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FORMAL DEFORMATION OF CURVES WITH GROUP SCHEME ACTION

by Stefan WEWERS

Introduction.

Let Y be a (not necessarily smooth) curve over an algebraically closed field k of characteristic $p > 0$. Let W be a complete local ring with residue field k. Furthermore, let G be a finite flat group scheme over W which acts faithfully on Y. We denote by $\mathrm{Def}(Y, G)$ the functor which associates to a local Artinian W-algebra R with residue field k the set of isomorphism classes of G-equivariant deformations of Y to R. The present paper is concerned with a study of the functor $\mathrm{Def}(Y, G)$, using cohomological methods. The special case where Y is smooth and G is a constant group scheme has been studied previously by Bertin and Mézard [3].

One of the motivations for studying the functor $\mathrm{Def}(Y, G)$ is the *lifting problem*. Suppose that Y is smooth and that G is a finite abstract group which acts faithfully on the curve Y. Let W be the ring of Witt vectors over k, and consider G as a constant group scheme over W. In this situation, the lifting problem asks the following question. Does there exist a finite extension R/W of complete discrete valuation rings and a G-equivariant lift of Y over R? For instance, if the deformation functor $\mathrm{Def}(Y, G)$ is unobstructed then the answer to this question is positive.

A conjecture of Oort predicts that the lifting problem has a positive solution if the group G is cyclic. However, even in the simplest nontrivial

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case $G = \mathbb{Z}/p$ (where Oort's conjecture is proved, see [21] and [6]) the functor $\mathrm{Def}(Y, G)$ is obstructed. In [3] these obstructions are identified as elements in a certain Galois cohomology group. However, they remain a bit mysterious. One of the motivations for generalizing the approach of Bertin–Mézard is the author's hope that this will lead to new insight into the nature of these obstruction, and the lifting problem in general.

Another (related) motivation comes from the study of the stable reduction of Galois covers of curves. Let R be a complete discrete valuation ring, with algebraically closed residue field k of characteristic p and fraction field K of characteristic 0. Let $Y_K \to X_K$ be a Galois cover of smooth projective curves over K, with Galois group G. After a finite extension of K, there exists a certain natural R-model $Y_R \to X_R$ of $Y_K \to X_K$, called the *stable model*, see [18] or [26] for a precise definition. The problem we are interested in is to understand this model and in particular its relation with the ramification of the prime p in the field K. It has become clear from recent work of several authors (e.g. [11], [19], [26]) that this problem naturally leads to the study of singular curves with an action of a finite group scheme, and of the deformation theory of such objects.

This paper is divided into two main parts. The first part (§1–3) is an exposition of certain cohomological methods for studying equivariant deformations of (not necessarily smooth) curves with group scheme action. Although the guiding principles are the same as in [3], we have to use much heavier technical machinery. For instance, it does not suffice to look at the equivariant cohomology of the tangent bundle of Y over k, as in [3]. Instead, one has to consider certain hyperext groups with values in the *equivariant cotangent complex* of Y over k. The latter is an object in the derived category of G-$\mathcal{O}_Y$-modules, and was first introduced by Grothendieck in [8].

In principal, everything one might want to known about the equivariant cotangent complex and its role in deformation theory can be found in Illusie's book [12]. However, the generality in which [12] is written makes it somewhat difficult to read and to work with in a concrete situation (at least for the author of this paper). In the literature there are a number of excellent and readable accounts of certain special cases (see e.g. [23] or [3], §2-3) but none seems to be sufficiently general to deal with the case we need. To improve this situation, the present paper contains a self-contained exposition of a special case of Illusie's theory, which should nevertheless be sufficiently general for the applications we have in mind.

In the second part of this paper (§4–5) we apply the general theory to a special case which is relevant for the study of three point covers with bad reduction. In particular, we prove a certain result which is a key ingredient for the main theorem of [26].

We start in §4 with a *multiplicative deformation datum*. To give an idea what this is, fix a smooth projective curve X over k. Then a multiplicative deformation datum over X is a pair (Z, V), where $Z \to X$ is a Galois cover of smooth projective curves over a field k of characteristic $p > 0$, with Galois group H of order prime-to-p, and $V \subset \Omega_{k(Z)/k}$ is an H-stable $\mathbb{F}_p$-vector space of logarithmic differential forms. To (Z, V) we associate a finite flat group scheme G over $W(k)$ and a (singular) curve Y over k with an action of G such that $X = Y/G$. Briefly, the group scheme G is of the form $\boldsymbol{\mu}_p^s \rtimes H$ and $Y \to Z$ is the $\boldsymbol{\mu}_p^s$-cover locally given by s Kummer equations $y_i^p = u_i$, where $\phi_i = \mathrm{d}u_i/u_i$, $i = 1, \ldots, s$, form a basis of V.

We study the deformation functor $\mathrm{Def}(Y, G)$ for such an action of a group scheme G and exhibit a number of its properties. Some of theses properties are specific to the action of G we deal with. They are in general very different from the properties enjoyed by the deformation functor studied in [3]. For instance, there is in general no such thing as a local-global principle, because the "local contribution" to the tangent space of the functor $\mathrm{Def}(Y, G)$ is not concentrated in a finite number of closed points. However, from another point of view things are really much easier than in [3], due to the fact that the "p-Sylow" of G is a multiplicative group scheme. Since multiplicative group schemes have trivial cohomology, the general machinery developed in the first sections shows that the deformation functor $\mathrm{Def}(Y, G)$ is unobstructed. Another nice property of $\mathrm{Def}(Y, G)$ is the existence of a natural morphism of deformation functors

$$\mathrm{Def}(Y, G) \longrightarrow \mathrm{Def}(X; \tau_j) \tag{1}$$

which sends an equivariant deformation of Y to its quotient by G. (Here we regard X as a marked curve, the marked points being the "branch points" $\tau_1, \ldots, \tau_n$ of the G-cover $Y \to X$.) In this respect, the G-cover $Y \to X$ behaves like a tamely ramified Galois cover. However, unlike in the case of tamely ramified Galois covers, the functor (1) is in general not an isomorphism.

In §5 we assume in addition that the curve X is the projective line and that the vector space V is an irreducible $\mathbb{F}_p[H]$-module which decomposes, after tensoring with $\bar{\mathbb{F}}_p$, into the direct sum of one dimensional modules.

Among all the multiplicative deformation data (Z, V) of this type, there are some which we call *special*. The definition of specialty is given in terms of certain numerical invariants attached to (Z, V). But philosophically, special deformation data are attached to three point Galois covers of the projective line with bad reduction to characteristic p. We refer to [25] and [26] for details on the case $\dim_{\mathbb{F}_p} V = 1$ and for a more satisfactory explanation of the connection to three point covers. Let us only mention that the deformation theory of the G-cover $Y \to X$ attached to a special deformation datum has a number of very nice and surprising properties:

- *The lifting property:* the morphism of deformation functors (1) is an isomorphism. In this respect, the G-cover $Y \to X$ behaves just like a tamely ramified Galois cover.
- *The local-global principle:* local deformations in formal neighborhoods of the ramification points (which satisfy a certain condition) can be interpolated by a *unique* global deformation of Y.
- *Rigidity:* If an equivariant deformation of Y in equal characteristic (*i.e.* over a local k-algebra) is again special then it is the trivial deformation. Therefore, there exist at most a finite number of special deformation data of a given type (up to isomorphism), and every special deformation datum can be defined over a finite field.

These properties are very particular to special deformation data. They reflect, in a rather subtle way, the connection to three point covers with bad reduction and in particular to the fact that three point covers are "rigid" objects.

At the end of the paper, the reader will find three appendices containing background material which the author found difficult to extract from the literature. This includes Picard stacks, the cohomology of affine group schemes, and two spectral sequences which are useful to compute equivariant hyperext groups.

1. The equivariant cotangent complex.

In [12] Illusie defines, for any morphism of schemes $Y \to S$, the cotangent complex $\mathcal{L}_{Y/S}$. This is a complex of flat $\mathcal{O}_Y$-modules, well defined up to canonical quasi-isomorphism, such that $H^0(\mathcal{L}_{Y/S}) = \Omega_{Y/S}$. If $Y \to S$ is smooth then $\mathcal{L}_{Y/S} = \Omega_{Y/S}$. Moreover, if $G \to S$ is a group scheme acting on Y, Illusie defines the *equivariant cotangent complex* as an object of the derived classifying topos $\mathfrak{D}^+(BG_{/X})$ whose underlying complex of $\mathcal{O}_Y$-modules is $\mathcal{L}_{Y/S}$.

In this section we give a more down-to-earth definition of the equivariant cotangent complex which, however, works well only if $Y \to S$ and the G-action on Y have certain good properties. We follow the original approach of Grothendieck [8]. This gives the "correct" cotangent complex only if $Y \to S$ is a local complete intersection morphism. We assume that Y admits locally an equivariant embedding into a formally smooth S-scheme with G-action. Under this assumption, it is much easier to endow the cotangent complex with a natural G-action.

1.1. — Let $S = \operatorname{Spec} R$ be an affine scheme, $G \to S$ a flat affine group scheme and $Y \to S$ an S-scheme with an action of G. By a *G-$\mathcal{O}_Y$-module* we mean a sheaf of $\mathcal{O}_Y$-modules $\mathcal{F}$, together with a lift of the G-action from Y to $\mathcal{F}$. A homomorphism between two G-$\mathcal{O}_Y$-modules $\mathcal{F}$ and $\mathcal{G}$ is a sheaf homomorphism which is both $\mathcal{O}_Y$-linear and G-equivariant. The group of such homomorphisms is denoted by $\operatorname{Hom}_G(\mathcal{F}, \mathcal{G})$. We denote by $\operatorname{Mod}(Y, G)$ the corresponding category of G-$\mathcal{O}_Y$-modules. See Appendix C.1 for more details on the category $\operatorname{Mod}(Y, G)$. For $* \in \{+, -, b\}$, we denote by $\mathfrak{K}^*(Y, G)$ the category of cochain complexes in $\operatorname{Mod}(Y, G)$, up to homotopy, which are bounded from below $(* = +)$, bounded from above $(* = -)$ or bounded in both directions $(* = b)$. We write $\mathfrak{D}^*(Y, G)$ for the derived category of $\mathfrak{K}^*(Y, G)$.

In this section we define the equivariant cotangent complex $\mathcal{L}_{Y/S}$ of the morphism $Y \to S$ as an object of $\mathfrak{D}^+(Y, G)$, assuming:

Assumption 1.1. — *Every point of Y is contained in an affine and G-stable open neighborhood $U \subset Y$ such that the following holds. There exists a formally smooth affine S-scheme $P \to S$ with G-action and a G-equivariant closed immersion $\varphi : U \hookrightarrow P$.*

Remark 1.2. — It is not clear to the author how restrictive Assumption 1.1 is. We expect that it can be verified in any concrete situation where one actually wants to apply our theory. For instance, in §4 we use the case where G is an extension of a constant by a multiplicative group scheme and acts freely on a dense open subset of Y. In this situation, Assumption 1.1 is easy to verify.

1.2. — A triple (U, P, φ) as in Assumption 1.1 is called a *local chart* for $Y \to S$. Often we will simply write φ instead of (U, P, φ). Given such a local chart, we denote by $\mathcal{I} \subset \mathcal{O}_P$ the sheaf of ideals defining the image of φ. We define the *cotangent complex* of the chart φ as the following complex of G-$\mathcal{O}_Y$-modules:

$$\mathcal{L}_\varphi := (\mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{P/S} \otimes \mathcal{O}_Y). \tag{2}$$

The two nontrivial terms of $\mathcal{L}_\varphi$ lie in degree -1 and 0. Note that there is a natural augmentation $\mathcal{L}_\varphi \to \Omega_{Y/S}$ which identifies $\Omega_{Y/S}$ with $H^0(\mathcal{L}_\varphi)$.

Remark 1.3.

(i) If Y/S is of finite type, then we may take P/S to be smooth. In this case, $\mathcal{L}_\varphi^0 = \Omega_{P/S} \otimes \mathcal{O}_Y$ is a locally free $\mathcal{O}_Y$-module of finite rank.

(ii) If, moreover, $Y \to S$ is a local complete intersection (in the sense of [2], VIII.1.1) then the embedding φ is regular. Recall that this means the following. For every point $y \in U$ the stalk $\mathcal{I}_y$ is an ideal generated by a regular sequence of the local ring $\mathcal{O}_{P,y}$. It follows that $\mathcal{L}_\varphi^{-1} = \mathcal{I}/\mathcal{I}^2$ is a locally free $\mathcal{O}_Y$-module of finite rank, too.

Let (U, P, φ) and (V, Q, ψ) be two local charts, and assume that $V \subset U$. A *morphism* from ψ to φ is a G-equivariant morphism of S-schemes $u : Q \to P$ such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\psi} & Q \\ \downarrow & & \downarrow u \\ U & \xrightarrow{\varphi} & P \end{array}$$

commutes. We use the notation $u : \psi \to \varphi$. Note that u induces a morphism of complexes of G-$\mathcal{O}_Y$-modules

$$u^* : \mathcal{L}_\varphi|_V \longrightarrow \mathcal{L}_\psi.$$

LEMMA 1.4.

(i) *The homotopy class of u^* is independent of the morphism u.*

(ii) *The morphism u^* is a quasi-isomorphism.*

Proof. — It is no restriction to assume that $U = V$. We may also assume that U is affine. Let P' denote the second infinitesimal neighborhood of U in P, *i.e.* the closed subscheme of P defined by the sheaf of ideals $\mathcal{I}^2$. Similarly, Q' denotes the second infinitesimal neighborhood of

Y in Q. It is defined by $\mathcal{J}^2$, where $\mathcal{J} \subset \mathcal{O}_Q$ is the sheaf of ideals defining the image of ψ. Let $v : Q \to P$ be another morphism of local charts, and set $u' := u|_{Q'}$ and $v' := v|_{Q'}$. It is clear that u^* (resp. v^*) only depends on the restriction u' (resp. on v'). An easy computation shows that the difference of the two pullback maps

$$(u')^* - (v')^* : \mathcal{O}_P \longrightarrow \mathcal{J}/\mathcal{J}^2$$

is an R-linear derivation. Hence it gives rise to an $\mathcal{O}_Y$-linear map $s : \Omega_P \otimes \mathcal{O}_Y \to \mathcal{J}/\mathcal{J}^2$, and one checks that s is the desired homotopy between u^* and v^*:

$$\begin{tikzcd} \mathcal{I}/\mathcal{I}^2 \arrow[r, "d"] \arrow[d, "u^*-v^*"'] & \Omega_{P/S} \otimes \mathcal{O}_Y \arrow[d, "u^*-v^*"] \arrow[dl, "s"] \\ \mathcal{J}/\mathcal{J}^2 \arrow[r, "d"] & \Omega_{Q/S} \otimes \mathcal{O}_Y \end{tikzcd}$$

This proves (i).

By assumption $Q \to S$ is formally smooth and U is affine. Hence there exists a morphism $w' : P' \to Q$ lifting $\psi : Y \hookrightarrow Q$. As in the proof of (i), one shows that there are homotopies

$$(w')^* \circ u^* \sim \mathrm{Id}_{\mathcal{L}_\varphi}, \qquad u^* \circ (w')^* \sim \mathrm{Id}_{\mathcal{L}_\psi} .$$

This proves (ii). □

1.3. — We are now ready to define the equivariant cotangent complex. By Assumption 1.1 there exists a covering $(U_i)_{i\in I}$ of Y by affine and G-stable opens $U_i \subset Y$, each admitting a local chart $\varphi_i : U_i \hookrightarrow P_i$. We choose, once and for all, a well-ordering on the set of indices of the covering (U_i). The datum (U_i, φ_i) is called an *atlas*.

For any $(n+1)$-tuple $\underline{i} = (i_0, \dots, i_n)$ we set

$$U_{\underline{i}} := U_{i_0} \cap \ldots \cap U_{i_n}, \qquad P_{\underline{i}} := P_{i_0} \times_S \ldots \times_S P_{i_n},$$

$$\varphi_{\underline{i}} := \varphi_{i_0} \times \cdots \times \varphi_{i_n} : U_{\underline{i}} \hookrightarrow P_{\underline{i}}.$$

Clearly, $\varphi_{\underline{i}}$ is a local chart and gives rise to a complex of G-$\mathcal{O}_{U_{\underline{i}}}$-modules $\mathcal{L}_{\varphi_{\underline{i}}}$. We denote by $\mathcal{L}_{\underline{i}}$ the push-forward of $\mathcal{L}_{\varphi_{\underline{i}}}$ to Y. Thus, $\mathcal{L}_{\underline{i}}$ is a flat and quasi-coherent G-$\mathcal{O}_Y$-module such that $\mathcal{L}_{\underline{i}}|_{U_i} = \mathcal{L}_{\varphi_{\underline{i}}}$.

For $\underline{i} = (i_0, \dots, i_n)$ as above and $0 \leqslant \nu \leqslant n$, let $p^\nu_{\underline{i}} : P_{\underline{i}} \to P_{\underline{i}'}$ denote the projection which leaves out the νth component (*i.e.* $\underline{i}' = (i_0, \dots, \widehat{i_\nu}, \dots, i_n)$); it is a morphism $\varphi_{\underline{i}} \to \varphi_{\underline{i}'}$ of local charts. The resulting morphism $(p_{\underline{i}}^{\nu})^* : \mathcal{L}_{\varphi_{\underline{i}'}}|_{U_{\underline{i}}} \to \mathcal{L}_{\varphi_{\underline{i}}}$ extends in a canonical way to a morphism of G-$\mathcal{O}_Y$-modules $\partial_{\underline{i}}^{\nu} : \mathcal{L}_{\underline{i}'} \to \mathcal{L}_{\underline{i}}$. Note that

$$\partial_{\underline{i}}^{\nu} \circ \partial_{\underline{i}'}^{\mu} = \partial_{\underline{i}}^{\mu} \circ \partial_{\underline{i}''}^{\nu-1} \tag{3}$$

holds for $\mu < \nu$, if we set $\underline{i}' := (\ldots, \widehat{i_\nu}, \ldots)$ and $\underline{i}'' := (\ldots, \widehat{i_\mu}, \ldots)$.

Definition 1.5. — The *equivariant cotangent complex* of the morphism $Y \to S$ (relative to the open covering (U_i) and the local charts φ_i) is the total complex

$$\mathcal{L}_{Y/S} := \mathrm{Tot}(\mathcal{K})$$

of the following double complex of G-$\mathcal{O}_Y$-modules:

$$\mathcal{K} : \left\{ \begin{array}{ccc} \prod_i \mathcal{L}_i^{-1} & \xrightarrow{d} & \prod_i \mathcal{L}_i^{0} \\ \big\downarrow \partial & & \big\downarrow \partial \\ \prod_{i<j} \mathcal{L}_{i,j}^{-1} & \xrightarrow{-d} & \prod_{i<j} \mathcal{L}_{i,j}^{0} \\ \big\downarrow \partial & & \big\downarrow \partial \\ \prod_{i<j<k} \mathcal{L}_{i,j,k}^{-1} & \xrightarrow{d} & \prod_{i<j<k} \mathcal{L}_{i,j,k}^{0} \\ \big\downarrow \partial & & \big\downarrow \partial \\ \cdots & & \cdots \end{array} \right.$$

The vertical differentials are defined as $\partial := \sum_{\nu=0}^{p} (-1)^{\nu} \prod_{\underline{i}} \partial_{\underline{i}}^{\nu} : \mathcal{K}^{p,q} \to \mathcal{K}^{p+1,q}$. The horizontal differentials are induced from the differentials of the complexes $\mathcal{L}_{\underline{i}}$. The columns of $\mathcal{K}$ start with degree 0, so $\mathcal{L}_{Y/S}$ starts with degree -1. Note that $\mathcal{L}_{Y/S}$ consists of flat and quasi-coherent sheaves.

Proposition 1.6. — *For all i there exist a quasi-isomorphism $\beta_i : \mathcal{L}_{Y/S}|_{U_i} \to \mathcal{L}_{\varphi_i}$. Moreover, for all $i < j$ we have a homotopy*

$$s_{i,j} : \partial_{i,j}^{1} \circ \beta_i \sim \partial_{i,j}^{0} \circ \beta_j.$$

which satisfies the cocycles relation

$$\partial_{i,j,k}^{0} \circ s_{j,k} - \partial_{i,j,k}^{1} \circ s_{i,k} + \partial_{i,j,k}^{2} \circ s_{i,j} = 0. \tag{4}$$

Proof. — The natural projections $\mathcal{K}^{0,q}|_{U_i} \to \mathcal{L}_i^q$ induce a morphism $\beta_i : \mathcal{L}_{Y/S}|_{U_i} \to \mathcal{L}_{\varphi_i}$. We define the homotopy $s_{i,j}$ as follows:

$$s_{i,j}^0 : \left\{ \begin{array}{ccc} (\mathcal{L}_{Y/S})^0 & \longrightarrow & \mathcal{L}_{i,j}^{-1} \\ (f_k; g_{k,l}) & \longmapsto & g_{i,j} \end{array} \right. , \qquad s_{i,j}^1 : \left\{ \begin{array}{ccc} (\mathcal{L}_{Y/S})^1 & \longrightarrow & \mathcal{L}_{i,j}^{0} \\ (f_{k,l}; g_{k,l,m}) & \longmapsto & f_{i,j} \end{array} \right. .$$

We leave it to the reader to check that $s_{i,j}$ is indeed a homotopy from $\partial^1_{i,j} \circ \beta_i$ to $\partial^0_{i,j} \circ \beta_j$ and satisfies the cocycle relation (4).

It remains to show that β_i is a quasi-isomorphism. Let i, j be a pair of indices. By definition and by Lemma 1.4 (ii) the restriction of $\partial^0_{i,j} : \mathcal{L}_j \to \mathcal{L}_{i,j}$ to $U_{i,j}$ is a quasi-isomorphism. Therefore, for $q = -1, 0$ we may define

$$\alpha^q_{i,j} := H^q(\partial^0_{i,j})^{-1} \circ H^q(\partial^1_{i,j}) : H^q(\mathcal{L}_i)|_{U_{i,j}} \xrightarrow{\sim} H^q(\mathcal{L}_j)|_{U_{i,j}}.$$

One checks that the cocycle relation $\alpha^q_{j,k} \circ \alpha^q_{i,j} = \alpha^q_{i,k}$ holds. Therefore, there exists a G-$\mathcal{O}_Y$-module $\mathcal{T}^q$ together with isomorphisms $\gamma^q_i : \mathcal{T}^q|_{U_i} \xrightarrow{\sim} H^q(\mathcal{L}_i)$ such that $\alpha^q_{i,j} = \gamma^q_j \circ (\gamma^q_i)^{-1}$. It is a bit tedious but elementary to define, for each $(n{+}1)$-tuple $\underline{i} = (i_0, \ldots, i_n)$ an isomorphism $\gamma^q_{\underline{i}} : \mathcal{T}^q|_{U_{\underline{i}}} \xrightarrow{\sim} H^q(\mathcal{L}_{\underline{i}})$ which identifies the complex

$$H^q(\mathcal{K}) = \Big(\prod_i H^q(\mathcal{L}_i) \longrightarrow \prod_{i<j} H^q(\mathcal{L}_{i,j}) \longrightarrow \cdots\Big) \tag{5}$$

with the Čech-resolution of the sheaf $\mathcal{T}^q$. We conclude that the complex (5) is exact. Now the spectral sequence $H^p(H^q(\mathcal{K})) \Rightarrow H^{p+q}(\mathcal{L}_{Y/S})$ identifies $\mathcal{T}^q$ with $H^q(\mathcal{L}_{Y/S})$ in such a way that γ^q_i is identified with $H^q(\beta_i)$. In particular, $H^q(\beta_i)$ is an isomorphism, which is what we wanted to show.□

Remark 1.7. — Let (U_i, φ_i) and (U'_i, φ'_i) be two atlases and $\mathcal{L}_{Y/S}$ and $\mathcal{L}'_{Y/S}$ the corresponding complexes, as defined above. Then the disjoint union of (U_i, φ_i) and (U'_i, φ'_i) is again an atlas and gives rise to a third complex $\mathcal{L}''_{Y/S}$, canonically equipped with quasi-isomorphisms $\mathcal{L}''_{Y/S} \to \mathcal{L}_{Y/S}$ and $\mathcal{L}''_{Y/S} \to \mathcal{L}'_{Y/S}$. In other words: the cotangent complex $\mathcal{L}_{Y/S}$, considered as an object of the derived category $\mathfrak{D}^+(Y, G)$, does not depend on the choice of the atlas (U_i, φ_i).

Remark 1.8. — By definition we have $H^0(\mathcal{L}_{Y/S}) = \Omega_{Y/S}$ and $H^q(\mathcal{L}_{Y/S}) = 0$ for $q \notin \{-1, 0\}$. It is also clear that $\mathcal{L}_{Y/S}$ has functorial properties similar to the sheaf of differentials $\Omega_{Y/S}$. Namely, if

$$\begin{array}{ccc} Y' & \xrightarrow{u} & Y \\ \downarrow & & \downarrow \\ S' = \operatorname{Spec} R' & \longrightarrow & S = \operatorname{Spec} R \end{array} \tag{6}$$

is a commutative and G-equivariant diagram of schemes (where Y/S and Y'/S' satisfy Assumption 1.1), then we have a natural homomorphism

$$u^* \mathcal{L}_{Y/S} \longrightarrow \mathcal{L}_{Y'/S'} \tag{7}$$

in $\mathfrak{D}^+(Y', G)$. The morphism (7) is an isomorphism in each of the following two cases:

(i) We have $S = S'$ and $Y' \to Y$ is an open immersion.

(ii) The diagram (6) is Cartesian and either $Y \to S$ or $S' \to S$ is flat.

Remark 1.9. — If $Y \to S$ is a local complete intersection, then $\mathcal{L}_{Y/S}$ agrees with Illusie's equivariant cotangent complex, up to canonical quasi-isomorphism. In general, $\mathcal{L}_{Y/S}$ is quasi-isomorphic to Illusie's equivariant cotangent complex, truncated at degree -1. In particular, if $\mathcal{F}$ is a G-$\mathcal{O}_Y$-module then the nth hyperext group $\mathbb{E}\mathrm{xt}^n_G(\mathcal{L}_{Y/S}, \mathcal{F})$ is the "correct" one only for $n \leqslant 1$. See [12], Chapitre III, Corollaire 1.2.9.1.

2. Extensions.

In this section we prove that G-equivariant extensions of the morphism $Y \to S$ by a quasi-coherent G-$\mathcal{O}_Y$-module $\mathcal{F}$ are classified by the group $\mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/S}, \mathcal{F})$. See Corollary 2.3. This result will be the basis for the results on equivariant deformations of $Y \to S$ in §3. Actually, instead of working with extensions of the scheme Y, we prefer to work with the opposite category of extensions of the sheaf $\mathcal{O}_Y$.

2.1. — Let $G \to S = \operatorname{Spec} R$ and $Y \to S$ be as in §1.1. We also fix a G-$\mathcal{O}_Y$-module $\mathcal{F}$ which is a quasi-coherent $\mathcal{O}_Y$-module.

DEFINITION 2.1. — An *equivariant extension of* $\mathcal{O}_Y$ *by* $\mathcal{F}$ is given by a short exact sequence of sheaves of R-modules on Y, of the form

$$0 \to \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Y \to 0,$$

together with a G-action and a structure of sheaf of R-algebras on $\mathcal{E}$ such that the following holds:

(i) The maps $\mathcal{F} \to \mathcal{E}$ and $\mathcal{E} \to \mathcal{O}_Y$ are G-equivariant.

(ii) The map $\mathcal{E} \to \mathcal{O}_Y$ is an R-algebra morphism.

(iii) The sheaf $\mathcal{F}$, considered as a subsheaf of $\mathcal{E}$, is a sheaf of ideals with square zero.

We denote by $\mathfrak{Ext}_G(\mathcal{O}_Y, \mathcal{F})$ the category of all such extensions. Morphisms between extensions are defined in the obvious manner. (The Five Lemma shows that all morphisms are in fact isomorphisms.)

Given an extension $\mathcal{E}$ of $\mathcal{O}_Y$ by $\mathcal{F}$, we get a morphism of S-schemes $Y' \to Y$ (which is a homeomorphism on the underlying topological spaces) such that $\mathcal{E} = \mathcal{O}_{Y'}$. The scheme Y' is called an extension of Y by $\mathcal{F}$.

By taking Baer sums of short exact sequences, one defines a bifunctor $(\mathcal{E}_1, \mathcal{E}_2) \mapsto \mathcal{E}_1 + \mathcal{E}_2$. Together with certain natural transformations $(\mathcal{E}_1+\mathcal{E}_2)+\mathcal{E}_3 \cong \mathcal{E}_1+(\mathcal{E}_2+\mathcal{E}_3)$ and $\mathcal{E}_1+\mathcal{E}_2 \cong \mathcal{E}_2+\mathcal{E}_1$, it gives $\mathfrak{Ext}_G(\mathcal{O}_Y, \mathcal{F})$ the structure of a (strictly commutative) Picard category, see Appendix A.

Theorem 2.2. — *Let $\mathcal{F}$ be a coherent sheaf of G-$\mathcal{O}_Y$-modules. We denote by $\mathfrak{Pic}(\mathbb{R}\mathrm{Hom}_G(\mathcal{L}_{Y/S}, \mathcal{F}))$ the Picard category associated to the derived complex $\mathbb{R}\mathrm{Hom}_G(\mathcal{L}_{Y/S}, \mathcal{F})$, see Appendix A.1 and C.3. Then there exists a natural isomorphism of Picard categories*

$$\mathfrak{Ext}_G(\mathcal{O}_Y, \mathcal{F}) \cong \mathfrak{Pic}(\mathbb{R}\mathrm{Hom}_G(\mathcal{L}_{Y/S}, \mathcal{F})).$$

We will sketch a proof of Theorem 2.2 in the rest of this section. The following corollary corresponds to Theorem 1.5.1 of [13].

Corollary 2.3. — *The group of isomorphism classes of equivariant extensions of Y by $\mathcal{F}$ is canonically isomorphic to $\mathrm{Ext}^1_G(\mathcal{L}_{Y/S}, \mathcal{F})$. Moreover, the group of automorphisms of any fixed equivariant extension of Y by $\mathcal{F}$ is canonically isomorphic to $\mathrm{Hom}_G(\Omega_{Y/S}, \mathcal{F})$.*

2.2. — In the following three subsections we prove a non-equivariant version of Theorem 2.2. To this end, we denote by $\mathfrak{Ext}(\mathcal{O}_Y, \mathcal{F})$ the Picard category of (non-equivariant) extensions of $\mathcal{O}_Y$ by $\mathcal{F}$.

Proposition 2.4. — *Suppose that Y is affine and admits a global chart $\varphi : Y \hookrightarrow P$. Let $\mathcal{L}_\varphi$ be the cotangent complex of φ, see §1.2. Then there exists an isomorphism of Picard categories*

$$F_\varphi : \mathfrak{Ext}(\mathcal{O}_Y, \mathcal{F}) \xrightarrow{\sim} \mathfrak{Pic}(\mathbb{R}\mathrm{Hom}_Y(\mathcal{L}_\varphi, \mathcal{F})).$$

Given a morphism $u : \psi \to \varphi$ of global charts, let

$$\widetilde{u} : \mathfrak{Pic}(\mathbb{R}\mathrm{Hom}_Y(\mathcal{L}_\psi, \mathcal{F})) \cong \mathfrak{Pic}(\mathbb{R}\mathrm{Hom}_Y(\mathcal{L}_\varphi, \mathcal{F}))$$

denote the isomorphism of Picard categories induced from the quasi-isomorphism $u^* : \mathcal{L}_\varphi \to \mathcal{L}_\psi$. There exists an isomorphism of additive functors

$$t_u : \widetilde{u} \circ F_\psi \xrightarrow{\sim} F_\varphi$$

such that the following holds. If $\chi \xrightarrow{v} \psi \xrightarrow{u} \varphi$ is the composition of two morphisms of global charts, then

$$t_{u\circ v} = t_u \circ \widetilde{u}(t_v). \tag{8}$$

Proof. — Using that Y is affine and that P/S is formally smooth one shows that $\mathcal{L}^0_\varphi = \Omega_{P/S} \otimes \mathcal{O}_Y$ is a projective $\mathcal{O}_Y$-module. This implies that

$$\mathbb{R}\mathrm{Hom}_Y(\mathcal{L}_\varphi, \mathcal{F})^{[0,1]} \cong \mathrm{Hom}^\bullet_Y(\mathcal{L}_\varphi, \mathcal{F})^{[0,1]}.$$

Therefore, we may replace $\mathbb{R}\mathrm{Hom}_Y(\mathcal{L}_\varphi, \mathcal{F})$ in the statement of the proposition by the complex $\mathrm{Hom}^\bullet_Y(\mathcal{L}_{Y/S}, \mathcal{F})$. We write $Y = \operatorname{Spec} A$ and $P = \operatorname{Spec} B$. Then φ corresponds to an ideal $I \lhd B$ such that $A = B/I$. We also write $\mathcal{F} = \widetilde{M}$ for some A-module M. With this notation, we have

$$\mathrm{Hom}^\bullet_Y(\mathcal{L}_\varphi, \mathcal{F}) = (\mathrm{Hom}_A(\Omega_{B/R} \otimes A, M) \xrightarrow{\circ d} \mathrm{Hom}_A(I/I^2, M)).$$

An object of $\mathfrak{Ext}(\mathcal{O}_Y, \mathcal{F})$ is given by an extension of R-modules $0 \to M \to E \to A \to 0$, where E carries in addition the structure of an R-algebra such that the following holds. Firstly, $E \to A$ is a homomorphism of R-algebras; secondly, $M^2 = 0$, considered as ideal of E. In the rest of the proof, we shall refer to such an object simply as an *extension*. Since B is formally smooth, there exists a homomorphism of R-algebras $\lambda : B \to E$ lifting the canonical map $B \to A$. Set $\nu := \lambda|_I \bmod I^2$. It is clear that ν is an A-linear morphism $I/I^2 \to M$. We consider ν as an object of $\mathfrak{Pic}(\mathrm{Hom}^\bullet_Y(\mathcal{L}_\varphi, \mathcal{F}))$ and set

$$F_\varphi(E) := \nu.$$

Let $0 \to M \to E' \to A \to 0$ be another extension and $f : E \xrightarrow{\sim} E'$ an isomorphism of extensions. Let $\lambda' : B \to E'$ be an R-algebra morphism lifting $B \to A$ and set $\nu' := \lambda'|_I \bmod I^2$. Then the map $\lambda' - f \circ \lambda : B \to M$ is easily seen to be an R-linear derivation which vanishes on I^2. It corresponds to an A-linear homomorphism $\theta : \Omega_{B/R} \otimes A \to M$ such that $\theta \circ d = \nu' - \nu$. In other words, θ is a homomorphism $\nu \to \nu'$ in $\mathfrak{Pic}(\mathrm{Hom}^\bullet_Y(\mathcal{L}_\varphi, \mathcal{F}))$. We set

$$F_\varphi(f) = \theta.$$

One checks that F_φ is a faithful additive functor.

Given an arbitrary A-linear map $\nu : I/I^2 \to M$, we define the extension E_ν as the pushout of B/I^2 along ν:

$$\begin{array}{ccccccc} 0 \to & I/I^2 & \longrightarrow & B/I^2 & \longrightarrow & A & \to 0 \\ & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle =} & \\ 0 \to & M & \longrightarrow & E_\nu & \longrightarrow & A & \to 0. \end{array} \tag{9}$$

It is easy to see that E_ν carries a unique R-algebra structure such that $M^2 = 0$ and λ is an R-algebra morphism. Moreover, we have $F_\varphi(E_\nu) = \nu$ by construction. Hence F_φ is essentially surjective.

Let $\theta : \nu \to \nu'$ be an isomorphism in $\mathfrak{Pic}(\mathrm{Hom}^{\bullet}_Y(\mathcal{L}_\varphi, \mathcal{F}))$. This means that $\nu' : I/I^2 \to M$ and $\theta : \Omega_{B/R} \otimes A \to M$ are A-linear maps such that $\theta \circ d = \nu' - \nu$. We may identify θ with the corresponding derivation $B/I^2 \to M$; then $\theta|_{I/I^2} = \nu' - \nu$. The universal property of the push-forward shows that there exists a unique R-linear map $f : E_\nu \to E_{\nu'}$ such that $f \circ \lambda = \lambda'$. By construction we have $F_\varphi(f) = \theta$. Hence F_φ is fully faithful and even an isomorphism of Picard categories.

Now let $\psi : Y \hookrightarrow Q$ be another global chart and $u : \psi \to \varphi$ a morphism of charts. We write $Q = \operatorname{Spec} B'$, $A = B'/I'^2$ and consider u as a morphism of R-algebras $B \to B'$. Let E be an extension and $\lambda : B/I^2 \to E$ (resp. $\lambda' : B'/I'^2 \to E$) a lift of $B/I^2 \to A$ (resp. of $B'/I'^2 \to A$). By definition we have

$$F_\varphi(E) = \lambda|_I \pmod{I^2}, \qquad \widetilde{u} \circ F_\psi(E) = \lambda' \circ u|_I \pmod{I^2}.$$

Again it is clear that $\lambda - \lambda' \circ u \pmod{I^2}$ is a derivation $B/I^2 \to M$, corresponding to an A-linear map $\theta : \Omega_{B/R} \otimes A \to M$ and representing a homomorphism $\widetilde{u} \circ F_\psi(E) \to F_\varphi(E)$. We set

$$t_u(E) = \theta.$$

A formal verification shows that t_u is a morphism of additive functors $\widetilde{u} \circ F_\psi \cong F_\varphi$ and that (8) holds. □

2.3. — Let $\underline{\mathfrak{Ext}}(\mathcal{O}_Y, \mathcal{F})$ denote the Y-stack whose fiber over a given open subset $U \subset Y$ is the Picard category $\mathfrak{Ext}(\mathcal{O}_U, \mathcal{F}|_U)$ (here Y-stack means a stack over the Zariski site of Y). It is clear that $\underline{\mathfrak{Ext}}(\mathcal{O}_Y, \mathcal{F})$ is a Picard stack, see Appendix A.2.

PROPOSITION 2.5. — *We assume that Y is affine and admits a global chart $\varphi : Y \hookrightarrow P$.*

(i) *Let $U \subset Y$ be an affine open. Then the natural functor*

$$\mathfrak{Pic}(\mathbb{R}\mathrm{Hom}_U(\mathcal{L}_\varphi|_U, \mathcal{F}|_U)) \longrightarrow \underline{\mathfrak{Pic}}(\mathbb{R}\mathcal{H}om_Y(\mathcal{L}_\varphi, \mathcal{F}))(U) \tag{10}$$

is an isomorphism.

(ii) *There exists a unique isomorphism of Picard stacks*

$$\underline{F}_\varphi : \underline{\mathfrak{Ext}}(\mathcal{O}_Y, \mathcal{F}) \xrightarrow{\sim} \underline{\mathfrak{Pic}}(\mathbb{R}\mathcal{H}om_Y(\mathcal{L}_\varphi, \mathcal{F}))$$

such that for each affine open $U \subset Y$ the restriction of $\underline{F}_\varphi$ to the fiber $\underline{\mathfrak{Ext}}(\mathcal{O}_Y, \mathcal{F})(U) = \mathfrak{Ext}(U, \mathcal{F}|_U)$ is equal (up to canonical isomorphism) to the composition of $F_{\varphi|_U}$ with (10).

Proof. — Part (i) follows from Proposition A.2 and the fact that the cohomology of the complex $\mathbb{R}\mathcal{H}om_Y(\mathcal{L}_\varphi, \mathcal{F})$ consists of quasi-coherent sheaves. Part (ii) is a formal consequence of (i) and is left to the reader. □

2.4. — We will now globalize the isomorphism of Picard stacks constructed in the previous two subsections. To this end, we will use the notation introduced in §1.3. In particular, $(U_i)_{i\in I}$ is a covering of Y by affine opens, admitting local charts $\varphi_i : U_i \hookrightarrow P_i$. By Proposition 1.6 we obtain, for each ordered pair $i < j$, an essentially commutative square of quasi-isomorphisms

$$\begin{array}{ccc} \mathcal{L}_{Y/S}|_{U_{i,j}} & \xrightarrow{\beta_i} & \mathcal{L}_{\varphi_i}|_{U_{i,j}} \\ \downarrow{\scriptstyle \beta_j} & & \downarrow{\scriptstyle \partial^1_{i,j}} \\ \mathcal{L}_{\varphi_j}|_{U_{i,j}} & \xrightarrow{\partial^0_{i,j}} & \mathcal{L}_{\varphi_{i,j}}, \end{array} \tag{11}$$

i.e. a homotopy $s_{i,j} : \partial^1_{i,j} \circ \beta_i \sim \partial^0_{i,j} \circ \beta_j$ such that the cocycle relation (4) holds. Set

$$\underline{\mathfrak{P}} := \underline{\mathfrak{Pic}}(\mathbb{R}\mathcal{H}om_Y(\mathcal{L}_{Y/S}, \mathcal{F})), \qquad \underline{\mathfrak{P}}_{\underline{i}} = \underline{\mathfrak{Pic}}(\mathbb{R}\mathcal{H}om_Y(\mathcal{L}_{\varphi_{\underline{i}}}, \mathcal{F})).$$

The diagram (11) yields an essentially commutative square of isomorphisms of Picard stacks

$$\begin{array}{ccc} \underline{\mathfrak{P}}_{i,j} & \xrightarrow{\widetilde{\partial}^1_{i,j}} & \underline{\mathfrak{P}}_i|_{U_{i,j}} \\ {\scriptstyle \widetilde{\partial}^0_{i,j}}\downarrow & & \downarrow{\scriptstyle \widetilde{\beta}_i} \\ \underline{\mathfrak{P}}_j|_{U_{i,j}} & \xrightarrow[\widetilde{\beta}_j]{} & \underline{\mathfrak{P}}|_{U_{i,j}}, \end{array} \tag{12}$$

i.e. an isomorphism of additive functors $\widetilde{s}_{i,j} : \widetilde{\beta}_i \circ \widetilde{\partial}^1_{i,j} \sim \widetilde{\beta}_j \circ \widetilde{\partial}^0_{i,j}$ such that

$$\widetilde{\partial}^0_{i,j,k}(\widetilde{s}_{j,k}) \circ \widetilde{\partial}^2_{i,j,k}(\widetilde{s}_{i,j}) = \widetilde{\partial}^1_{i,j,k}(\widetilde{s}_{i,k}) \tag{13}$$

for all triples $i < j < k$.

PROPOSITION 2.6. — *There exists an isomorphism of Picard stacks*

$$\underline{F} : \underline{\mathfrak{Ext}}(\mathcal{O}_Y, \mathcal{F}) \xrightarrow{\sim} \underline{\mathfrak{P}} = \underline{\mathfrak{Pic}}(\mathbb{R}\mathcal{H}om_Y(\mathcal{L}_{Y/S}, \mathcal{F}))$$

and for each index i an isomorphism of additive functors $u_i : \underline{F}|_{U_i} \cong \widetilde{\beta}_i \circ \underline{F}_{\varphi_i}$.

Proof. — Let $i < j$. By Proposition 2.4 we obtain two natural isomorphisms of additive functors

$$F_{\varphi_i}|_{U_{i,j}} \cong \widetilde{\partial}^1_{i,j} \circ F_{\varphi_{i,j}}, \qquad F_{\varphi_j}|_{U_{i,j}} \cong \widetilde{\partial}^0_{i,j} \circ F_{\varphi_{i,j}}.$$

Using the essentially commutative square (12) they can be extended to an isomorphism

$$u_{i,j} : \widetilde{\beta}_j \circ F_{\varphi_j}|_{U_{i,j}} \cong \widetilde{\beta}_j \circ \widetilde{\partial}^0_{i,j} \circ F_{\varphi_{i,j}} \cong \widetilde{\beta}_i \circ \widetilde{\partial}^1_{i,j} \circ F_{\varphi_{i,j}} \cong \widetilde{\beta}_i \circ F_{\varphi_i}|_{U_{i,j}}.$$

A tedious but elementary verification, using (8) and (13), shows that $u_{i,j}$ satisfies the obvious cocycle relation. The proposition follows. □

Using the canonical isomorphism

$$\mathbb{R}\mathrm{Hom}_Y(\mathcal{L}_{Y/S}, \mathcal{F})) \cong \mathbb{R}\Gamma(Y, \mathbb{R}\mathcal{H}om_Y(\mathcal{L}_{Y/S}, \mathcal{F}))$$

and Proposition A.2, we obtain a non-equivariant version of Theorem 2.2:

Corollary 2.7. — *There is a natural isomorphism of Picard categories*

$$F : \mathfrak{Ext}(\mathcal{O}_Y, \mathcal{F}) \xrightarrow{\sim} \mathfrak{Pic}(\mathbb{R}\mathrm{Hom}_Y(\mathcal{L}_{Y/S}, \mathcal{F})).$$

2.5. — We are now going to prove Theorem 2.2 in full generality. In the sequel, R' will always denote a *flat* R-algebra, and a prime stands for base change with respect to $R \to R'$; for instance $Y' := Y \otimes_R R'$. By Remark 1.8 we have natural isomorphisms

$$\mathcal{L}_{Y/S} \otimes_R R' \cong \mathcal{L}_{Y'/S'}. \tag{14}$$

and

$$\mathbb{R}\mathrm{Hom}_Y(\mathcal{L}_{Y/S}, \mathcal{F}) \otimes_R R' \cong \mathbb{R}\mathrm{Hom}_{Y'}(\mathcal{L}_{Y'/S'}, \mathcal{F}'). \tag{15}$$

To simplify the notation, we will henceforth write

$$A := \mathbb{R}\mathrm{Hom}_Y(\mathcal{L}_{Y/S}, \mathcal{F})^{[0,1]}.$$

Note that A is a complex of G-R-modules of amplitude $[0, 1]$, well defined up to canonical isomorphism in $\mathcal{D}^{[0,1]}(Y, G)$.

Let R' be a flat R-algebra and $\sigma \in G(R')$. The automorphism $A' \xrightarrow{\sim} A'$, $a \mapsto a^\sigma$ induces an isomorphism of Picard categories $\widetilde{\sigma}$: $\mathfrak{Pic}(A') \xrightarrow{\sim} \mathfrak{Pic}(A')$. Given an extension $0 \to \mathcal{F}' \to \mathcal{E}' \to \mathcal{O}_{Y'} \to 0$ (*i.e.* an object of $\mathfrak{Ext}(Y', \mathcal{F})^{\mathrm{op}}$), let $\mathcal{E}^\sigma$ be the extension

$$0 \to \mathcal{F}' \cong \sigma^*\mathcal{F} \longrightarrow \sigma^*\mathcal{E} \longrightarrow \sigma^*\mathcal{O}_{Y'} \cong \mathcal{O}_{Y'} \to 0$$

Here the isomorphisms $\mathcal{F}' \cong \sigma^*\mathcal{F}$ and $\sigma^*\mathcal{O}_{Y'} \cong \mathcal{O}_{Y'}$ come from the G-action on $\mathcal{F}$ and $\mathcal{O}_Y$. Given an isomorphism $f : \mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2$ of extensions, then f^σ, as defined in §C.1, is an isomorphism $\mathcal{E}_1^\sigma \xrightarrow{\sim} \mathcal{E}_2^\sigma$. One checks that the association $\mathcal{E} \mapsto \mathcal{E}^\sigma$ is an automorphism of Picard categories

$$\widetilde{\sigma} : \mathfrak{Ext}(Y', \mathcal{F}') \xrightarrow{\sim} \mathfrak{Ext}(Y', \mathcal{F}').$$

One checks:

LEMMA 2.8. — *We have an essentially commutative diagram*

$$\begin{array}{ccc} \mathfrak{Pic}(A') & \xrightarrow{F'} & \mathfrak{Ext}(Y',\mathcal{F}) \\ {\scriptstyle\tilde\sigma}\Big\downarrow & & \Big\downarrow{\scriptstyle\tilde\sigma} \\ \mathfrak{Pic}(A') & \xrightarrow{F'} & \mathfrak{Ext}(Y',\mathcal{F}) \end{array}$$

(the isomorphism F' is given by Corollary 2.7).

It follows from Proposition B.2 that

$$\mathbb{R}\mathrm{Hom}_G(\mathcal{L}_{Y/S},\mathcal{F})^{[0,1]} \cong \mathrm{Tot}(K)^{[0,1]}, \tag{16}$$

where K is the double complex

$$K:\ \left\{\begin{array}{ccc} A^0 & \xrightarrow{d} & A^1 \\ \Big\downarrow\partial & & \Big\downarrow\partial \\ C^1(G,A^0) & \xrightarrow{-d} & C^1(G,A^1) \\ \Big\downarrow\partial & & \Big\downarrow\partial \\ C^2(G,A^0) & \xrightarrow{d} & C^2(G,A^1) \\ \Big\downarrow\partial & & \Big\downarrow\partial \\ \cdots & & \cdots \end{array}\right.$$

We are now going to construct an isomorphism of Picard categories

$$F^G : \mathfrak{Ext}_G(\mathcal{O}_Y,\mathcal{F}) \xrightarrow{\sim} \mathfrak{Pic}(\mathrm{Tot}(K)). \tag{17}$$

Together with (16), this will complete the proof of Theorem 2.2.

An object of $\mathfrak{Ext}_G(\mathcal{O}_Y,\mathcal{F})$ is an object $\mathcal{E}$ of $\mathfrak{Ext}(\mathcal{O}_Y,\mathcal{F})$, together with an action of G on $\mathcal{E}$ such that the maps $\mathcal{F}\to\mathcal{E}$ and $\mathcal{E}\to\mathcal{O}_Y$ are G-equivariant. Such an action is determined by the following data. For each flat R-algebra R' and group element σ we get an isomorphism $f_\sigma:\mathcal{E}' \xrightarrow{\sim} \mathcal{E}^\sigma$ in $\mathfrak{Ext}(\mathcal{O}_{Y'},\mathcal{F}')$ such that

$$f_{\sigma\tau} = f_\sigma^\tau \circ f_\tau \tag{18}$$

holds for all pairs $\sigma,\tau\in G(R')$. Let F be the isomorphism of Corollary 2.7 and set $\nu := F(\mathcal{E}) \in A^1$, $\theta_\sigma := F(f_\sigma) \in (A^0)'$. By Lemma 2.8, θ_σ is an isomorphism $\nu' \xrightarrow{\sim} \nu^\sigma$, i.e.

$$d(\theta_\sigma) = \nu^\sigma - \nu'. \tag{19}$$

Equation (18) shows that

$$\theta_{\sigma\tau} = \theta_\sigma^\tau + \theta_\tau.$$

In other words, the association $\sigma \mapsto \theta_\sigma$ corresponds to a 1-cocycle θ, *i.e.* an element of $Z^1(G, A^0) = \mathrm{Ker}(C^1(G, A^0) \xrightarrow{\partial} C^2(G, A^0))$, see §B.3. Also, Equation (19) means that $d(\theta) = \partial(\nu)$. We have shown that the pair (ν, θ) lies in $Z^1(\mathrm{Tot}(K))$, *i.e.* represents an object of $\mathfrak{Pic}(\mathrm{Tot}(K))$. We set

$$F^G(\mathcal{E}) := (\nu, \theta).$$

Now let $g : \mathcal{E}_1 \xrightarrow{\sim} \mathcal{E}_2$ be an isomorphism in $\mathfrak{Ext}_G(Y, \mathcal{F})$. Set $F^G(\mathcal{E}_1) := (\nu_1, \theta_1)$, $F^G(\mathcal{E}_2) := (\nu_2, \theta_2)$ and $\eta := F(g)$. Then $d(\eta) = \nu_2 - \nu_1$. By definition, g is G-equivariant, so the diagram

$$\begin{array}{ccc} \mathcal{E}_1' & \xrightarrow{g'} & \mathcal{E}_2' \\ {\scriptstyle f_{1,\sigma}}\downarrow & & \downarrow{\scriptstyle f_{2,\sigma}} \\ \mathcal{E}_1^\sigma & \xrightarrow{g^\sigma} & \mathcal{E}_2^\sigma \end{array}$$

commutes for all $\sigma \in G(R')$. By Lemma 2.8 this means that

$$\theta_{2,\sigma} - \theta_{1,\sigma} = \eta^\sigma - \eta', \tag{20}$$

or, equivalently, $\theta_2 - \theta_1 = \partial(\eta)$. It follows that η corresponds to an isomorphism $(\nu_1, \theta_1) \xrightarrow{\sim} (\nu_2, \theta_2)$ in $\mathfrak{Pic}(\mathrm{Tot}(K))$. We set

$$F^G(g) := \eta.$$

We leave it to the reader to check that F^G is indeed an isomorphism of Picard categories. Now the proof of Theorem 2.2 is complete. □

3. Deformations.

In this section we show how one can classify equivariant deformations of $Y \to S$ along an infinitesimal extension $S \hookrightarrow S'$, using the equivariant cotangent complex. The main result is Theorem 3.3. In §3.3 we discuss how this result behaves under localization to a formal neighborhood of a point (Theorem 3.11).

3.1. — Let R' be a commutative ring and $\mathfrak{a} \lhd R'$ an ideal with $\mathfrak{a}^2 = 0$. We set $R := R'/\mathfrak{a}$, $S' := \mathrm{Spec}\, R'$ and $S := \mathrm{Spec}\, R$. Furthermore, let $G' \to S'$ be a flat affine group scheme and $Y \to S$ a flat morphism together with an S-linear action of $G := G' \times_{S'} S$ on Y.

Definition 3.1. — An *equivariant deformation* of $Y \to S$ to S' is a flat morphism $Y' \to S'$ together with an S'-linear action of G' on Y' and a G-equivariant isomorphism of S-schemes $Y \cong Y' \times_{S'} S$. An isomorphism of deformations is a G'-equivariant isomorphism of S'-schemes $Y_1' \cong Y_2'$ which induces the identity on Y.

Theorem 3.3 below shows how to classify isomorphism classes of equivariant deformations of $Y \to S$ using the equivariant cotangent complex $\mathcal{L}_{Y/S}$. However, in the proof of Theorem 3.3 we will also use the cotangent complex of the composed morphism $Y \to S \hookrightarrow S'$. Therefore, to be able to use the definition of $\mathcal{L}_{Y/S'}$ in §1, we make the following assumption.

Assumption 3.2. — Every point of Y is contained in an affine and G-stable open $U \subset Y$ such that the following holds. There exists a smooth affine S'-scheme $P' \to S'$, an S'-linear action of G' on P' and a G-equivariant closed immersion $\varphi : U \hookrightarrow P'$. In other words, Assumption 1.1 holds for the composed morphism $Y \to S'$ and the group scheme G'.

Under this assumption we can prove:

Theorem 3.3.

(i) *There exists an element (called the obstruction)*

$$\omega = \omega(Y/S, S') \in \mathbb{E}\mathrm{xt}^2_G(\mathcal{L}_{Y/S}, \mathcal{O}_Y) \otimes_R \mathfrak{a},$$

depending functorially on $Y \to S \hookrightarrow S'$, whose vanishing is necessary and sufficient for the existence of an equivariant deformation of $Y \to S$ to S'.

(ii) *Suppose that $\omega = 0$. Then the set of isomorphism classes of deformations of $Y \to S$ to S' is, in a natural way, a principal homogeneous space under the Abelian group*

$$\mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/S}, \mathcal{O}_Y) \otimes_R \mathfrak{a}.$$

This corresponds to Proposition 2.3 of [13]. However, if $Y \to S$ is not a local complete intersection, then our definition of $\mathcal{L}_{Y/S}$ does not always give the same hyperext group $\mathbb{E}\mathrm{xt}^2_G(\mathcal{L}_{Y/S}, \mathcal{O}_Y)$ as in [12] and [13]. In particular, our obstruction does not necessarily agree with the obstruction constructed in [12] and [13], simply because it does not lie in the same group. See also Remark 3.6 below.

3.2. Proof of Theorem 3.3. — Let $\mathcal{F} := \mathcal{O}_Y \otimes_R \mathfrak{a}$. Since $Y \to S$ is flat we have natural isomorphisms

$$\mathbb{E}\mathrm{xt}^n_G(\mathcal{L}_{Y/S}, \mathcal{F}) \cong \mathbb{E}\mathrm{xt}^n_G(\mathcal{L}_{Y/S}, \mathcal{O}_Y) \otimes_R \mathfrak{a} \tag{21}$$

for all n. By Assumption 3.2 and Definition 1.5, the equivariant cotangent complexes $\mathcal{L}_{Y/S}$ and $\mathcal{L}_{Y/S'}$ are defined as complexes of G-$\mathcal{O}_Y$-modules and we have a natural G-equivariant morphism $\mathcal{L}_{Y/S'} \to \mathcal{L}_{Y/S}$.

Lemma 3.4. — *There is a natural exact sequence in $\mathfrak{D}^+(Y,G)$*

$$0 \longrightarrow \mathcal{F}[1] \longrightarrow \mathcal{L}_{Y/S'} \longrightarrow \mathcal{L}_{Y/S} \longrightarrow 0. \tag{22}$$

More precisely, the natural morphism $\mathcal{L}_{Y/S'} \to \mathcal{L}_{Y/S}$ is surjective in all degrees, and there exists a G-equivariant quasi-isomorphism $\mathcal{F}[1] \to \mathrm{Ker}(\mathcal{L}_{Y/S'} \to \mathcal{L}_{Y/S})$. (Recall that $\mathcal{F}[1]$ denotes the complex where $\mathcal{F}$ is placed in degree -1.)

Proof. — Let $\varphi' : U \hookrightarrow P'$ be a local chart for the morphism $Y \to S'$ and $\mathcal{I}' \subset \mathcal{O}_{P'}$ the corresponding sheaf of ideals. Then φ' gives rise to a local chart $\varphi : U \hookrightarrow P := P \times_{S'} S$ for the morphism $Y \to S$. The corresponding sheaf of ideals is $\mathcal{I} := \mathcal{I}'/\mathcal{F}$. It is clear that $\Omega_{P'/S'} \otimes \mathcal{O}_Y \cong \Omega_{P/S} \otimes \mathcal{O}_Y$. Moreover, we have a short exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I}'/\mathcal{I}'^2 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow 0. \tag{23}$$

Hence it follows from Definition 1.5 that $\mathcal{L}_{Y/S'} \to \mathcal{L}_{Y/S}$ is surjective in all degrees and that its kernel is isomorphic to the Čech-resolution of $\mathcal{F}[1]$ (with respect to the open covering (U_i) used to define $\mathcal{L}_{Y/S}$). This proves the lemma. □

Let $\mathcal{G}$ be a G-$\mathcal{O}_Y$-module. The short exact sequence (22) gives rise to the following long exact sequence

$$\begin{aligned} 0 \to \mathrm{Ext}^1_G(\mathcal{L}_{Y/S},\mathcal{G}) \longrightarrow \mathrm{Ext}^1_G(\mathcal{L}_{Y/S'},\mathcal{G}) &\longrightarrow \mathrm{Hom}_G(\mathcal{F},\mathcal{G}) \\ &\xrightarrow{\partial} \mathrm{Ext}^2_G(\mathcal{L}_{Y/S},\mathcal{G}). \end{aligned} \tag{24}$$

This applies in particular to the case $\mathcal{G} := \mathcal{F}$. We define the obstruction $\omega := \omega(Y/S,S')$ as the image of the identity map $\mathrm{Id} : \mathcal{F} \to \mathcal{F}$ under the boundary map ∂ in (24). Now Theorem 3.3 follows from Corollary 2.3, the exactness of (24) and the following proposition.

Proposition 3.5. — *There is a natural bijection between*

(a) *deformations of $Y \to S$ to S', up to isomorphism, and*

(b) *elements of $\mathrm{Ext}^1_G(\mathcal{L}_{Y/S'},\mathcal{F})$ which are mapped to $\mathrm{Id}_{\mathcal{F}}$ (by the middle arrow in* (24)*).*

Proof. — Let $\mathcal{G}$ be a quasi-coherent G-$\mathcal{O}_Y$-module. By Corollary 2.3, an element of $\mathrm{Ext}^1_G(\mathcal{L}_{Y/S'},\mathcal{G})$ corresponds to an equivariant extensions of Y

by $\mathcal{G}$, *i.e.* a closed equivariant embedding $Y \hookrightarrow Y'$ of S'-schemes defined by an ideal $\mathcal{J} \subset \mathcal{O}_{Y'}$, together with an isomorphism $\mathcal{G} \cong \mathcal{J}$ of G-$\mathcal{O}_Y$-modules. We obtain a morphism of G-$\mathcal{O}_Y$-modules

$$\mathcal{F} = \mathcal{O}_Y \otimes_R \mathfrak{a} \longrightarrow \mathcal{J} \cong \mathcal{G}. \tag{25}$$

By reexamination of the proof of Theorem 2.2 one shows that the middle arrow of the sequence (24) maps the element of $\mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/S'}, \mathcal{G})$ corresponding to the extension Y' to the morphism (25). Also, the local criterion of flatness (see [15], Theorem 49) shows that the morphism $Y' \to S'$ is flat if and only if (25) is an isomorphism.

The proposition follows easily from these arguments. First, an equivariant extension of Y by $\mathcal{F}$ for which (25) is the identity on $\mathcal{F}$ gives rise to an equivariant deformation of $Y \to S$ to S'. Conversely, let $Y' \to S'$ be an equivariant deformation of $Y \to S$, and let $\mathcal{J} \subset \mathcal{O}_{Y'}$ be the sheaf of ideals corresponding to the embedding $Y \hookrightarrow Y'$. Since $Y' \to S'$ is flat by assumption, the natural map $\mathcal{F} \to \mathcal{J}$ is an isomorphism. Using this isomorphism, we can see Y' as an equivariant extension of Y by $\mathcal{F}$ for which (25) is the identity on $\mathcal{F}$. This concludes the proof of the proposition and hence of Theorem 3.3. □

Remark 3.6. — The short exact sequence of Lemma 3.4 should be compared with the *transitivity triangle* attached to the composition of morphisms $Y \to S \hookrightarrow S'$ in [12]:

$$\begin{tikzcd} & \mathcal{L}^I_{Y/S} \arrow[dl] & \\ \mathcal{L}^I_{S/S'} \otimes \mathcal{O}_Y \arrow[rr] & & \mathcal{L}^I_{Y/S'} \arrow[ul] \end{tikzcd} \tag{26}$$

Here $\mathcal{L}^I_{Y/S}$ denotes the cotangent complex in the sense of Illusie.

Now suppose that $Y \to S$ is a local complete intersection. Then $\mathcal{L}_{Y/S} \cong \mathcal{L}^I_{Y/S}$. We also have natural morphisms $\mathcal{L}^I_{Y/S'} \to \mathcal{L}_{Y/S'}$ and $\mathcal{L}^I_{S/S'} \otimes \mathcal{O}_Y \to \mathcal{F}[1]$, but they are quasi-isomorphisms only if $S \hookrightarrow S'$ is a local complete intersection (which is typically not the case). Nevertheless, one can show that the obstruction ω in Theorem 3.3 is the same as the obstruction obtained by Illusie's theory (via the canonical isomorphism $\mathbb{E}\mathrm{xt}^2_G(\mathcal{L}_{Y/S}, \mathcal{O}_Y) \cong \mathbb{E}\mathrm{xt}^2_G(\mathcal{L}^I_{Y/S}, \mathcal{O}_Y)$).

3.3. Localization. — Keeping the notation introduced before, we now impose the following finiteness conditions.

Assumption 3.7.

(i) *The affine scheme $S = \mathrm{Spec}\, R$ is local, Artinian and Noetherian.*

(ii) *The group scheme G is finite and flat over S.*

(iii) *The scheme Y is either of finite type over S or the localization of something of finite type over S.*

It follows from Part (i) and (iii) of the assumption that Y is Noetherian.

By Assumption 1.1 the action of G on Y is admissible; hence the quotient scheme $X := Y/G$ exists. It follows from Assumption 3.7 (ii) that the projection $\pi : Y \to X$ is finite. Let $x \in X$ be a point. Let $\widehat{X} = \operatorname{Spec} \widehat{O}_{X,x}$ denote the completion of X at x and set $\widehat{Y} := Y \times_X \widehat{X}$. Since $\pi : Y \to X$ is finite, $\widehat{Y}$ is naturally isomorphic to the completion of Y along the fiber $\pi^{-1}(x)$. The action of G on Y induces an action of G on $\widehat{Y}$. Since $\widehat{X} \to X$ is flat, we have $\widehat{Y}/G = \widehat{X}$.

Let $u : \widehat{Y} \to Y$ denote the canonical map. By Remark 1.8 we have a canonical morphism of complexes of G-$\mathcal{O}_{\widehat{Y}}$-modules

$$u^*\mathcal{L}_{Y/S} \longrightarrow \mathcal{L}_{\widehat{Y}/S}. \tag{27}$$

A technical complication arises from the fact that (27) is in general not a quasi-isomorphism. However, the next proposition shows that this does not really matter to us.

Proposition 3.8. — *Let $\mathcal{F}$ be a coherent sheaf of G-$\mathcal{O}_Y$-modules. There exists an isomorphism of Picard categories*

$$F_x : \mathfrak{Ext}_G(\mathcal{O}_{\widehat{Y}}, u^*\mathcal{F}) \xrightarrow{\sim} \mathfrak{Pic}(\mathbb{R}\mathrm{Hom}_G(u^*\mathcal{L}_{Y/S}, u^*\mathcal{F}))$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{Ext}_G(\mathcal{O}_Y, \mathcal{F}) & \longrightarrow & \mathfrak{Ext}_G(\mathcal{O}_{\widehat{Y}}, u^*\mathcal{F}) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F_x} V \\ \mathfrak{Pic}(\mathbb{R}\mathrm{Hom}_G(\mathcal{L}_{Y/S}, \mathcal{F})) & \longrightarrow & \mathfrak{Pic}(\mathbb{R}\mathrm{Hom}_G(u^*\mathcal{L}_{Y/S}, u^*\mathcal{F})). \end{array} \tag{28}$$

Here the upper horizontal arrow is the functor which sends an extension Y' of Y by $\mathcal{F}$ to the completion of Y' along the fiber $\pi^{-1}(x)$. The left vertical arrow is the isomorphism from Theorem 2.2. The lower horizontal arrow is the natural pullback map.

Proof. — The construction of the equivalence F_x is very similar to the construction of F in the proof of Theorem 2.2. An essential difference appears only in the first step, see §2.2. We will therefore assume for the rest of the proof that Y is affine and that $G = 1$.

Replacing Y by its localization at any point $y \in \pi^{-1}(x)$ we may assume that $Y = \operatorname{Spec} A$ is local. We have $\widehat{Y} = \operatorname{Spec} \widehat{A}$, where $\widehat{A}$ is the completion of A. The coherent sheaf $\mathcal{F}$ is given by a finite A-module M; the pullback $u^*\mathcal{F}$ corresponds to the $\widehat{A}$-module $\widehat{M} := M \otimes_A \widehat{A}$. Since M is a finite A-module, $\widehat{M}$ is the $\mathfrak{m}_A$-adic completion of M. By Assumption 3.7 (iii) we can write $A = B/I$, where B is the localization of a polynomial ring over R and $I \lhd B$ is an ideal. By Assumption 3.7 (i) the ring B is Noetherian and hence I is finitely generated. Moreover, $\widehat{A} = \widehat{B}/\widehat{I}$, where $\widehat{B}$ is the completion of B at its maximal ideal and $\widehat{I} := I\widehat{B}$. The ring $\widehat{B}$ is a power series ring over R. In general, B is not formally smooth over R but only $\mathfrak{m}_B$-smooth (see [16]; note that "formal smoothness" is called "0-smoothness" in loc.cit.).

The complex $u^*\mathcal{L}_{Y/S}$ corresponds to the complex of $\widehat{A}$-modules

$$\widehat{L} := (\widehat{I}/\widehat{I}^2 \longrightarrow \Omega_{B/R} \otimes_B \widehat{A}).$$

The canonical map $\Omega_{B/R} \otimes_B \widehat{A} \to \Omega_{\widehat{B}/R} \otimes_{\widehat{B}} \widetilde{A}$ is injective but in general not surjective. However, $\Omega_{B/R} \otimes \widehat{A}$ is mapped isomorphically onto $\Omega^{\mathrm{cont}}_{\widehat{B}/R} \otimes \widehat{A}$, where

$$\Omega^{\mathrm{cont}}_{\widehat{B}/R} := \Omega_{\widehat{B}/R} / (\cap_n \mathfrak{m}^n_{\widehat{B}} \cdot \Omega_{\widehat{B}/R})$$

denotes the module of *continuous differentials*. Since $\Omega_{B/R} \otimes \widehat{A}$ is a free $\widehat{A}$-module, we have

$$\begin{aligned} (29) \qquad & \mathbb{R}\mathrm{Hom}_Y(u^*\mathcal{L}_{Y/S}, u^*\mathcal{F})^{[0,1]} \\ & = \left(\mathrm{Hom}_{\widehat{A}}(\Omega_{B/R} \otimes \widehat{A}, \widehat{M}) \longrightarrow \mathrm{Hom}_{\widehat{A}}(\widehat{I}/\widehat{I}^2, \widehat{M})\right). \end{aligned}$$

An object of the Picard category $\mathfrak{Ext}(\widehat{Y}, u^*\mathcal{F})$ is given by an extension $\widehat{M} \to E \to \widehat{A}$ of R-modules, with $\widehat{M}^2 = 0$. In what follows we will refer to such an object simply as an *extension*.

Lemma 3.9. — *Let $\widehat{M} \to E \to \widehat{A}$ be an extension and let $\mathfrak{m}_E \lhd E$ denote the inverse image of the maximal ideal $\mathfrak{m}_{\widehat{A}}$ of $\widehat{A}$.*

(i) *The ring E is complete with respect to the ideal $\mathfrak{m}_E$.*

(ii) *There exists a continuous lift $\lambda : \widehat{B} \to E$ of the canonical map $\widehat{B} \to \widehat{A}$.*

Proof. — Look at the following ladder with exact rows:

$$(30) \qquad \begin{array}{ccccccccc} 0 & \to & \widehat{M} & \to & E & \to & \widehat{A} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \varprojlim \widehat{M}/(\mathfrak{m}^n_E \cap \widehat{M}) & \to & \varprojlim E/\mathfrak{m}^n_E & \to & \varprojlim \widehat{A}/\widehat{\mathfrak{m}}^n & \to & 0. \end{array}$$

The vertical arrow on the right is an isomorphism by definition. An argument similar to the one used in the proof of the Artin–Rees Lemma (see [16], Theorem 8.5) shows that there exists a constant $c > 0$ such that
$$\mathfrak{m}_E^n \cap \widehat{M} \subset \mathfrak{m}_{\widehat{A}}^{n-c} \cdot \widehat{M}$$
for all $n \geqslant c$ (here we use that A and hence $\widehat{A}$ is Noetherian). Therefore, the vertical arrow in (30) on the left is an isomorphism. Now the Five-Lemma implies that the vertical arrow in the middle is an isomorphism, too. This proves (i). Part (ii) of the lemma follows from Part (i) and the $\mathfrak{m}_B$-smoothness of $\widehat{B}$. □

Using this lemma, the construction of the equivalence F_x is essentially the same as in the proof of Proposition 2.5. It is also clear from this construction that the diagram (28) commutes. There are two points one has to pay attention to. The first is to consider only continuous lifts $\lambda : \widehat{B} \to E$. The second is this: if E' is another extension, $\lambda' : \widehat{B} \to E'$ a lift and $f : E \xrightarrow{\sim} E'$ an isomorphism of extensions, then $\lambda' - f \circ \lambda : \widehat{B} \to \widehat{M}$ is a *continuous* R-linear derivation which vanishes on $\widehat{I}^2$; it therefore corresponds to an $\widehat{A}$-linear map $\theta : \Omega_{B/R} \otimes \widehat{A} \to \widehat{M}$. Moreover, any R-linear derivation $\widehat{B} \to \widehat{M}$ is automatically continuous because $\widehat{M}$ is complete and hence separated with respect to the $\mathfrak{m}_A$-adic topology. This completes the proof of the proposition. □

Remark 3.10. — The proposition is essentially equivalent with the statement that the homomorphism
$$\mathbb{E}\mathrm{xt}^n_G(\mathcal{L}_{\widehat{Y}/S}, u^*\mathcal{F}) \longrightarrow \mathbb{E}\mathrm{xt}^n_G(u^*\mathcal{L}_{\widehat{Y}/S}, u^*\mathcal{F})$$
induced by (27) is an isomorphism for $n = 0, 1$. I suspect that this is true for $n > 1$ as well, but I don't know how to prove this.

For $n \geqslant 0$ we write $\mathcal{E}xt^n_G(\mathcal{L}_{Y/S}, \mathcal{F})^{\widehat{}}_x$ for the $\widehat{O}_{X,x}$-module $\mathcal{E}xt^n_G(\mathcal{L}_{Y/S}, \mathcal{F})_x \otimes \widehat{O}_{X,x}$. It follows from flatness of $\widehat{X} \to X$ that
$$\mathcal{E}xt^n_G(\mathcal{L}_{Y/S}, \mathcal{F})^{\widehat{}}_x = \mathbb{E}\mathrm{xt}^n_G(u^*\mathcal{L}_{Y/S}, u^*\mathcal{F}).$$
The local-global spectral sequence from §C.4 gives rise to a *localization map*
$$\mathbb{E}\mathrm{xt}^n_G(\mathcal{L}_{Y/S}, \mathcal{F}) \longrightarrow \mathcal{E}xt^n_G(\mathcal{L}_{Y/S}, \mathcal{F})^{\widehat{}}_x.$$

THEOREM 3.11. — *Let $S \hookrightarrow S' = \operatorname{Spec} R'$ be a small extension, with $R = R'/\mathfrak{a}$. Let ω be the obstruction for lifting Y to S'. Also, let ω_x denote the image of ω under the localization map*
$$\mathbb{E}\mathrm{xt}^2_G(\mathcal{L}_{Y/S}, \mathcal{O}_Y) \otimes \mathfrak{a} \longrightarrow \mathcal{E}xt^2_G(\mathcal{L}_{Y/S}, \mathcal{O}_Y)^{\widehat{}}_x \otimes \mathfrak{a}.$$

(i) *There exists an equivariant deformation of* $\widehat{Y}$ *to* S' *if and only if* $\omega_x = 0$.

(ii) *If* $\omega_x = 0$ *then the set of isomorphism classes of deformations of* $\widehat{Y}$ *to* S' *is a principal homogeneous space under the group* $\mathcal{E}xt^1_G(\mathcal{L}_{Y/S}, \mathcal{O}_Y)\widehat{_x} \otimes \mathfrak{a}$.

(iii) *If* $\omega = 0$ *then the action of* $\mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/S}, \mathcal{F})$ *on the set of isomorphism classes of deformations of* Y *to* S' *is compatible with the action of* $\mathcal{E}xt^1_G(\mathcal{L}_{Y/S}, \mathcal{O}_Y)\widehat{_x} \otimes \mathfrak{a}$ *on deformations of* $\widehat{Y}$, *with respect to the localization map*

$$\mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/S}, \mathcal{O}_Y) \otimes \mathfrak{a} \longrightarrow \mathcal{E}xt^1_G(\mathcal{L}_{Y/S}, \mathcal{O}_Y)\widehat{_x} \otimes \mathfrak{a}.$$

Proof. — This is proved in the same way as Theorem 3.3 except that the exact sequence (24) is replaced by the sequence

$$(31) \quad 0 \to \mathcal{E}xt^1_G(\mathcal{L}_{Y/S}, \mathcal{F})\widehat{_x} \to \mathcal{E}xt^1_G(\mathcal{L}_{Y/S'}, \mathcal{F})\widehat{_x} \to \mathcal{H}om_G(\mathcal{F}, \mathcal{F})\widehat{_x} \xrightarrow{\partial} \mathcal{E}xt^2_G(\mathcal{L}_{Y/S}, \mathcal{F})\widehat{_x}$$

and we use Proposition 3.8 in addition to Theorem 2.2. The compatibility statement (iii) follows from the commutativity of the diagram (28) and the fact that the localization maps define a homomorphism between the exact sequences (24) and (31). □

4. Multiplicative deformation data.

Let X be a smooth projective curve, defined over an algebraically closed field k of characteristic $p > 0$. A multiplicative deformation datum over X is a pair (Z, V), where $Z \to X$ is a Galois cover, with Galois group H of order prime to p, and an H-stable $\mathbb{F}_p$-vector space V of logarithmic differential forms on Z. In §4.1, we associate to the pair (Z, V) a singular curve Y together with an action of a finite group scheme G such that $X = Y/G$. Essentially, G is a semi-direct product $\boldsymbol{\mu}_p^s \rtimes H$ (where $s := \dim_{\mathbb{F}_p} V$) and $Y \to Z$ is generically a $\boldsymbol{\mu}_p^s$-torsor determined by a basis $\phi_1, \ldots, \phi_s$ of V.

As an application of the general theory developed in the previous sections, we study equivariant deformations of Y. Even though the cover $Y \to X$ is inseparable, its deformation theory is in some sense similar to the deformation theory of a tame cover. For instance, we get a morphism of deformation functors

$$\mathrm{Def}(Y,G) \longrightarrow \mathrm{Def}(X;\tau_j),$$

see §4.2 for a precise definition. In §4.3 we give a criterion when this morphism is an isomorphism. The reason for this relatively nice behavior of $\mathrm{Def}(Y,G)$ is that the "p-Sylow subgroup" of G is a *multiplicative* group scheme, whose cohomology is trivial. Thus, all the contribution to the hyperext groups $\mathbb{E}\mathrm{xt}^n_G(\mathcal{L}_{Y/k},\mathcal{O}_Y)$ comes from the cohomology of a certain coherent sheaf on X, and there is no group cohomology involved.

4.1. The G-cover associated to a deformation datum. — Fix an algebraically closed field k of characteristic $p>0$ and a smooth k-curve X. Let H be a finite group of prime-to-p order and χ a character of H with values in $\mathbb{F}_p$.

Definition 4.1. — A *(multiplicative) deformation datum* on X of type (H,χ) is a pair (Z,V), where

- $\pi: Z\to X$ is a finite, tamely ramified Galois cover with Galois group H, and
- $V\subset\Omega_{k(Z)/k}$ is an H-stable and finite dimensional $\mathbb{F}_p$-vector space consisting of logarithmic differential forms on Z. Let V_k denote the k-linear hull of V in $\Omega_{k(Z)/k}$. We demand that $\dim_k V_k=\dim_{\mathbb{F}_p} V$ and that H acts on V with character χ.

Recall that a differential form $\phi\in\Omega_{k(Z)/k}$ is called *logarithmic* if it can be written as $\phi=\mathrm{d}u/u$ for some rational function $u\in k(Z)$.

If $\dim_{\mathbb{F}_p} V=1$ then Definition 4.1 agrees with Definition 1.5 of [26]. In this paper we shall only consider multiplicative deformation data (as opposed to *additive* deformation data), so we omit from now on the adjective "multiplicative".

Let us fix a deformation datum (Z,V) of type (H,χ). For the moment, we will consider V simply as a (right) $\mathbb{F}_p[H]$-module. Let $W(k)$ denote the ring of Witt vectors over k and $W(k)[V]$ the group ring of V over $W(k)$ (here we consider V as an Abelian group). Then

$$G_0 := \mathrm{Spec}\, W(k)[V]$$

is a finite flat and commutative group scheme over $W(k)$. In fact, G_0 represents the group functor (on the category of $W(k)$-algebras)

$$R\longmapsto G_0(R)=\mathrm{Hom}_{\mathrm{gr}}(V,R^\times).$$

Groups schemes of this form are called *diagonalizable* in [9], Exposé I. We shall write $\underline{\zeta} = (\zeta_\phi)_{\phi \in V}$ for an element of $G_0(R)$. Here $\zeta_\phi \in R^\times$ such that $\zeta_{\phi_1}\zeta_{\phi_2} = \zeta_{\phi_1+\phi_2}$. In particular, $\zeta_\phi^p = 1$. Therefore, the choice of an $\mathbb{F}_p$-basis of V gives rise to an isomorphism $G_0 \cong \boldsymbol{\mu}_p^n$, where $n = \dim_{\mathbb{F}_p} V$.

An element $\beta \in H$ induces an automorphism $G_0 \xrightarrow{\sim} G_0$ of group schemes which sends $\underline{\zeta} = (\zeta_\phi)_{\phi \in V} \in G_0(R)$ to

$$\beta(\underline{\zeta}) := (\phi \mapsto \zeta_{\beta^*\phi}) \in G_0(R).$$

This gives an action of H on G_0 from the left. We define the group scheme G as the semidirect product $G_0 \rtimes H$; it represents the group functor

$$R \longmapsto G(R) := G_0(R) \rtimes H.$$

The multiplication on the right hand is determined by the rule

$$(\underline{\zeta}_1, \beta_1) \cdot (\underline{\zeta}_2, \beta_2) := (\underline{\zeta}_1 \cdot \beta_1(\underline{\zeta}_2), \beta_1\beta_2).$$

Note that the subgroup scheme $G_0 \subset G$ is equal to the local part of G.

Let R be a $W(k)$-algebra and M a G-R-module. The induced action of G_0 on M is given by a map $\mu : M \to R[V] \otimes_R M$. It gives rise to a V*-grading*, *i.e.* a direct sum decomposition

$$M = \bigoplus_{\phi \in V} M_\phi, \qquad M_\phi := \{\, m \in M \mid \mu(m) = \phi \otimes m \,\}.$$

One checks that a G-R-module is the same as an R-module together with a V-grading and an R-linear action of H from the right such that $M_\phi^\beta = M_{\beta^*\phi}$ for all $\beta \in H$ and $\phi \in V$. See also [9], Exposé I. Using the assumption that the order of H is prime to p one shows:

Lemma 4.2. — *Let R be a $W(k)$-algebra and M a G-R-module. Then*

$$H^n(G, M) = \begin{cases} M_0^H & \text{for } n = 0, \\ 0 & \text{for } n > 0. \end{cases}$$

Construction 4.3. — Let (Z, V) be a deformation datum of type (H, χ) over X. We shall construct a curve Y over k and a G-action on Y such that $Z = Y/G_0$ and $X = Y/G$. The definition of Y and the G-action will depend, up to canonical isomorphism, only on the deformation datum (Z, V) but not on the choices we make during the construction. Therefore, it suffices to give the construction locally on X. Hence, we may assume that $Z = \operatorname{Spec} A$ is affine. Let us also choose an $\mathbb{F}_p$-basis $\phi_1, \ldots, \phi_s$ of V. Since ϕ_i is logarithmic, we have $\phi_i = \mathrm{d}u_i/u_i$ for some rational function u_i

on Z. After shrinking Z and replacing ϕ_i by a suitable $\mathbb{F}_p$-multiple of itself, we may assume that u_i lies in A and has at most simple zeros on Z. Set

$$B := A[y_1, \ldots, y_s \mid y_i^p = u_i], \qquad Y := \operatorname{Spec} B.$$

The A-algebra B has a unique V-grading such that $B_0 = A$ and $y_i \in B_{\phi_i}$. It gives rise to an action of G_0 on Y such that $Z = Y/G_0$. One checks that there is a unique way to extend the action of H on A to an action on B such that $\beta^* B_\phi = B_{\beta^*\phi}$. Whence an action of G on Y such that $Z = Y/G_0$ and $X = Y/G$. This finishes the construction of Y.

Definition 4.4. — Let (Z, V) be a deformation datum of type (H, χ), and let Y be the k-curve with G-action from Construction 4.3. Let $\tau \in X$ be a closed point and choose a point $\xi \in Z$ above τ. We say that τ is

(i) a *tame branch point* if it is a branch point of the tame cover $Z \to X$,

(ii) a *wild branch point* if there exists $\phi \in V$ such that $\operatorname{ord}_\xi \phi = -1$,

(iii) a *critical point* if it is a branch point (tame or wild) or if

$$\min_{\phi \in V} (\operatorname{ord}_\xi \phi) \neq 0.$$

Note that these conditions do not depend on the choice of ξ and that a branch point can be wild and tame at the same time.

Notation 4.5. — Let $(\tau_j)_{j\in B}$ denote the set of critical points for (Z, V), indexed by the finite set B. Let $B_{\rm tame}$ (resp. $B_{\rm wild}$) denote the subset of B corresponding to the tame (resp. wild) branch points; set $B_{\rm ram} := B_{\rm tame} \cup B_{\rm wild}$. We have a divisor on Z

$$D := \sum_{\xi \in Z} (\min_{\phi \in V} \operatorname{ord}_\xi \phi) \cdot \xi.$$

We can write D as the difference of two disjoint effective divisors in a unique way:

$$D = D_0 - D_\infty.$$

Note that the image of D (resp. of D_∞) on X has support in the set of critical points (resp. in the set of wild branch points).

Remark 4.6.

(i) The map $Y \to X$ is finite and flat. It is a G-torsor precisely outside the set of branch points.

(ii) The curve Y is generically smooth over k if and only if $\dim_{\mathbb{F}_p} V=1$. If this is the case then the singular points of Y are precisely the points lying over a critical point which is not a wild branch point.

4.2. Equivariant deformations of Y. — Let $\mathfrak{C}_k$ denote the category of local Artinian $W(k)$-algebras. A *G-equivariant deformation* of Y over an object R of $\mathfrak{C}_k$ is a flat R-scheme Y_R together with an action of G and a G-equivariant isomorphism $Y \cong Y_R \otimes k$ (compare with Definition 3.1). We are concerned with the deformation functor

$$R \longmapsto \mathrm{Def}(Y, G)(R)$$

which sends R to the set of isomorphism classes of G-equivariant deformations of Y over R. The next lemma follows easily from Construction 4.3:

LEMMA 4.7. — *Let Y_R be an equivariant deformation of Y over R. Furthermore, let $R' \to R$ be a small extension, i.e. $R = R'/\mathfrak{a}$ for an ideal $\mathfrak{a} \triangleleft R'$ such that $\mathfrak{a} \cdot \mathfrak{m}_{R'} = 0$. Then the morphism $Y_R \to \mathrm{Spec}\, R'$ satisfies Assumption* 1.1.

The lemma shows that the equivariant cotangent complex $\mathcal{L}_{Y/k}$ is defined and that we can apply Theorem 3.3 to classify the set of liftings of the deformation Y_R to R'. Let $k[\epsilon]$ denote the ring of dual numbers. We call $T^1(Y, G) := \mathrm{Def}(Y, G)(k[\epsilon])$ the *tangent space* of the deformation functor $\mathrm{Def}(Y, G)$. Theorem 3.3 says in particular that there is a canonical isomorphism

$$T^1(Y, G) \cong \mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y). \tag{32}$$

Moreover, Theorem 3.3 together with standard arguments (see e.g. [20] or [23]) implies:

THEOREM 4.8. — *Suppose that $n := \dim_k \mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y)$ is finite (this holds, for instance, if X is projective). Then Y admits a versal deformation over a ring of the form*

$$R_{\mathrm{univ}} = W(k)[[t_1, \ldots, t_n]]/\langle f_1, \ldots, f_m \rangle.$$

If, moreover, $\mathbb{E}\mathrm{xt}^2_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y) = 0$ then $\mathrm{Def}(Y, G)$ is formally smooth and we have $R_{\mathrm{univ}} = W(k)[[t_1, \ldots, t_n]]$.

Let Y_R be an equivariant deformation of Y over R. Then the quotient schemes $Z_R := Y_R/G_0$ and $X_R := Y_R/G$ are deformations of Z and X, respectively. Let $\mathrm{Def}(X; \tau_j)$ denote the functor which classifies deformations of the *marked* curve $(X;\tau_j \mid j\in B_{\rm ram})$, *i.e.* deformations X_R of X together with sections $\tau_{j,R}:\operatorname{Spec} R\to X_R$ lifting the points τ_j.

We claim that the association $Y_R\mapsto X_R:=Y_R/G$ gives rise to a morphism of deformation functors

$$\mathrm{Def}(Y,G)\longrightarrow \mathrm{Def}(X;\tau_j\mid j\in B_{\rm ram}). \tag{33}$$

To prove the claim we have to endow the curve X_R with sections $\tau_{j,R}$: $\operatorname{Spec} R\to X_R$ lifting the branch points τ_j, for all $j\in B_{\rm ram}$. This is obvious for $j\in B_{\rm tame}$: the G-action on Y_R induces an action of H on Z_R such that $X_R=Z_R/H$ and such that the map $Z_R\to X_R$ is a tame H-cover lifting $Z\to X$. It follows that the branch locus of $Z_R\to X_R$ is the disjoint union of sections $\tau_{j,R}:\operatorname{Spec} R\to X_R$ lifting the tame branch points τ_j (for $j\in B_{\rm tame}$). Now let $j\in B_{\rm wild}$ and let $\xi\in Z$ be a point above the wild branch point τ_j. Let $\phi_1,\ldots,\phi_s$ be a basis of V. We can choose this basis in such a way that ϕ_1 has a simple pole in ξ and that $\phi_2,\ldots,\phi_s$ generate the kernel of the residue map $\operatorname{res}_\xi: V\to\mathbb{F}_p$. If we further replace ϕ_i by a multiple of itself then we may assume that $\phi_i=\mathrm{d}u_i/u_i$, with $\operatorname{ord}_\xi u_1=1$ and $\operatorname{ord}_\xi u_i=0$ for $i>1$. In a neighborhood of ξ, the cover $Y\to Z$ is (locally at ξ) given by s Kummer equations $y_i^p=u_i$, see Construction 4.3. Hence the deformation $Y_R\to Z_R$ of $Y\to Z$ is (locally at ξ) given by s Kummer equations $y_i^p=u_{i,R}$, where $u_{i,R}$ lifts u_i. The equation $u_{1,R}=0$ defines a section $\xi_R:\operatorname{Spec} R\to Z_R$ which lifts the point ξ. We define $\tau_{j,R}:\operatorname{Spec} R\to X_R$ to be the image of ξ_R. Using the H-action, one also checks that the definition of $\tau_{j,R}$ for $j\in B_{\rm wild}$ agrees with the definition of $\tau_{j,R}$ for $j\in B_{\rm tame}$, in case that $j\in B_{\rm tame}\cap B_{\rm wild}$. This proves the claim.

It is well known (see e.g. [4]) that the tangent space of the deformation functor $\mathrm{Def}(X;\tau_j)$ is given by

$$T^1(X;\tau_j\mid j\in B_{\rm ram})\cong H^1(X,\mathcal{T}_X(-\sum_{j\in B_{\rm ram}}\tau_j)).$$

Here $\mathcal{T}_X$ is the sheaf of tangent vectors of X. Hence the morphism (33) induces a k-linear map

$$\mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/k},\mathcal{O}_Y)\longrightarrow H^1(X,\mathcal{T}_X(-\sum_{j\in B_{\rm ram}}\tau_j)). \tag{34}$$

In the next subsection we will analyze this map in more detail.

4.3. Analysis of $\mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/S},\mathcal{O}_Y)$. — By Lemma 4.2 and the spectral sequence (79) we have

$$\mathcal{E}xt^n_G(\mathcal{L}_{Y/k},\mathcal{O}_Y)=\mathcal{E}xt^n_{G_0}(\mathcal{L}_{Y/k},\mathcal{O}_Y)^H=\mathcal{E}xt^n_Y(\mathcal{L}_{Y/k},\mathcal{O}_Y)^G. \tag{35}$$

The analogous statement for $\mathbb{E}\mathrm{xt}^n_G$ holds as well. In other words, we do not have to worry about group cohomology. In the following, we shall use this sort of argument over and over again, sometimes without mentioning it explicitly.

A *V-derivation* is an $\mathbb{F}_p$-linear function $\theta : V \to k(Z)$. We say that θ is integral at a point $\xi \in Z$ if $\theta(\phi) \in \mathcal{O}_{Z,\xi}$ for all $\phi \in V$. Let $\mathcal{M}$ be the sheaf of integral V-derivations; for an open subset $U \in Z$ the group $\Gamma(U, \mathcal{M})$ of sections over U is the set of V-derivations θ which are integral at each point $\xi \in U$. Obviously, $\mathcal{M}$ is a locally free $\mathcal{O}_Z$-module whose rank is equal to $s = \dim_{\mathbb{F}_p} V$. There is a natural H-action on $\mathcal{M}$, *i.e.* a structure of H-$\mathcal{O}_Z$-module, such that $\mathcal{M}^H$ is the sheaf of H-equivariant V-derivations on X.

Let $\mathcal{T}_Z = \mathrm{Hom}_Z(\Omega_{Z/k}, \mathcal{O}_Z)$ denote the sheaf of tangent vectors on Z. We write $\mathcal{T}_Z(D) := \mathcal{T}_Z \otimes \mathcal{O}_Z(D)$ etc. There is a natural injection of H-$\mathcal{O}_Z$-modules

$$\mathcal{T}_Z(D) \hookrightarrow \mathcal{M}$$

which sends a vector field θ to its restriction to V. From the definition of the divisor D it is clear that this is well defined and that the quotient $\mathcal{M}/\mathcal{T}_Z(D)$ is torsion free.

LEMMA 4.9. — *There is a natural isomorphism of H-$\mathcal{O}_Z$-modules*

$$\mathcal{M} \xrightarrow{\sim} \mathrm{Hom}_{G_0}(\Omega_{Y/k}, \mathcal{O}_Y) := \mathcal{T}_Y^{G_0}. \tag{36}$$

Furthermore, we have a short exact sequence of H-$\mathcal{O}_Z$-modules

$$0 \longrightarrow \mathcal{T}_Z(-D_\infty) \longrightarrow \mathcal{M} \longrightarrow \mathcal{E}xt^1_{G_0}(\mathcal{L}_{Y/k}, \mathcal{O}_Y) \longrightarrow 0. \tag{37}$$

Proof. — It suffices to prove this locally on Z. Hence we may assume that $Z = \mathrm{Spec}\, A$ is affine and that there is a basis $\phi_1, \ldots, \phi_s$ of V such that $\phi_i = \mathrm{d}u_i/u_i$ with $u_i \in A$ and such that u_i has at most simple zeros. By construction we have

$$Y = \mathrm{Spec}\, B, \quad B = C/I,$$

where $C = A[y_1, \ldots, y_s]$ is the polynomial algebra over A in s variables (with V-grading such that $y_i \in C_{\phi_i}$) and I is generated by the polynomials $u_i - y_i^p$. One checks that the B-module I/I^2 is free, with G_0-invariant generators $[u_i - y_i^p]$.

The cotangent complex $\mathcal{L}_{Y/k}$ may be identified with the complex of G-$\mathcal{O}_Y$-modules associated to the complex $L := (I/I^2 \to \Omega_{C/k} \otimes B)$ of V-graded B-modules with H-action. The differential of this complex sends the generator $[u_i - y_i^p]$ to the 1-form $\mathrm{d}u_i$. It follows that $\Omega_{B/k} = H^0(L)$ is the direct sum of the free B-module generated by $\mathrm{d}y_i$ and the torsion module

$$(\Omega_{B/k})_{\mathrm{tors}} = \frac{\Omega_{A/k}}{\langle \mathrm{d}u_i\rangle} \otimes_A B.$$

Let $\theta : V \to A$ be an everywhere integral V-derivation. It gives rise to a G_0-equivariant derivation $\eta : \Omega_{B/k} \to B$ which is zero on $(\Omega_{B/k})_{\mathrm{tors}}$ and such that

$$\eta(\mathrm{d}y_i) := \theta(\phi_i)\, y_i.$$

One checks that the association $\theta \mapsto \eta$ defines an isomorphism of H-$\mathcal{O}_Z$-modules (36).

Since both nontrivial terms of the complex L are locally free B-modules, we have

$$\mathbb{E}\mathrm{xt}^n_{G_0}(\mathcal{L}_{Y/k}, \mathcal{O}_Y) = H^n(\mathrm{Hom}^\bullet_{G_0}(L, B)).$$

For $n = 1$ this gives the exact sequence

$$\mathrm{Hom}_{G_0}(\Omega_{C/k}, B) \longrightarrow \mathrm{Hom}_{G_0}(I/I^2, B) \longrightarrow \mathbb{E}\mathrm{xt}^1_{G_0}(\mathcal{L}_{Y/k}, \mathcal{O}_Y) \to 0. \tag{38}$$

Let $\theta : V \to A$ be a global section of $\mathcal{M}$. There exists a unique B-linear and G_0-equivariant map $\nu : I/I^2 \to B$ such that

$$\nu([u_i - y_i^p]) = u_i\, \theta(\phi_i)$$

for all i. This defines an A-linear map

$$H^0(Z, \mathcal{M}) \longrightarrow \mathrm{Hom}_{G_0}(I/I^2, B). \tag{39}$$

From (38) and (39) we obtain the sequence (37). It is easy to see that this sequence is H-equivariant and does not depend on the choice of the basis of V. It remains to show that (37) is exact.

Exactness on the left is obvious; exactness in the middle follows easily from the exactness of (38). To prove exactness on the right, let $\nu : I/I^2 \to B$ be a B-linear and G_0-equivariant homomorphism. We can define a V-derivation $\theta : V \to k(Z) = \mathrm{Frac}(A)$ by setting

$$\theta(\phi_i) = \frac{\nu([u_i - y_i^p])}{u_i}.$$

By construction the images of θ and of ν in $\mathbb{E}\mathrm{xt}^1_{G_0}(\mathcal{L}_{Y/k}, \mathcal{O}_Y)$ are equal. If θ was integral everywhere then we would be done. However, if $\xi \in Z$ is a point in the support of D_∞ then θ may not be integral at ξ. In this case we may suppose that $\mathrm{ord}_\xi\, u_1 = 1$ and that $\mathrm{ord}_\xi\, u_i = 0$ for $i > 1$. After shrinking Z to a sufficiently small neighborhood of ξ we may suppose that

$$\theta' = \theta - \nu([u_1 - y_1^p])\, \partial/\partial u_1|_V$$

is integral everywhere. But since (38) is exact in the middle, the image of θ' in $\mathbb{E}\mathrm{xt}^1_{G_0}(\mathcal{L}_{Y/k}, \mathcal{O}_Y)$ is the same as the image of θ. This finishes the proof of the lemma. □

Applying the exact functor $\mathcal{F} \mapsto \mathcal{F}^H$ to (37) we obtain a short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{T}_Z(-D_\infty)^H \longrightarrow \mathcal{M}^H \longrightarrow \mathcal{E}xt^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y) \longrightarrow 0. \tag{40}$$

The following proposition identifies the first boundary map associated to (40) with the differential of the morphism of deformation functors (33).

Proposition 4.10. — *The following diagram commutes:*

$$\begin{array}{ccc} \mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y) & \longrightarrow & H^0(X, \mathcal{E}xt^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y)) \\ \downarrow & & \downarrow \partial \\ H^1(X, \mathcal{T}_X(-\sum_{j\in B_{\mathrm{ram}}} \tau_j)) & \xrightarrow{\cong} & H^1(X, \mathcal{T}_Z(-D_\infty)^H). \end{array} \tag{41}$$

Here the upper horizontal arrow is deduced from the local-global spectral sequence (82)*. The left vertical arrow is the tangent map of the morphism* (33)*. The right vertical arrow is the boundary map of the short exact sequence* (40)*. The lower horizontal arrow comes from the canonical isomorphism*

$$\mathcal{T}_Z(-D_\infty)^H \cong \mathcal{T}_X(-\sum_{j\in B_{\mathrm{ram}}} \tau_j).$$

Proof. — Let us denote by $\mathrm{Def}(Z, H, D_\infty)$ the functor which classifies H-equivariant deformations of the marked curve (Z, D_∞) (here we identify D_∞ with its support, which consists of the points of Z lying above the wild branch points). By [3] the tangent space of $\mathrm{Def}(Z, H, D_\infty)$ is canonically isomorphic to

$$H^1(Z, \mathcal{T}_Z(-D_\infty)^H) \cong H^1(X, \mathcal{T}_X(-\sum_{j\in B_{\mathrm{ram}}} \tau_j)).$$

Using this fact, the proposition is easily reduced to the case $H = 1$ and $Z = X$.

Let Y' be a G-equivariant deformation of Y over $R = k[\epsilon]$ and $Z' := Y'/G$ the induced deformation of Z. We have seen in the last subsection that Z' is naturally endowed with a lift D'_∞ of the divisor D_∞. We denote by $e(Y')$ the global section of $\mathcal{E}xt^1_{G_0}(\mathcal{L}_{Y/k}, \mathcal{O}_Y)$ corresponding to Y', see §2 and §3. Similarly, we denote by $e(Z', D'_\infty) \in H^1(Z, \mathcal{T}_Z(-D_\infty))$

the cohomology class representing (Z', D'_∞). We have to show that $e(Z')$ is the image of $e(Y')$ under the boundary map ∂.

To prove this, we will first recall the definition of $e(Y')$ and $e(Z', D'_\infty)$. Let (U_μ) be a covering of Z by sufficiently small affine open subsets $U_\mu = \operatorname{Spec} A_\mu$. Let $W_\mu = \operatorname{Spec} B_\mu \subset Y$ be the inverse image of U_μ. Also, let $U'_\mu = \operatorname{Spec} A'_\mu \subset Z'$ (resp. $W'_\mu = \operatorname{Spec} B'_\mu \subset Y'$) be the induced deformation of U (resp. the induced G-equivariant deformation of W).

Since Z is smooth over k there exists, for all μ, a (non-canonical) isomorphism of R-algebras

$$\sigma_\mu : A'_\mu \xrightarrow{\sim} A_\mu \otimes_k R$$

which lifts the identity on A_μ. For each pair of indices μ, λ we set $U_{\mu,\lambda} := U_\mu \cap U_\lambda = \operatorname{Spec} A_{\mu,\lambda}$. Then the equality

$$\sigma_\lambda \circ \sigma_\mu^{-1} = \mathrm{Id}_{A_{\mu,\lambda}} + \epsilon \cdot \theta_{\mu,\lambda}$$

defines a vector field $\theta_{\mu,\lambda} \in \Gamma(U_{\mu,\lambda}, \mathcal{T}_Z)$. The 1-cocycle $(\theta_{\mu,\lambda})$ represents the cohomology class $e(Z', D'_\infty)$.

We may assume that

$$B_\mu = A[\, y_i \mid u_{\mu,i} - y_{\mu,i}^p \,],$$

with $u_{\mu,i} \in A_\mu$ such that $\phi_i = \mathrm{d}u_{\mu,i}/u_{\mu,i}$. There is a V-grading on B_μ such that $y_{\mu,i} \in (B_\mu)_{\phi_i}$. Let $y'_{\mu,i} \in (B'_\mu)_{\phi_i}$ be a lift of $y_{\mu,i}$. Then $u'_{\mu,i} := (y'_{\mu,i})^p \in A_\mu$. Note that $u'_{\mu,i}$ is independent of the choice of the lift $y'_{\mu,i}$. Set

$$\sigma_\mu(u'_{\mu,i}) = u_{\mu,i} + \epsilon \cdot v_{\mu,i}.$$

Let ν_μ be the section of the sheaf $\mathcal{M}$ over U_μ such that

$$\nu_\mu(\phi_i) = v_{\mu,i}.$$

Using the definition of $e(Y')$ via Theorem 3.3, together with the proof of Proposition 4.10, one checks that the image of ν_μ under the second map in (39) is equal to the restriction of $e(Y')$ to U_μ.

A straightforward computation shows that

$$\theta_{\mu,\lambda}(\phi_i) = \frac{\theta_{\mu,\lambda}(u_{\lambda,i})}{u_{\lambda,i}} = v_{\mu,i} - v_{\lambda,i}$$

for all μ, λ, i. This means that $\theta_{\mu,\lambda}$ is mapped to $\nu_\mu - \nu_\lambda \in \Gamma(U_{\mu,\lambda}, \mathcal{M})$ under the first map in (39). Therefore, $e(Z', D'_\infty)$ is the image of $e(Y')$ under the boundary map ∂. This is what we wanted to prove. $\square$

Theorem 4.11. — *Suppose $H^n(X,\mathcal{M}^H)=0$ for $n=0,1$. Then the morphism*

$$\mathrm{Def}(Y,G)\longrightarrow \mathrm{Def}(X;\tau_j)$$

is an isomorphism. In particular, the deformation functor $\mathrm{Def}(Y,G)$ *is unobstructed.*

Proof. — The hypothesis implies that the boundary map

$$\partial : H^0(X,\mathcal{E}xt^1_G(\mathcal{L}_{Y/k},\mathcal{O}_Y)) \xrightarrow{\sim} H^1(X,\mathcal{T}_Z(-D_\infty)^H) \tag{42}$$

deduced from the short exact sequence (40) is an isomorphism. The local-global spectral sequence for $\mathbb{E}\mathrm{xt}^n_G$ gives rise to a short exact sequence

$$0\to H^1(X,\mathcal{T}_Y^G)\longrightarrow \mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/k},\mathcal{O}_Y)\longrightarrow H^0(X,\mathcal{E}xt^1_G(\mathcal{L}_{Y/k},\mathcal{O}_Y))\to 0.$$

But (36) and the hypothesis show that $H^1(X,\mathcal{T}_Y^G)=0$. Therefore, it follows from Proposition 4.10 that the morphism $\mathrm{Def}(Y,G)\to\mathrm{Def}(X;\tau_j)$ induces an isomorphism on tangent spaces. The theorem would follow if we knew that $\mathrm{Def}(Y,G)$ is unobstructed.

The local global spectral sequence for $\mathbb{E}\mathrm{xt}^n_G$ also shows that

$$\mathbb{E}\mathrm{xt}^2_G(\mathcal{L}_{Y/k},\mathcal{O}_Y)=H^1(X,\mathcal{E}xt^1_G(\mathcal{L}_{Y/k},\mathcal{O}_Y)). \tag{43}$$

Using again the long exact cohomology sequence deduced from (42) and the hypothesis we see that (40) is zero. Hence $\mathrm{Def}(Y,G)$ is unobstructed by Theorem 3.3. This concludes the proof of the theorem. □

Remark 4.12. — Suppose that all elements of V are regular, *i.e.* $B_{\mathrm{wild}}=\emptyset$. Then we may regard V as an $\mathbb{F}_p$-subvector space of the χ-isotypical part of $J_Z[p](k)$. It can be shown that the hypothesis $H^n(X,\mathcal{M}^H)=0$ of Theorem 4.11 is equivalent to the condition that the χ-isotypical part of the group scheme $J_Z[p]$ is étale. Using this fact one can give a different proof of Theorem 4.11. In the special case $\dim_{\mathbb{F}_p}V=1$ this is the approach taken in [25].

5. Special deformation data.

In this section we suppose that $X=\mathbb{P}^1_k$. We begin by defining a certain class of multiplicative deformation data over X, which we call *special*. The definition of specialty may seem a little bit ad hoc. However, we show that the deformation functor $\mathrm{Def}(Y,G)$ associated to (Z,V) in the last section has some very nice properties if (Z,V) is special. These are the *lifting property* (Theorem 5.7), the *local-global principle* (Theorem 5.11) and *rigidity* (Theorem 5.14). At a deeper level, these properties are explained by the way special deformation data arise in the study of three point covers with bad reduction, see [25] and [26].

We also prove a technical result (Proposition 5.15) which is used in [26].

5.1. — Let p be a prime, H a finite group of order prime to p and $\chi_0 : H \to \bar{\mathbb{F}}_p^\times$ a one dimensional character on H with values in the algebraic closure of $\mathbb{F}_p$. The values of χ_0 generate a finite field $\mathbb{F}_q$ with $q = p^s$ elements. Set

$$\chi_i := \chi_0^{p^i}, \qquad \chi = \sum_{i=0}^{s-1} \chi_i.$$

Then χ is an irreducible $\mathbb{F}_p$-valued character.

Let k be an algebraically closed field of characteristic p and set $X := \mathbb{P}^1_k$. Let (Z,V) be a (multiplicative) deformation datum of type (H,χ) over X. Choose a basis $\omega_0, \ldots, \omega_{s-1}$ of $V \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p$ consisting of eigenvectors, such that

$$\alpha^* \omega_i = \chi_i(\alpha)\, \omega_i, \tag{44}$$

for all $\alpha \in H$ and $i = 0, \ldots, s-1$. Let $\mathcal{C}$ denote the *Cartier operator*. Since $\mathcal{C}$ is p^{-1}-linear and is the identity on V, we have

$$\mathcal{C}(\omega_{i+1}) = c_i\, \omega_i \tag{45}$$

for a constant $c_i \neq 0$ in $\bar{\mathbb{F}}_p$. (Here and for the rest of this section we will consider the index i modulo s.) After multiplying the ω_i with a constant in $\bar{\mathbb{F}}_p$ we may assume that $c_i = 1$.

As in the previous section, we denote by τ_j, $j \in B$, the critical points on X. Choose $j \in B$ and a point $\xi \in Z$ above τ_j, and set

$$m_j := |\mathrm{Stab}_H(\xi)|, \qquad h_j^{(i)} := \mathrm{ord}_\xi\, \omega_i + 1, \qquad \sigma_j^{(i)} := \frac{h_j^{(i)}}{m_j}.$$

The tuple $(\sigma_j^{(i)})_{i,j}$ is called the *signature* of the deformation datum (Z,V). For a rational number w, we let $\langle w \rangle$ denote the fractional part of w (such that $0 \leqslant \langle w \rangle < 1$ and $w - \langle w \rangle \in \mathbb{Z}$).

Lemma 5.1. — *For all i we have*

(i) $\sum_{j\in B}(\sigma_j^{(i)}-1)=-2$,

(ii) $\langle\sigma_j^{(i)}\rangle=\langle p^i\sigma_j^{(0)}\rangle$.

Proof. — Part (i) follows from a straightforward computation using the Riemann-Hurwitz formula. To prove (ii), let $\xi\in Z$ be a point above τ_j and z a local coordinate at ξ. The inertia character $\psi_\xi : H_\xi\to k^\times$ is determined by the congruence

$$\alpha^* z\equiv\psi_\xi(\alpha)\,z \pmod{z^2} \tag{46}$$

for all $\alpha\in H_\xi:=\mathrm{Stab}_H(\xi)$. Now (44) and the definition of $h_j^{(i)}$ imply that

$$\chi_0^{p^i}=\psi_\xi^{h_j^{(i)}}.$$

Part (ii) of the lemma follows. □

Definition 5.2. — The deformation datum (Z,V) is called *pure* if for all i we have

$$\sum_{j\in B}\langle\sigma_j^{(i)}\rangle=1.$$

Lemma 5.3. — *Let $\mathcal{M}$ be the sheaf of H-$\mathcal{O}_Z$-modules defined in* §4.3. *The deformation datum (Z,V) is pure if and only if $H^n(X,\mathcal{M}^H)=0$ for $n=0,1$.*

Proof. — Let $\mathcal{O}_{Z,\chi_i}$ denote the χ_i-isotypical part of the sheaf $\pi_*\mathcal{O}_Z$. Let $\theta: V\to k(Z)$ be an H-equivariant V-derivation and extend it k-linearly to V_k. Then $f_i:=\theta(\omega_i)$ is a meromorphic section of $\mathcal{O}_{Z,\chi_i}$; it is holomorphic at $\tau\in X$ if and only if θ is integral at all points $\xi\in Z$ above τ. Therefore, the rule $\theta\mapsto(f_i)$ defines an isomorphism

$$\mathcal{M}^H\xrightarrow{\sim}\bigoplus_{i=0}^{s-1}\mathcal{O}_{Z,\chi_i}$$

of $\mathcal{O}_X$-modules. A local calculation as in the proof of Lemma 5.1 shows that

$$\deg\,\mathcal{O}_{Z,\chi_i}=-\sum_{j\in B}\langle\sigma_j^{(i)}\rangle.$$

Hence the lemma follows from the Riemann–Roch formula. □

5.2. — In order to discuss special deformation data we need some more notation:

NOTATION 5.4. — Let (Z, V) be a deformation datum of type (H, χ) and signature $(\sigma_j^{(i)})$. Set

$$\nu_j^{(i)} := \lfloor \sigma_j^{(i)} \rfloor = \sigma_j^{(i)} - \langle \sigma_j^{(i)} \rangle, \qquad \nu_j := \min_i \nu_j^{(i)},$$

and

$$a_j^{(i)} := m_j \cdot \langle \sigma_j^{(i)} \rangle, \qquad a_j := \min_i a_j^{(i)}.$$

DEFINITION 5.5. — The deformation datum (Z, V) is called *special* if $\sigma_j^{(i)} \neq 1$ for all i and j and if the following holds. There exists a subset $B_0 \subset B$ with exactly three elements such that

$$\nu_j = \begin{cases} 0, & j \in B_0 \\ 1, & j \notin B_0. \end{cases}$$

A special deformation datum (Z, V) is called *normalized* if $\{ \tau_j \mid j \in B_0 \} = \{0, 1, \infty\} \subset X = \mathbb{P}^1$.

For the rest of this section we assume that the deformation datum (Z, V) is special. Whenever it is convenient, we may also assume that (Z, V) is normalized. (However, sometimes it more convenient to have $\tau_j \neq \infty$ for all $j \in B$.) Since $\sigma_j^{(i)} \neq 1$ we conclude that $B_{\mathrm{wild}} \subset B_0$ and that $B_{\mathrm{ram}} = B$. We set

$$B_{\mathrm{new}} := B - B_0, \qquad B_{\mathrm{prim}} := B_0 - B_{\mathrm{wild}}\,.$$

For an explanation of the terminology, see [26].

LEMMA 5.6. — *Suppose (Z, V) is special. Then the following holds.*

(i) *The deformation datum (Z, V) is pure.*

(ii) *We have $\nu_j^{(i)} = \nu_j$ for all $j \in B$.*

(iii) *Let $j \in B - B_{\mathrm{wild}}$ and let $\xi \in Z$ be a point above τ_j. Then for all $\phi \in V$ we have*

$$\mathrm{ord}_\xi\, \phi = \nu_j m_j + a_j - 1.$$

Proof. — By Lemma 5.1 (i) we have

$$1 = 3 + \sum_{j \in B} (\nu_j^{(i)} - 1) + \sum_{j \in B} \langle \sigma_j^{(i)} \rangle \geqslant \sum_{j \in B} \langle \sigma_j^{(i)} \rangle. \tag{47}$$

Suppose that $\sum_j \langle \sigma_j^{(i)} \rangle = 0$. Since $\sigma_j^{(i)} \neq 1$, it would follow that $B = B_0$ and $\sigma_j^{(i)} = 0$ for all i and j. But then $\sum_j (\sigma_j^{(i)} - 1) = -3$, contradicting Lemma 5.1. We conclude that $\sum_j \langle \sigma_j^{(i)} \rangle = 1$, proving (i). We also conclude that the inequality (47) is an equality, which means that $\nu_j^{(i)} = \nu_j$, proving (ii).

It follows from Lemma 5.1 (ii) that $\operatorname{ord}_\xi \omega_i = m_j \sigma_j^{(i)} - 1$ takes pairwise distinct values for all i. Every element $\phi \in V$ can be written as $\sum_i c_i \omega_i$, with $c_i \in k$. Using $\mathcal{C}(\phi) = \phi$ and Equations (44) and (45), one shows that $c_i \neq 0$ for all i. Therefore,

$$\operatorname{ord}_\xi \phi = \min_i (\operatorname{ord}_\xi \omega_i) = \min_i (m_j \nu_j^{(i)} + a_j^{(i)} - 1).$$

Now (iii) follows from (ii). □

Putting Theorem 4.11, Lemma 5.3 and Lemma 5.6 (i) together, we get:

THEOREM 5.7 (Lifting property). — *Let Y be the curve with G-action corresponding to the special deformation datum (Z, V), as defined in §4. The homomorphism of deformation functors*

$$\operatorname{Def}(Y, G) \longrightarrow \operatorname{Def}(X; \tau_j)$$

is an isomorphism.

PROBLEM 5.8. — Let (H, χ) be as in the beginning of this section. Let $(\sigma_j^{(i)})$ be a tuple of rational numbers (indexed by $j \in B$ and $i \in \mathbb{Z}/s$) such that the statements of Lemma 5.1 and of Definition 5.5 hold. Furthermore, let $(\tau_j)_{j \in B}$ be a B-tuple of closed points of $X = \mathbb{P}^1_k$. Does there exists a special deformation datum (Z, V) of type (H, χ) with signature $(\sigma_j^{(i)})$ and critical points (τ_j)?

PROPOSITION 5.9. — *With assumptions as in Problem 5.8:*

(i) *Suppose that the character $\chi_0 : H \to \mathbb{F}_q^\times$ is injective. Then if it exists, the special deformation datum (Z, V) is uniquely determined (up to isomorphism) by the datum $(H, \chi, \sigma_j^{(i)}, \tau_j)$.*

(ii) *Fix $(H, \chi, \sigma_j^{(i)})$. The set of all tuples (τ_j) such that there exists a special deformation datum (Z, V) with critical points (τ_j) is a locally closed subset of $(\mathbb{P}^1_{\mathbb{F}_p})^B$.*

We will see later (Theorem 5.14) that the set of tuples (τ_j) in (ii) is actually finite.

Proof. — (Compare with [25], §3.5.) Suppose that (Z, V) exists. Let $\bar{H} := H/\operatorname{Ker}(\chi_0)$, $\bar{Z} := Z/\operatorname{Ker}(\chi_0)$ and $\bar{\chi}$ the restriction of χ to $\bar{H}$. The subvector space $V \subset \Omega_{k(Z)/k}$ descends to a subvector space $\bar{V} \subset \Omega_{k(\bar{Z})/k}$. One checks that $(\bar{Z}, \bar{V})$ is again a special deformation datum, of type $(\bar{H}, \bar{\chi})$. The signature $(\sigma_j^{(i)})$ and the set (τ_j) of critical points remain unchanged during this descent. Therefore, we may assume that χ_0 is injective, even for the proof of (ii). (For (i) the assumption of injectivity is necessary because the cover $Z \to \bar{Z}$ is not unique if $\operatorname{Ker}(\chi_0) \neq 1$.)

If χ_0 is injective then H is cyclic of order m where m is a positive integer such that $\mathbb{F}_q = \mathbb{F}_p[\zeta_m]$; in particular, $m|q-1$. Set $a_j^{(i)} := m\,\langle\sigma_j^{(i)}\rangle$. Then the $a_j^{(i)}$ are integers with $0 \leqslant a_j^{(i)} < m$, $\sum_j a_j^{(i)} = m$ and $a_j^{(i+1)} \equiv p^i\, a_j^{(i)} \pmod{m}$. The proof of Lemma 5.1 shows that the $a_j^{(i)}$ determine the ramification type of the m-cyclic cover $\pi : Z \to X = \mathbb{P}^1_k$. This can be made more explicit with Kummer theory. In fact, for all i there exists a rational section z_i of $\mathcal{O}_{Z,\chi_i}$ which satisfies the equation

$$z_i^m = \prod_{j \in B} (x - \tau_j)^{a_j^{(i)}}. \tag{48}$$

Here x denotes the standard coordinate on $X = \mathbb{P}^1$ and we assume, without loss of generality, that $\tau_j \neq \infty$. The curve Z is the smooth projective model of the plane curve with equation (48) (for any i).

We claim that the eigenvector ω_i of V_k is of the form

$$\omega_i = \epsilon_i \, \frac{z_i \, \mathrm{d}x}{\prod_{j \in B_0} (x - \tau_j)} \tag{49}$$

for some constant $\epsilon_i \in k$. Indeed, a local calculation shows that the right hand side of (49) has everywhere the right order of poles and zeros compatible with the signature $(\sigma_j^{(i)})$ and the set of critical points (τ_j). This proves the claim. If we plug in (49) into the equation

$$\mathcal{C}(\omega_{i+1}) = c_i \, \omega_i \tag{50}$$

and look at Taylor series (say in x) on both sides, we obtain a set of algebraic equations with coefficients in $\mathbb{F}_p$ which are satisfied by the tuple (τ_j). These equations define a Zariski closed subset of $(\mathbb{P}^1_{\mathbb{F}_p})^B$. The conditions $c_i \neq 0$ define an open subset of this closed subset. We have shown that the set of tuples (τ_j) coming from a special deformation datum with given type and signature are contained in a certain locally closed subset of $(\mathbb{P}^1_{\mathbb{F}_p})^B$. It is clear that (Z, V) is uniquely determined by the datum $(H, \chi, \sigma_j^{(i)}, \tau_j)$.

Conversely, let (τ_j) be a B-tuple of k-rational points of $\mathbb{P}^1$ which is contained in the locally closed subset constructed above. This means that if we define an H-cover $\pi : Z \to X = \mathbb{P}^1$ by equation (48) and define differentials ω_i on Z by equation (49) then (50) holds with certain constants $c_i \neq 0$. After multiplying the ω_i by suitable constants we may assume that $c_i = 1$. Let $V' \subset \Omega_{k(Z)/k}$ be the $\mathbb{F}_q$-linear subspace spanned by the ω_i. The Cartier operator $\mathcal{C}$ stabilizes V' and acts semi-simply on it. A well known lemma in p^{-1}-linear algebra shows that the stabilizer V of $\mathcal{C}$ inside V' is an $\mathbb{F}_p$-vector space of dimension $s = \dim_{\mathbb{F}_q} V'$. Here we can be more explicit: if $\alpha \in H$ is an element such that $\chi_0(\alpha)$ generates $\mathbb{F}_q$ then

$$\phi_l := \sum_i \chi_{i+l}(\alpha) \cdot \omega_i, \qquad l = 0, \ldots, s-1 \tag{51}$$

gives a basis for V. By construction, (Z, V) is a special deformation datum of type (H, χ), signature $(\sigma_j^{(i)})$ and with critical points (τ_j). This concludes the proof of the proposition. □

5.3. The local-global principle. — For $j \in B$, let $\widehat{Y}_j$ denote the completion of Y at the critical point τ_j, see §3.3. Given an equivariant deformation Y_R of Y, we denote by $\widehat{Y}_{j,R}$ the completion of Y_R at τ_j; this is an equivariant deformation of $\widehat{Y}_j$. We obtain a morphism

$$\Phi :\ \mathrm{Def}(Y, G) \longrightarrow \prod_{j \in B} \mathrm{Def}(\widehat{Y}_j, G)$$

which maps a deformation Y_R to the tuple $(\widehat{Y}_{j,R})_j$. Following [3], we call Φ the *local-global morphism*. By the results of §3.3 we can identify the natural morphism arising from the local-global spectral sequence

$$\mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y) \longrightarrow \bigoplus_{j \in B} \mathcal{E}xt^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y)_{\widehat{\tau_j}} \tag{52}$$

with the differential of Φ. In contrast to the situation studied in [3], Φ is not formally smooth unless $s = 1$. In fact, if $s > 1$ then the groups $\mathcal{E}xt^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y)_{\widehat{\tau_j}}$ are not finite-dimensional over k. However, if we restrict our attention to the image of Φ, then we obtain a *local-global-principle*, comparable to [3], Théorème 3.3.4.

Lemma 5.10. — *The map* (52) *is injective. Its image is the direct sum*

$$\bigoplus_{j \in B_{\mathrm{new}}} \mathcal{T}_{X,\tau_j} \otimes k(\tau_j) \subset \bigoplus_{j \in B} \mathcal{E}xt^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y)_{\widehat{\tau_j}}.$$

Proof. — We have already seen in the proof of Theorem 4.11 that the natural map

$$\mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y) \longrightarrow H^0(X, \mathcal{E}xt^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y))$$

is an isomorphism. (Note that the hypothesis of Theorem 4.11 is verified by Lemma 5.6 (i).) Furthermore, we have an isomorphism

$$H^0(X, \mathcal{E}xt^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y)) \cong H^1\Big(X, \mathcal{T}_X\Big(-\sum_{j\in B}\tau_j\Big)\Big). \tag{53}$$

The k-dimension of (53) is $|B|-3 = |B_{\mathrm{new}}|$ by Riemann-Roch.

Let $\mathcal{E}_{\mathrm{tor}} \subset \mathcal{E}xt^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y)$ be the maximal sub-$\mathcal{O}_X$-module which is torsion. The sequence (40) and a local computation shows

$$\mathcal{E}_{\mathrm{tor}} \cong \frac{\mathcal{T}_Z(D)^H}{\mathcal{T}_Z(-D_\infty)^H} \cong \frac{\mathcal{T}_X(-\sum_{j\in B}(1-\nu_j)\tau_j)}{\mathcal{T}_X(-\sum_{j\in B}\tau_j)}.$$

Therefore,

$$H^0(X, \mathcal{E}_{\mathrm{tor}}) \cong \bigoplus_{j\in B_{\mathrm{new}}} \mathcal{T}_{X,\tau_j}\otimes k(\tau_j). \tag{54}$$

Comparing dimensions, we find that $H^0(X,\mathcal{E}_{\mathrm{tor}}) \hookrightarrow H^0(X, \mathcal{E}xt^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y))$ is an isomorphism. This proves the lemma. □

Let $\mathrm{Def}(\widehat{Y}_j, G)^\dagger \subset \mathrm{Def}(\widehat{Y}_j, G)$ denote the image of $\mathrm{Def}(Y,G)$ under the localization map. In other words, $\mathrm{Def}(\widehat{Y}_j, G)^\dagger$ classifies those equivariant deformations of $\widehat{Y}_j$ which arise as the completion of a global deformation of Y. We denote by

$$\Phi^\dagger : \mathrm{Def}(Y,G) \longrightarrow \mathrm{Def}(Y,G)^{\mathrm{loc}} := \prod_{j\in B}\mathrm{Def}(\widehat{Y}_j, G)^\dagger$$

the restriction of Φ onto its image. By Lemma 5.10, the differential of $\Phi^\dagger$ is the isomorphism

$$\mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/S}, \mathcal{O}_Y) \xrightarrow{\sim} \bigoplus_{j\in B_{\mathrm{new}}} \mathcal{T}_{X,\tau_j}\otimes k(\tau_j).$$

THEOREM 5.11 (Local-global principle).

(i) *The functor* $\mathrm{Def}(\widehat{Y}_j, G)^\dagger$ *admits a versal deformation over the ring*

$$\widetilde{R}_j := \begin{cases} W(k), & \text{for } j\in B_0,\\ W(k)[[t_j]], & \text{for } j \in B_{\mathrm{new}}.\end{cases}$$

(ii) *The functor* $\mathrm{Def}(Y,G)$ *admits an effective universal deformation. Let* $\widetilde{R}$ *be the universal deformation ring.*

(iii) *The restricted local-global morphism* $\Phi^\dagger$ *is an isomorphism. Therefore, we have*

$$\widetilde{R} \cong \widehat{\otimes}_{W(k)}\widetilde{R}_j = W(k)[[\,t_j \mid j\in B_{\mathrm{new}}\,]].$$

Proof. — By Theorem 4.8 the functor $\mathrm{Def}(Y,G)$ admits a versal deformation. By Theorem 4.11 it is unobstructed. The space of "infinitesimal automorphisms" of $\mathrm{Def}(Y,G)$ is isomorphic to $H^0(X,\mathcal{T}_Y^G)$. By Lemma 4.9 we have

$$H^0(X,\mathcal{T}_Y^G)\cong H^0(X,\mathcal{M}^H)=0.$$

Therefore, the versal deformation of $\mathrm{Def}(Y,G)$ is also universal, see[20]. It is effective by Grothendieck's existence theorem. This proves (ii). The functors $\mathrm{Def}(\widehat{Y}_j,G)^\dagger$ are unobstructed, because the same holds for $\mathrm{Def}(Y,G)$. Using Theorem 3.11 and Lemma 5.10, it is easy to verify Schlessinger's axioms[20], showing that $\mathrm{Def}(\widehat{Y}_j,G)^\dagger$ admits a versal deformation over the ring $\widetilde{R}_j$, as claimed in (i). Finally, Lemma 5.10 together with the argument used in the proof of [3], Théorème 3.3.4, shows that $\Phi^\dagger$ is an isomorphism. This finishes the proof of the theorem. □

Remark 5.12. — If $s=\dim_{\mathbb{F}_p}V=1$ then $\Phi^\dagger=\Phi$.

5.4. Rigidity. — In this subsection (Z,V) and Y will be as before. Let R be an Artinian local k-algebra with residue field k. Since R has characteristic p, an equivariant deformation of Y over R corresponds to a deformation datum (Z_R,V_R) over R which lifts (Z,V). By this we mean that $\pi_R:Z_R\to X_R=\mathbb{P}^1_R$ is a tamely ramified H-Galois cover lifting $\pi:Z\to X$ and $V_R\subset H^0(Z_R,\Omega_{Z_R/R}(D_{\infty,R}))$ is an H-stable $\mathbb{F}_p$-vector space of logarithmic differentials lifting V (here $D_{\infty,R}\subset Z_R$ is a relative Cartier divisor lifting D_∞).

Let Y_R be an equivariant deformation of Y and (Z_R,V_R) the corresponding deformation datum. Choose $j\in B_{\mathrm{new}}$ and a point $\xi\in Z$ lying above τ_j. By the theory of tame ramification, there exists a local parameter z for Z_R at ξ such that $\mathcal{O}_{Z_R,\xi}=R[[z]]$ and $\alpha^*z=\psi(\alpha)\cdot z$ for some character $\psi:H_\xi\to R^\times$. We say that the deformation Y_R is *j-special* if every element $\phi\in V_R$ is of the form

$$\phi=z^{m_j+a_j-1}(c_0+c_1z+\ldots)\,\mathrm{d}z$$

with $c_i\in R$ and $c_0\in R^\times$. Note that this condition is independent of the choice of z.

Lemma 5.13. — *The equivariant deformation Y_R is trivial (i.e. isomorphic to $Y\otimes_k R$) if and only if it is j-special for all $j\in B_{\mathrm{new}}$.*

Proof. — One direction of the claim follows immediately from Lemma 5.6 (iii). To prove the other direction, suppose that Y_R is j-special for every $j \in B_{\rm new}$. We have to show that Y_R is the trivial deformation. By Theorem 5.11 (iii) it suffices to show that the completion $\widehat{Y}_{j,R}$ of Y_R at τ_j is the trivial deformation of $\widehat{Y}_j$, for all $j \in B_{\rm new}$. Fix one index j. Since R is Artinian, we may prove triviality of $\widehat{Y}_{j,R}$ by induction: suppose that $\widehat{Y}_{j,R'} := \widehat{Y}_{j,R} \otimes_R R'$ is trivial, where $R' := R/\mathfrak{m}_R^n$ for some $n \geqslant 1$. Then we want to conclude that $\widehat{Y}_{j,R''}$ is trivial, where $R'' := R/\mathfrak{m}_R^{n+1}$. To simplify the notation, we may even assume that $R = R''$.

The "difference" between $\widehat{Y}_{j,R}$ and the trivial deformation $\widehat{Y}_j \otimes_k R$, considered as lifts of the trivial deformation $\widehat{Y}_j \otimes_k R'$, is measured by an element $\bar\theta_j$ in

$$\mathcal{E}xt^1_G(\mathcal{L}_{Y/S}, \mathcal{O}_Y)\widehat{_{\tau_j}} \otimes_k \mathfrak{m}_R^n \cong (\mathcal{M}^H/\mathcal{T}_Z(-D_\infty)^H)\widehat{_{\tau_j}} \otimes_k \mathfrak{m}_R^n,$$

see Theorem 3.11 (ii). Since $\widehat{Y}_{j,R}$ lies in the image of the local-global morphism Φ the element $\bar\theta_j$ lies in the subspace

$$(\mathcal{T}_Z(D)^H/\mathcal{T}_Z(-D_\infty)^H)\widehat{_{\tau_j}} \otimes_k \mathfrak{m}_R^n \cong \mathcal{T}_{X,\tau_j} \otimes k(\tau_j) \otimes_k \mathfrak{m}_R^n,$$

see the proof of Lemma 5.10. In other words, we may regard $\bar\theta_j$ as a tangent vector at τ_j, with values in the k-vector space $\mathfrak{m}_R^n$. We have to show that $\bar\theta_j = 0$.

Choose a point $\xi \in Z$ above τ_j. Let z be a local parameter of Z_R at ξ such that $\widehat{O}_{Z_R,\xi} = R[[z]]$ and $\alpha^* z = \psi_\xi(\alpha)\cdot z$ for a character $\psi_\xi : H_\xi \to R^\times$. It follows that $\widehat{O}_{X_R,\tau_j} = R[[x]]$, where $x := z^{m_j}$ and $m_j := |H_\xi|$. Note that the fiber product $\widehat{Y}_{\xi,R} := Y_R \times_{Z_R} \operatorname{Spec} R[[z]]$ is a connected component of $\widehat{Y}_{j,R}$. Let $\phi_1, \dots, \phi_s$ be a basis of V_R. We have $\phi_i = \mathrm{d}u_i/u_i$ for a unit $u_i \in R[[z]]^\times$ which is unique up to multiplication by a pth power. The G_0-torsor $\widehat{Y}_{\xi,R} \to \operatorname{Spec} R[[z]]$ is given by the Kummer equations

$$y_i^p = u_i, \qquad i = 1, \dots, s.$$

By our induction hypothesis, the induced deformation $\widehat{Y}_{j,R'}$ is trivial. This means that, for a suitable choice of the parameter z and the units u_i, the image of u_i in $R'[[z]]$ actually lies in the subalgebra $k[[z]] \subset R'[[z]]$. In other words, we have

$$u_i = \bar u_i + v_i, \qquad \bar u \in k[[z]]^\times, \qquad v_i \in k[[z]] \otimes_k \mathfrak{m}_R^n.$$

Claim. — The tangent vector $\bar\theta_j$ extends to a vector field $\theta_j \in \mathcal{T}_{X,\tau_j} \otimes \mathfrak{m}_R^n$ in a neighborhood of τ_j, with values in $\mathfrak{m}_R^n$, such that

$$v_i = \theta_j(\mathrm{d}\bar u_i), \tag{55}$$

for $i = 1, \dots, s$.

Let us prove this claim. The class in $\mathcal{E}xt^1_G(\mathcal{L}_{Y/S}, \mathcal{O}_Y)\widehat{_{\tau_j}} \otimes_k \mathfrak{m}_R^n$ corresponding to $\bar{\theta}_j$ lifts to a local section θ'_j of the sheaf $\mathcal{M}^H$ in a neighborhood of τ_j, via the exact sequence (40). We consider θ'_j as an $\mathbb{F}_p$-linear and H-equivariant map $\theta_j : V \to k[[z]] \otimes \mathfrak{m}_R^n$. By the definition of $\bar{\theta}_j$ in terms of the deformation $\widehat{Y}_{j,R}$, we have $\theta'_j(\bar{\phi}_i) = v_i/\bar{u}_i$ (compare with the proof of Proposition 4.10). But since $\widehat{Y}_{j,R}$ lies in the image of the local-global morphism, θ'_j is actually the restriction to V of a vector field θ_j on X which is regular in a neighborhood of τ_j (compare with the proof of Lemma 5.10). The claim follows.

We can now finish the proof of the lemma. The vector field θ_j appearing in the claim we have just proved can be written as follows:

$$\theta_j = (b_0 + b_1 x + \ldots)\,\frac{\mathrm{d}}{\mathrm{d}x} = \frac{1}{m_j}(b_0 z^{1-m_j} + b_1 z + \ldots)\,\frac{\mathrm{d}}{\mathrm{d}z},$$

with $b_\mu \in \mathfrak{m}_R^n$. Since by assumption the deformation Y_R is j-special we have $\mathrm{d}u_i = z^{m_j+a_j-1}(c_0 + c_1 z + \ldots)\,\mathrm{d}z$ with $\bar{c}_0 \neq 0$. From (55) we get

$$v_i = \frac{z^{a_j}}{m_j}(\,\bar{c}_0 b_0 \;+\; (\bar{c}_0 b_1 + \bar{c}_1 b_0) z^{m_j} + \cdots). \tag{56}$$

But since $\mathrm{d}u_i = \mathrm{d}\bar{u}_i + \mathrm{d}v_i$ is divisible by $z^{m_j+a_j-1}$ it follows that

$$a_j\,\bar{c}_0\,b_0 = 0.$$

But a_j is prime to p and $\bar{c}_0 \neq 0$, hence $b_0 = 0$. We conclude that $\bar{\theta}_j = 0$, which completes the proof of the lemma. □

Theorem 5.14 (Rigidity). — *There exist, up to isomorphism, at most a finite number of special deformation data of given type (H, χ). Moreover, every special deformation datum can be defined over a finite field.*

Proof. — For a fixed type (H, χ) there exists at most a finite number of possibilities for the signature $(\sigma_j^{(i)})$ of a special deformation datum. Therefore, we may also fix the signature $(\sigma_j^{(i)})$. Let $U \subset (\mathbb{P}^1)^B$ be the locally closed subset from Proposition 5.9 (ii). Let $U' \subset (\mathbb{P}^1)^{B_{\rm new}}$ denote the intersection of U with the closed subset $\{(0,1,\infty)\} \times (\mathbb{P}^1)^{B_{\rm new}} \subset (\mathbb{P}^1)^B$. Thus, a point on U' corresponds to the branch locus of a *normalized* special deformation datum. To prove the theorem, it suffices to show that U' has pure dimension 0. Suppose that U' has an irreducible component of dimension > 0. Then there exists an algebraically closed field k of characteristic p and a nonconstant morphism $\varphi : \operatorname{Spec} R \to U'$, with $R = k[[t]]$. We will show that φ is constant, which gives a contradiction.

Going again through the proof of Proposition 5.9, we see that φ corresponds to a deformation datum (Z_R, V_R) defined over R. Moreover, the special and the generic fiber of (Z_R, V_R) are special deformation data. Applying Lemma 5.6 to the generic fiber of (Z_R, V_R), we see that the pullback $(Z_{R'}, V_{R'})$ of (Z_R, V_R) over $R' := R/t^n$ is a deformation of its special fiber which is j-special, for all $j \in B_{\mathrm{new}}$ and for all n. Hence it follows from Lemma 5.13 that the curve $Y_{R'}$ corresponding to $(Z_{R'}, V_{R'})$ is the trivial deformation of its special fiber. By Theorem 5.7, this implies that the branch locus of the induced G-cover $Y_{R'} \to X_{R'} = \mathbb{P}^1_{R'}$ is constant, for all n. We conclude that $\varphi : \operatorname{Spec} R \to U'$ is constant, which proves the theorem. □

5.5. — In this last section we prove a proposition which links two of our previous results on special deformation data: the lifting property (Theorem 5.7) and the local-global principle (Theorem 5.11). This proposition is a key ingredient for the proof of the main result of [26].

Let $\mathcal{Y}$ be the universal equivariant deformation of Y over $\widetilde{R}$, see Theorem 5.11. The quotient scheme $\mathcal{X} := \mathcal{Y}/G$ is naturally equipped with sections $\tau_{j,\widetilde{R}} : \operatorname{Spec} \widetilde{R} \to \mathcal{X}$ lifting the critical points τ_j. We may suppose that $\mathcal{X} = \mathbb{P}^1_{\widetilde{R}}$ and that $\{\tau_{j,\widetilde{R}} \mid j \in B_0\} = \{0, 1, \infty\}$. With this normalization, we may regard the sections $\tau_{j,\widetilde{R}}$ for $j \in B_{\mathrm{new}}$ simply as elements of the ring $\widetilde{R}$. Let $[\tau_j] \in W(k)$ denote the Teichmüller lift of $\tau_j \in k$ and set $T_j := \tau_{j,\widetilde{R}} - [\tau_j]$. By the lifting property (Theorem 5.7) we have

$$\widetilde{R} = W(k)[[t_j \mid j \in B_{\mathrm{new}}]] = W(k)[[T_j \mid j \in B_{\mathrm{new}}]].$$

A priori, it is not clear that these two sets of coordinates of $\widetilde{R}$ are in any way related. However, we have:

Proposition 5.15. — *For all $j \in B_{\mathrm{new}}$ there exists a unit $w_j \in \widetilde{R}^\times$ such that*

$$T_j \equiv w_j \cdot t_j \pmod{p}.$$

Proof. — Let $R := k[\epsilon]$ denote the ring of dual numbers. Fix some $j_0 \in B_{\mathrm{new}}$ and let $\kappa : \widetilde{R} \to R$ be the unique $W(k)$-algebra morphism which sends t_{j_0} to ϵ and t_j to 0 for $j \neq j_0$. Set $\tau_{j,R} := \kappa(\tau_{j,\widetilde{R}})$. Then $\tau_{j,R} = \tau_j + \epsilon \cdot \delta_j$

for an element $\delta_j \in k$. To prove the proposition it suffices to show that

$$\delta_j \neq 0 \quad \text{if and only if} \quad j = j_0. \tag{57}$$

Let Y_R denote the equivariant deformation of Y obtained from pulling back the universal deformation $\mathcal{Y}$ along κ. The isomorphism class of Y_R corresponds to a class in $\mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y)$. Via the isomorphism

$$\mathbb{E}\mathrm{xt}^1_G(\mathcal{L}_{Y/k}, \mathcal{O}_Y) \xrightarrow{\sim} \bigoplus_{j \in B_{\mathrm{new}}} \mathcal{T}_{X,\tau_j} \otimes k(\tau_j),$$

this class may be represented by a tuple $(\bar{\theta}_j)$, where $\bar{\theta}_j$ is a tangent vector in τ_j (see the proof of Lemma 5.10). By the choice of the indeterminates t_j and the homomorphism κ we have

$$\bar{\theta}_j \neq 0 \quad \text{if and only if} \quad j = j_0. \tag{58}$$

On the other hand, it follows from Proposition 4.10 that the tuple $(\bar{\theta}_j)$, considered as a class in

$$H^1(X, \mathcal{T}_X(-\sum_{j\in B} \tau_j)) \cong H^0(X, \frac{\mathcal{T}_X(-\sum_{j\in B_0} \tau_j)}{\mathcal{T}_X(-\sum_{j\in B} \tau_j)}) \cong \bigoplus_{j \in B_{\mathrm{new}}} \mathcal{T}_{X,\tau_j} \otimes k(\tau_j),$$

represents the isomorphism class of the deformation $(X_R; \tau_{j,R})$. Therefore,

$$\bar{\theta}_j = \delta_j \cdot \frac{\mathrm{d}}{\mathrm{d}x}\Big|_{x=\tau_j}. \tag{59}$$

Now (58) and (59) together imply (57). The proposition is proved. $\square$

To finish, let us explain briefly the motivation behind Proposition 5.15. Let R be a complete discrete valuation ring of mixed characteristic $(0, p)$. Let k be the residue field of R (which we assume algebraically closed) and K its fraction field. Let (Z, V) be a special deformation datum over $X = \mathbb{P}^1_k$ and $Y \to X$ the associated G-cover. Furthermore, let $\tau_{j,R} \in R$ be points on $X_R = \mathbb{P}^1_R$ which lift the branch points (τ_j) of $Y \to X$. By the lifting property, there exists a unique lift $Y_R \to X_R$ of $Y \to X$ with branch points $(\tau_{j,R})$. Assuming that $\zeta_p \in R$, the generic fiber $Y_K \to X_K = \mathbb{P}^1_K$ is a tame Galois cover with Galois group

$$G(K) \cong (\mathbb{Z}/p)^s \rtimes H.$$

By construction, the cover $Y_K \to X_K$ has bad reduction: the special fiber Y is singular and the induced map $Y \to X$ is not separable. However, after some blowing up we can find a certain nice model $\widetilde{Y}_R \to \widetilde{X}_R$ over R of $Y_K \to X_K$, called the *stable model*, see [18] and [26].

What can we say about the stable model, and how does it depend on the choice of the branch points $\tau_{j,R}$? Let us say that the stable reduction of

$Y_K \to X_K$ is *nice* if the vanishing cycles of Y_R are resolved by blowing up $X_R = \mathbb{P}^1_R$ in certain *disjoint* closed disks with center $\tau_{j,R}$, for $j \in B - B_{\mathrm{wild}}$ (in [25] this property is called *special*). If the stable reduction is nice, then the special fiber $\widetilde{X}$ of $\widetilde{X}_R$ is a *comb*. More precisely, $\widetilde{X}$ is a semistable curve consisting of the central component X and, for each index $j \in B - B_{\mathrm{new}}$, a tail X_j meeting X in τ_j. Using Proposition 5.15 one can show the following.

Result 5.16. — The stable reduction of $Y_K \to X_K$ is nice if and only if the branch points $\tau_{j,R} \in R$ are "sufficiently close" to the Teichmüller lift $[\tau_j] \in W(k) \subset R$.

In the case $s = \dim_{\mathbb{F}_p} V = 1$ the "if"-direction of this result was proved in [25], using a very different kind of argument. In [26] and still under the condition $s = 1$, both directions of the above result are proved, using Proposition 5.15. The case $s > 1$ is similar but a bit more involved and will be dealt with in a subsequent paper.

Roughly speaking, the singularities of Y_R can be described in terms of the image of the parameters t_j in R (under the classifying map $\widetilde{R} \to R$ of the deformation Y_R). Therefore, Proposition 5.15 provides a link between the singularities of Y_R and the position of the branch points of $Y_R \to X_R$. It is somewhat surprising that such a relation exists at all, because the dependence of the cover $Y_R \to X_R$ on the branch points $\tau_{j,R}$ seems to be of a more global nature.

A. Picard categories and Picard stacks.

In this first appendix we recall some basic facts about Picard categories and Picard stacks. The main result we need is Proposition A.2. References are [1], Exposé XVIII and [22].

A.1. — A *(strictly commutative) Picard category* is a nonempty monoid $\mathfrak{P}$, together with a functor

$$+ : \mathfrak{P} \times \mathfrak{P} \longrightarrow \mathfrak{P}, \qquad (x, y) \longmapsto x + y$$

and two functorial isomorphisms

$$\sigma : (x + y) + z \cong x + (y + z), \qquad \tau : x + y \cong y + x$$

such that the following holds.

(i) The isomorphisms σ and τ make $+$ an *associative* and *strictly commutative* functor, in the sense of [1], Exposé XVIII, §1.4.1.

(ii) For any object y of $\mathfrak{P}$, the functor $x \mapsto x+y$ is an equivalence of categories.

Given two Picard categories $\mathfrak{P}_1, \mathfrak{P}_2$, an *additive functor* from $\mathfrak{P}_1$ to $\mathfrak{P}_2$ is a functor $F : \mathfrak{P}_1 \to \mathfrak{P}_2$ together with a functorial isomorphism

$$F(x+y) \xrightarrow{\sim} F(x)+F(y)$$

which is compatible with the associativity and the commutativity laws, see [1], Exposé XVIII, §1.4.6. Given two additive functors $F, G : \mathfrak{P}_1 \to \mathfrak{P}_2$, a *morphism of additive functors* $u : F \to G$ is a morphism of functors (automatically an isomorphism) such that the diagram

$$\begin{array}{ccc} F(x+y) & \xrightarrow{u_{x+y}} & G(x+y) \\ \downarrow & & \downarrow \\ F(x)+F(y) & \xrightarrow{u_x+u_y} & G(x)+G(y) \end{array}$$

commutes. We denote by $\mathfrak{Hom}(\mathfrak{P}_1, \mathfrak{P}_2)$ the corresponding category of additive functors and by $\mathrm{Hom}(\mathfrak{P}_1, \mathfrak{P}_2)$ its set of isomorphism classes. One can show that $\mathfrak{Hom}(\mathfrak{P}_1, \mathfrak{P}_2)$ carries a natural structure of Picard category.

Let A be a complex of Abelian groups. We define a Picard category $\mathfrak{Pic}(A)$ as follows. Objects of $\mathfrak{Pic}(A)$ are 1-cocycles, *i.e.* elements of $Z^1(A) = \mathrm{Ker}(A^1 \xrightarrow{d} A^2)$. Given two objects $x, y \in Z^1(A)$, the set of morphisms $\mathrm{Hom}(x,y)$ is the set of elements $f \in A^0$ such that $d(f) = y - x$, modulo 0-coboundaries, *i.e.* elements of $B^0(A) = \Im(A^{-1} \xrightarrow{d} A^0)$. The composition of two morphisms $f : x \to y$ and $g : y \to z$ is the sum $f+g$. The functor $+$ is induced from the addition law of A^1. It follows immediately from this definition that the group of automorphisms of the "neutral object" of $\mathfrak{Pic}(A)$ is identified with $H^0(A)$, whereas the group of isomorphism classes of $\mathfrak{Pic}(A)$ is identified with $H^1(A)$. Note also that $\mathfrak{Pic}(A) = \mathfrak{Pic}(A^{[0,1]})$, where $A^{[0,1]}$ denotes the complex of amplitude $[0,1]$ deduced from A such that $H^n(A^{[0,1]}) = H^n(A)$ for $n = 0, 1$.

Given two complexes of Abelian groups A, B, a homomorphism of complexes $\varphi : A \to B$ gives rise to an additive functor $\mathfrak{Pic}(\varphi) : \mathfrak{Pic}(A) \to \mathfrak{Pic}(B)$. The functor $\mathfrak{Pic}(\varphi)$ is an equivalence of categories if and only if $H^n(\varphi)$ is an isomorphism for $n = 0, 1$. Given two homomorphisms $\varphi, \psi : A \to B$, a homotopy $\varphi \sim \psi$ gives rise to an isomorphism of additive functors $\mathfrak{Pic}(\varphi) \cong \mathfrak{Pic}(\psi)$. Therefore, the association $A \mapsto \mathfrak{Pic}(A)$ gives rise to a functor from the derived category $\mathfrak{D}^{\geqslant 0}(\mathfrak{Ab})$ to the category of all (small) Picard categories (morphisms in the latter category are isomorphism classes of additive functors). It is shown in [1], Exposé XVIII, that this functor becomes an equivalence of categories when restricted to the full subcategory $\mathfrak{D}^{[0,1]}(\mathfrak{Ab})$ of complexes of amplitude $[0,1]$.

Remark A.1. — Our definition of $\mathfrak{Pic}(A)$ is a bit different from the definition used in [1], Exposé XVIII. In *loc.cit.*, $\mathfrak{Pic}(A)$ is only defined for a complex of amplitude $[-1,0]$. For the application of Picard categories in this paper, it seemed more convenient to shift degrees by 1 and to allow arbitrary complexes.

A.2. — Let X be a topological space. (Actually, without changing anything essential, we could let X be an arbitrary site.) We denote by $\mathfrak{Ab}(X)$ the category of sheaves of Abelian groups on X. The total right derived functor of the global section functor $\Gamma(X,\cdot)$ is denoted by $\mathbb{R}\Gamma(X,\cdot)$. For generalities about stacks, see [5] or [14].

A *Picard stack* over X is a stack $\underline{\mathfrak{P}}$ over X, together with a morphism of X-stacks

$$+ : \underline{\mathfrak{P}} \times_X \underline{\mathfrak{P}} \longrightarrow \underline{\mathfrak{P}}, \qquad (x,y) \longmapsto x+y$$

and two functorial isomorphisms

$$\sigma : (x+y)+z \cong x+(y+z), \qquad \tau : x+y \cong y+x$$

such that the following holds.

(i) For each open subset $U \subset X$, the fiber $\underline{\mathfrak{P}}(U)$, together with the restrictions of $+$, σ and τ to U, is a (strictly commutative) Picard category.

(ii) For each inclusion $U \subset V$ of open subsets, the restriction functor $\underline{\mathfrak{P}}(U) \to \underline{\mathfrak{P}}(V)$ is a morphism of Picard categories.

Given two Picard stacks $\underline{\mathfrak{P}}_1, \underline{\mathfrak{P}}_2$, an additive functor from $\underline{\mathfrak{P}}_1$ to $\underline{\mathfrak{P}}_2$ is an X-functor $F : \underline{\mathfrak{P}}_1 \to \underline{\mathfrak{P}}_2$ together with functorial isomorphisms $F(x+y) \xrightarrow{\sim} F(x)+F(y)$ whose restriction to each fiber is an additive functor. Given two additive functors $F,G : \underline{\mathfrak{P}}_1 \to \underline{\mathfrak{P}}_2$, a morphism of additive functors is a morphism of X-functors $u : F \to G$ (automatically an isomorphism) whose restriction to all fibers is a morphism of additive functors. We denote by $\mathfrak{Hom}(\underline{\mathfrak{P}}_1, \underline{\mathfrak{P}}_2)$ the corresponding category of additive functors and by $\mathrm{Hom}(\underline{\mathfrak{P}}_1, \underline{\mathfrak{P}}_2)$ its set of isomorphism classes. It is easy to equip $\mathfrak{Hom}(\underline{\mathfrak{P}}_1, \underline{\mathfrak{P}}_2)$ with a natural structure of a Picard category. Moreover, one can show that the X-groupoid

$$U \longmapsto \mathfrak{Hom}(\underline{\mathfrak{P}}_1|_U, \underline{\mathfrak{P}}_2|_U)$$

is itself a Picard stack, see [1], Exposé XVIII.

Let $\mathcal{A}$ be a complex of Abelian sheaves on X. The association

$$U \longmapsto \mathfrak{Pic}(\Gamma(U,\mathcal{A}))$$

(where $U \subset X$ runs over all open subsets of X) gives rise to a prestack $\underline{\mathfrak{P}}'(\mathcal{A})$ over X. Let $\underline{\mathfrak{Pic}}(\mathcal{A})$ be the stack over X associated to this prestack, see e.g. [14], Lemme (3.2). One checks that $\underline{\mathfrak{Pic}}(\mathcal{A})$ is a Picard stack, in a natural way. For each open subset $U \subset X$, the natural functor

$$\mathfrak{Pic}(\Gamma(U,\mathcal{A})) \longrightarrow \underline{\mathfrak{Pic}}(\mathcal{A}) \tag{60}$$

is an additive functor. In general, it is not an isomorphism.

A homomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ of Abelian sheaves gives rise to a morphism $\mathfrak{Pic}(\varphi) : \underline{\mathfrak{Pic}}(\mathcal{A}) \to \underline{\mathfrak{Pic}}(\mathcal{B})$ of Picard stacks. Moreover, a homotopy $\varphi \sim \psi$ gives rise to a isomorphism of additive functors $\mathfrak{Pic}(\varphi) \cong \mathfrak{Pic}(\psi)$. Therefore, the association $\mathcal{A} \mapsto \underline{\mathfrak{Pic}}(\mathcal{A})$ gives rise to a functor from the derived category $\mathfrak{D}(X)$ to the category of all (small) Picard stacks on X (morphisms in the latter category are isomorphism classes of additive functors).

Proposition A.2. — *Let X be a topological space and $\mathcal{A}$ a sheaf of Abelian groups on X such that $H^n(\mathcal{A}) = 0$ for $n < 0$. Then we have a natural equivalence of Picard categories*

$$\underline{\mathfrak{Pic}}(\mathcal{A})(X) \cong \mathfrak{Pic}(\mathbb{R}\Gamma(X,\mathcal{A})).$$

This proposition seems to be well known. Since it is an important step in the proof of Theorem 2.2 and we could not find a suitable reference, we give a proof.

Proof. — The hyper-cohomology spectral sequence and the assumption $H^n(\mathcal{A}) = 0$ for $n < 0$ show that

$$\mathbb{H}^n(X,\mathcal{A}^{[0,1]}) = \mathbb{H}^n(X,\mathcal{A}) \quad \text{for } n = 0,1.$$

We may therefore assume that $\mathcal{A} = \mathcal{A}^{[0,1]}$. Let $\mathcal{U} = (U_i)_{i\in I}$ be an open covering of X. We choose a well-ordering on the index set I. Let $K_{\mathcal{U}} := C^\bullet(\mathcal{U},\mathcal{A})$ be the double complex whose nth column (for $n = 0,1$) is the Čech cochain complex of $\mathcal{A}^n$ with respect to $\mathcal{U}$:

$$K_{\mathcal{U}} : \left\{ \begin{array}{ccc} \prod_i \Gamma(U_i, \mathcal{A}^0) & \xrightarrow{d} & \prod_i \Gamma(U_i, \mathcal{A}^1) \\ \downarrow \partial & & \downarrow \partial \\ \prod_{i<j} \Gamma(U_{i,j}, \mathcal{A}^0) & \xrightarrow{-d} & \prod_{i<j} \Gamma(U_{i,j}, \mathcal{A}^1) \\ \downarrow \partial & & \downarrow \partial \\ \prod_{i<j<k} \Gamma(U_{i,j,k}, \mathcal{A}^0) & \xrightarrow{d} & \prod_{i<j<k} \Gamma(U_{i,j,k}, \mathcal{A}^1) \\ \downarrow \partial & & \downarrow \partial \\ \cdots & & \cdots \end{array} \right.$$

We define a morphism of Picard categories

$$\mathfrak{Pic}(\mathrm{Tot}(K_{\mathcal{U}})) \xrightarrow{\sim} \underline{\mathfrak{Pic}}(\mathcal{A})(X), \tag{61}$$

as follows. An object of $\mathfrak{Pic}(\mathrm{Tot}(K_{\mathcal{U}}))$ is a datum $(f_i; g_{i,j})$, with $f_i \in \Gamma(U_i, Z^1(\mathcal{A}))$ and $g_{i,j} \in \Gamma(U_{i,j}, \mathcal{A}^0)$, such that

$$d(g_{i,j}) = f_j|_{U_{i,j}} - f_i|_{U_{i,j}} \tag{62}$$

for all $i < j$ and

$$g_{i,j} - g_{i,k} + g_{j,k} = 0 \tag{63}$$

for all $i < j < k$. Let $\widetilde{f}_i$ denote the object of $\underline{\mathfrak{Pic}}(\mathcal{A})(U_i)$ corresponding to f_i. By (62), $g_{i,j}$ corresponds to an isomorphism $\widetilde{g}_{i,j} : \widetilde{f}_i|_{U_{i,j}} \xrightarrow{\sim} \widetilde{f}_j|_{U_{i,j}}$. Now (63) means that these isomorphisms satisfy the cocycle relation $\widetilde{g}_{j,k} \circ \widetilde{g}_{i,j} = \widetilde{g}_{i,k}$. In other words, $(\widetilde{f}_i; \widetilde{g}_{i,j})$ is a patching datum with values in $\underline{\mathfrak{Pic}}(\mathcal{A})$. Since $\underline{\mathfrak{Pic}}(\mathcal{A})$ is a stack, there exists an object $\widetilde{f}$ of $\underline{\mathfrak{Pic}}(\mathcal{A})(X)$ together with isomorphisms $\alpha_i : \widetilde{f}|_{U_i} \xrightarrow{\sim} \widetilde{f}_i$ such that $\widetilde{g}_{i,j} = \alpha_j \circ \alpha_i^{-1}$. By definition, $\widetilde{f}$ is the image of $(f_i, g_{i,j})$ under (61).

Let $(f'_i, g'_{i,j})$ be another object of $\mathfrak{Pic}(\mathrm{Tot}(K_{\mathcal{U}}))$, and let $\widetilde{f}'$ be the corresponding object of $\underline{\mathfrak{Pic}}(\mathcal{A})(X)$. A homomorphism from $(f_i, g_{i,j})$ to $(f'_i, g'_{i,j})$ is a datum (h_i), with $h_i \in \Gamma(U_i, \mathcal{A}^0)$, such that

$$d(h_i) = f'_i - f_i \tag{64}$$

for all i and

$$h_j|_{U_{i,j}} - h_i|_{U_{i,j}} = g'_{i,j} - g_{i,j} \tag{65}$$

for all $i < j$. Equation (64) shows that h_i corresponds to an isomorphism $\widetilde{h}_i : \widetilde{f}_i \xrightarrow{\sim} \widetilde{f}_j$. Moreover, by (65) the diagram

$$\begin{array}{ccc} \widetilde{f}_i|_{U_{i,j}} & \xrightarrow{\widetilde{h}_i} & \widetilde{f}'_i|_{U_{i,j}} \\ {\scriptstyle \widetilde{g}_{i,j}} \downarrow & & \downarrow {\scriptstyle \widetilde{g}'_{i,j}} \\ \widetilde{f}_j|_{U_{i,j}} & \xrightarrow{\widetilde{h}_j} & \widetilde{f}'_j|_{U_{i,j}} \end{array}$$

commutes for all $i < j$. In other words, $(\widetilde{h}_i)$ is an isomorphism of patching data. Using again that $\underline{\mathfrak{Pic}}(\mathcal{A})$ is a stack, we see that there exists a unique isomorphism $\widetilde{h} : \widetilde{f} \xrightarrow{\sim} \widetilde{f}'$ such that $\widetilde{h}_i \circ \alpha_i = \alpha_i' \circ \widetilde{h}|_{U_i}$. By definition, $\widetilde{h}$ is the image of (h_i) under (61). This finishes the definition of (61) as a functor. We leave it to the reader to check that (61) is indeed a morphism of Picard categories.

Clearly, the definition of (61) is compatible with taking refinements of the covering $\mathcal{U}$. Therefore, we obtain a morphism of Picard categories

$$\varinjlim_{\mathcal{U}} \mathfrak{Pic}(\mathrm{Tot}(K_{\mathcal{U}})) \longrightarrow \underline{\mathfrak{Pic}}(\mathcal{A})(X). \tag{66}$$

We claim that (66) is an isomorphism. Indeed, the discussion of the previous paragraph, leading to the definition of (61), shows that $\mathfrak{Pic}(\mathrm{Tot}(K_{\mathcal{U}}))$ is isomorphic to the category of patching data for the covering $\mathcal{U}$, with values in the prestack $\underline{\mathfrak{Pic}}(\mathcal{A})'$. On the other hand, the category $\underline{\mathfrak{Pic}}(\mathcal{A})(X)$ is the direct limit over the categories of such patching data, where the limit is taken over all possible coverings $\mathcal{U}$; this follows from the construction of a stack associated to a prestack, see e.g. [14], §3. This proves the claim.

To finish the proof of the proposition, it suffices to show that the natural morphisms

$$\mathrm{Tot}(K_{\mathcal{U}}) \longrightarrow \mathbb{R}\Gamma(X, \mathcal{A})$$

induces isomorphisms on cohomology

$$\varinjlim_{\mathcal{U}} H^n(\mathrm{Tot}(K_{\mathcal{U}})) \xrightarrow{\sim} \mathbb{H}^n(X, \mathcal{A})$$

for $n = 0, 1$. This is proved in two steps. First, one compares the two spectral sequences which compute the cohomology of $\mathrm{Tot}(K_{\mathcal{U}})$ on the one hand and the hyper-cohomology groups $\mathbb{H}^n(X, \mathcal{A})$ on the other hand. Then one uses the well known fact that Čech-cohomology agrees with ordinary sheaf cohomology in degree $n = 0, 1$ (see e.g. [10], Ex. III.4.4). We omit the details. □

B. Group cohomology for affine flat group schemes.

We show how to compute the cohomology of an affine group flat group scheme in terms of cocycles and coboundaries, just as for abstract groups. Reference is [9], Exposé I.

B.1. — Throughout this section, we fix a commutative ring and an affine flat R-group scheme $G = \operatorname{Spec} \mathcal{O}_G$. We denote by $\Delta : \mathcal{O}_G \to \mathcal{O}_G \otimes_R \mathcal{O}_G$ the comultiplication and by $e : \mathcal{O}_G \to R$ the counit of G.

A *(right) G-R-module* is an R-module M together with an R-linear morphism $\mu_M : M \to \mathcal{O}_G \otimes_R M$ such that $(\mathrm{Id}_{\mathcal{O}_G} \otimes \mu_M) \circ \mu_M = (\Delta \otimes \mathrm{Id}_M) \circ \mu_M$ and $(e \otimes \mathrm{Id}_M) \circ \mu_M = \mathrm{Id}_M$. For each R-algebra R' and $\sigma \in G(R')$ we obtain an R'-linear automorphism $m \mapsto m^\sigma$ of $M' := M \otimes_R R'$ such that $m^{\sigma\tau} = (m^\sigma)^\tau$. We shall denote by $\mathrm{Mod}(R, G)$ the category of G-R-modules, by $\mathfrak{K}^+(R, G)$ the (triangulated) category of bounded below cochain complex in $\mathrm{Mod}(R, G)$ (modulo homotopy) and by $\mathfrak{D}^+(R, G)$ the derived category of $\mathfrak{K}^+(R, G)$.

Given a G-R-module M, the *invariant* R-submodule M^G is the set of all $m \in M$ such that $\mu_M(m) = 1 \otimes m$, or, what is equivalent, $m^\sigma = m$ for all R' and $\sigma \in G(R')$. The functor $M \mapsto M^G$ from $\mathrm{Mod}(R, G)$ to the category of Abelian groups is obviously additive and left exact. We denote its nth right derived functor by $H^n(G, \cdot\,)$ and its total right derived functor by $\mathbb{R}^G$. (For the existence of enough injectives in $\mathrm{Mod}(G, R)$, see the proof of Lemma Bbelow.)

B.2. — Given an R-module M, we set

$$\widetilde{M} := \mathcal{O}_G \otimes_R M.$$

The map $\Delta \otimes \mathrm{Id}_M : \widetilde{M} \to \mathcal{O}_G \otimes_R \widetilde{M}$ gives $\widetilde{M}$ the structure of a (right) G-R-module. A G-R-module which is isomorphic to $\widetilde{M}$ for some R-module M is called *coinduced*. If G is a finite group then this agrees with the usual definition of coinduced modules.

Let M be an R-module M, P a G-R-module and $\varphi : P \to M$ an R-linear morphism. Then $\widetilde{\varphi} := (\mathrm{Id}_{\mathcal{O}_G} \otimes \varphi) \circ \mu_P : P \to \widetilde{M}$ is easily checked to be G-equivariant. One checks that this construction yields a natural isomorphism

$$\mathrm{Hom}_R(P, M) \xrightarrow{\sim} \mathrm{Hom}_G(P, \widetilde{M}). \tag{67}$$

(The inverse of (67) is defined as follows: given a G-equivariant homomorphism $\psi : P \to \widetilde{M}$, $\varphi := (e \otimes \mathrm{Id}_M) \circ \psi : P \to M$ is an R-linear morphism such that $\psi = \widetilde{\varphi}$.) Moreover, the isomorphism (67) makes the functor $M \mapsto \widetilde{M}$ a right adjoint of the forgetful map from $\mathrm{Mod}(R, G)$ to $\mathrm{Mod}(R)$.

Lemma B.1. — *For any R-module M we have*

$$H^n(G, \widetilde{M}) = \begin{cases} M & \text{for } n = 0, \\ 0 & \text{for } n > 0. \end{cases}$$

Proof. — For $n = 0$, the lemma is equivalent to the exactness of the sequence

$$0 \to M \longrightarrow \widetilde{M} \longrightarrow \mathcal{O}_G \otimes_R \widetilde{M} \tag{68}$$

(the first arrow sends m to $1 \otimes m$ and the second $a \otimes m$ to $\Delta(a) \otimes m - 1 \otimes a \otimes m$). Now (68) is exact on the left because $R \to \mathcal{O}_G$ is flat, by assumption. Exactness in the middle is proved using the properties of the counit $e : \mathcal{O}_G \to R$. Hence the lemma holds for $n = 0$.

Choose an injective resolution $M \to I^0 \to I^1 \to \cdots$ of the R-module M. The functor $M \mapsto \widetilde{M}$, being the right adjoint of an exact functor, preserves injectives, see e.g. [24], Proposition 2.3.11. Therefore, $\widetilde{M} \to \widetilde{I}^0 \to \widetilde{I}^1 \to \cdots$ is an injective resolution of the G-R-module $\widetilde{M}$. Now the general case of the lemma follows from the case $n = 0$. □

B.3. — Let M be a G-R-module. In order to compute the cohomology groups $H^n(G, M)$, it suffices to write down a resolution of M by coinduced G-R-modules. We do this as follows. Set $B^{-1}(G, M) := M$ and define inductively $B^n(G, M) := (B^{n-1}(M))^\sim$ for all $n \geqslant 0$. As R-modules, we simply get

$$B^n(G, M) = \underbrace{\mathcal{O}_G \otimes_R \cdots \otimes_R \mathcal{O}_G}_{(n+1)\times} \otimes_R M.$$

We define differentials $\partial : B^n(G, M) \to B^{n+1}(G, M)$ by setting

$$\partial(a_0 \otimes \cdots \otimes a_n \otimes m) := \sum_{\nu=0}^{n} (-1)^\nu a_0 \otimes \cdots \otimes \Delta(a_\nu) \otimes \cdots \otimes a_n \otimes m$$
$$+(-1)^{n+1} a_0 \otimes \cdots \otimes a_n \otimes \mu_M(m).$$

Note that $\partial : M = B^{-1}(G, M) \to \widetilde{M} = B^0(G, M)$ equals μ_M. It is easy to check that

$$M \xrightarrow{\mu_M} B^0(G, M) \xrightarrow{\partial} B^1(G, M) \xrightarrow{\partial} \cdots \tag{69}$$

is an (augmented) complex a G-R-modules. (In fact, (69) is the (augmented) cochain complex associated to M and the pair of adjoint functors $\mathrm{Mod}(G, R) \rightleftarrows \mathrm{Mod}(R)$, see [24], §8.6.) Moreover, (69) is exact. This follows immediately from the existence of the homotopy

$$s : \begin{cases} B^{n+1}(G, M) & \longrightarrow & B^n(G, M) \\ a_0 \otimes \cdots \otimes a_n \otimes m & \longmapsto & e(a_0) a_1 \otimes \cdots \otimes a_n \otimes m \end{cases}.$$

Applying the functor $M \mapsto M^G$ to the resolution (69) defines an augmented complex of R-modules

$$M^G \longrightarrow C^0(G, M) \xrightarrow{\partial} C^1(G, M) \xrightarrow{\partial} \cdots \tag{70}$$

Elements of $C^n(G,M)$ are called *n-cochains* with values in M. Likewise, elements of $Z^n(G,M) := \mathrm{Ker}(\partial)$ (resp. of $B^n(G,M) := \Im(\partial)$) are called *cocycles* (resp. *coboundaries*). Note that we have an isomorphism of R-modules

$$C^n(G,M) \cong \underbrace{\mathcal{O}_G \otimes_R \cdots \otimes_R \mathcal{O}_G}_{n\times} \otimes_R M,$$

such that the canonical injection $C^n(G,M) \hookrightarrow B^n(G,M)$ sends the element $a_1 \otimes \cdots \otimes a_n \otimes m$ to the element $1 \otimes a_1 \otimes \cdots \otimes a_n \otimes m$. Given an R-algebra R', an n-cochain $\varphi \in C^n(G,M)$ gives rise to a function

$$G(R') \times \cdots \times G(R') \longrightarrow M' = M \otimes_R R', \qquad \underline{\sigma} = (\sigma_1, \ldots, \sigma_n) \longmapsto \varphi_{\underline{\sigma}}.$$

Now φ is a cocycles (resp. a coboundary) if and only if this function is a cocycle (resp. a coboundary) in the traditional sense, for all R-algebras R' (again, it suffices to take R' flat over R). For instance, a 1-cochain φ is a cocycle if and only if

$$\varphi_{\sigma,\tau} = \varphi_\sigma^\tau + \varphi_\tau$$

holds for all $\sigma, \tau \in G(R')$. It is a coboundary if and only if for all R' there exists an element $m \in M'$ such that $\varphi_\sigma = m^\sigma - m$ holds for all $\sigma \in G(R')$.

It follows from Lemma B.1 that

$$H^n(G,M) = H^n(C^\bullet(G,M)) \tag{71}$$

for all G-R-modules M and all $n \geqslant 0$. The next proposition is a slight generalization of (71).

Proposition B.2. — *Let $M^\bullet \in \mathfrak{K}^+(R,G)$ be a bounded below complex of G-R-modules. Then we have a natural isomorphism of derived complexes*

$$\mathbb{R}^G(M^\bullet) \cong \mathrm{Tot}(C^\bullet(G,M^\bullet)).$$

Proof. — Let K denote the double complex $B^\bullet(G,M^\bullet)$. The qth row of K is exact except at degree $p=0$, where the cohomology is M^q. Therefore, the spectral sequence associated to K (filtered by rows) shows that the augmentation $M^\bullet \to K$ gives rise to a quasi-isomorphism

$$M^\bullet \longrightarrow \mathrm{Tot}(K).$$

By definition, the complex $\mathrm{Tot}(K)$ consists entirely of coinduced G-R-modules, which are acyclic with respect to taking G-invariants, by Lemma B.1. Therefore, [24], Theorem 10.5.9 implies

$$\mathbb{R}^G(M^\bullet) = \mathrm{Tot}(K)^G = \mathrm{Tot}(C^\bullet(G,M^\bullet)).$$

This finishes the proof of the proposition. □

C. Sheaves of G-$\mathcal{O}_Y$-modules.

The goal of this last appendix is to review the definition of equivariant hyperext groups and the construction of the two spectral sequences (79) and (82). The standard reference is [7].

C.1. — Let $G \to S$ and $Y \to S$ be as in §1. Let $\lambda : G \times_S Y \to Y$ (resp. $p : G \times_S Y \to Y$) denote the morphism defining the action of G on Y (resp. the second projection). Given a sheaf of $\mathcal{O}_Y$-modules $\mathcal{F}$, a *lift* of the G-action from Y to $\mathcal{F}$ is given by an isomorphism $\lambda^*\mathcal{F} \to p^*\mathcal{F}$ which satisfies certain obvious axioms, see e.g. [17], §III.12.

Let $\mathcal{F}$ and $\mathcal{G}$ be G-$\mathcal{O}_Y$-modules. Let $\mathrm{Hom}_Y(\mathcal{F}, \mathcal{G})$ denote the R-module of $\mathcal{O}_Y$-linear (but not necessarily G-equivariant) homomorphisms from $\mathcal{F}$ to $\mathcal{G}$. It carries a natural structure of G-R-module, defined as follows. Let R' be a flat R-algebra and $\sigma \in G(R')$. Since R' is flat over R we have a natural isomorphism

$$\mathrm{Hom}_Y(\mathcal{F}, \mathcal{G}) \otimes_R R' \cong \mathrm{Hom}_{Y'}(\mathcal{F}', \mathcal{G}').$$

Given $f : \mathcal{F}' \to \mathcal{G}' \in \mathrm{Hom}_{Y'}(\mathcal{F}', \mathcal{G}')$, we define f^σ via the following commutative diagram:

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{f^\sigma} & \mathcal{G}' \\ {\scriptstyle\varphi_\sigma}\downarrow & & \downarrow{\scriptstyle\varphi_\sigma} \\ \sigma^*\mathcal{F}' & \xrightarrow{\sigma^* f} & \sigma^*\mathcal{G}' \end{array}$$

Note that an $\mathcal{O}_Y$-linear morphism $f : \mathcal{F} \to \mathcal{G}$ is G-equivariant if and only if it is invariant under the G-action just defined, *i.e.*

$$\mathrm{Hom}_G(\mathcal{F}, \mathcal{G}) = \mathrm{Hom}_Y(\mathcal{F}, \mathcal{G})^G.$$

C.2. — Let $\mathcal{F}$ be an $\mathcal{O}_Y$-module. We set $\widetilde{\mathcal{F}} := \mu_* p^* \mathcal{F}$ and claim that $\widetilde{\mathcal{F}}$ carries a natural structure of G-$\mathcal{O}_Y$-module. To define a G-action on $\widetilde{\mathcal{F}}$, let $S' = \mathrm{Spec}\, R'$ be an affine S-scheme and $\sigma \in G(S')$. As usual, we indicate base change to S' with a prime (e.g. $Y' = Y \times_S S'$) and identify σ with the automorphism of Y' induced from σ via the action of G on Y. Also, we let $t_\sigma : G' \times_{S'} Y' \xrightarrow{\sim} G' \times_{S'} Y'$ denote the automorphism induced from left translation by σ on the first factor. Since $p' \circ t_\sigma = p'$, we have a natural isomorphism

$$p'^*\mathcal{F}' \xrightarrow{\sim} t_\sigma^* p'^* \mathcal{F}'. \tag{72}$$

Using the commutative diagram

$$\begin{array}{ccc} G' \times_{S'} Y' & \xrightarrow{t_\sigma} & G' \times_{S'} Y' \\ {\scriptstyle \mu'}\big\downarrow & & \big\downarrow{\scriptstyle \mu'} \\ Y' & \xrightarrow{\sigma} & Y' \end{array} \tag{73}$$

we obtain an isomorphism $\varphi_\sigma : \widetilde{\mathcal{F}}' \xrightarrow{\sim} \sigma^* \widetilde{\mathcal{F}}'$ as follows:

$$\widetilde{\mathcal{F}}' = \mu'_* p'^* \mathcal{F}' \xrightarrow{\sim} \mu'_* t_\sigma^* p'^* \mathcal{F}' \xrightarrow{\sim} \sigma^* \mu'_* p'^* \mathcal{F}' = \sigma^* \widetilde{\mathcal{F}}'. \tag{74}$$

One checks that the definition of φ_σ is functorial in S' and satisfies the rule $\varphi_{\sigma\tau} = (\tau^* \varphi_\sigma) \circ \varphi_\tau$. This defines a lift of the G-action from Y to $\widetilde{\mathcal{F}}$. we may therefore regard $\widetilde{\mathcal{F}}$ as a G-$\mathcal{O}_Y$-module. A G-$\mathcal{O}_Y$-module isomorphic to $\widetilde{\mathcal{F}}$ for some $\mathcal{O}_Y$-module $\mathcal{F}$ is called *coinduced*. Compare with §B.2.

Proposition C.1.

(i) *Given an $\mathcal{O}_Y$-module $\mathcal{F}$ and a G-$\mathcal{O}_Y$-module $\mathcal{G}$, we have a natural isomorphism of G-R-modules*

$$\mathrm{Hom}_Y(\mathcal{G}, \widetilde{\mathcal{F}}) \cong \widetilde{\mathrm{Hom}_Y(\mathcal{G}, \mathcal{F})}$$

(the right hand side is the G-R-module coinduced from the R-module $\mathrm{Hom}_Y(\mathcal{G}, \mathcal{F})$).

(ii) *The functor $\mathcal{F} \longmapsto \widetilde{\mathcal{F}}$ is the right adjoint of the forgetful functor $\mathrm{Mod}(Y, G) \to \mathrm{Mod}(Y)$.*

Proof. — Given a G-$\mathcal{O}_Y$-module $\mathcal{G}$, we have natural isomorphisms of R-modules

$$\begin{aligned} \mathrm{Hom}_Y(\mathcal{G}, \mathcal{F}) &\cong \mathrm{Hom}_{G\times Y}(\mu^*\mathcal{G}, p^*\mathcal{F}) \cong \mathrm{Hom}_{G\times Y}(p^*\mathcal{G}, p^*\mathcal{F}) \\ &\cong \widetilde{\mathrm{Hom}_Y(\mathcal{G}, \mathcal{F})}. \end{aligned} \tag{75}$$

Here the first isomorphism is obtained from the adjointness of μ_* and μ^*, the second isomorphism from the G-action on $\mathcal{G}$ and the third isomorphism exists because $G \to S$ is flat. Note that both the first and the last term in (75) carry a natural structure of G-R-module. One checks that (75) defines an isomorphism of G-R-modules. This proves (i). Taking G-invariants and using Lemma B.1 for $n = 0$, we obtain an isomorphism of R-modules

$$\mathrm{Hom}_G(\mathcal{G}, \widetilde{\mathcal{F}}) \cong \mathrm{Hom}_Y(\mathcal{G}, \mathcal{F}). \tag{76}$$

A tedious but elementary verification shows that this isomorphism makes $\mathcal{F} \mapsto \widetilde{\mathcal{F}}$ the right adjoint of the forgetful functor $\mathrm{Mod}(Y, G) \to \mathrm{Mod}(Y)$. □

Corollary C.2.

(i) *The category* $\mathrm{Mod}(Y,G)$ *has enough injectives.*

(ii) *Given a* G-$\mathcal{O}_Y$*-module* $\mathcal{G}$*, the functor*

$$\mathrm{Hom}_Y(\mathcal{G},\cdot):\mathrm{Mod}(Y,G)\longrightarrow \mathrm{Mod}(R,G)$$

sends injective G-$\mathcal{O}_Y$*-modules to* G-R*-modules which are acyclic with respect to the functor* $M\mapsto M^G$.

C.3. — Let $\mathcal{A},\mathcal{B}$ be complexes of G-$\mathcal{O}_Y$-modules. We assume that $\mathcal{A}$ is bounded below and that $\mathcal{B}$ is bounded in both directions. Then the *total Hom complex* $\mathrm{Hom}_G^\bullet(\mathcal{A},\mathcal{B})$ is a (bounded above) complex of Abelian groups whose nth cohomology group is isomorphic to the group of homomorphisms $\mathcal{A}\to\mathcal{B}[-n]$ up to homotopy, *i.e.*

$$H^n(\mathrm{Hom}_G^\bullet(\mathcal{A},\mathcal{B}))=\mathrm{Hom}_{\mathfrak{K}^+(Y,G)}(\mathcal{A},\mathcal{B}[-n]).$$

See e.g. [24], 2.7.4. Let $\mathfrak{K}^-(\mathfrak{Ab})$ denote the category of bounded above complexes of Abelian groups, up to homotopy, and $\mathfrak{D}^-(\mathfrak{Ab})$ its derived category. The functor $\mathrm{Hom}_G^\bullet(\mathcal{A},\,\cdot\,):\mathfrak{K}^b(G,Y)\to\mathfrak{K}^-(\mathfrak{Ab})$ is a morphism of triangulated categories and has a total right derived functor, which we denote by $\mathbb{R}\mathrm{Hom}_G(\mathcal{A},\,\cdot\,):\mathfrak{D}^b(Y,G)\to\mathfrak{D}^-(\mathfrak{Ab})$. The nth *hyperext* of $\mathcal{A}$ and $\mathcal{B}$ is defined as

$$\mathbb{E}\mathrm{xt}_G^n(\mathcal{A},\mathcal{B}):=H^n(\mathbb{R}\mathrm{Hom}_G(\mathcal{A},\mathcal{B}))$$

see e.g. [24], §10.7. It follows from standard arguments that the functor $\mathbb{R}\mathrm{Hom}_G(\,\cdot\,,\mathcal{B}):\mathfrak{D}^+(Y,G)\to\mathfrak{D}^-(\mathfrak{Ab})$ is a morphism of triangulated categories.

Ignoring the G-action, we may as well define the total Hom complex $\mathrm{Hom}_Y^\bullet(\mathcal{A},\mathcal{B})$. By §C.1, the terms of $\mathrm{Hom}_Y^\bullet(\mathcal{A},\mathcal{B})$ carry a natural structure of G-R-modules such that

$$\mathrm{Hom}_G^\bullet(\mathcal{A},\mathcal{B})=\mathrm{Hom}_Y^\bullet(\mathcal{A},\mathcal{B})^G. \tag{77}$$

This formula displays the functor $\mathrm{Hom}_G^\bullet(\mathcal{A},\cdot)$ as the composition of two morphisms of triangulated categories

$$\mathfrak{K}^b(Y,G)\longrightarrow\mathfrak{K}^-(R,G)\longrightarrow\mathfrak{K}^-(\mathfrak{Ab}).$$

It follows from Corollary C.2 and [24], Theorem 10.8.2 that

$$\mathbb{R}\mathrm{Hom}_G(\mathcal{A},\mathcal{B})\cong\mathbb{R}^G(\mathbb{R}\mathrm{Hom}_Y(\mathcal{A},\mathcal{B})). \tag{78}$$

Therefore, the hyperext group $\mathbb{E}\mathrm{xt}_G^n(\mathcal{A},\mathcal{B})$ can be computed via the Grothendieck spectral sequence

$$E_2^{p,q}:=H^p(G,\mathbb{E}\mathrm{xt}_Y^q(\mathcal{A},\mathcal{B}))\Longrightarrow\mathbb{E}\mathrm{xt}_G^{p+q}(\mathcal{A},\mathcal{B}). \tag{79}$$

C.4. — Now assume that $G \to S$ is finite and that Y can be covered by affine G-stable open subsets. Then by [17], Theorem 12.1, the quotient scheme $X := Y/G$ exists and the natural projection $\pi : Y \to X$ is finite. Given a G-$\mathcal{O}_Y$-module $\mathcal{F}$, one defines a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}^G$ such that

$$\Gamma(V, \mathcal{F}^G) = \Gamma(\pi^{-1}(V), \mathcal{F})^G.$$

Let $\mathcal{F}$, $\mathcal{G}$ be two G-$\mathcal{O}_Y$-modules. In view of §C.1 it is clear that the sheaf $\mathcal{H}om_Y(\mathcal{F}, \mathcal{G})$ is endowed with a natural action of G, i.e.with a structure of G-$\mathcal{O}_Y$-module. We set

$$\mathcal{H}om_G(\mathcal{F}, \mathcal{G}) := \mathcal{H}om_Y(\mathcal{F}, \mathcal{G})^G.$$

By definition, we have

$$\mathrm{Hom}_G(\mathcal{F}, \mathcal{G}) = \Gamma(X, \mathcal{H}om_G(\mathcal{F}, \mathcal{G})). \tag{80}$$

Lemma C.3. — *The additive functor $\mathcal{H}om_G(\mathcal{F}, \,\cdot\,)$ sends injective G-$\mathcal{O}_Y$-modules to $\Gamma(X, \,\cdot\,)$-acyclic $\mathcal{O}_X$-modules.*

Proof. — By the construction of injective objects of $\mathrm{Mod}(Y, G)$ in §C.2, it suffices to proof the lemma for G-$\mathcal{O}_Y$-modules of the form $\widetilde{\mathcal{I}}$, where $\mathcal{I}$ is an injective $\mathcal{O}_Y$-module. Using Proposition C.1 (ii) one shows that

$$\mathcal{H}om_G(\mathcal{F}, \widetilde{\mathcal{I}}) = \pi_* \,\mathcal{H}om_Y(\mathcal{F}, \mathcal{I}).$$

It is well known that $\mathcal{H}om_Y(\mathcal{F}, \,\cdot\,)$ sends injective to $\Gamma(Y, \,\cdot\,)$-acyclic $\mathcal{O}_Y$-modules. But since π is finite the functor π_* is exact and so $\pi_* \,\mathcal{H}om_Y(\mathcal{F}, \mathcal{I})$ is $\Gamma(X, \,\cdot\,)$-acyclic. This proves the lemma. □

Given two complexes of G-$\mathcal{O}_Y$-modules $\mathcal{A}$ and $\mathcal{B}$ (with $\mathcal{A}$ bounded below and $\mathcal{B}$ bounded), one defines the (bounded above) complex of $\mathcal{O}_X$-modules $\mathcal{H}om^\bullet_G(\mathcal{A}, \mathcal{B})$. We let $\mathbb{R}\mathcal{H}om_G(\mathcal{A}, \mathcal{B})$ denote the total right derived functor of $\mathcal{H}om^\bullet_G(\mathcal{A}, \,\cdot\,)$, evaluated at $\mathcal{B}$, and set

$$\mathcal{E}xt^n_G(\mathcal{A}, \mathcal{B}) := H^n(\mathbb{R}\mathcal{H}om_G(\mathcal{A}, \mathcal{B})).$$

As in the previous subsection, it follows from (80) and Lemma C.3 that

$$\mathbb{R}\mathrm{Hom}_G(\mathcal{A}, \mathcal{B}) \cong \mathbb{R}\Gamma(X, \mathbb{R}\mathcal{H}om_G(\mathcal{A}, \mathcal{B})). \tag{81}$$

In particular, there exists a spectral sequence

$$H^p(X, \mathcal{E}xt^q_G(\mathcal{A}, \mathcal{B})) \Longrightarrow \mathbb{E}\mathrm{xt}^{p+q}_G(\mathcal{A}, \mathcal{B}). \tag{82}$$

Following [7], we call (82) the *local-global spectral sequence* for $\mathbb{E}\mathrm{xt}^*_G$.

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