 4.1 that there are $A, B > 0$ with

$$\|\,\mathrm{d}\,F_{(t,x)}\|^2 \leqslant A \cdot \mathrm{d}_\Sigma(x_0,x)^2 + B$$

for all $(t,x) \in [1,2] \times \Sigma$. We deduce that

$$\left| \int_{[0,t]\times\partial K_n} (F^*\omega_{\mathrm{WP}}) \right| \leqslant \int_{[0,t]\times\partial K_n} \| \mathrm{d}\, F_{(t,X)} \|^2 \nu_{[0,t]\times\partial K_n}$$
$$\leqslant \left(A \cdot \max_{x\in\partial K_n} \mathrm{d}_\Sigma(x,x_0)^2 + B \right) \mathrm{vol}_\Sigma(\partial K_n)\,.$$

This last quantity tends to 0 as $n \to \infty$ by points (2) and (3) above. Having proved (4.1), we are done with the proof of the Imayoshi–Shiga theorem. □

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