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Bas LEMMENS & Mark ROELANDS

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UNIQUE GEODESICS FOR THOMPSON'S METRIC

by Bas LEMMENS & Mark ROELANDS (*)

ABSTRACT. — In this paper a geometric characterization of the unique geodesics in Thompson's metric spaces is presented. This characterization is used to prove a variety of other geometric results. Firstly, it will be shown that there exists a unique Thompson's metric geodesic connecting x and y in the cone of positive self-adjoint elements in a unital C^*-algebra if, and only if, the spectrum of $x^{-1/2}yx^{-1/2}$ is contained in $\{1/\beta, \beta\}$ for some $\beta \geqslant 1$. A similar result will be established for symmetric cones. Secondly, it will be shown that if C° is the interior of a finite-dimensional closed cone C, then the Thompson's metric space (C°, d_C) can be quasi-isometrically embedded into a finite-dimensional normed space if, and only if, C is a polyhedral cone. Moreover, (C°, d_C) is isometric to a finite-dimensional normed space if, and only if, C is a simplicial cone. It will also be shown that if C° is the interior of a strictly convex cone C with $3 \leqslant \dim C < \infty$, then every Thompson's metric isometry is projectively linear.

RÉSUMÉ. — Nous présentons une caractérisation géométrique des géodésiques uniques des espaces métriques de Thompson. Nous utilisons cette caractérisation pour démontrer plusieurs autres résultats géométriques. D'abord, nous démontrons qu'il existe une géodésique unique de la métrique de Thompson entre x and y dans le cône d'éléments positifs autoadjoints dans une C^*-algèbre unitale si et seulement s'il existe $\beta \geqslant 1$ tel que le spectre de $x^{-1/2}yx^{-1/2}$ soit contenu dans $\{1/\beta, \beta\}$. Un résultat similaire est établi pour des cônes symétriques. Ensuite, nous démontrons que si C° est l'intérieur d'un cône fermé C de dimension finie, il existe un plongement quasi-isométrique de l'espace métrique de Thompson (C°, d_C) dans un espace normé de dimension finie si et seulement si C est un cône polyédrale. De plus, (C°, d_C) est isométrique à un espace normé de dimension finie si et seulement si C est un cône simplicial. Par ailleurs, il est établi que pour C° l'intérieur d'un cône C strictement convexe avec $3 \leqslant \dim C < \infty$, chaque isométrie de la métrique de Thompson est projectivement linéaire.

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1. Introduction

In [4] Birkhoff showed that one can use Hilbert's (projective) metric and the contraction mapping principle to prove the existence and uniqueness of a positive eigenvector for a large class of linear operators that leave a closed cone C in a Banach space invariant. An alternative to Hilbert's metric was introduced by Thompson in [31]. Thompson's (part) metric, denoted here by d_C, has the advantage that it is a metric on each part of a cone C rather than a metric between pairs of rays in each part. It has found numerous applications in the analysis of linear and nonlinear operators on cones, see for instance [1, 13, 17, 26] and the references therein. Thompson's metric is also used to study the geometry of cones of positive operators [2, 8, 9, 22] and symmetric cones [16, 18, 19, 21], where it provides an alternative to the usual Riemannian metric. It also appears in the analysis of order-isomorphisms on cones, see [24, 25].

Despite the frequent use of Thompson's metric spaces in mathematical analysis, there are still many interesting aspects of their geometry that remain to be explored. A number of individual results exist. For example, it is known that Thompson metric spaces are Finsler manifolds, see [27]. Furthermore, on the cones of positive self-adjoint elements in unital C^*-algebras and symmetric cones, Thompson's metric possesses certain non-positive curvature properties, see [2, 16]. On general closed cones Thompson's metric is semi-hyperbolic, see [28]. It is also known [17, Section 2.2] that if C° is the interior of a closed polyhedral cone in a vector space V, then (C°, d_C) can be isometrically embedded into $(\mathbb{R}^m, \|\cdot\|_\infty)$, where $\|z\|_\infty = \max_i |z_i|$ is the *sup-norm* and m is the number of facets of C. Moreover, if C is an n-dimensional *simplicial cone* in V, that is to say, there exist linearly independent vectors $v_1, \ldots, v_n \in V$ such that $C = \{\sum_i \lambda_i v_i \colon \lambda_i \geqslant 0 \text{ for all } i\}$, then (C°, d_C) is isometric to $(\mathbb{R}^n, \|\cdot\|_\infty)$. Furthermore if $\Lambda_{n+1} = \{(s,x) \in \mathbb{R}\times\mathbb{R}^n \colon s^2 - x_1^2 - \cdots - x_n^2 \geqslant 0 \text{ and } s \geqslant 0\}$ is the *Lorentz cone*, then $(\Lambda_{n+1}^\circ, d_{\Lambda_{n+1}})$ contains an isometric copy of the real n-dimensional hyperbolic space. In fact, on the upper sheet of the hyperboloid $H = \{(s,x) \in \mathbb{R}\times\mathbb{R}^n \colon s^2 - x_1^2 - \cdots - x_n^2 = 1\}$, Thompson's metric coincides with the hyperbolic distance, see [19] or [17, Section 2.3].

One of the main objectives of this paper is to give a geometric characterization of the unique geodesics in Thompson's metric spaces. This characterization is subsequently used to prove a variety of other results.

In particular, we show in Section 5 that if A_+° is the interior of the cone of positive self-adjoint elements in a unital C^*-algebra A, then there exists a unique Thompson metric geodesic connecting x and y in A_+° if, and only if, $\sigma(x^{-1/2}yx^{-1/2}) \subseteq \{\beta, 1/\beta\}$ for some $\beta \geqslant 1$. Here $\sigma(z)$ denotes the spectrum of z. It turns out that a similar result holds for elements in a symmetric cone. In fact, we will prove in Section 6 that there exists a unique Thompson metric geodesic connecting x and y in a symmetric cone if, and only if, $\sigma(P(y^{-1/2})x) \subseteq \{\beta, 1/\beta\}$ for some $\beta \geqslant 1$. Here P is the quadratic representation. These results generalize [20, Theorem 5.2] by Lim, who showed the equivalence for the cone of positive definite Hermitian matrices.

The characterization will also be used to prove a number of geometric properties of Thompson's metric spaces. For example we prove in Section 7 that if C is a finite-dimensional closed cone with nonempty interior, then (C°, d_C) can be quasi-isometrically embedded into a finite-dimensional normed space if, and only if, C is a polyhedral cone. Furthermore we show that a Thompson's metric space (C°, d_C) is isometric to an n-dimensional normed space if, and only if, C is an n-dimensional simplicial cone. Analogous results for Hilbert's metric spaces were obtained by Colbois and Verovic [6], and by Foertsch and Karlsson [12], see also [3]. Our method of proof is similar to theirs, but interesting adaptations need to be made to make the arguments work.

In the final section it will be shown that if C is a strictly convex cone with nonempty interior and $3 \leqslant \dim C < \infty$, then every isometry of (C°, d_C) is projectively linear. This result complements recent work by Bosché [5] who determined the isometries for Thompson's metric on symmetric cones, and work by Molnár [23] on Thompson's metric isometries on the cone of positive self-adjoint operators on a Hilbert space. In [10] de la Harpe proved a similar result for strictly convex Hilbert's metric spaces. Our proof will appeal to his result.

In the next section we recall some basic concepts and results.

2. Thompson's metric

Let C be a cone in a vector space V. So, C is convex, $\lambda C \subseteq C$ for all $\lambda \geqslant 0$, and $C \cap (-C) = \{0\}$. The cone C induces a partial ordering $\leqslant_C$ on V by $x \leqslant_C y$ if $y - x \in C$. For $x, y \in C$, we say that y *dominates* x if there exists $\beta > 0$ such that $x \leqslant_C \beta y$. Given $x, y \in C$ we write $x \sim_C y$ if y dominates x, and x dominates y. In other words, $x \sim_C y$ if and only if there exist $0 < \alpha \leqslant \beta$ such that $\alpha y \leqslant_C x \leqslant_C \beta y$. It is easy to verify that $\sim_C$ is an equivalence relation on C. The equivalence classes are called *parts* of C. If C is a finite-dimensional closed cone, then the parts are precisely the relative interiors of the faces of C, see [17, Lemma 1.2.2]. Recall that a nonempty convex set $F \subseteq C$ is a *face* of C if $x, y \in C$ and $\lambda x + (1-\lambda)y \in F$ for some $0 < \lambda < 1$ implies $x, y \in F$. The *relative interior* of a convex set $S \subset V$ is its interior in the affine span of S.

Given $x, y \in C$ such that $x \sim_C y$, we define

$$M(x/y;C) = \inf\{\beta > 0 \colon x \leqslant_C \beta y\} \text{ and } m(x/y;C) = \sup\{\alpha > 0 \colon \alpha y \leqslant_C x\}.$$

We simply write $M(x/y)$ and $m(x/y)$ if C is clear from the context. Note that $m(x/y) = M(y/x)^{-1}$.

Definition 2.1. — *On a cone C in a vector space V,* Thompson's metric*, $d_C \colon C \times C \to [0, \infty]$, is defined by*

$$d_C(x, y) = \log\Big(\max\{M(x/y), M(y/x)\}\Big)$$

for $x \sim_C y$ in C, and $d_C(x, y) = \infty$ otherwise.

This metric was introduced by Thompson in [31], who showed that d_C is a metric on each part of C, when C is a closed cone in a normed space. Furthermore, he showed that if C is a closed cone in a Banach space $(V, \|\cdot\|)$, and C is a normal cone, i.e., there exists $\kappa > 0$ such that $\|x\| \leqslant \kappa\|y\|$ whenever $x \leqslant_C y$, then (P, d_C) is a complete metric space for each part P of C, and the topology coincides with the norm topology on P. In particular, Thompson's metric topology on the interior of a closed finite-dimensional cone coincides with the norm topology.

It can be shown, see [17, Appendix A.2], that d_C is a metric on each part if C is an *almost Archimedean* cone, i.e., if $x \in V$ and there exists $y \in V$ such that $-\epsilon y \leqslant_C x \leqslant_C \epsilon y$ for all $\epsilon > 0$, then $x = 0$. Almost Archimedean cones can be characterized by their intersections with finite-dimensional linear subspaces. To state this result the following notation is convenient.

Given an almost Archimedean cone C in a vector space V and $S \subseteq V$, we let $V(S) = \operatorname{span}\{S\}$. If $\dim V(S) < \infty$, then we define $C(S) = \overline{C \cap V(S)}$, where the topology is the unique topology that turns $V(S)$ into a Hausdorff topological vector space. We denote the interior of $C(S)$ in $V(S)$ by $C(S)^\circ$, and its boundary in $V(S)$ by $\partial C(S)$. Now the characterization of almost Archimedean cones can be stated as follows.

Lemma 2.2. — *A cone C in a vector space V is almost Archimedean if and only if for each finite dimensional subspace W of V we have that $C(W)$ is a cone.*

Proof. — From [17, Proposition A.2.2] we know that C is almost Archimedean if and only if for each 2-dimensional subspace W of V we have that $C(W)$ is a cone. Thus, it remains to show that the condition is necessary. So, let C be an almost Archimedean cone and let W be a finite-dimensional subspace of V. We need to show that $C(W)$ is a cone. It is clear that $C(W)$ is convex and $\lambda C(W) \subseteq C(W)$ for all $\lambda \geqslant 0$. Suppose that there exists $x \neq 0$ such that x and $-x$ in $C(W)$. Note that we can replace W by $C(W) - C(W)$ and assume that the span of $C(W)$ is W. As W is finite-dimensional, this implies that $C(W)^\circ$ is nonempty.

Select $y \in C(W)^\circ$ and $\delta > 0$ such that $B_\delta(y) \subseteq C(W)^\circ$, where $B_\delta(w)$ denotes the δ-ball around w in W. Let $\epsilon > 0$. There exists $z \in B_\delta(0) \cap C(W)$ such that $x + \epsilon z \in C(W)$. Using the convexity of $C(W)$ we see that $\frac{1}{1+\epsilon}x + (1 - \frac{1}{1+\epsilon})y = \frac{1}{1+\epsilon}(x + \epsilon z) + (1 + \frac{1}{1+\epsilon})(y - z) \in C(W)$. These points lie in $\mathrm{span}\{x, y\}$ and converge to x as $\epsilon \to 0$. In the same way we can find points in $\mathrm{span}\{x, y\}$ converging to $-x$. This implies that x and $-x$ are in $C(x, y)$, which is impossible by [17, Proposition A.2.2]. $\square$

A useful variant of Thompson's metric, which will also play a role here, is *Hilbert's (projective) metric*,

$$\delta_C(x, y) = \log\Big(M(x/y)M(y/x)\Big)$$

for $x \sim_C y$ in C, and $d_C(x, y) = \infty$ otherwise. Hilbert's metric is only a metric on the rays in each part of C, as $\delta_C(\lambda x, \mu y) = \delta_C(x, y)$ for all $\lambda, \mu > 0$ and $x \sim_C y$ in C.

Given a cone C in V we denote the *dual cone* by $C^* = \{\varphi \in V^* : \varphi(x) \geqslant 0 \text{ for all } x \in C\}$. A linear functional $\varphi \in C^*$ is said to be *strictly positive* if $\varphi(x) > 0$ for all $x \in C \setminus \{0\}$. It is well know, see for example, [17, Theorem 2.1.2], that if C is a closed cone with nonempty interior in a finite-dimensional vector space V, then C^* is also a closed cone with nonempty interior. Moreover, for each strictly positive $\varphi \in C^*$ the set $\Sigma_\varphi^\circ = \{x \in C^\circ : \varphi(x) = 1\}$ is a bounded convex set on which δ_C coincides with *Hilbert's cross-ratio metric*,

$$\kappa(x, y) = \log\Big(\frac{\|x' - y\|}{\|x' - x\|}\frac{\|y' - x\|}{\|y' - y\|}\Big),$$

where x' and y' are the points of intersection of the straight line through x and y and $\partial\Sigma_\varphi^\circ$ such that x is between x' and y, and y is between y' and x.

We will be interested in the geodesics in (C, d_C). Recall that a map γ from an (open, closed, bounded, or, unbounded) interval $I \subseteq \mathbb{R}$ into a metric space (X, d_X) is called a *geodesic path* if

$$d_X(\gamma(s), \gamma(t)) = |s - t| \quad \text{for all } s, t \in I.$$

The image of γ is called a *geodesic segment* in (X, d_X). It said to be a *geodesic line* in (X, d_X) if $I = \mathbb{R}$.

It is known, see for example [17, Theorem 2.6.9], that if P is a part of C, then (P, d_C) is a *geodesic* metric space, i.e., for each $x, y \in P$ there exists a geodesic path $\gamma\colon [a,b] \to P$ with $\gamma(a) = x$ and $\gamma(b) = y$. In general there can be more than one geodesic segment connecting x and y in (P, d_C). One of the main objectives is to characterize those x and y in (C, d_C) that are connected by a **unique** geodesic segment. The following elementary result will be useful. We leave the proof to the reader.

Lemma 2.3. — *If x and y are distinct points in a geodesic metric space (X, d_X) and $\gamma\colon [a,b] \to X$ is a geodesic path with $\gamma(a) = x$ and $\gamma(b) = y$, then the image of γ is a unique geodesic segment connecting x and y if and only if for each $z \in X$ with $d_X(x,y) = d_X(x,z) + d_X(z,y)$, we have that $z = \gamma(t)$ for some $t \in [a,b]$.*

3. Two dimensional cones

The following elementary lemma is useful.

Lemma 3.1. — *Let C be an almost Archimedean cone. If $x \sim_C y$ in C, then $x, y \in C(x,y)^\circ$ and $d_C(w,z) = d_{C(x,y)}(w,z)$ for all $w, z \in C(x,y)^\circ$.*

Proof. — The statements are trivial for $x = y = 0$. If $x = \mu y$ for some $\mu > 0$ and $x \neq 0$, then $C(x,y) = \{\lambda x\colon \lambda \geqslant 0\}$ and hence $x, y \in C(x,y)^\circ$. Obviously, for $w = \alpha x$ and $z = \beta x$ with $0 < \alpha \leqslant \beta$ we have $d_C(w,z) = \log \beta/\alpha = d_{C(x,y)}(w,z)$.

If $x \sim_C y$ are linearly independent, then $C(x,y)$ is a 2-dimensional closed cone in $V(x,y)$. By [17, Theorem A.5.1] we know that there exists linearly independent vector u and v in $V(x,y)$ such that

$$C(x,y) = \{su + tv\colon s, t \geqslant 0\}.$$

It follows that $C(x,y)$ has 4 parts: $\{0\}$, $\{su\colon s > 0\}$, $\{tu\colon t > 0\}$, and $C(x,y)^\circ$. As x and y are linearly independent, x and y must be in $C(x,y)^\circ$. Moreover, it follows from [17, Corollary A.5.2] that

$$M(w/z; C) = M(w/z; C \cap V(x,y)) = M(w/z; C(x,y))$$

for all $w, z \in C(x,y)^\circ$, which proves the final assertion. $\square$

Lemma 3.1 has the following basic consequence.

Corollary 3.2. — *If $x \sim_C y$ are connected by a unique geodesic segment γ in (C, d_C), then γ lies in $C(x,y)^\circ$ and γ is a unique geodesic segment connecting x and y in $(C(x,y)^\circ, d_{C(x,y)})$.*

Thus, we need to first analyze the problem in two dimensions. If K is a closed cone with nonempty interior in a 2-dimensional vector space W, then there exists $u, v \in \partial K$ linearly independent such that

$$K = \{\alpha u + \beta v \colon \alpha, \beta \geqslant 0\},$$

see [17, Theorem A.5.1]. Alternatively, there exists linearly independent functionals ψ_1 and ψ_2 on W such that

$$K = \{x \in W \colon \psi_1(x) \geqslant 0 \text{ and } \psi_2(x) \geqslant 0\}.$$

Lemma 3.3. — *Let $K \subseteq W$ be a closed cone with nonempty interior in a 2-dimensional normed space W. If $x, y \in K^\circ$, then there exists a unique geodesic segment connecting x and y in (K°, d_K) if and only if either*

(i) $M(x/y) = M(y/x)$, *or,*
(ii) $M(x/y) = M(y/x)^{-1}$, *in which case $x = \lambda y$ for some $\lambda > 0$.*

In particular, through each $x \in K^\circ$ there are precisely two unique geodesics.

Proof. — Define a map $\Psi \colon K^\circ \to \mathbb{R}^2$ by $\Psi(x) = (\log \psi_1(x), \log \psi_2(x))$. Since $x \leqslant y$ if and only if $\psi_i(x) \leqslant \psi_i(y)$ for $i = 1, 2$, it follows that

$$M(x/y) = \max_{i=1,2} \frac{\psi_i(x)}{\psi_i(y)}$$

on K°. So, for $x, y \in K^\circ$ the equalities

$$d_K(x,y) = \max_{i=1,2} \left| \log \frac{\psi_i(x)}{\psi_i(y)} \right| = \|\Psi(x) - \Psi(y)\|_\infty,$$

hold, where $\|z\|_\infty = \max_i |z_i|$ is the sup-norm. This implies that Ψ is an isometry from (K°, d_K) onto $(\mathbb{R}^2, \|\cdot\|_\infty)$. In $(\mathbb{R}^2, \|\cdot\|_\infty)$ there are precisely two unique geodesic lines through each point z, namely

$$\ell_I = \{z + t(1,-1) \colon t \in \mathbb{R}\} \quad \text{and} \quad \ell_{II} = \{z + t(1,1) \colon t \in \mathbb{R}\},$$

which proves the last assertion of the lemma.

It follows that there exists a unique geodesic segment connecting x and y in (K°, d_K) if and only if either

$$\Psi(x) - \Psi(y) = s(1,-1) \quad \text{or} \quad \Psi(x) - \Psi(y) = s(1,1)$$

for some $s \in \mathbb{R}$. The first equality is equivalent to

$$\log \frac{\psi_1(x)}{\psi_1(y)} = s = \log \frac{\psi_2(y)}{\psi_2(x)},$$

which holds if and only if $M(x/y) = M(y/x)$. The second equality is equivalent to

$$\log \frac{\psi_1(x)}{\psi_1(y)} = s = \log \frac{\psi_2(x)}{\psi_2(y)},$$

which holds if and only if $M(x/y) = M(y/x)^{-1}$. Finally note that as K is closed, $M(y/x)^{-1}y \leqslant_K x \leqslant_K M(x/y)y$. So, if $M(x/y) = M(y/x)^{-1}$, then $x = M(x/y)y$. □

As an immediate consequence we obtain the following result.

Corollary 3.4. — *Suppose that C is an almost Archimedean cone in a vector space V. If $x \sim_C y$ are linearly independent elements of C and there exists a unique geodesic segment connecting x and y in (C, d_C), then $M(x/y) = M(y/x)$.*

It will be convenient to make the following definition.

Definition 3.5. — *Let C be an almost Archimedean cone in a vector space V. If $x \sim_C y$ are linearly independent elements in C and $M(x/y) = M(y/x)$, then we call the unique geodesic segment (line) through x and y in $(C(x,y)^\circ, d_{C(x,y)})$* a type I geodesic segment (line) *in (C, d_C). For $x \in C \setminus \{0\}$ we call a segment of the ray, $\{tx \colon t > 0\}$, through x a* type II geodesic segment *in (C, d_C).*

Remark 3.6. — Note that if u and v are points on a type I geodesic segment, then $M(u/v) = M(v/u)$.

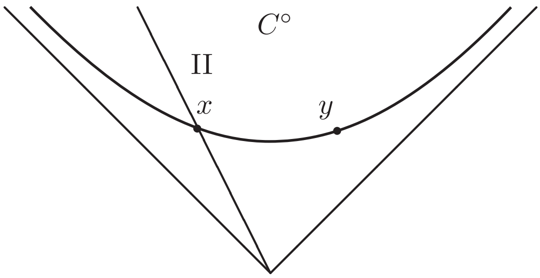

Figure 3.1. Type I and type II geodesic segments

Lemma 3.7. — *Let $K = \{\alpha u + \beta v \in W \colon \alpha, \beta \geqslant 0\}$ be a closed cone with nonempty interior in a 2-dimensional vector space W. Every type I geodesic line in (K°, d_K) is of the form*

$$\{\alpha(e^t u + e^{-t} v) \colon t \in \mathbb{R}\}$$

for some $\alpha > 0$. Moreover, for each $\alpha > 0$, the map $\gamma\colon t \mapsto \alpha(e^t u + e^{-t} v)$, $t \in \mathbb{R}$, is a geodesic path and its image is a type I geodesic line in (K°, d_K).

Proof. — Let $x \in K^\circ$. As u and v are linearly independent, there exist unique $a, b > 0$ such that $x = au + bv$. A simple linear algebra argument shows that $a = \alpha e^t$ and $b = \alpha e^{-t}$ has a unique solution with $\alpha > 0$ and $t \in \mathbb{R}$. Thus there exist unique $\alpha > 0$ and $t \in \mathbb{R}$ such that $x = \alpha(e^t u + e^{-t} v)$.

As K is 2-dimensional, there is exactly one type I geodesic line through x in (K°, d_K). So, it suffices to show for $\alpha > 0$ that the image of $\gamma\colon \mathbb{R} \to (K^\circ, d_K)$ given by,

$$\gamma(t) = \alpha(e^t u + e^{-t} v) \quad \text{for } t \in \mathbb{R},$$

is a type I geodesic line. Let $t > s$ and note that

$$e^{t-s}\gamma(s) - \gamma(t) = \alpha(e^{t-2s} - e^{-t})v \in \partial K,$$

so that $M(\gamma(t)/\gamma(s)) = e^{t-s}$. Likewise,

$$\gamma(t) - e^{s-t}\gamma(s) = \alpha(e^t - e^{2s-t})u \in \partial K$$

implies that $M(\gamma(s)/\gamma(t)) = e^{t-s}$. Thus, $d_K(\gamma(t), \gamma(s)) = t - s$ and

$$M(\gamma(t)/\gamma(s)) = M(\gamma(s)/\gamma(t)) \quad \text{for all } t > s.$$

This shows that $\gamma(\mathbb{R})$ is a unique type I geodesic line in (K°, d_K). $\square$

4. A characterization of unique geodesics

In this section we prove a geometric characterization of the unique geodesic segments in (C, d_C). As we shall see, it is quite easy to show that a type II geodesic segment is always a unique geodesics segment in the whole space (C, d_C). In general, however, additional assumptions are needed for a type I geodesic to be unique in the whole space.

Proposition 4.1. — *Let C be an almost Archimedean cone in a vector space V, If $x \in C\setminus\{0\}$ and $y = \lambda x$ for some $\lambda > 1$, then the type II geodesic segment, $\{\lambda^t x \colon 0 \leqslant t \leqslant 1\}$, connecting x and y is a unique geodesic segment in (C°, d_C).*

Proof. — Suppose that $z \in C$ is such that

$$d_C(x,z) = s d_C(x,y) \quad \text{and} \quad d_C(z,y) = (1-s) d_C(x,y).$$

As $\lambda > 1$, $d_C(x,y) = \log M(y/x) = \log \lambda$. Thus, $M(z/x) \leqslant \lambda^s$ and $M(y/z) \leqslant \lambda^{(1-s)}$. It follows from the first inequality that $z \leqslant_C (\lambda^s + \epsilon)x$

for all $\epsilon > 0$. The second inequality gives $y \leqslant_C (\lambda^{1-s}+\epsilon)z$ for all $\epsilon > 0$. As $y = \lambda x$ we find that

$$\frac{\lambda}{\lambda^{1-s}+\epsilon}x \leqslant_C z \leqslant_C (\lambda^s+\epsilon)x$$

for all $\epsilon > 0$. This implies that $z = \lambda^s x$, as C is almost Archimedean. □

Before we analyze the type I geodesic segments, we prove the following basic lemma.

Lemma 4.2. — *Let C be an almost Archimedean cone in a vector space V. If $x \sim_C y$ are linearly independent elements in C and $M(x/y) = M(y/x)$, then the straight line through x and y intersects $\partial C(x,y)$ in precisely two points.*

Proof. — Suppose that $x \sim_C y$ are linearly independent and $M(x/y) = M(y/x)$. Write $\beta = M(x/y)$. Note that $d_{C(x,y)}(x,y) = d_C(x,y) = \log\beta > 1$ by Lemma 3.1. So, $x \leqslant_{C(x,y)} \beta y$ and $y \leqslant_{C(x,y)} \beta x$, as $C(x,y)$ is closed. This implies that $x - \frac{1}{\beta}y \in \partial C(x,y)$ and $y - \frac{1}{\beta}x \in \partial C(x,y)$. Thus,

$$x' = \frac{\beta}{\beta-1}x - \frac{1}{\beta-1}y \in \partial C(x,y) \quad \text{and} \quad y' = \frac{\beta}{\beta-1}y - \frac{1}{\beta-1}x \in \partial C(x,y).$$

Obviously, x' and y' also lie on the straight line through x and y. □

Theorem 4.3. — *Let C be an almost Archimedean cone in a vector space V. Suppose that $x \sim_C y$ are linearly independent elements of C and $M(x/y) = M(y/x)$. Let $x', y' \in \partial C(x,y)$ be the points of intersection of the straight line through x and y such that x is between x' and y and y is between y' and x. The type I geodesic segment connecting x and y is a unique geodesic segment in (C, d_C) if and only if there exist no $z \in V\setminus\{0\}$ and $\epsilon > 0$ such that $x' + tz \in \partial C(x,y,z)$ and $y' + tz \in \partial C(x,y,z)$ for all $|t| < \epsilon$.*

Proof. — Suppose that $z \in V\setminus\{0\}$ and $\epsilon > 0$ are such that $x' + tz \in \partial C(x,y,z)$ and $y' + tz \in \partial C(x,y,z)$ whenever $|t| < \epsilon$. Let γ be the type I geodesic segment connecting x and y. Further let ζ be the point on γ with the property

$$d_C(x,\zeta) = \frac{1}{2}d_C(x,y) = d_C(\zeta,y).$$

For $\delta > 0$ define $\zeta_\delta = \zeta + \delta z$. Note that as ζ lies on γ, $M(x/\zeta) = M(\zeta/x)$. By Lemma 4.2 the straight line through x and ζ intersects $\partial C(x,y)$ in two points $\tilde{x}$ and ζ', as in Figure 4.1. Note that $\tilde{x}$ is a positive multiple of x' and ζ' is a positive multiple of y', as $\zeta \in \operatorname{span}\{x,y\}$.

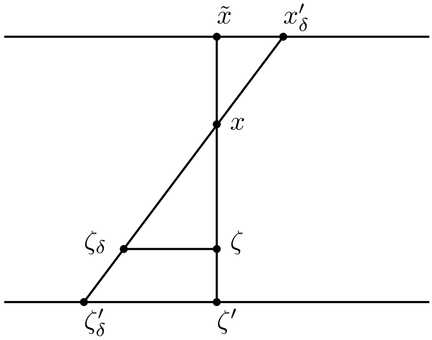

Figure 4.1. The points in the boundary

If $s > 1$ is such that $sx + (1-s)\zeta = \tilde{x}$, then

$$sx + (1-s)\zeta_\delta = sx + (1-s)\zeta + (1-s)\delta z = \tilde{x} + (1-s)\delta z.$$

As $\tilde{x}$ is a multiple of x', $\tilde{x} + \lambda z \in \partial C(x,y,z)$ for all $\lambda \in \mathbb{R}$ with $|\lambda|$ small. Thus for all $\delta > 0$ sufficiently small $x'_\delta = \tilde{x} + (1-s)\delta z \in \partial C(x,y,z)$. Similarly, if we let $t < 0$ be such that $\zeta' = tx + (1-t)\zeta$, then

$$tx + (1-t)\zeta_\delta = tx + (1-t)\zeta + (1-t)\delta z = \zeta' + (1-t)\delta z.$$

As ζ' is a multiple of y', the point $\zeta'_\delta = \zeta' + (1-t)\delta z \in \partial C(x,y,z)$ for all $\delta > 0$ small. Note also that if $x'_\delta, \zeta'_\delta \in \partial C(x,y,z)$, then $\zeta_\delta \in C(x,y,z)$.

Recall that $\tilde{x} = sx - (1-s)\zeta$ and $\zeta' = tx - (1-t)\zeta$. Using similarity of triangles in Figure 4.2, we see that

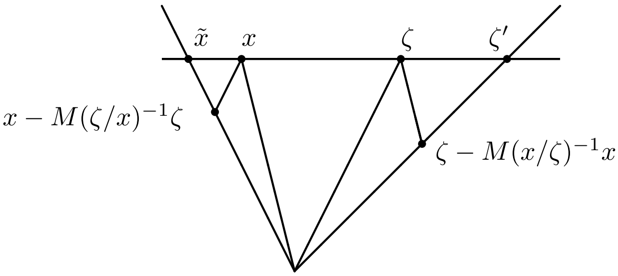

Figure 4.2. Identities

$$M(\zeta/x) = \frac{s}{s-1} \quad \text{and} \quad M(x/\zeta) = \frac{t}{t-1}.$$

Since $x'_\delta = sx - (1-s)\zeta_\delta$ and $\zeta'_\delta = tx - (1-t)\zeta_\delta$, we can derive in the same way that

$$M(\zeta_\delta/x) = \frac{s}{s-1} \quad \text{and} \quad M(x/\zeta_\delta) = \frac{t}{t-1}.$$

This implies that $d_C(x,\zeta_\delta) = d_C(x,\zeta) = \frac{1}{2}d_C(x,y)$. Analogously, for $\delta > 0$ small enough we have $d_C(\zeta_\delta, y) = d_C(\zeta, y) = \frac{1}{2}d_C(x,y)$. It now follows from Lemma 2.3 that γ is not a unique geodesic segment.

Conversely, suppose that γ is not a unique geodesic segment connecting x and y in (C, d_C). It follows from Lemmas 2.3 and 3.3 that there exists an element $\zeta \in C \setminus C(x,y)^\circ$ such that $d_C(x,\zeta) + d_C(\zeta,y) = d_C(x,y)$. As

$$\begin{aligned} d_C(x,y) &= \log M(x/y) \\ &\leqslant \log(M(x/\zeta)M(\zeta/y)) \\ &= \log M(x/\zeta) + \log M(\zeta/y) \\ &\leqslant d_C(x,\zeta) + d_C(\zeta,y) \\ &= d_C(x,y), \end{aligned}$$

we have that $d_C(x,\zeta) = \log M(x/\zeta)$ and $d_C(\zeta,y) = \log M(\zeta/y)$. Also, since $M(x/y) = M(y/x)$, we have $d_C(x,\zeta) = \log M(\zeta/x)$ and $d_C(\zeta,y) = \log M(y/\zeta)$. Write $K = C(x,y,\zeta)$. Then K is a 3-dimensional closed cone in $W = \operatorname{span}\{x,y,\zeta\}$, with x, y and ζ in its interior. Let $\varphi\colon W \to \mathbb{R}$ be a strictly positive functional. Such a functional exists, since K is a finite dimensional closed cone, see [17, Lemma 1.2.4]. Consider the bounded convex set $\Sigma^\circ_\varphi = \{w \in K^\circ : \varphi(w) = 1\}$.

Now, for the Hilbert metric δ_K on Σ°_φ and the elements $[x] = x/\varphi(x)$, $[y] = y/\varphi(y)$ and $[\zeta] = \zeta/\varphi(\zeta)$, our previous findings together with the scalar invariance of δ_K imply that

$$\begin{aligned} \delta_K([x],[y]) &\leqslant \delta_K([x],[\zeta]) + \delta_K([\zeta],[y]) \\ &= \delta_K(x,\zeta) + \delta_K(\zeta,y) \\ &= 2d_C(x,\zeta) + 2d_C(\zeta,y) \\ &= 2d_C(x,y) \\ &= \delta_K(x,y) \\ &= \delta_K([x],[y]). \end{aligned}$$

Straight line segments are geodesic segments in $(\Sigma^\circ_\varphi, \delta_K)$, see for example [29, Section 5.6]. So, it follows from the previous equality and Lemma 2.3 that there exists more than one geodesic segment in $(\Sigma^\circ_\varphi, \delta_K)$ connecting $[x]$ and $[y]$. This implies that there exists two straight line segments I_x and I_y in $\partial\Sigma^\circ_\varphi$ such that the endpoints $u \in \partial K$ and $v \in \partial K$ of the straight line segment through $[x]$ and $[y]$ lie in the relative interiors of I_x and I_y, respectively, see for example [29, Theorem 5.6.7]. Thus, u and v lie in the relative interiors of two distinct 2-dimensional faces of K. Since $W = \mathrm{span}\{x, y, \zeta\}$ is 3-dimensional, it follows that the intersection of the span of these two faces is non-trivial. To that end, let $z \neq 0$ be a point in the intersection of the spans of the faces of u and v in W. Then there exists $\eta > 0$ such that $u + \mu z \in \partial K$ and $v + \mu z \in \partial K$ whenever $|\mu| < \eta$.

As $x' = \alpha u$ and $y' = \beta v$ for some $\alpha, \beta > 0$, we conclude that there exists an $\epsilon > 0$ such that $x' + tz \in \partial K$ and $y' + tz \in \partial K$ whenever $|t| < \epsilon$. To finish the proof, it remains to be shown that $K = C(x, y, z)$. To establish this equality, we argue by contradiction that $z \notin \mathrm{span}\{x, y\}$. We know that there exists linearly independent functionals ψ_1 and ψ_2 on $V(x, y)$ such that

$$C(x, y) = \{w \in V(x, y) \colon \psi_1(w) \geqslant 0 \text{ and } \psi_2(w) \geqslant 0\}$$

and $\psi_1(x') = 0 = \psi_2(y')$. So, $\psi_1(x' + tz) = t\psi_1(z) \geqslant 0$ and $\psi_2(y' + tz) = t\psi_2(z) \geqslant 0$ for all $|t| < \epsilon$. So, $\psi_1(z) = 0 = \psi_2(z)$, and hence $z = 0$, as ψ_1 and ψ_2 are linearly independent, which is impossible. □

Note that if C is a closed cone with nonempty interior in a normed space V, then C° is a part of C. In that case, if $x, y \in C^\circ$ and $M(x/y) = M(y/x)$ then the type I geodesic segment connecting x and y is unique if and only if there exists no $z \in V \setminus \{0\}$ and $\epsilon > 0$ such that $x' + tz \in \partial C$ and $y' + tz \in \partial C$ for all $|t| < \epsilon$.

The type I unique geodesics in (C, d_C) are closely related to unique Hilbert's metric geodesics as the following lemma shows.

Lemma 4.4. — *Let C be an almost Archimedean cone in a vector space V. Suppose that $\varphi \in V^*$ is a strictly positive functional and let $\Sigma_\varphi = \{x \in C \colon \varphi(x) = 1\}$. If $x \sim_C y$ are linearly independent elements in C and $M(x/y) = M(y/x)$, then the type I geodesic connecting x and y is unique in (C, d_C) if and only if the straight line segment connecting $[x] = x/\varphi(x)$ and $[y] = y/\varphi(y)$ is a unique geodesic in $(\Sigma_\varphi \cap P_x, \delta_C)$, where P_x is the part of x.*

Proof. — It is known, see [26, Proposition 1.9], that straight lines are geodesic segments in $(\Sigma_\varphi \cap P_x, \delta_C)$. Now suppose that the type I geodesic segment connecting x and y in (C, d_C) is not unique. Then there exists $z \in P_x$ with $z \notin \mathrm{span}\{x, y\}$ and

$$d_C(x, y) = d_C(x, z) + d_C(z, y).$$

As $M(x/y) = M(y/x)$, we have

$$\begin{aligned} \log M(x/y) &= d_C(x,y) \\ &= d_C(x,z) + d_C(z,y) \\ &\geqslant \log M(x/z) + \log M(z/y) \\ &\geqslant \log M(x/y), \end{aligned}$$

so that $d_T(x,z) = \log M(x/z)$ and $d_T(z,y) = \log M(z/y)$. Using the fact that $\log M(y/x) = d_T(x,y)$, it can be shown in the same way that $d_T(x,z) = \log M(z/x)$ and $d_T(z,y) = \log M(y/z)$. Thus,

$$M(x/z) = M(z/x) \quad \text{and} \quad M(y/z) = M(z/y). \tag{4.1}$$

Writing $[u] = u/\varphi(u)$ for $u \in C \setminus \{0\}$, it now follows from (4.1) that

$$\delta_C([x],[y]) = 2d_C(x,y) = 2d_C(x,z) + 2d_C(z,y) = \delta_C([x],[z]) + \delta_C([z],[y]).$$

As $z \notin \operatorname{span}\{x,y\}$, $[z]$ is not on the straight line segment connecting $[x]$ and $[y]$. It now follows from Lemma 2.3 that there is more than one geodesic segment connecting $[x]$ and $[y]$ in $(\Sigma_\varphi \cap P_x, \delta_C)$.

Conversely, suppose that the straight line segment connecting $[x]$ and $[y]$ is not a unique Hilbert's metric geodesic in $\Sigma_\varphi \cap P_x$. Then there exists $w \in \Sigma_\varphi \cap P_x$ with $w \notin \operatorname{span}\{x,y\}$ such that

$$\delta_C([x],[y]) = \delta_C([x],w) + \delta_C(w,[y]). \tag{4.2}$$

Recall that $M(x/y) = M(y/x)$. So, for

$$\lambda = M(x/w)^{1/2} M(w/x)^{-1/2} \quad \text{and} \quad \mu = M(y/w)^{1/2} M(w/y)^{-1/2}$$

we have that

$$M(x/\lambda w) = M(\lambda w/x) \quad \text{and} \quad M(y/\mu w) = M(\mu w/y). \tag{4.3}$$

We will show by contradiction that $\lambda = \mu$. Without loss of generality assume that $\lambda < \mu$. Note that $2d_C(x,\lambda w) = \delta_C([x],w)$ and $2d_C(\mu w,y) = \delta_C(w,[y])$, so that

$$d_C(x,\lambda w) + d_C(y,\mu w) = d_C(x,y) \tag{4.4}$$

by (4.2). As $\lambda < \mu$, $M(\lambda w/\mu w) = \lambda/\mu < 1$, and hence it follows from (4.3) and (4.4) that

$$\begin{aligned} \log M(x/y) &\leqslant \log\left(M(x/\lambda w)M(\lambda w/\mu w)M(\mu w/y)\right) \\ &< \log M(x/\lambda w) + \log M(\mu w/y) \\ &= d_C(x,\lambda w) + d_C(y,\mu w) \\ &= d_C(x,y) \\ &= \log M(x/y), \end{aligned}$$

which is absurd, and hence $\lambda = \mu$.

This implies that $d_C(x,y) = d_C(x,\lambda w) + d_C(\lambda w, y)$. As $\lambda w \notin \operatorname{span}\{x,y\}$ and (C, d_C) contains the type I geodesic segment in $C(x,y)$ connecting x and y, it follows from Lemma 2.3 that this type I geodesic segment is not a unique geodesic segment in (C, d_C). □

5. Unique geodesics in unital C^*-algebras

In this section A will denote a unital C^*-algebra and $\Re(A)$ will be the real vector space of the self-adjoint elements in A. For standard results in the theory of C^*-algebras we refer the reader to [7]. In $\Re(A)$ all elements have real spectra, which yields a closed cone $A_+ = \{a \in \Re(A)\colon \sigma(a) \subseteq [0,\infty)\}$, where $\sigma(a)$ denotes the spectrum of a. It is well known that the interior, A_+°, of A_+ is the set of those $a \in A_+$ that are invertible. Moreover, A_+° is a part of A_+ and for $a, b \in A_+^\circ$ we have that

$$d_{A_+}(a,b) = \|\log(b^{-1/2}ab^{-1/2})\|,$$

see for example [2].

For $x \in A_+^\circ$ we define the linear map $\psi_x\colon \Re(A) \to \Re(A)$ by $\psi_x(a) = x^{-1/2}ax^{-1/2}$. Note that if $a \in A_+$, then $\psi_x(a) = (x^{-1/2}a^{1/2})(x^{-1/2}a^{1/2})^*$, so that $\psi_x(a) \in A_+$, and hence $\psi_x(A_+) \subseteq A_+$. In fact, ψ_x is an invertible linear map that maps A_+ onto itself. It follows from [17, Corollary 2.1.4] that ψ_x is a Thompson's metric isometry on A_+°. This isometry will be useful in the sequel.

For $a \in A_+^\circ$, we write

$$\lambda_+(a) = \max\{\lambda\colon \lambda \in \sigma(a)\} \quad \text{and} \quad \lambda_-(a) = \min\{\lambda\colon \lambda \in \sigma(a)\}.$$

Using this notation, we have for $a, b \in A_+^\circ$ that $a \leqslant \beta b$ if and only if $b^{-1/2}ab^{-1/2} \leqslant \beta e$, where e is the unit in A. So,

$$M(a/b) = \inf\{\beta > 0\colon \sigma(\beta e - b^{-1/2}ab^{-1/2}) \subseteq [0,\infty)\} = \lambda_+(b^{-1/2}ab^{-1/2}).$$

Likewise $\alpha b \leqslant a$ is equivalent to $\sigma(b^{-1/2}ab^{-1/2} - \alpha e) \subseteq [0,\infty)$, so that $M(b/a) = m(a/b)^{-1} = \lambda_-(b^{-1/2}ab^{-1/2})^{-1}$. So, for $a,b \in A_+^\circ$ we have that

$$d_{A_+}(a,b) = \log\left(\max\{\lambda_+(b^{-1/2}ab^{-1/2}), \lambda_-(b^{-1/2}ab^{-1/2})^{-1}\}\right).$$

We have the following characterization for the unique geodesic in A_+°.

Theorem 5.1. — *Let A be a unital C^*-algebra. If x ad y are linearly independent elements of A_+°, then there exists a unique geodesic segment connecting x and y in (A_+°, d_{A_+}) if and only if $\sigma(x^{-1/2}yx^{-1/2}) = \{\beta^{-1},\beta\}$ for some $\beta > 1$.*

Proof. — Note that there is a unique geodesic segment connecting x and y in A_+° if and only if there is a unique geodesic segment connecting $\psi_x(x) = e$ and $\psi_x(y) = x^{-1/2}yx^{-1/2}$, as ψ_x is an isometry. Thus, it suffices to show that there is a unique geodesic segment connecting e and $z \in A_+^\circ$ if and only if $\sigma(z) = \{\beta^{-1},\beta\}$ for some $\beta > 1$ whenever e and z are linearly independent.

Suppose first that there exists a unique geodesic segment connecting z and e in A_+°, where z and e are linearly independent. It follows from Corollary 3.4 that $\lambda_+(z) = M(z/e) = M(e/z) = \lambda_-(z)^{-1}$. This yields the inclusions,

$$\{\lambda_+(z)^{-1}, \lambda_+(z)\} \subseteq \sigma(z) \subseteq [\lambda_+(z)^{-1}, \lambda_+(z)].$$

Suppose that there exists $\lambda \in \sigma(z)$ such that $\lambda_+(z)^{-1} < \lambda < \lambda_+(z)$. Let $\delta > 0$ be such that $\lambda_+(z)^{-1} < \lambda - \delta < \lambda + \delta < \lambda_+(z)$, then there is a continuous function $f_\delta\colon [\lambda_+(z)^{-1}, \lambda_+(z)] \to [0,1]$ with $f_\delta(\lambda) = 1$ and $\mathrm{supp}(f_\delta) \subseteq [\lambda - \delta, \lambda + \delta]$. Furthermore, let $g\colon [\lambda_+(z)^{-1}, \lambda_+(z)] \to [\lambda_+(z)^{-1}, \lambda_+(z)]$ be the identity map. Using the functional calculus $\phi_z\colon C(\sigma(z)) \to C^*(z,e)$ we can define $\zeta_\delta = \phi_z(f_\delta) \in C^*(z,e)$, where $C^*(z,e)$ is the C^*-algebra generated by z and e. For $\epsilon > 0$ we have that $\phi_z(g + \epsilon f_\delta) = z + \epsilon\zeta_\delta$ with $g + \epsilon f_\delta \geqslant 0$. So,

$$\phi_z((g+\epsilon f_\delta)^{1/2}) = (z + \epsilon\zeta_\delta)^{1/2}$$

and $\sigma((z+\epsilon\zeta_\delta)^{1/2}) = \{(g(t)+\epsilon f_\delta(t))^{1/2} : t \in \sigma(z)\}$ by the spectral mapping theorem. So, by choosing $\epsilon > 0$ such that $\lambda + \delta + \epsilon < \lambda_+(z)$, we find that

$$\{\lambda_+(z)^{-1/2}, \lambda_+(z)^{1/2}\} \subseteq \sigma((z+\epsilon\zeta_\delta)^{1/2}) \subseteq [\lambda_+(z)^{-1/2}, \lambda_+(z)^{1/2}]$$

and $d_{A_+}((z+\epsilon\zeta_\delta)^{1/2}, e) = \frac{1}{2}\log(\lambda_+(z))$ for all $\epsilon > 0$ sufficiently small. As $\zeta_\delta \in C^*(z,e)$,

$$d_{A_+}((z+\epsilon\zeta_\delta)^{1/2}, z) = d_{A_+}(z^{-1}(z+\epsilon\zeta_\delta)^{1/2}, e).$$

Now, define $\xi \in C(\sigma(z))$ by $\xi(t) = t^{-1}(t + \epsilon f_\delta(t))^{1/2}$. Again by the functional calculus we have $\xi(z) = z^{-1}(z + \epsilon\zeta_\delta)^{1/2}$. Moreover,

$$\xi(t) = \begin{cases} t^{-1/2} & t \in [\lambda_+(z)^{-1}, \lambda_+(z)] \setminus [\lambda - \delta, \lambda + \delta], \\ t^{-1/2}\left(1 + \frac{\epsilon f_\delta(t)}{t}\right)^{1/2} & t \in [\lambda - \delta, \lambda + \delta], \end{cases}$$

and hence $\xi(t) \leqslant \max\{\lambda_+(z)^{1/2}, (\lambda - \delta)^{-1/2}(1 + \frac{\epsilon}{\lambda - \delta})^{1/2}\}$ for all $\lambda_+(z)^{-1} \leqslant t \leqslant \lambda_+(z)$. For sufficiently small $\epsilon > 0$ we can ensure the inequality $\xi(t) \leqslant \lambda_+(z)^{1/2}$ on $\sigma(z)$. As $\xi(t) \geqslant t^{-1/2}$ and $\xi(\lambda_+(z)^{-1}) = \lambda_+(z)^{1/2} = \xi(\lambda_+(z))^{-1}$,

$$\{\lambda_+(z)^{-1/2}, \lambda_+(z)^{1/2}\} \subseteq \sigma(\xi(z)) \subseteq [\lambda_+(z)^{-1/2}, \lambda_+(z)^{1/2}].$$

This implies that $d_{A_+}((z + \epsilon\zeta_\delta)^{1/2}, z) = d_{A_+}(\xi(z), e) = \frac{1}{2}\log(\lambda_+(z))$ for all $\epsilon > 0$ sufficiently small. Note that $\zeta_\delta \neq 0$, as $f_\delta \neq 0$, which contradicts the fact that there is a unique geodesic segment connecting z and e by Lemma 2.3. We conclude that $\sigma(z) = \{\lambda_+(z)^{-1}, \lambda_+(z)\}$.

Conversely, if $z \in A_+^\circ$ and e are linearly independent and $\sigma(z) = \{\beta^{-1}, \beta\}$ for some $\beta > 1$, then the function $f : \sigma(z) \to \{0, 1\}$ defined by $f(\beta^{-1}) = 1$ and $f(\beta) = 0$ is continuous, and $\beta^{-1}f + \beta(\mathbf{1} - f)$ is the identity function on $\sigma(z)$. So, for $\pi = \phi_z(f)$, it follows that $\beta^{-1}\pi + \beta(e - \pi) = z$ by the functional calculus $\phi_z : C(\sigma(z)) \to C^*(z, e)$. Now consider the 2-dimensional closed cone $A_+ \cap \operatorname{span}\{e, z\}$, which we can identify with $\mathbb{R}^2_+$. It follows that

$$z - m(z/e)e = z - \beta^{-1}e = (\beta - \beta^{-1})(e - \pi) \in \partial\left(A_+ \cap \operatorname{span}\{e, z\}\right)$$

and

$$e - m(e/z)z = e - \beta^{-1}z = (1 - \beta^{-2})\pi \in \partial\left(A_+ \cap \operatorname{span}\{e, z\}\right).$$

So, for some $\alpha_1, \alpha_2 > 0$ we have that

$$e' = \alpha_1(1 - \beta^{-2})\pi \quad \text{and} \quad z' = \alpha_2(\beta - \beta^{-1})(e - \pi)$$

are the endpoints in $\partial\left(A_+ \cap \operatorname{span}\{e, z\}\right)$ of the straight line segment through e and z. Suppose that there is a $v \in \Re(A)$ and an $\epsilon > 0$ such that $e' + tv \in \partial A_+$ and $z' + tv \in \partial A_+$ for $|t| < \epsilon$, or equivalently, there is a $\delta > 0$ such that $|t| < \delta$ implies $\pi + tv \in \partial A_+$ and $(e - \pi) + tv \in \partial A_+$. By the Gelfand-Naimark theorem, we can view A as a C^*-subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. So, let $P\colon \mathcal{H} \to \mathcal{H}$ be the projection representing π and $V\colon \mathcal{H} \to \mathcal{H}$ be the operator representing v. We now have the identities

$$P = \begin{pmatrix} 0 & 0 \\ 0 & I_2 \end{pmatrix}, \quad I - P = \begin{pmatrix} I_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} V_1 & V_2 \\ V_2^* & V_4 \end{pmatrix}$$

relative to $\mathcal{H} = \ker(P) \oplus \operatorname{ran}(P)$. Since $P - tV \geqslant 0$, it follows that for each $x_1 \oplus x_2 \in \ker(P) \oplus \operatorname{ran}(P)$ we have

$$-t\langle V_1(x_1), x_1\rangle - t\langle V_2(x_2), x_1\rangle - t\langle V_2^*(x_1), x_2\rangle - t\langle V_4(x_2), x_2\rangle + \|x_2\|^2 \geqslant 0$$

whenever $|t| < \delta$. If we take $0 \neq x_1 \in \ker(P)$ and $x_2 = 0$, then $\langle V_1(x_1), x_1\rangle = 0$, and hence $V_1 = 0$, since V_1 is self-adjoint. Similarly, the inequality obtained from $(I - P) - tV \geqslant 0$ for all $|t| < \delta$ implies that $V_4 = 0$. Now let $0 \neq x_2 \in \operatorname{ran}(P)$ and $x_1 = \alpha V_2(x_2)$, which is an element of $\ker(P)$ for an arbitrary $\alpha \in \mathbb{R}$. Then our findings yield

$$-2t\alpha\langle V_2(x_2), V_2(x_2)\rangle + \|x_2\|^2 = -2t\alpha\|V_2(x_2)\|^2 + \|x_2\|^2 \geqslant 0$$

whenever $|t| < \delta$. It follows that $V_2 = 0$, and therefore also $V_2^* = 0$. We conclude from Theorem 4.3 that the geodesic segment connecting e and z is unique. So, if $x, y \in A_+^\circ$ are linearly independent with $\sigma(x^{-1/2}yx^{-1/2}) = \{\beta^{-1}, \beta\}$ for some $\beta > 1$, we have shown that the geodesic segment connecting e and $x^{-1/2}yx^{-1/2}$ is unique, and hence the geodesic segment connecting $x = \psi_{x^{-1}}(e)$ and $\psi_{x^{-1}}(x^{-1/2}yx^{-1/2}) = y$ is unique. ☐

We have a similar result for Hilbert's metric geodesic segments in Σ_φ° for some strictly positive functional φ on A. Such a functional exists if A is separable. Indeed, in that case, the *state space*,

$$\mathcal{S}_A = \{\psi \in A^* : \psi \geqslant 0 \text{ and } \|\psi\| = 1\}$$

is w^*-metrizable and therefore, since it is also w^*-compact by the Banach-Alaoglu's theorem, we must have that $\mathcal{S}_A$ is separable. For a w^*-dense sequence $(\varphi_n)_n$ in $\mathcal{S}_A$ we can define the functional

$$\varphi = \sum_{n=1}^{\infty} 2^{-n}\varphi_n.$$

Clearly, this defines a positive functional with $\|\varphi\| = 1$. Since we have

$$\|x\| = \sup\{\psi(x) : \psi \in \mathcal{S}_A\}$$

for all $x \geqslant 0$, it follows that φ is strictly positive. For more details, see [7, §5.1] and [7, §5.15].

Theorem 5.2. — *Let A be a unital C^*-algebra with a strictly positive functional φ. For distinct $x, y \in \Sigma_\varphi^\circ$, there exists a unique Hilbert's metric geodesic segment connecting x and y in Σ_φ° if and only if $\sigma(x^{-1/2}yx^{-1/2}) = \{\alpha, \beta\}$ for some $\beta > \alpha > 0$.*

Proof. — Suppose the straight line segment connecting x and y is the unique Hilbert's metric geodesic segment in Σ_φ°. For

$$\lambda = M(x/y)^{1/2} M(y/x)^{-1/2}$$

we have

$$M(x/\lambda y) = M(x/y)^{1/2} M(y/x)^{1/2} = M(\lambda y/x).$$

So, there exists a unique type I geodesic segment connecting x and λy in $A_+^\circ \cap \operatorname{span}\{x, y\}$. By Lemma 4.4 this geodesic segment is unique in (A_+°, d_{A_+}). Now Theorem 5.1 implies that

$$\sigma(x^{-1/2}(\lambda y)x^{-1/2}) = \lambda\sigma(x^{-1/2}yx^{-1/2}) = \{\beta^{-1}, \beta\}$$

for some $\beta > 1$, or equivalently, $\sigma(x^{-1/2}yx^{-1/2}) = \{\lambda^{-1}\beta^{-1}, \lambda^{-1}\beta\}$.

Conversely, if $\sigma(x^{-1/2}yx^{-1/2}) = \{\alpha, \beta\}$ for some $\beta > \alpha > 0$. Then for $\mu = \sqrt{\alpha\beta}$ and $\xi = \sqrt{\beta/\alpha}$, we can write

$$\sigma(x^{-1/2}(\mu^{-1}y)x^{-1/2}) = \{\xi^{-1}, \xi\}.$$

So, Theorem 5.1 implies that the Thompson's metric geodesic segment connecting x and $\mu^{-1}y$ in A_+° is unique. So, by Lemma 4.4 the straight line segment connecting x and y in Σ_φ° is the unique Hilbert's metric geodesic segment. □

6. Unique geodesics in symmetric cones

Recall that the interior K° of a closed cone K in a finite-dimensional inner-product space $(V, \langle\cdot,\cdot\rangle)$ is called a *symmetric cone* if K the *dual cone*, $K^* = \{y \in V : \langle y, x\rangle \geqslant 0 \text{ for all } x \in K\}$ satisfies $K^* = K$, and $\operatorname{Aut}(K) = \{A \in \operatorname{GL}(V) : A(K) = K\}$ acts transitively on K°. A prime example is the cone of positive definite Hermitian matrices. In this section we prove a characterization of the unique Thompson metric geodesics in symmetric cones K°, which is similar to the one given in Theorem 5.1.

It is well known that the symmetric cones in finite dimensions are precisely the interiors of the cones of squares of Euclidean Jordan algebras. This fundamental result is due to Koecher [15] and Vinberg [32]. A detailed exposition of the theory of symmetric cones can be found in the book by Faraut and Korányi [11]. We will follow their notation and terminology. Recall that a *Euclidean Jordan algebra* is a finite-dimensional real inner-product space $(V, \langle\cdot,\cdot\rangle)$ equipped with a bilinear product $(x, y) \mapsto x \bullet y$ from $V \times V$ into V such that for each $x, y \in V$:

(1) $x \bullet y = y \bullet x$,

(2) $x \bullet (x^2 \bullet y) = x^2 \bullet (x \bullet y)$, and
(3) for each $x \in V$, the linear map $L(x)\colon V \to V$ given by $L(x)y = x \bullet y$ satisfies

$$\langle L(x)y, z\rangle = \langle y, L(x)z\rangle \quad \text{for all } y, z \in V.$$

In general a Euclidean Jordan algebra is not associative, but it is commutative. We denote the *unit* in a Euclidean Jordan algebra by e. An element $c \in V$ is called an *idempotent* if $c^2 = c$. A set $\{c_1, \dots, c_k\}$ is called a *complete system of orthogonal idempotents* if

(1) $c_i^2 = c_i$ for all i,
(2) $c_i \bullet c_j = 0$ for all $i \neq j$, and
(3) $c_1 + \cdots + c_k = e$.

The spectral theorem [11, Theorem III.1.1] says that for each $x \in V$ there exist unique real numbers $\lambda_1, \dots, \lambda_k$, all distinct, and a complete system of orthogonal idempotents $c_1, \dots, c_k$ such that $x = \lambda_1 c_1 + \cdots + \lambda_k c_k$. The numbers λ_i are called the *eigenvalues* of x. The *spectrum* of x is denoted by $\sigma(x) = \{\lambda\colon \lambda \text{ eigenvalue of } x\}$, and we write

$$\lambda_+(x) = \max\{\lambda\colon \lambda \in \sigma(x)\} \quad \text{and} \quad \lambda_-(x) = \min\{\lambda\colon \lambda \in \sigma(x)\}.$$

It is known, see for example [11, Theorem III.2.2], that $x \in K^\circ$ if and only if $\sigma(x) \subseteq (0, \infty)$, which is equivalent to $L(x)$ being positive definite. So, one can use the spectral decomposition, $x = \lambda_1 c_1 + \cdots + \lambda_k c_k$, of $x \in K^\circ$, to define a spectral calculus, e.g.,

$$x^{-1/2} = \lambda_1^{-1/2} c_1 + \cdots + \lambda_k^{-1/2} c_k.$$

For $x \in V$ the linear mapping, $P(x) = 2L(x)^2 - L(x^2)$, is called the *quadratic representation* of x. Note that $P(x^{-1/2})x = e$ for all $x \in K^\circ$. It is known that $P(x^{-1}) = P(x)^{-1}$ for all $x \in K^\circ$ and $P(x) \in \mathrm{Aut}(K)$ whenever $x \in K^\circ$, see [11, Proposition III.2.2]. So, $P(x)$ is an isometry of (K°, d_K) if $x \in K^\circ$ by [17, Corollary 2.1.4]. For $x, y \in K^\circ$ we write

$$\lambda_+(x, y) = \lambda_+(P(y^{-1/2})x) \quad \text{and} \quad \lambda_-(x, y) = \lambda_-(P(y^{-1/2})x).$$

Note that for $x, y \in K^\circ$, $x \leqslant \beta y$ if and only if $0 \leqslant \beta e - P(y^{-1/2})x$, and hence

$$M(x/y) = \lambda_+(x, y).$$

Similarly, $\alpha y \leqslant x$ is equivalent with $0 \leqslant P(y^{-1/2})x - \alpha e$, and hence

$$M(y/x)^{-1} = m(x/y) = \lambda_-(x, y).$$

So, for $x, y \in K^\circ$ the Thompson metric distance is given by

$$d_K(x, y) = \log\left(\max\{\lambda_+(x, y), \lambda_-(x, y)^{-1}\}\right).$$

The following lemma is Exercise 3.3 in [11]. For the sake of completeness, we will give a proof.

Lemma 6.1. — *Let V be a Euclidean Jordan algebra with symmetric cone K°. For $x, y \in K$ we have $\langle x, y\rangle = 0$ if and only if $x \bullet y = 0$.*

Proof. — Without loss of generality, we may assume that $x, y \in K \setminus \{0\}$. Suppose that $\langle x, y\rangle = 0$. Write $y = v^2$ for some $v \in V$. It follows that

$$\langle x, v \bullet v\rangle = \langle L(v)x, v\rangle = \langle L(x)v, v\rangle = 0.$$

Since $L(x) : V \to V$ is a self-adjoint positive semi-definite linear map, we know that $L(x)^{1/2}$ is well defined, which yields

$$\|L(x)^{1/2}v\|^2 = \langle L(x)^{1/2}v, L(x)^{1/2}v\rangle = 0.$$

It follows that $L(x)v = L(x)^{\frac{1}{2}}(L(x)^{\frac{1}{2}}v) = 0$. However, $L(y)$ and $L(v)$ commute, so that

$$\begin{aligned} \langle L(x)y, y\rangle &= \langle L(x)y, L(v)v\rangle \\ &= \langle L(v)(L(y)x), v\rangle \\ &= \langle L(y)(L(v)x), v\rangle \\ &= \langle L(y)(L(x)v), v\rangle = 0. \end{aligned}$$

Using the same argument as above, we deduce that $L(x)y = x \bullet y = 0$.

Obviously, if $x \bullet y = 0$, then $\langle x, y\rangle = \langle e, L(x)y\rangle = \langle e, x \bullet y\rangle = 0$. □

We can now prove the analogue of Theorem 5.1 for symmetric cones.

Theorem 6.2. — *Let V be a Euclidean Jordan algebra with symmetric cone K°. If $x, y \in K^\circ$ are linearly independent, then there exists a unique geodesic segment connecting x and y in (K°, d_K) if and only if $\sigma(P(y^{-\frac{1}{2}})x) = \{\beta^{-1}, \beta\}$ for some $\beta > 1$.*

Proof. — Suppose that there exists a unique geodesic segment connecting two linearly independent elements $x, y \in K^\circ$. As $y^{-1/2} \in K^\circ$, $P(y^{-1/2}) \in \mathrm{Aut}(K)$ (see [11, Theorem III.2.2]), and hence $P(y^{-1/2})$ is an isometry of (K°, d_K) by [17, Corollary 2.1.4]. Thus, there exists a unique geodesic segment connecting x and y in K° if and only if there is a unique geodesic connecting $P(y^{-1/2})y = e$ and $P(y^{-1/2})x$. So, it suffices to show that if there exists a unique geodesic segment connecting e and $z \in K^\circ$ with e and z linearly independent, then $\sigma(z) = \{\beta, 1/\beta\}$ for some $\beta > 1$.

Let $z = \lambda_1 c_1 + \cdots + \lambda_k c_k$ be the spectral decomposition of z. We have that

$$\lambda_+(z) = M(z/e) \quad \text{and} \quad \lambda_-(z) = m(z/e) = M(e/z)^{-1}.$$

As there exists a unique geodesic segment connecting z and e, we have that $M(z/e) = M(e/z)$ by Corollary 3.4. So, if we write $r = \lambda_+(z)$, then

$$\{1/r, r\} \subseteq \sigma(z) \subseteq [1/r, r].$$

Note that $d_K(e, z) = \log M(z/e) > 0$, and hence $r > 1$.

Suppose there exists $\lambda_i \in \sigma(z)$ with $1/r < \lambda_i < r$. For $\epsilon > 0$ we define $z_\epsilon = (z + \epsilon c_i)^{1/2}$. If $\lambda_i + \epsilon < r$, we can use the spectral decomposition of z to find that $d_K(e, z_\epsilon) = \frac{1}{2}\log r$. Also, note that $d_K(z_\epsilon, z) = d_K(P(z^{-1/2})z_\epsilon, e)$ and

$$P(z^{-1/2})z_\epsilon = \lambda_1^{-1/2}c_1 + \cdots + \frac{(\lambda_i + \epsilon)^{1/2}}{\lambda_i}c_i + \cdots + \lambda_k^{-1/2}c_k.$$

As $0 < 1/r < \lambda_i < r$, we have that $r^{-1/2} < \lambda_i^{-1/2} < (\lambda_i + \epsilon)^{1/2}/\lambda_i$ and $r\lambda_i^2 - \lambda_i > 0$. So, for $0 < \epsilon < r\lambda_i^2 - \lambda_i$,

$$r^{-1/2} < \frac{(\lambda_i + \epsilon)^{1/2}}{\lambda_i} < r^{1/2}.$$

Thus, $d_K(z_\epsilon, z) = \frac{1}{2}\log r$ and $d_K(e, z_\epsilon) = \frac{1}{2}\log r$ for all $0 < \epsilon < \min\{r - \lambda_i, r\lambda_i^2 - \lambda_i\}$. This is impossible by Lemma 2.3 and therefore $\sigma(z) = \{1/r, r\}$.

Conversely, suppose that z and e in K° are linearly independent, and $\sigma(z) = \{1/\beta, \beta\}$ for some $\beta > 1$. Then we have the spectral decomposition $z = \beta^{-1}c_1 + \beta c_2$. Note that, as $\sigma(z) = \{\beta, 1/\beta\}$, $M(z/e) = M(e/z) = \beta > 1$. So, the straight line through e and z intersects ∂K in 2 points e' and x' by Lemma 4.2. In fact, $(\beta - 1/\beta)c_1 = \beta e - z \in \partial K$ and $(\beta^2 - 1)c_2 = \beta z - e \in \partial K$, so that $e' = \lambda_1 c_1$ and $z' = \lambda c_2$ for some $\lambda_1, \lambda_2 > 0$. Suppose there exist $\epsilon > 0$ and $v \in V$ such that $c_1 + tv \in \partial K$ and $c_2 + tv \in \partial K$ whenever $|t| < \epsilon$. So, for $|t| < \epsilon$ the operator $L(c_1 + tv)$ is positive semi-definite, which yields

$$0 \leqslant \langle L(c_1 + tv)c_2, c_2\rangle = t\langle v \bullet c_2, c_2\rangle = t\langle v, c_2\rangle,$$

and hence $\langle v, c_2\rangle = 0$. It follows from Lemma 6.1 that $v \bullet c_2 = 0$. In a similar way we find that $v \bullet c_1 = 0$, so $v = v \bullet e = v \bullet (c_1 + c_2) = 0$ and we conclude from Theorem 4.3 that the geodesic segment connecting e and z in (K°, d_K) is unique. We have shown that if $x, y \in K^\circ$ are linearly independent and $\sigma(P(y^{-1/2})x) = \{\beta, 1/\beta\}$ for some $\beta > 1$, that there exists a unique geodesic segment connecting e and $P(y^{-1/2})x$ in (K°, d_K). Equivalently, there is a unique geodesic segment connecting $y = P(y^{1/2})e$ and $x = P(y^{1/2})(P(y^{-1/2})x)$. □

As for the characterization of unique geodesic segments in $(\Sigma_\varphi^\circ, \delta_K)$ for some strictly positive functional φ on V, we also have an analogue of Theorem 5.2. The proof is completely analogous and is left to the reader.

Theorem 6.3. — *Let V be a Euclidean Jordan algebra with symmetric cone K°, $\varphi \in K^\circ$, and $\Sigma_\varphi^\circ = \{x \in K^\circ \colon \langle \varphi, x\rangle = 1\}$. For distict $x, y \in \Sigma_\varphi^\circ$ there exists a unique geodesic segment connecting x and y in $(\Sigma_\varphi^\circ, \delta_K)$ if and only if $\sigma(P(y^{-1/2})x) = \{\alpha, \beta\}$ for some $\beta > \alpha > 0$.*

7. Quasi-isometric embeddings into normed spaces

In this section we will study isometric and quasi-isometric embeddings of Thompson's metric spaces (C°, d_C), where C° is the interior of a finite-dimensional closed cone, into finite-dimensional normed spaces. Recall that a map f from a metric space (X, d_X) into a metric space (Y, d_Y) is a called a *quasi-isometric embedding* if there exist constants $\alpha \geqslant 1$ and $\beta \geqslant 0$ such that

$$\frac{1}{\alpha} d_X(x,y) - \beta \leqslant d_Y(f(x), f(y)) \leqslant \alpha d_X(x,y) + \beta \quad \text{for all } x, y \in X.$$

It is known that if C is a polyhedral cone with N facets, then (C°, d_C) can be isometrically embedded into $(\mathbb{R}^N, \|\cdot\|_\infty)$, see [17, Lemma 2.2.2]. We will show that polyhedral cones are the only ones that allow a quasi-isometric embedding into a finite-dimensional normed space.

A similar result exists for Hilbert's metric spaces and was proved by Colbois and Verovic in [6]. The idea of their proof can be traced back to [12] and relies on properties of the Gromov product in Hilbert's metric spaces proved in [14, Theorem 5.2]. It turns out that the usual Gromov product does not have the right behavior in Thompson's metric spaces. The following generalized Gromov product, however, will be useful.

Definition 7.1. — *Let (X, d_X) be a metric space. For $p \in X$ and $\eta > 0$ the* generalized Gromov product *on $X \times X$ is given by*

$$(x|y)_{p,\eta} = \frac{1}{2}\left(d_X(x,p) + d_X(y,p) - \eta d_X(x,y)\right).$$

Note that for $\eta = 1$ we recover the usual Gromov product. It turns out that for Thompson's metric the generalized Gromov product where $\eta = 2$ is relevant.

The following lemma is a slight generalization of [6, Proposition 2.1].

Lemma 7.2. — *Let (X, d_X) be a metric space that can be quasi-isometrically embedded into a finite-dimensional normed space $(V, \|\cdot\|)$. If there exist $p \in X$, a constant $\eta > 0$, and sequences $(x_k^i)_k$ for $i = 1, \dots, m$ such that $d_X(x_k^i, p) = k$ for all $i = 1, \dots, m$ and $k \geqslant 1$, and*

$$\limsup_{k\to\infty} (x_k^i | x_k^j)_{p,\eta} \leqslant C_{ij} < \infty$$

for all $i \neq j$*, then there exist* $v^1, \ldots, v^m \in V$ *satisfying*

$$\|v^i - v^j\| \geqslant \frac{2}{\alpha\eta} \quad \text{for all } i \neq j$$

and $1/\alpha \leqslant \|v^i\| \leqslant \alpha$ *for all* i*, where* $\alpha \geqslant 1$ *is the constant from the quasi-isometry.*

Proof. — Let $f\colon X \to V$ be a quasi-isometric embedding. We may as well assume that $f(0) = 0$, as the map $g(x) = f(x) - f(p)$ is also a quasi-isometric embedding. Now for $i \neq j$ there exists a number $N \geqslant 1$ and a constant $R < \infty$ such that

$$d_X(x_k^i, x_k^j) \geqslant \frac{d_X(x_k^i, p) + d_X(x_k^j, p) - R}{\eta} = \frac{2k - R}{\eta}$$

whenever $k \geqslant N$. Define the vectors $u_k^i = \frac{1}{k} f(x_k^i) \in V$ for all $k \geqslant 1$ and $i = 1, \ldots, m$. It follows that

$$\|u_k^i - u_k^j\| = \frac{1}{k}\|f(x_k^i) - f(x_k^j)\| \geqslant \frac{1}{\alpha k} d_X(x_k^i, x_k^j) - \frac{\beta}{k} \geqslant \frac{2}{\alpha\eta} - \frac{1}{k}\left(\frac{R}{\alpha\eta} + \beta\right)$$

whenever $k \geqslant 1$ and $i \neq j$. Also, we have that

$$\frac{1}{\alpha} - \frac{\beta}{k} \leqslant \|u_k^i\| = \frac{1}{k}\|f(x_k^i) - f(p)\| \leqslant \alpha + \frac{\beta}{k}$$

for all $k \geqslant 1$ and $1 \leqslant i \leqslant m$. Since V is finite-dimensional, there are convergent subsequences $(u_{k_j}^i)_j$ with limits v^i for $i = 1, \ldots, m$. The vectors v^i have the desired properties. □

Lemma 7.2 has the following consequence.

Corollary 7.3. — *If* (X, d_X) *is a metric space and there exist* $p \in X$*, a constant* $\eta > 0$*, and sequences* $(x_k^i)_k$ *in* X *for* $i = 1, 2, \ldots$ *such that* $d(x_k^i, p) = k$ *for all* $i \geqslant 1$ *and* $k \geqslant 1$*, and*

$$\limsup_{k \to \infty} (x_k^i | x_k^j)_{p,\eta} \leqslant C_{ij} < \infty$$

for all $i \neq j$*, then* (X, d_X) *cannot be quasi-isometrically embedded into a finite-dimensional normed space.*

Proof. — Suppose that (X, d_X) can be quasi-isometrically embedded into a finite-dimensional normed space $(V, \|\cdot\|)$ using a quasi-isometry with constants $\alpha \geqslant 1$ and $\beta \geqslant 0$. As the set $S = \{x \in V\colon 1/\alpha \leqslant \|x\| \leqslant \alpha\}$ is compact, the maximum number of points in S whose pairwise distance is at least $2/(\alpha\eta)$ is bounded by a constant $M_{\alpha,\eta} < \infty$. This contradicts Lemma 7.2. □

We will see that if C is not a polyhedral cone, then we can find infinitely many sequences $(x^i_k)_k$ in (C°, d_C) satisfying the conditions in Corollary 7.3 with $\eta = 2$. We will need the following auxiliary lemma.

Lemma 7.4. — *Let C be a closed cone with nonempty interior in a finite-dimensional normed space $(V, \|\cdot\|)$. If $S \subseteq C^\circ$ is a norm compact subset and $(x_k)_k$ is a sequence in C° such that $x_k \to x \in \partial C$, then there exists $N \geqslant 1$ such that $d_C(x_k, s) = \log M(s/x_k)$ for all $s \in S$ and all $k \geqslant N$.*

Proof. — Let $u \in C^\circ$ and $\Sigma^*_u = \{\varphi \in C^* \colon \varphi(u) = 1\}$. As C^* is a closed cone with nonempty interior in V^*, we know from [17, Lemma 1.2.4] that Σ^*_u is a compact set of V^*, and hence there exists a constant $M_1 > 0$ such that $\|\varphi\| \leqslant M_1$ for all $\varphi \in \Sigma^*_u$. Define functions $f \colon C^\circ \to \mathbb{R}$ and $g \colon C^\circ \to \mathbb{R}$ by

$$f(x) = \min_{\varphi\in\Sigma^*_u} \varphi(x) \quad \text{and} \quad g(x) = \max_{\varphi\in\Sigma^*_u} \varphi(x) \quad \text{for } x \in C^\circ.$$

The topology on C° generated by d_C is the same as the norm topology by [17, Corollary 2.5.6]. Note that there exists a constant $M_2 > 0$ such that $\|s\| \leqslant M_2$ for all $s \in S$, as S is compact. Thus, $g(s) \leqslant \max_{\varphi\in\Sigma^*_u} \|\varphi\|\|s\| \leqslant M_1M_2$ for all $s \in S$. Also, if $|f(x) - f(y)| = f(x) - f(y)$ and $f(y) = \psi(y)$ with $\psi \in \Sigma^*_u$, then $|f(x) - f(y)| = f(x) - f(y) \leqslant \psi(x) - \psi(y) \leqslant \|\psi\|\|x - y\| \leqslant M_1\|x - y\|$. Thus, f is a continuous function, and hence $\delta = \min_{s\in S} f(s) > 0$.

For $x_k \in C^\circ$ we know, by [17, Lemma 1.2.1], that

$$\sup_{s\in S} M(x_k/s) = \sup_{s\in S} \left(\max_{\varphi\in\Sigma^*_u} \frac{\varphi(x_k)}{\varphi(s)} \right) \leqslant \frac{g(x_k)}{\delta} \leqslant \frac{M_1M_2}{\delta}.$$

As $x \in \partial C$, there exists $\rho \in \Sigma^*_u$ such that $\rho(x) = 0$. This implies that there exists $N \geqslant 1$ such that $\delta/\rho(x_k) > M_1M_2/\delta$ for all $k \geqslant N$, and hence

$$M(s/x_k) = \max_{\varphi\in\Sigma^*_u} \frac{\varphi(s)}{\varphi(x_k)} \geqslant \max_{\varphi\in\Sigma^*_u} \frac{f(s)}{\phi(x_k)} \geqslant \frac{\delta}{\rho(x_k)} > M(x_k/s)$$

for all $s \in S$ and $k \geqslant N$. Thus, $d_C(x_k, s) = \log M(s/x_k)$ for all $s \in S$ whenever $k \geqslant N$. $\square$

The following result is the analogue of [14, Theorem 5.2] for Thompson's metric.

Proposition 7.5. — *Let C be a closed cone with nonempty interior in a finite-dimensional normed space $(V, \|\cdot\|)$, $\varphi \in C^*$ strictly positive, and $\Sigma^\circ_\varphi = \{x \in C^\circ \colon \varphi(x) = 1\}$. Suppose that $(x_n)_n$ and $(y_n)_n$ are convergent sequences in Σ°_φ with $x_n \to x \in \partial C$ and $y_n \to y \in \partial C$. If $tx + (1-t)y \in C^\circ$*

for all $0 < t < 1$ *and* $p \in C^\circ$*, then*

$$\limsup_{k\to\infty} (x_k|y_k)_{p,2} < \infty.$$

Proof. — For $k \geqslant 1$ let $z_k = \frac{1}{2}(x_k + y_k)$ and $z = \frac{1}{2}(x+y) \in C^\circ$. Note that $z_k \to z$ as $k \to \infty$. Let $\epsilon > 0$ be such that the closed norm ball, B_ϵ, with radius ϵ and center z is contained in C°. There exists a number $M \geqslant 1$ such that $z_k \in B_\epsilon$ for all $k \geqslant M$. By Lemma 7.4 there also exists a number $N \geqslant M$ such that $d_C(x_k, s) = \log M(s/x_k)$ and $d_C(y_k, s) = \log M(s/y_k)$ for all $k \geqslant N$ and all $s \in B_\epsilon$. Let x'_k and y'_k be the points of intersection of the straight line through x_k and y_k with ∂C such that x_k is between x'_k and y_k, and y_k is between y'_k and x_k. As was shown in the proof of Theorem 4.3, we have that

$$\log M(x_k/y_k) = \log \frac{\|x_k - y'_k\|}{\|y_k - y'_k\|} \quad \text{and} \quad \log M(y_k/x_k) = \log \frac{\|y_k - x'_k\|}{\|x_k - x'_k\|}.$$

It follows that

$$\begin{aligned} d_C(x_k, y_k) \geqslant \log M(x_k/y_k) \quad &= \quad \log \frac{\|x_k - y'_k\|}{\|y_k - y'_k\|} \\ &\geqslant \quad \log \frac{\|z_k - y'_k\|}{\|y_k - y'_k\|} = \log M(z_k/y_k) = d_C(z_k, y_k). \end{aligned}$$

Similarly, $d_C(x_k, y_k) \geqslant d_C(x_k, z_k)$. As the norm topology coincides with the Thompson's metric topology on C°, these inequalities, finally imply that

$$\begin{aligned} \limsup_{k\to\infty} 2(x_k|y_k)_{p,2} &\leqslant \limsup_{k\to\infty} d_C(x_k,p) - d_C(x_k,z_k) + d_C(y_k,p) - d_C(y_k,z_k) \\ &\leqslant \limsup_{k\to\infty}\ 2d_C(z_k,p) \\ &\leqslant 2d_C(z,p). \end{aligned}$$

□

Recall that if C is a closed polyhedral cone with nonempty interior in a finite-dimensional vector space V, the dual cone is also a polyhedral cone. Indeed, as $C^{**} = C$ whenever C is a closed finite-dimensional cone with nonempty interior, we know that if C is non-polyhedral, then C^* is also non-polyhedral, see [30, Corollary 19.2.2]. The following notions play a role in the proof of the next result. A face F of a closed cone C is called an *extreme ray* if $\dim F = 1$. An extreme ray F of C is said to be an *exposed ray* if there exists $\varphi \in C^*$ such that $F = \{x \in C \colon \varphi(x) = 0\}$. The cone version of Strazewicz's theorem [30, p.167] says that in a finite-dimensional closed cone C the exposed rays are dense in the extreme rays, i.e., the norm closure of $\{x \in C\colon x \text{ on an exposed ray of } C\}$ coincides with the norm closure of $\{x \in C\colon x \text{ on an extreme ray of } C\}$.

Theorem 7.6. — *If C is a closed finite-dimensional cone with nonempty interior, then (C°, d_C) can be quasi-isometrically embedded into a finite-dimensional normed space if and only if C is a polyhedral cone.*

Proof. — It is known that if C is a closed polyhedral cone with nonempty interior, then (C°, d_C) can be isometrically embedded into $(\mathbb{R}^m, \|\cdot\|_\infty)$, where m is the number of facets of C, see [17, Lemma 2.2.2].

To prove the converse, let C be a closed non-polyhedral cone with nonempty interior in a finite-dimensional vector space V. As C is a closed non-polyhedral cone with nonempty interior, C^* is a also a closed non-polyhedral cone with nonempty interior in V^*. So, C^* has infinitely many extreme rays. By the cone version of Strazewicz's theorem [30, p.167], C^* has infinitely many exposed rays. Let $\xi \in C^*$ be a strictly positive functional and let $\Sigma_\xi^\circ = \{x \in C^\circ\colon \xi(x) = 1\}$.

For each integer $i \geqslant 1$, select distinct $\psi_i \in \partial C^*$ such that $F_i = \{\mu\psi_i\colon \mu \geqslant 0\}$ is an exposed ray of C^*. This means that there exists $w_i \in V^{**} = V$ with $w_i \in \partial C$ with $\xi(w_i) = 1$ such that $F_i = \{\varphi \in C^*\colon \varphi(w_i) = 0\}$. So, $\varphi(w_i) > 0$ whenever $\varphi \in C^*$ and $\varphi \neq \mu\psi_i$ for all $\mu \geqslant 0$.

Clearly, if $i \neq j$ and $0 < \lambda < 1$, then $\varphi(\lambda w_i + (1-\lambda)w_j) > 0$ for all $\varphi \in C^* \setminus \{0\}$. This implies that $\lambda w_i + (1-\lambda)w_j \in C^\circ$ for all $i \neq j$ and $0 < \lambda < 1$, see [30, Theorem 11.2].

Take $p \in \Sigma_\xi^\circ$ fixed. For $i \geqslant 1$ and $0 < t < 1$ let

$$\gamma_i(t) = tw_i + (1-t)p.$$

As the norm topology of d_C coincides with the topology on C°, the maps, $t \mapsto d_C(\gamma_i(t), p)$, are continuous on $(0,1)$ for all $i \geqslant 1$. Moreover, $d_C(\gamma_i(t), p) \to \infty$ as $t \to 1$. Thus, for each $i \geqslant 1$, there exists a strictly increasing sequence $(t_k^i)_k$ in $(0,1)$ with $t_k^i \to 1$ as $k \to \infty$ such that $d_C(\gamma_i(t_k^i), p) = k$ for all $k \geqslant 1$. If we let $x_k^i = \gamma_i(t_k^i)$ in Σ_ξ°, the sequences $(x_k^i)_k$ in (C°, d_C) satisfy the conditions of Corollary 7.3 by Proposition 7.5, and hence (C°, d_C) cannot be quasi-isometrically embedded into a finite-dimensional normed space. □

We can use Theorems 4.3 and 7.6 to prove the following characterization of simplicial cones, which is the analogue of [12, Theorem 2] for Thompson's metric spaces.

Theorem 7.7. — *If C is a closed finite-dimensional cone with nonempty interior, then (C°, d_C) is isometric to a finite-dimensional normed space if and only if C is a simplicial cone.*

Proof. — Suppose that that C is not simplicial and that f is an isometry of (C°, d_C) onto a finite-dimensional normed space $(V, \|\cdot\|)$. Let $\varphi \in C^*$ be strictly positive and $\Sigma_\varphi^\circ = \{x \in C^\circ \colon \varphi(x) = 1\}$. By Theorem 7.6 we have that C is a polyhedral cone, so Σ_φ° is the interior of a polytope. Since C is not simplicial, it follows that Σ_φ° is not the interior of an $(n-1)$-simplex, where $n = \dim(V)$. This implies that there exist vertices v_1 and v_2 in $\partial\Sigma_\varphi^\circ$ and $u \in \partial\Sigma_\varphi^\circ$ such that $tv_1 + (1-t)u \in \Sigma_\varphi^\circ$ and $tv_2 + (1-t)u \in \Sigma_\varphi^\circ$ for all $0 < t < 1$ and u is not a vertex. The situation is depicted in the Figure 7.1.

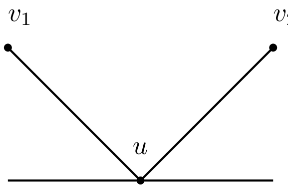

Figure 7.1. Vertices

Let $\gamma_1(t) = e^t u + e^{-t} v_1$ and $\gamma_2(t) = e^t u + e^{-t} v_2$, for $t \in \mathbb{R}$, be type I geodesics in (C°, d_C), see Lemma 3.7. As v_1 and v_2 are vertices of Σ_φ, it follows from Theorem 4.3 that both γ_1 and γ_2 are unique geodesic lines in (C°, d_C). This implies that the images of γ_1 and γ_2 under the isometry f, which we will denote by ℓ_1 and ℓ_2, respectively, are straight lines in V, since unique geodesic lines are mapped to unique geodesic lines by f.

Now, fix $\xi \in C^\circ$ and let $\Sigma_\xi^* = \{\varphi \in C^* : \varphi(\xi) = 1\}$. For $x, y \in C^\circ$ we have that

$$M(x/y) = \sup_{\varphi \in \Sigma_\xi^*} \frac{\varphi(x)}{\varphi(y)},$$

see [17, p.34]. Note that for $\varphi \in \Sigma_\xi^*$ and $t \in \mathbb{R}$ we have

$$\frac{\varphi(\gamma_1(t))}{\varphi(\gamma_2(t))} = \frac{e^t\varphi(u) + e^{-t}\varphi(v_1)}{e^t\varphi(u) + e^{-t}\varphi(v_2)} = \frac{\varphi(u) + e^{-2t}\varphi(v_1)}{\varphi(u) + e^{-2t}\varphi(v_2)}.$$

So, if $\varphi(u) = 0$, then neither $\varphi(v_1)$ nor $\varphi(v_2)$ can be 0, and we find that

$$\frac{\varphi(\gamma_1(t))}{\varphi(\gamma_2(t))} = \frac{\varphi(v_1)}{\varphi(v_2)} < \infty$$

for all $t \in \mathbb{R}$. On the other hand, if $\varphi(u) \neq 0$, then

$$\frac{\varphi(\gamma_1(t))}{\varphi(\gamma_2(t))} \leqslant \frac{\varphi(u) + \varphi(v_1)}{\varphi(u)} < \infty$$

for all $t \geqslant 0$. Thus,

$$\limsup_{k\to\infty} M(\gamma_1(t_k)/\gamma_2(t_k)) = \limsup_{k\to\infty} \left(\sup_{\varphi \in \Sigma_\xi^*} \frac{\varphi(\gamma_1(t_k))}{\varphi(\gamma_2(t_k))} \right) < \infty$$

for all sequences $(t_k)_k$ with $t_k \to \infty$. Interchanging the roles of γ_1 and γ_2 yields an analogous result, from which we deduce that

$$\limsup_{k\to\infty} d_C(\gamma_1(t_k), \gamma_2(t_k)) < \infty$$

for all sequences $(t_k)_k$ with $t_k \to \infty$. These findings imply that for each $k \geqslant 1$ there exist x_k on ℓ_1 and y_k on ℓ_2 with $\sup_{k\geqslant 1} \|x_k - y_k\| < \infty$ such that $\|x_k\| \to \infty$ and $\|y_k\| \to \infty$ as $k \to \infty$. This implies that ℓ_1 and ℓ_2 are parallel.

As the extreme rays of the polyhedral cone C are exposed, there exists a functional $\vartheta \in C^*$ such that $\vartheta(\mu v_2) = 0$ for all $\mu \geqslant 0$ and $\vartheta(x) > 0$ for all $x \in C \setminus \{\mu v_2 \colon \mu \geqslant 0\}$. After scaling with an appropriate factor, we have $\vartheta \in \Sigma^*_\xi$. Now it follows that

$$\frac{\vartheta(\gamma_1(t))}{\vartheta(\gamma_2(t))} = \frac{e^t\vartheta(u) + e^{-t}\vartheta(v_1)}{e^t\vartheta(u) + e^{-t}\vartheta(v_2)} = \frac{\vartheta(u) + e^{-2t}\vartheta(v_1)}{\vartheta(u)} \to \infty$$

as $t \to -\infty$, and hence $M(\gamma_1(t)/\gamma_2(t)) \to \infty$ as $t \to -\infty$. This, however, implies that

$$\lim_{t\to-\infty} d_T(\gamma_1(t), \gamma_2(t)) = \infty$$

and hence ℓ_1 and ℓ_2 are not parallel, which is absurd. Thus, C must be a simplicial cone.

Conversely, if C is a simplicial cone in a, say n-dimensional vector space X, then there are linearly independent $v_1, \dots, v_n \in X$ such that $C = \{\sum_{k=1}^n \alpha_k v_k \colon \alpha_k \geqslant 0 \text{ for } 1 \leqslant k \leqslant n\}$. The map $T : X \to \mathbb{R}^n$ given by $\sum_{k=1}^n \alpha_k x_k \mapsto (\alpha_1, \dots, \alpha_k)$ is a bijective linear map with $T(C) = \mathbb{R}^n_+$. and hence T is an isometry of (C°, d_C) onto $((\mathbb{R}^n_+)^\circ, d_{\mathbb{R}^n_+})$. Recall that $((\mathbb{R}^n_+)^\circ, d_{\mathbb{R}^n_+})$ is isometric to $(\mathbb{R}^n, \|\cdot\|_\infty)$, see [17, Proposition 2.2.1]. In fact, the reader can easily check that the coordinatewise log function is an isometry. □

8. Isometries on strictly convex cones

In this section we will analyze the isometries of (C°, d_C) when C is a closed strictly convex cone with nonempty interior in a finite-dimensional vector space V. Recall that C is *strictly convex* if for each linearly independent $x, y \in \partial C$ we have that

$$\lambda x + (1 - \lambda)y \in C^\circ \quad \text{for all } 0 < \lambda < 1.$$

If C is a closed cone with nonempty interior in a normed space V and $T\colon V \to V$ is an invertible linear map with $T(C) = C$, then T is an isometry of (C°, d_C). Given a closed cone with nonempty interior in a finite-dimensional vector space V, we let $\mathrm{Aut}(C) = \{T \in \mathrm{GL(V)}\colon T(C) = C\}$ and we let $\mathrm{Isom}(C)$ be the set of maps $g\colon C^\circ \to C^\circ$ such that g is an isometry of (C°, d_C). So, $\mathrm{Aut}(C)$ is a subgroup of $\mathrm{Isom}(C)$. It is known [5] that $\mathrm{Aut}(C) \neq \mathrm{Isom}(C)$, even if C is a strictly convex cone. Consider, for example, the cone, $\Pi_2(\mathbb{R})$, of positive semi-definite matrices in the space of 2×2 symmetric matrices. This is a 3-dimensional, strictly convex, closed cone. In fact, $\Pi_2(\mathbb{R})$ is order-isomorphic with the 3-dimensional Lorentz cone, see [17, p. 44]. The map $h\colon \Pi_2(\mathbb{R})^\circ \to \Pi_2(\mathbb{R})^\circ$ given by $h(A) = A^{-1}$ is an isometry under Thompson's metric, as h is an order-reversing homogeneous degree -1 involution, see [17, Corollary 2.1.5]. Obviously, $h \notin \mathrm{Aut}(C)$. It turns out, however, that h is projectively linear

Definition 8.1. — *A map $f\colon C^\circ \to C^\circ$ is* projectively linear *if there exists $T \in \mathrm{Aut}(C)$ such that for each $x \in C^\circ$,*

$$f(x) = \lambda_x T(x) \quad \textit{for some } \lambda_x > 0.$$

Note that in the example above, if

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \in \Pi_2(\mathbb{R})^\circ,$$

then

$$h(A) = A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} c & -b \\ -b & a \end{bmatrix},$$

which shows that h is projectively linear.

Theorem 8.2. — *If C is a closed strictly convex cone with nonempty interior in an n-dimensional vector space V and $n \geqslant 3$, then every $f \in \mathrm{Isom}(C)$ is projectively linear.*

Proof. — Let $f \in \mathrm{Isom}(C)$. We will first show that f maps type I geodesic lines to type I geodesic lines, and type II geodesic lines to type II geodesic lines. Suppose, by way of contradiction, that γ is a type I geodesic line that is mapped to a type II geodesic line under f. Then $K = \mathrm{span}\{\gamma\} \cap C$ is closed 2-dimensional cone. So, by [17, Lemma A.5.1] there exists linearly independent $u_0, v_0 \in \partial C$ such that

$$K = \{\alpha u_0 + \beta v_0 \colon \alpha, \beta \geqslant 0\}.$$

From Lemma 3.7 we know that, after rescaling u_0 and v_0, we can write γ as the image of

$$\gamma(t) = \frac{1}{2}(e^t u_0 + e^{-t} v_0)$$

where $t \in \mathbb{R}$. Let $x = \gamma(0)$ and $\varphi \in (C^*)^\circ$ with $\varphi(u_0) = 1 = \varphi(v_0)$. As $\dim C \geqslant 3$, $\Sigma_\varphi = \{v \in C \colon \varphi(v) = 1\}$ is a compact convex set with $\dim \Sigma_\varphi \geqslant 2$. Thus, there exists a sequence $(u_k)_k$ in ∂C with $\varphi(u_k) = 1$ and $u_k \neq u_0$ for all $k \geqslant 1$ such that $\|u_k - u_0\| \to 0$ as $k \to \infty$. Let $v_k \in \partial C$ be the point of intersection of the straight line through u_k and x and ∂C. So, $\|v_k - v_0\| \to 0$ as $k \to \infty$, since $x = \frac{1}{2}(u_0 + v_0)$. For each $k \geqslant 1$ there exists $0 < \alpha_k < 1$ such that $x = \alpha_k u_k + (1 - \alpha_k) v_k$. Note that $\alpha_k \to 1/2$ as $k \to \infty$, and hence the geodesic paths,

$$\gamma_k(t) = \alpha_k e^t u_k + (1 - \alpha_k) e^{-t} v_k \quad \text{for } t \in \mathbb{R},$$

are type I geodesics by Lemma 3.7 and $\gamma_k(t) \to \gamma(t)$ pointwise as $k \to \infty$.

Now fix $y = \gamma(t)$ with $t \neq 0$ and consider the sequence $(y_k)_k$ with $y_k = \gamma_k(t)$. As the norm topology coincides with the Thompson's metric topology on C°, we have that $d_C(y_k, y) \to 0$ as $k \to \infty$. For $z \in C \setminus \{0\}$, write $[z] = z/\varphi(z)$. So,

$$\delta_C([y], [y_k]) = \delta_C(y, y_k) \leqslant 2 d_C(y, y_k) \to 0$$

as $k \to \infty$. Note that there is only one type II geodesic through $f(x)$, and hence $f(\gamma_k)$ must be a type I geodesic line for all $k \geqslant 1$, as f is an isometry. So,

$$\begin{aligned} \delta_C([x], [y_k]) = \delta_C(x, y_k) &= 2 d_C(x, y_k) \\ &= 2 d_C(f(x), f(y_k)) = \delta_C([f(x)], [f(y_k)]). \end{aligned}$$

As γ is mapped to a type II geodesic, the previous equality implies that $0 < \delta_C([x], [y]) = \delta_C([f(x)], [f(y)]) = 0$, which is impossible. Thus f maps type I geodesic lines to type I geodesic lines. Also, f has to map type II geodesic lines to type II geodesic lines, as otherwise $f^{-1} \in \operatorname{Isom}(C)$ maps a type I geodesic line to a type II geodesic line.

Let $\Sigma_\varphi^\circ = \{x \in C^\circ \colon \varphi(x) = 1\}$. Next we will show that $g \colon \Sigma_\varphi^\circ \to \Sigma_\varphi^\circ$ given by,

$$g(x) = \frac{f(x)}{\varphi(f(x))} \quad \text{for all } x \in \Sigma_\varphi^\circ,$$

is an isometry under δ_C. For x and y in Σ_φ° distinct there exists a $\lambda > 0$ such that x and λy lie on a type I geodesic in (C°, d_C). Write $\xi = \lambda y$. As $M(x/\xi) = M(\xi/x)$, we have that

$$2d_C(x,\xi) = \delta_C(x,\xi) = \delta_C(x,y).$$

Now using the fact that f maps type I geodesic lines to type I geodesic lines we get that $M(f(x)/f(\xi)) = M(f(\xi)/f(x))$. Also, as f maps type II geodesic lines to type II geodesic lines,

$$g(\xi) = \frac{f(\xi)}{\varphi(f(\xi))} = \frac{f(\lambda y)}{\varphi(f(\lambda y))} = \frac{f(y)}{\varphi(f(y))} = g(y).$$

Thus,

$$2d_C(x,\xi) = 2d_C(f(x), f(\xi)) = \delta_C(g(x), g(\xi)) = \delta_C(g(x), g(y)).$$

It now follows that g is an isometry under δ_C. As Σ_φ° is a strictly convex set, we deduce from [10, Proposition 3] that f is a projectively linear map. $\square$

If C is not strictly convex, $f \in \mathrm{Isom}(C)$ need not be projectively linear. Indeed, the map $(x,y,z) \mapsto (x,y,z^{-1})$ on the interior of the standard positive cone $\mathbb{R}^3_+ = \{(x,y,z)\colon x,y,z \geqslant 0\}$ is an isometry under Thompson's metric but not projectively linear. It would be interesting to characterize those finite-dimensional closed cones C for which all Thompson's metric isometries are projectively linear. It would also be interesting to know for which cones C we have $\mathrm{Aut}(C) = \mathrm{Isom}(C)$.

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Bas LEMMENS & Mark ROELANDS
School of Mathematics, Statistics & Actuarial Science,
Cornwallis Building, University of Kent,
Canterbury, Kent CT2 7NF,
UK.
B.Lemmens@kent.ac.uk
mark.roelands@gmail.com