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BOUNDS OF RIESZ TRANSFORMS ON L^p SPACES FOR SECOND ORDER ELLIPTIC OPERATORS

by Zhongwei SHEN

1. Introduction.

Consider the second order elliptic operator of divergence form

$$\mathcal{L} = -\operatorname{div}\big(A(x)\nabla\big) \quad \text{on } \Omega, \tag{1.1}$$

where $\Omega = \mathbb{R}^n$ or a bounded open set of $\mathbb{R}^n$. In the case of bounded domains, we impose the Dirichlet boundary condition $u = 0$ on $\partial\Omega$. Throughout this paper, we assume that $A(x) = (a_{jk}(x))$ is an $n \times n$ symmetric matrix with real-valued, bounded measurable entries satisfying the uniform ellipticity condition,

$$\mu|\xi|^2 \le \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \le \frac{1}{\mu}|\xi|^2, \tag{1.2}$$

for all $\xi, x \in \mathbb{R}^n$ and some $\mu > 0$.

Under these assumptions, it is known that the Riesz transform $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^p(\Omega)$ for $1 < p < 2+\varepsilon$ where $\varepsilon = \varepsilon(n,\mu) > 0$, and is of weak type $(1,1)$ (see e.g. [4], [7]). Moreover, the range of p is sharp. The main purpose of this paper is to investigate the L^p boundedness of the Riesz transform for $p > 2$, as well as the closely related boundedness on weighted L^2 spaces, under some *additional* conditions. For any fixed $p > 2$, we obtain a necessary and sufficient condition for the boundedness of the Riesz transform on $L^p(\Omega)$. Armed with this condition, for elliptic operators with VMO coefficients, we are able to establish the L^p boundedness of Riesz transforms on Lipschitz domains for the optimal range of p.

Keywords: Riesz transform, elliptic operator, Lipschitz domain.
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Theorems A, B and C below are the main results of the paper.

Theorem A. — *Let $\mathcal{L}$ be a second-order uniform elliptic operator of divergence form with real, symmetric, bounded measurable coefficients on $\mathbb{R}^n$, $n \geq 2$. For any fixed $p > 2$, the following statements are equivalent.*

(i) *There exists a constant $C > 0$ such that for any ball $B = B(x_0,r)$ and any $W^{1,2}_{\mathrm{loc}}$ weak solution of $\mathcal{L}u = 0$ in $3B = B(x_0,3r)$, one has $|\nabla u| \in L^p(B)$ and*

$$\left(\frac{1}{|B|}\int_B |\nabla u|^p \,\mathrm{d}x\right)^{1/p} \leq C\left(\frac{1}{|2B|}\int_{2B} |\nabla u|^2 \,\mathrm{d}x\right)^{1/2}. \tag{1.3}$$

(ii) *There exists $\varepsilon > 0$ such that the Riesz transform $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^q(\mathbb{R}^n,\mathrm{d}x)$ for any $1 < q < p + \varepsilon$.*

(iii) *There exists $\delta > 0$ such that if ω is an A_s weight with $s = 2(1 - 1/p) + \delta$, then*

$$\left\|\nabla(\mathcal{L})^{-1/2} f\right\|_{L^2(\mathbb{R}^n,\,\mathrm{d}x/\omega)} \leq C\,\|f\|_{L^2(\mathbb{R}^n,\,\mathrm{d}x/\omega)}, \tag{1.4}$$

where C depends on the A_s bound of ω.

In particular, $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^p(\mathbb{R}^n,\mathrm{d}x)$ if and only if condition (i) holds for the same p. Consequently, the set of exponents $p \in (1,\infty)$ for which $\nabla(\mathcal{L})^{1/2}$ is bounded on $L^p(\mathbb{R}^n,\mathrm{d}x)$ is an open interval $(1,p_0)$ with $2 < p_0 \leq \infty$.

We remark that condition (ii) follows directly from (iii) by an extrapolation theorem, due to Rubio de Francia. It is also not hard to see that condition (i) follows from (ii) by a standard localization argument, since the L^p boundedness of the Riesz transform yields the $W^{1,p}$ estimates for $\mathcal{L}$. The rest of the proof of Theorem A, however, is much more involved. To prove that condition (i) implies (ii), we use a new and refined version of the celebrated Calderón-Zygmund Lemma. See Theorem 3.1. This theorem, formulated by the author in [17], was inspired by a paper of Caffarelli and Peral [6] as well as a recent work of L. Wang [19]. For any fixed $p > 2$, it gives a sufficient condition for an L^2 bounded sublinear operator to be bounded on L^q for all $2 < q < p$. It enables us to show that the operator $\nabla\mathcal{L}^{-1}\,\mathrm{div}$ is bounded on L^p under condition (i). The boundedness of the Riesz transform then follows from the fact that $\|\mathcal{L}^{1/2}f\|_q \leq C\,\|\nabla f\|_q$ for any $1 < q < \infty$ [4]. To show that condition (i) implies (iii), the basic observation is that condition (i) leads to an L^p estimate on the kernel function of the Riesz transform. Using this estimate on the kernel as well as the L^p boundedness established above, we show that the sharp function of the adjoint of the Riesz transform can be dominated by the Hardy-Littlewood maximal function. The desired estimate (1.4) then follows from the weighted norm inequalities for the sharp and maximal functions.

It is not very difficult to extend the argument above to the case of bounded Lipschitz domains. This gives us the following.

THEOREM B. — *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let $\mathcal{L}$ be a second-order uniform elliptic operator of divergence form on Ω, subject to Dirichlet boundary condition. For any fixed $p > 2$, the following statements are equivalent.*

(i) *There exist constants $C > 1$, $\alpha_2 > \alpha_1 > 1$ and $r_0 > 0$ such that for any ball $B(x_0,r)$ with the property that $0 < r < r_0$ and either $x_0 \in \partial\Omega$ or $B(x_0,\alpha_2 r) \subset \Omega$, and for any weak solution of $\mathcal{L}u = 0$ in $\Omega \cap B(x_0,\alpha_2 r)$ and $u = 0$ on $B(x_0,\alpha_2 r) \cap \partial\Omega$ (if $x_0 \in \partial\Omega$), one has $|\nabla u| \in L^p(\Omega \cap B(x_0,r))$ and*

$$\left(\frac{1}{r^n}\int_{\Omega\cap B(x_0,r)} |\nabla u|^p \,\mathrm{d}x\right)^{1/p} \leq C\left(\frac{1}{r^n}\int_{\Omega\cap B(x_0,\alpha_1 r)} |\nabla u|^2 \,\mathrm{d}x\right)^{1/2}. \tag{1.5}$$

(ii) *There exists $\varepsilon > 0$ such that the Riesz transform $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^q(\Omega, \mathrm{d}x)$ for any $1 < q < p + \varepsilon$.*

(iii) *There exists $\delta > 0$ such that if ω is an $A_s(\mathbb{R}^n)$ weight with $s = 2(1 - 1/p) + \delta$, then*

$$\left\|\nabla(\mathcal{L})^{-1/2} f\right\|_{L^2(\Omega,\,\mathrm{d}x/\omega)} \leq C\,\|f\|_{L^2(\Omega,\,\mathrm{d}x/\omega)}, \tag{1.6}$$

where C depends on the A_s bound of ω.

In particular, $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^p(\Omega,\mathrm{d}x)$ if and only if condition (i) *holds for the same p. Consequently, the set of exponents $p \in (1,\infty)$ for which $\nabla(\mathcal{L})^{1/2}$ is bounded on $L^p(\Omega,\mathrm{d}x)$ is an open interval $(1,p_0)$ with $2 < p_0 \leq \infty$.*

A few remarks are in order.

Remark 1.7. — Let $\Omega = \mathbb{R}^n$ or a bounded Lipschitz domain. It follows from Theorems A and B that $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^p(\Omega, \mathrm{d}x)$

for any $p \in (1,\infty)$ if and only if it is bounded on $L^2(\Omega, \omega\,\mathrm{d}x)$ for any $\omega \in A_2(\mathbb{R}^n)$. To see this, we note that $\omega \in A_2$ implies that $\omega \in A_q$ for some $q < 2$, and $1/\omega \in A_2$. For the classical Riesz transform $\nabla(-\Delta)^{-1/2}$, the boundedness on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and on $L^2(\mathbb{R}^n, \omega\,\mathrm{d}x)$ with $\omega \in A_2(\mathbb{R}^n)$ is well known (see e.g. [18], [10]).

Remark 1.8. — For a general second order elliptic operator $\mathcal{L}$ with real, symmetric, bounded measurable coefficients, $\nabla(\mathcal{L})^{-1/2}$ is bounded on L^p for $1 < p < 2+\varepsilon$. The range of p was shown to be optimal by C. Kenig (see [4], pp. 119–121). It is worth mentioning that the boundedness of $\nabla(\mathcal{L})^{-1/2}$ on L^p is equivalent to the inequality $\|\nabla f\|_p \le C\,\|\mathcal{L}^{1/2} f\|_p$. The reverse inequality $\|\mathcal{L}^{1/2} f\|_p \le C\,\|\nabla f\|_p$, nevertheless, holds for all $1 < p < \infty$ [4,5]. The proof of Theorems A and B depends on this fact.

Remark 1.9. — By a simple geometric observation, one may see that condition (i) in Theorem B is equivalent to the following. There exist $C_1 > 0$, $\alpha_4 > \alpha_3 > 1$ and $r_1 > 0$ such that for any $D = B(x_0, r) \cap \Omega \ne \emptyset$ with $x_0 \in \mathbb{R}^n$, $0 < r < r_1$, and for any weak solution of $\mathcal{L}u = 0$ in $\Omega \cap B(x_0, \alpha_4 r)$ and $u = 0$ on $B(x_0, \alpha_4 r) \cap \partial\Omega$ (if it's not empty), one has $|\nabla u| \in L^p(D)$ and

$$\left\{\frac{1}{r^n}\int_D |\nabla u|^p\,\mathrm{d}x\right\}^{1/p} \le C_1 \left\{\frac{1}{r^n}\int_{\Omega\cap B(x_0,\alpha_3 r)} |\nabla u|^2\,\mathrm{d}x\right\}^{1/2}. \tag{1.10}$$

By the reverse Hölder inequality estimates [12], pp. 122–123, this implies that condition (i) in Theorem B has the self-improvement property. That is, if $\mathcal{L}$ satisfies condition (i) in Theorem B for some $p > 2$, then it satisfies condition (i) for some $\bar{p} > p$. Clearly, the same can be said about the condition (i) in Theorem A. It follows that the set of exponents $p \in (1,\infty)$ for which $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^p(\Omega)$ is an open interval.

Let $\mathcal{L} = -\Delta$ on a bounded Lipschitz domain Ω, subject to Dirichlet boundary condition. Using the solvability of the L^2 regularity problem and the boundary Hölder estimates (see [14]), it is not hard to show that condition (i) in Theorem B holds for $p = 3$ if $n \ge 3$, and for $p = 4$ in the case $n = 2$ (see Lemma 4.1). It follows that for $n \ge 3$, the Riesz transform $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^p(\Omega)$ for $1 < p < 3+\varepsilon$, and on $L^2(\Omega, \mathrm{d}x/\omega)$ with $\omega \in A_{\frac{4}{3}+\delta}(\mathbb{R}^n)$. If $n = 2$, $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^p(\Omega)$ for $1 < p < 4+\varepsilon$, and on $L^2(\Omega, \mathrm{d}x/\omega)$ with $\omega \in A_{\frac{3}{2}+\delta}(\mathbb{R}^2)$. The ranges of p are known to be sharp [13]. In the case that Ω is a C^1 domain, $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^p(\Omega)$ for $1 < p < \infty$. We should point out that although our weighted L^2 bounds are new, the boundedness of $\nabla(-\Delta)^{-1/2}$ on $L^p(\Omega)$ for Lipschitz or C^1 domains was proved already in [13], by the method of complex interpolation. The direct extension of this method to the case of continuous coefficients fails, since it relies on the solvabilities of the L^2 Dirichlet and regularity problems. However, Theorem B in this paper allows us to perturb the operator $\mathcal{L}$. This leads to the following theorem.

Theorem C. — *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. Let $\mathcal{L}$ be a second-order elliptic operator of divergence form with real, symmetric, bounded measurable coefficients on Ω, subject to Dirichlet boundary condition. Assume that the coefficients $a_{jk}(x)$ are in $\mathrm{VMO}(\mathbb{R}^n)$. Then there exists $\varepsilon > 0$ such that $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^p(\Omega)$ for $1 < p < 3+\varepsilon$ if $n \geq 3$, and for $1 < p < 4+\varepsilon$ in the case $n = 2$. Consequently, there exists $\delta > 0$ such that $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^2(\Omega, \mathrm{d}x/\omega)$ where $\omega \in A_{\frac{4}{3}+\delta}(\mathbb{R}^n)$ if $n \geq 3$, and $\omega \in A_{\frac{3}{2}+\delta}(\mathbb{R}^2)$ in the case $n = 2$. If Ω is a C^1 domain, $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^p(\Omega)$ for any $1 < p < \infty$, and on $L^2(\Omega, \omega\, \mathrm{d}x)$ for any $\omega \in A_2(\mathbb{R}^n)$.*

Remark 1.11. — For divergence form elliptic equations on $C^{1,1}$ domains with VMO coefficients, the $W^{1,p}$ estimates were obtained in [11] for any $1 < p < \infty$. The result was extended in [3] to the case of C^1 domains, for operators with complex coefficients. Our approach to Theorem C, which is very different from that in [11,3], is based on Theorem B and a perturbation argument found in [6]. Indeed, by Theorem B, it suffices to show that solutions of $\mathcal{L}u = 0$ satisfies condition (i) in Theorem B for some $p > 3$ if $n \geq 3$, and $p > 4$ in the case $n = 2$. To do this, we approximate u on each ball by a solution of a second order elliptic equation with constant coefficients (Lemma 4.7). The desired estimate for ∇u follows from an approximation theorem (Theorem 4.13), which is essentially proved in [6].

The paper is organized as follows. Sections 2 and 3 are devoted to the proof of Theorems A and B. Theorem C is proved in section 4. Finally in Section 5 we give the proof of Theorems 3.1 and 3.2.

Acknowledgment. — After this paper was submitted, the author was informed kindly by S. Hofmann of two recent preprints [1], [2] on the study of Riesz transforms. In these two papers necessary and sufficient conditions are obtained for the L^p boundedness of Riesz transforms on manifolds [2], Theorem 1.3, and of Riesz transforms associated to

second order elliptic operators with complex coefficients on $\mathbb{R}^n$ [1], Proposition 5.6. The conditions are given in terms of the L^p boundedness of the operators $\sqrt{t}\nabla \mathrm{e}^{-t\mathcal{L}}$ uniformly for all $t > 0$. It is interesting to point out that the key step in the proof of sufficiency of the conditions in [1], [2] uses a result similar to Theorem 3.1 of the present paper (see Theorem 2.2 in [1]).

The author thanks S. Hofmann for pointing out the relevance of the results in [1], [2]. The author also would like to thank the referee for several valuable comments.

2. Some preliminaries.

In this section we will prove that condition (iii) in Theorems A and B implies (ii) which, in turn, implies condition (i). We will also show that condition (i) leads to an L^p estimate on the kernel function of the resolvent $(\mathcal{L}+\lambda)^{-1}$ for $\lambda > 0$.

The following proposition is essentially due to Rubio de Francia [16].

Proposition 2.1. — *Let T be a bounded operator on $L^2(E)$ where E is a measurable subset of $\mathbb{R}^n$. Let $0 < \delta \le 1$. Suppose that*

$$\int_E |Tf|^2 \frac{\mathrm{d}x}{\omega} \le C \int_E |f|^2 \frac{\mathrm{d}x}{\omega} \quad \text{for any } \omega \in A_{1+\delta}(\mathbb{R}^n), \tag{2.2}$$

where C depends only on the $A_{1+\delta}$ bound of ω. Then T is bound on $L^p(E)$ for $1 < p < 2/(1-\delta)$.

Proof. — By considering the operator $\widetilde{T}(f) = \chi_E T(f\chi_E)$, we may assume that $E = \mathbb{R}^n$. It follows from assumption (2.2) that for any $\omega \in A_1$, T is bounded on $L^2(\mathbb{R}^n, \mathrm{d}x/\omega)$ and $L^2(\mathbb{R}^n, \omega^\delta\,\mathrm{d}x)$. It is known that the boundedness of T on $L^2(\mathbb{R}^n, \mathrm{d}x/\omega)$ for any $\omega \in A_1$ implies its boundedness on $L^p(\mathbb{R}^n)$ for $1 < p < 2$, while the boundedness of T on $L^2(\mathbb{R}^n, \omega^\delta \mathrm{d}x)$ for any $\omega \in A_1$ implies its boundedness on $L^p(\mathbb{R}^n)$ for $2 < p < 2/(1-\delta)$. We refer the reader to [10], pp. 141–142, for a simple and elegant proof of these facts. $\square$

Using Proposition 2.1, it is easy to see that condition (iii) in Theorem A or B implies condition (ii). Next we show that the $W^{1,p}$ estimate follows from the L^p boundedness of the Riesz transform.

PROPOSITION 2.3. — *Suppose that operator $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for some $p > 2$. For $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$ where $1/p = 1/q - 1/n$, let $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^n)$ be a weak solution of $\mathcal{L}u = \operatorname{div} f + g$ in $\mathbb{R}^n$. If*

$$\lim_{R\to\infty} R^{n/p-n/2}\big\{\|\nabla u\|_{L^2(R\le|x|\le 2R)} + R^{-1}\|u\|_{L^2(R\le|x|\le 2R)}\big\} = 0,$$

then $\|\nabla u\|_p \le C\big\{\|f\|_p + \|g\|_q\big\}$.

Proof. — The proof is rather standard. Let φ be a smooth cut-off function such that $\varphi = 1$ on $B(0,R)$, $\varphi = 0$ outside of $B(0,2R)$, and $|\nabla\varphi| \le C/R$. Then $\mathcal{L}(u\varphi) = \operatorname{div}(\widetilde{f}) + \widetilde{g}$ in $\mathbb{R}^n$, where $\widetilde{f} = f\varphi - a_{jk}\,u\,\partial_k\varphi$ and $\widetilde{g} = -f\nabla\varphi + g\varphi - a_{jk}\,\partial_j u\,\partial_k\varphi$. Since $u\varphi$, $\widetilde{f}$ and $\widetilde{g}$ all have compact supports, we may write $u\varphi = \mathcal{L}^{-1}(\operatorname{div}\widetilde{f} + \widetilde{g})$.

Suppose now that $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$ for some $p > 2$. Since $\nabla(\mathcal{L})^{-1/2}$ is always bounded on $L^t(\mathbb{R}^n)$ for any $1 < t \le 2$, it follows from duality that $\nabla\mathcal{L}^{-1}\operatorname{div}$ is bounded on $L^t(\mathbb{R}^n)$ for $2 \le t \le p$. This implies that

$$\big\|\nabla(u\varphi)\big\|_t \le C\big\{\|\widetilde{f}\|_t + \|(\mathcal{L})^{-1/2}\widetilde{g}\|_t\big\} \le C\big\{\|\widetilde{f}\|_t + \|\widetilde{g}\|_s\big\}, \tag{2.4}$$

where $1/t = 1/s - 1/n$. We remark that the second inequality in (2.4) follows from the fact that the kernel function $K(x,y)$ of the operator $(\mathcal{L})^{-1/2}$ is bounded by $C|x-y|^{1-n}$. Hence,

$$\begin{aligned}\|\nabla u\|_{L^t(B(0,R))} \le C\big\{\|f\|_{L^t(B(0,2R))} + \|g\|_{L^s(B(0,2R))}\big\} \\ + C R^{-1}\big\{\|u\|_{L^t(B(0,2R)\setminus B(0,R))} \\ + \|\nabla u\|_{L^s(B(0,2R)\setminus B(0,R))}\big\}.\end{aligned} \tag{2.5}$$

By an iteration argument and Sobolev imbedding, this yields that

$$\begin{aligned}\|\nabla u\|_{L^p(B(0,R))} \le C\big\{\|f\|_{L^p(B(0,CR))} + \|g\|_{L^q(B(0,CR))}\big\} \\ + C R^{n/p-n/2}\big\{R^{-1}\,\|u\|_{L^2(R\le|x|\le CR)} \\ + \|\nabla u\|_{L^2(R\le|x|\le CR)}\big\}.\end{aligned} \tag{2.6}$$

Letting $R \to \infty$ in (2.6), one obtains the desired estimate. □

LEMMA 2.7. — *In Theorem A or B, condition* (ii) *implies condition* (i).

Proof. — Let $p > 2$, and suppose that $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^p(\mathbb{R}^n)$. Then the operator is bounded on $L^t(\mathbb{R}^n)$ for $1 < t \le p$. Let u be a weak solution of $\mathcal{L}u = 0$ in $3B = B(x_0, 3r)$. For $1 < \gamma_1 < \gamma_2 < \frac{3}{2}$, choose a smooth cut-off function φ such that $\varphi = 1$ on $\gamma_1 B$, $\varphi = 0$ outside of $\gamma_2 B$, and $|\nabla\varphi| \le C/r$. Note that $\mathcal{L}(u\varphi) = -\partial_j(a_{jk}\, u\, \partial_k\varphi) - a_{jk}\,\partial_k u\,\partial_j\varphi$ in $\mathbb{R}^n$. By Proposition 2.3, if $|\nabla u| \in L^s(\gamma_2 B)$, then $|\nabla u| \in L^t(\gamma_1 B)$ and

$$\|\nabla u\|_{L^t(\gamma_1 B)} \le C r^{-1}\big\{\|u\|_{L^t(\gamma_2 B)} + \|\nabla u\|_{L^s(\gamma_2 B)}\big\}, \tag{2.8}$$

where $2 < t \le p$ and $1/t = 1/s - 1/n$. Since $u - c$ is also a weak solution in $3B$, we may use the L^∞ estimate and Poincaré inequality to obtain

$$\begin{aligned}\Big(\frac{1}{|B|}\int_{\gamma_1 B}|\nabla u|^t\,\mathrm{d}x\Big)^{1/t} \\ \le C\Big\{\Big(\frac{1}{|2B|}\int_{2B}|\nabla u|^2\,\mathrm{d}x\Big)^{1/2} + \Big(\frac{1}{|B|}\int_{\gamma_2 B}|\nabla u|^s\,\mathrm{d}x\Big)^{1/s}\Big\}.\end{aligned} \tag{2.9}$$

From this, estimate (1.3) in Theorem A follows by an iteration argument, starting with $s = 2$.

In the case of Theorem B, we first choose $r_0 > 0$ so small that for any $P \in \partial\Omega$, $\Omega \cap B(P, 3r_0)$ is given by the intersection of the region above a Lipschitz graph and $B(P, 3r_0)$, after a possible rotation of the coordinate system. Given $B(x_0, r)$ with $0 < r < r_0$, consider two cases: 1) $B(x_0, 3r) \subset \Omega$, and 2) $x_0 \in \partial\Omega$. The first case may be treated exactly as in Theorem A. In the second case, instead of replacing u by $u - c$ and using Poincaré inequality, one applies the Poincaré inequality on $\Omega \cap B(x_0, \gamma_2 r)$ for functions which vanish on $B(x_0, 3r) \cap \partial\Omega$. The rest is the same. □

To complete the proof of Theorems A and B, it remains to show that condition (i) implies conditions (ii) and (iii). To this end, we need to estimate the kernel function $\Gamma_\lambda(x, y)$ of the resolvent $(\mathcal{L} + \lambda)^{-1}$ for $\lambda > 0$. We begin with a size estimate for $n \ge 3$:

$$\big|\Gamma_\lambda(x, y)\big| \le C\,\mathrm{e}^{-c\sqrt{\lambda}\,|x-y|}\cdot\frac{1}{|x-y|^{n-2}}, \tag{2.10}$$

which follows directly from the formula $(\mathcal{L} + \lambda)^{-1} = \int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{e}^{-t\mathcal{L}}\,\mathrm{d}t$ and the well known upper bound for the heat kernel of $\mathcal{L}$ [9]. In the case $n = 2$, one needs to replace $1/|x - y|^{n-2}$ by $|\ln(\sqrt{\lambda}\,|x - y|)| + 1$. The rest of this section is devoted to the proof of the following theorem. We remark that estimates similar to (2.12)–(2.13) may be found in [10].

THEOREM 2.11. — *Suppose that operator* $\mathcal{L}$ *in* (1.1) *satisfies condition* (i) *in Theorem B for some* $p > 2$. *Then, if* $n \geq 3$,

$$\left(\frac{1}{r^n}\int_{\{x\in\Omega:\, r\leq|x-y|\leq 2r\}}\big|\nabla_x\Gamma_\lambda(x,y)\big|^p\,\mathrm{d}x\right)^{1/p} \leq C\,\mathrm{e}^{-c\sqrt{\lambda}r}\,\frac{1}{r^{n-1}}, \tag{2.12}$$

$$\left(\frac{1}{r^n}\int_{\{x\in\Omega:\, r\leq|x-y|\leq 2r\}}\big|\nabla_x\Gamma_\lambda(x,y)-\nabla_x\Gamma_\lambda(x,y+h)\big|^p\,\mathrm{d}x\right)^{1/p} \leq C\left(\frac{|h|}{r}\right)^{\eta}\mathrm{e}^{-c\sqrt{\lambda}r}\,\frac{1}{r^{n-1}}, \tag{2.13}$$

where $0 < r < cr_0$, $y,y+h \in \Omega$, $|h| < cr$, *and* $\eta = \eta(n,\mu,\Omega) > 0$. *If* $n = 2$, *one needs to replace* $1/r^{n-1}$ *in* (2.12)–(2.13) *by* $(|\ln(\sqrt{\lambda}r)|+1)/r$. *If* $\Omega = \mathbb{R}^n$ *and* $\mathcal{L}$ *satisfies condition* (i) *in Theorem A, above estimates hold for any* $0 < r < \infty$.

Note that $\Gamma_\lambda(x,y) = \Gamma_\lambda(y,x)$, and $\Gamma_\lambda(\cdot,y)$ is a weak solution of $\mathcal{L}u+\lambda u = 0$ in $\Omega\setminus\{y\}$. Using size estimate (2.10) and Hölder estimates, it is easy to see that Theorem 2.11 is a consequence of the following lemma.

LEMMA 2.14. — *Assume that* $\mathcal{L}$ *satisfies condition* (i) *in Theorem B for some* $p > 2$. *Then there exist constants* $r_1 > 0$, $\alpha > 1$ *and* $C > 0$ *independent of* $\lambda > 0$, *such that if* u *is a weak solution of* $\mathcal{L}u+\lambda u = 0$ *in* $B(x_0,\alpha r)\cap\Omega$ *for some* $x_0 \in \overline{\Omega}$, $0 < r < r_1$ *and* $u = 0$ *on* $B(x_0,\alpha r)\cap\partial\Omega$, *then*

$$\left(\frac{1}{r^n}\int_{B(x_0,r)\cap\Omega}|\nabla u|^p\,\mathrm{d}x\right)^{1/p} \leq \frac{C}{r}\left(\frac{1}{r^n}\int_{B(x_0,\alpha r)\cap\Omega}|u|^2\,\mathrm{d}x\right)^{1/2} \tag{2.15}$$

If $\Omega = \mathbb{R}^n$ *and* $\mathcal{L}$ *satisfies condition* (i) *in Theorem A, above statement holds for* $r_1 = \infty$.

Proof. — Let u be a weak solution of $\mathcal{L}u+\lambda u = 0$ in $B(x_0,\alpha r)\cap\Omega$ and $u = 0$ on $B(x_0,\alpha r)\cap\partial\Omega$, where $\alpha = 2\alpha_2$. We only consider the case $x_0 \in \partial\Omega$.

Let $D = B(x_0,r)\cap\Omega$ and $tD = B(x_0,tr)\cap\Omega$. Let v be a weak solution of $\mathcal{L}v = 0$ in $\alpha_2 D$ such that $w \equiv u - v \in H_0^1(\alpha_2 D)$. Using condition (i), we have

$$\left(\frac{1}{r^n}\int_D|\nabla u|^p\,\mathrm{d}x\right)^{1/p} \leq C\left(\frac{1}{r^n}\int_{\alpha_1 D}|\nabla u|^2\,\mathrm{d}x\right)^{1/2} + C\left(\frac{1}{r^n}\int_{\alpha_1 D}|\nabla w|^p\,\mathrm{d}x\right)^{1/p}. \tag{2.16}$$

To estimate ∇w on $\alpha_1 D$, observe that $\mathcal{L}w = -\lambda u$ in $\alpha_2 D$. Hence we may write

$$w(x) = -\lambda \int_{\alpha_2 D} G(x,y) u(y)\,\mathrm{d}y, \tag{2.17}$$

where $G(x,y)$ is the Green's function for $\mathcal{L}$ on $\alpha_2 D$. It follows that

$$\begin{aligned} \big|\nabla w(x)\big| &\leq \lambda \|u\|_{L^\infty(\alpha_2 D)} \int_{\alpha_2 D} \big|\nabla_x G(x,y)\big|\,\mathrm{d}y \\ &\leq \frac{C}{r^2}\Big(\frac{1}{r^n}\int_{2\alpha_2 D} |u|^2\,\mathrm{d}x\Big)^{1/2} h(x), \end{aligned} \tag{2.18}$$

where $h(x) = \int_{\alpha_2 D} |\nabla_x G(x,y)|\,\mathrm{d}y$, and we have used the Cacciopoli inequality

$$\lambda \int_{\frac{3}{2}\alpha_2 D} |u|^2\,\mathrm{d}x + \int_{\frac{3}{2}\alpha_2 D} |\nabla u|^2\,\mathrm{d}x \leq \frac{C}{r^2}\int_{2\alpha_2 D} |u|^2\,\mathrm{d}x. \tag{2.19}$$

Note that

$$\|h\|_{L^p(\alpha_1 D)} = \sup_{\|g\|_{p'}\leq 1} \Big|\int_{\alpha_1 D} h(x)\,g(x)\,\mathrm{d}x\Big| \leq \sup_{\|g\|_{p'}\leq 1} \int_{\alpha_2 D} \big|Tg(y)\big|\,\mathrm{d}y, \tag{2.20}$$

where

$$Tg(y) = \int_{\alpha_1 D} \big|\nabla_x G(x,y)\big| g(x)\,\mathrm{d}x. \tag{2.21}$$

Since $G(\cdot,y)$ is a weak solution of $Lu = 0$ in $\alpha_2 D \setminus \{y\}$, it follows from estimate (1.10) that

$$\Big(\int_{E_j} |\nabla_x G(x,y)|^p\,\mathrm{d}x\Big)^{1/p} \leq C\,(2^{-j}r)^{1-\frac{n}{p'}}, \tag{2.22}$$

where $E_j = E_j(y) = \Big\{x \in \alpha_1 D \colon 2^{-j-1}r \leq |x-y| \leq 2^{-j}r\Big\}$ for $j \geq -3$. Thus, by Hölder inequality,

$$\begin{aligned} \big|Tg(y)\big| &\leq \sum_{j=-3}^{\infty} \Big(\int_{E_j} \big|\nabla_x G(x,y)\big|^p\,\mathrm{d}x\Big)^{1/p}\Big(\int_{E_j} |g|^{p'}\,\mathrm{d}x\Big)^{1/p'} \\ &\leq Cr\big\{M(|g|^{p'})(y)\big\}^{1/p'}, \end{aligned} \tag{2.23}$$

where M denotes the Hardy-Littlewood maximal function. By Kolmogorov's Lemma [10], p. 102, this implies that

$$\int_{\alpha_2 D} \big|Tg(y)\big| \,\mathrm{d}y \leq Cr|\alpha_2 D|^{1/p} \|g\|_{p'}. \tag{2.24}$$

In view of (2.18) and (2.20), we obtain

$$\begin{aligned} \|\nabla w\|_{L^p(\alpha_1 D)} &\leq \frac{C}{r^2}\Big(\frac{1}{r^n}\int_{2\alpha_2 D} |u|^2 \,\mathrm{d}x\Big)^{1/2} \|h\|_{L^p(\alpha_1 D)} \\ &\leq C r^{n/p-1}\Big(1/r^n \int_{2\alpha_2 D} |u|^2 \,\mathrm{d}x\Big)^{1/2}. \end{aligned} \tag{2.25}$$

The desired estimate (2.15) with $\alpha = 2\alpha_2$ now follows from (2.16) and (2.25). □

3. Proof of Theorems A and B.

In this section we show that condition (i) in Theorem A or B implies conditions (ii) and (iii). This, together with Proposition 2.1 and Lemma 2.7, completes the proof of Theorems A and B.

The proof of condition (i) implying (ii) relies on Theorem 3.1, which may be considered as a refined (and dual) version of the well known Calderón-Zygmund Lemma. Its proof as well as the proof of Theorem 3.3 will be given in Section 5.

THEOREM 3.1. — *Let T be a bounded sublinear operator on $L^2(\mathbb{R}^n)$. Let $p > 2$. Suppose that there exist constants $\alpha_2 > \alpha_1 > 1$, $N > 1$ such that*

$$\begin{aligned} \Big\{\frac{1}{|B|}\int_B |Tf|^p \,\mathrm{d}x\Big\}^{1/p} \leq N\Big\{\Big(\frac{1}{|\alpha_1 B|}\int_{\alpha_1 B} |Tf|^2 \,\mathrm{d}x\Big)^{1/2} \\ + \sup_{B' \supset B}\Big(\frac{1}{|B'|}\int_{B'} |f|^2 \,\mathrm{d}x\Big)^{1/2}\Big\}, \end{aligned} \tag{3.2}$$

for any ball $B \subset \mathbb{R}^n$, and any bounded measurable function f with compact $\operatorname{supp}(f) \subset \mathbb{R}^n \setminus \alpha_2 B$. *Then T is bounded on $L^q(\mathbb{R}^n)$ for any $2 < q < p$.*

Theorem 3.1 may be extended to the case of bounded Lipschitz domains.

THEOREM 3.3. — *Let T be a bounded sublinear operator on $L^2(\Omega)$, where Ω is a bounded Lipschitz domain in $\mathbb{R}^n$. Let $p > 2$. Suppose that there exist constants $r_0 > 0$, $N > 1$ and $\alpha_2 > \alpha_1 > 1$ such that for any bounded measurable function f with* $\operatorname{supp}(f) \subset \Omega \setminus \alpha_2 B$,

$$\left\{\frac{1}{r^n}\int_{\Omega\cap B} |Tf|^p \,\mathrm{d}x\right\}^{1/p} \le N\left\{\left(\frac{1}{r^n}\int_{\Omega\cap\alpha_1 B} |Tf|^2 \,\mathrm{d}x\right)^{1/2} + \sup_{B'\supset B}\left(\frac{1}{|B'|}\int_{B'} |f|^p \,\mathrm{d}x\right)^{1/p}\right\}, \tag{3.4}$$

where $B = B(x_0,r)$ is a ball with the property that $0 < r < r_0$ and either $x_0 \in \partial\Omega$ or $B(x_0,\alpha_2 r) \subset \Omega$. Then T is bounded on $L^q(\Omega)$ for any $2 < q < p$.

LEMMA 3.5. — *In Theorem A or B, condition* (i) *implies* (ii).

Proof. — We first consider the case $\Omega = \mathbb{R}^n$. Assume that operator $\mathcal{L}$ satisfies condition (i) in Theorem A for some $p > 2$. By Theorem 3.1, the linear operator $T = \nabla(\mathcal{L})^{-1}\operatorname{div}$ is bounded on $L^q(\mathbb{R}^n)$ for $2 \le q < p$. Indeed, T is clearly bounded on $L^2(\mathbb{R}^n)$. To verify (3.2), we let $u = (\mathcal{L})^{-1}\operatorname{div}(f)$ where f is a bounded measurable function with compact $\operatorname{supp}(f) \subset \mathbb{R}^n \setminus 4B$. Observe that $\mathcal{L}u = 0$ in $3B$. Thus inequality (3.2) follows directly from condition (i). By Theorem 3.1 and duality, $\nabla(\mathcal{L})^{-1}\operatorname{div}$ is bounded on L^q for $p' < q < p$.

Next, since $\|L^{1/2}f\|_q \le C\|\nabla f\|_q$ for any $1 < q < \infty$ (see [4], p. 114), we have

$$\begin{aligned}\left\|(\mathcal{L})^{-1/2}\operatorname{div} f\right\|_q &= \left\|\mathcal{L}^{1/2}(\mathcal{L})^{-1}\operatorname{div} f\right\|_q \\ &\le C\,\|\nabla(\mathcal{L})^{-1}\operatorname{div} f\|_q \le C\,\|f\|_q\end{aligned} \tag{3.6}$$

where $p' < q < p$. Consequently, by duality, $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^q(\mathbb{R}^n)$ for $1 < q < p$. Finally by the self-improvement property of condition (i) (see Remark 1.9), we may conclude that $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^q(\mathbb{R}^n)$ for $1 < q < p + \varepsilon$.

The proof is similar in the case of Theorem B. In the place of Theorem 3.1, we use Theorem 3.3. Also we note that for a bounded Lipschitz domain, the inequality $\|\mathcal{L}^{1/2}f\|_q \le C\,\|\nabla f\|_q$ has been established in [5]. The proof is finished. □

To show that condition (i) implies the L^2 weighted norm inequality for the Riesz transform, we use the functional calculus formula

$$(\mathcal{L})^{-1/2} = \frac{1}{\pi}\int_0^\infty \lambda^{-1/2}(\mathcal{L}+\lambda)^{-1}\,\mathrm{d}\lambda \tag{3.7}$$

to write

$$\nabla(\mathcal{L})^{-1/2}f(x)=\int_\Omega K(x,y)f(y)\,\mathrm{d}y, \tag{3.8}$$

where

$$K(x,y)=\frac{1}{\pi}\int_0^\infty \lambda^{-1/2}\nabla_x\Gamma_\lambda(x,y)\,\mathrm{d}\lambda, \tag{3.9}$$

and $\Gamma_\lambda(x,y)$ is the Green's function for $\mathcal{L}+\lambda$.

Lemma 3.10. — *Suppose that $\mathcal{L}$ satisfies condition* (i) *in Theorem B for some $p>2$. Then*

$$\Big(\frac{1}{r^n}\int_{\{x\in\Omega:\ r\le|x-y|\le 2r\}}\big|K(x,y)\big|^p\,\mathrm{d}x\Big)^{1/p}\le\frac{C}{r^n}, \tag{3.11}$$

$$\Big(\frac{1}{r^n}\int_{\{x\in\Omega:\ r\le|x-y|\le 2r\}}\big|K(x,y)-K(x,y+h)\big|^p\,\mathrm{d}x\Big)^{1/p}, \le\frac{C}{r^n}\Big(\frac{|h|}{r}\Big)^\eta \tag{3.12}$$

where $0<r<r_1$, $y,y+h\in\Omega$, $|h|\le cr$, and $\eta=\eta(n,\mu,\Omega)>0$. If $\Omega=\mathbb{R}^n$ and $\mathcal{L}$ satisfies condition (i) *in Theorem A, then estimates* (3.11)–(3.12) *hold for all $0<r<\infty$.*

Proof. — In view of (3.9), estimates (3.11) and (3.12) follow directly from (2.14) and (2.15) respectively by integration. □

To use the estimates in Lemma 3.10 effectively, we consider the adjoint operator of $\nabla(\mathcal{L})^{-1/2}$:

$$Sf(x)=\int_\Omega K(y,x)f(y)\,\mathrm{d}y. \tag{3.13}$$

Recall that the sharp function of f is defined by

$$f^\#(x)\equiv\sup_{B\ni x}\inf_{\beta\in\mathbb{R}}\frac{1}{|B|}\int_B|f(y)-\beta|\,\mathrm{d}y. \tag{3.14}$$

Lemma 3.15. — *If operator $\mathcal{L}$ satisfies condition* (i) *in Theorem A for some $p>2$, then*

$$(Sf)^\#(x)\le C\big\{M(|f|^{p'})(x)\big\}^{1/p'} \tag{3.16}$$

for any $x\in\mathbb{R}^n$. If $\mathcal{L}$ satisfies condition (i) *in Theorem B, and Sf is defined to be zero outside of Ω, then* (3.16) *holds for any $x\in\Omega$.*

Proof. — We first consider the case of Theorem A. Suppose $x \in B = B(x_0, r)$. Let $f = g + h$ where $g = f\chi_{4B}$. Since S is bounded on $L^{p'}(\mathbb{R}^n)$ by Lemma 3.4 and duality, we have

$$\begin{aligned} \frac{1}{|B|}\int_B \big|S(g)\big|\,\mathrm{d}y &\leq \Big(\frac{1}{|B|}\int_B |S(g)|^{p'}\,\mathrm{d}y\Big)^{1/p'} \\ &\leq C\Big(\frac{1}{|B|}\int_{4B} |f|^{p'}\,\mathrm{d}y\Big)^{1/p'} \leq C\big\{M(|f|^{p'})(x)\big\}^{1/p'}. \end{aligned} \tag{3.17}$$

Next, let $\beta = S(h)(x_0)$. It follows from Hölder inequality and estimate (3.12) that, for $y \in B$,

$$\begin{aligned} &\big|S(h)(y) - \beta\big| \leq \int_{\mathbb{R}^n\setminus 4B} \big|K(z,y) - K(z,x_0)\big| \cdot \big|f(z)\big|\,\mathrm{d}z \\ &\leq \sum_{j=2}^{\infty}\Big(\int_{2^{j+1}B\setminus 2^jB}\big|K(z,y) - K(z,x_0)\big|^p\,\mathrm{d}z\Big)^{1/p}\Big(\int_{2^{j+1}B\setminus 2^jB}\big|f(z)\big|^{p'}\,\mathrm{d}z\Big)^{1/p'} \\ &\leq C\big\{M(|f|^{p'})(x)\big\}^{1/p'}\sum_{j=2}^{\infty} 2^{-j\eta} \leq C\big\{M(|f|^{p'})(x)\big\}^{1/p'}. \end{aligned}$$

This, together with (3.17), gives

$$\frac{1}{|B|}\int_B \big|Sf(y) - \beta\big|\,\mathrm{d}y \leq C\big\{M(|f|^{p'})(x)\big\}^{1/p'}, \tag{3.18}$$

from which (3.16) follows.

In the case of Theorem B, we may use the same argument as above to show that for any $x \in \Omega$, estimate (3.18) holds for any ball $B = B(x_0, r) \ni x$ with $0 < r < r_1$. If $r \geq r_1$, we use the boundedness of S on $L^{p'}(\Omega)$ to obtain

$$\begin{aligned} \frac{1}{|B|}\int_{B\cap\Omega} |Sf|\,\mathrm{d}y &\leq C_{r_1}\Big\{\int_\Omega |Sf|^{p'}\,\mathrm{d}y\Big\}^{1/p'} \\ &\leq C\Big\{\int_\Omega |f|^{p'}\,\mathrm{d}y\Big\}^{1/p'} \leq C\big\{M(|f|^{p'})(x)\big\}^{1/p'}. \end{aligned} \tag{3.19}$$

The proof is complete. $\square$

Proposition 3.20. — *Let $f \in L^1_{\mathrm{loc}}(E)$ where E is a measurable set of $\mathbb{R}^n$. Suppose $\omega \in A_2(\mathbb{R}^n)$ and $f \in L^2(E, \omega\,\mathrm{d}x)$. Define f to be zero on $\mathbb{R}^n \setminus E$. Then*

$$\int_E \big|f(x)\big|^2\omega\,\mathrm{d}x \leq C\int_E \big|f^{\#}(x)\big|^2\omega\,\mathrm{d}x, \tag{3.21}$$

where C depends only on n and the A_2 bound of ω.

Proof. — The case $E=\mathbb{R}^n$ is well known. It was proved in [10], using the following good-λ inequality

$$\omega\bigl\{x\in\mathbb{R}^n\colon M_df(x)>2\lambda, f^{\#}(x)\le\gamma\lambda\bigr\} \\ \le C\gamma^\delta\omega\bigl\{x\in\mathbb{R}^n\colon M_df(x)>\lambda\bigr\}, \tag{3.22}$$

where $\omega\in A_\infty$, and M_df denotes the dyadic maximal function of f. In general, we use (3.22) to obtain

$$\omega\bigl\{x\in E\colon M_df(x)>\lambda\bigr\}\le\omega\bigl\{x\in E\colon f^{\#}(x)>\gamma\lambda\bigr\} \\ +C\gamma^\delta\omega\bigl\{x\in\mathbb{R}^n\colon M_df(x)>\lambda\bigr\}. \tag{3.23}$$

By integration, this gives

$$\begin{aligned}\int_E\bigl|f(x)\bigr|^2\omega\,\mathrm{d}x&\le C_\gamma\int_E\bigl|f^{\#}(x)\bigr|^2\omega\,\mathrm{d}x+C\gamma^\delta\int_{\mathbb{R}^n}\bigl|M_df(x)\bigr|^2\omega\,\mathrm{d}x\\&\le C_\gamma\int_E\bigl|f^{\#}(x)\bigr|^2\omega\,\mathrm{d}x+C\gamma^\delta\int_E|f(x)|^2\omega\,\mathrm{d}x,\end{aligned} \tag{3.24}$$

where we have used $\omega\in A_2$ and the weighted norm inequality for M_d. Inequality (3.21) now follows by choosing γ so small that $C\gamma^\delta<\frac12$. □

We are now in a position to complete the proof of Theorems A and B.

Lemma 3.25. — *In Theorem A or B, condition* (i) *implies* (iii).

Proof. — We give the proof for the case of Theorem A. The case of Theorem B is similar.

Suppose that operator $\mathcal{L}$ satisfies condition (i) in Theorem A for some $p>2$. It follows from Proposition 3.20, Lemma 3.15 and the weighted norm inequality for the Hardy-Littlewood maximal function that

$$\begin{aligned}\int_{\mathbb{R}^n}|Sf|^2\omega\,\mathrm{d}x&\le C\int_{\mathbb{R}^n}\bigl|(Sf)^{\#}\bigr|^2\omega\,\mathrm{d}x\\&\le C\int_{\mathbb{R}^n}\bigl\{M(|f|^{p'})\bigr\}^{2/p'}\omega\,\mathrm{d}x\le C\int_{\mathbb{R}^n}|f|^2\omega\,\mathrm{d}x,\end{aligned} \tag{3.26}$$

where f is a bounded function with compact support, and $\omega\in A_{2/p'}(\mathbb{R}^n)$. We remark that the first inequality in (3.26) requires $Sf\in L^2(\mathbb{R}^n,\omega\,\mathrm{d}x)$. To see this, let us assume that $\mathrm{supp}(f)\subset B(0,R)$. Since S is bounded on $L^q(\mathbb{R}^n, \mathrm{d}x)$ for $q \geq p'$, $Sf \in L^2(B(0,2R), \omega\,\mathrm{d}x)$ by Hölder inequality. If $|x| \geq 2R$, using (3.11), we may show that

$$\big|Sf(x)\big| \leq C\|f\|_{p'} \frac{1}{|x|^{n(1-1/p)}}. \tag{3.27}$$

This is enough to assure that Sf belongs to $L^2(\mathbb{R}^n \setminus B(0,2R), \omega\,\mathrm{d}x)$ for any $\omega \in A_{2/p'}$.

Finally, by (3.26) and duality, $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^2(\mathbb{R}^n, \mathrm{d}x/\omega)$ for any $\omega \in A_{2/p'}(\mathbb{R}^n)$. Condition (iii) now follows by the self improvement property of condition (i). □

4. Operators with VMO coefficients on Lipschitz domains.

In this section we will prove Theorem C stated in the Introduction. To do this, we first show that operators with constant coefficients satisfy condition (i) in Theorem B. We then use an approximation argument found in [6] to prove that operators with VMO coefficients also satisfy condition (i).

Lemma 4.1. — *Suppose operator $\mathcal{L}$ in* (1.1) *has constant coefficients. Then it satisfies condition* (i) *in Theorem B for some $p > 3$ if $n \geq 3$, and for some $p > 4$ in the case $n = 2$. Moreover, p depends only on Ω, n and the ellipticity constant μ of $\mathcal{L}$.*

Proof. — We may assume that $\mathcal{L} = -\Delta$. If $B(x_0, 3r) \subset \Omega$, inequality (1.5) for any $p > 2$ follows easily from the interior estimates. Suppose $x_0 \in \partial\Omega$. We may assume that $x_0 = 0$ and

$$D(0,r_1) = \Omega \cap B(0,r_1) = \big\{(x', x_n) \in \mathbb{R}^n : x_n > \psi(x')\big\} \cap B(0,r_1), \tag{4.2}$$

where $\psi(x')$ is a Lipschitz function on $\mathbb{R}^{n-1}$.

Let $\alpha_2 > \alpha_1 > 1$. Suppose $0 < r < r_0 = cr_1$, where $c > 0$ is sufficiently small. Let $u \in H^1(D(0,\alpha_2 r))$ be a harmonic function in $D(0,\alpha_2 r)$ such that $u = 0$ on $B(0,\alpha_2 r) \cap \partial\Omega$. By the boundary Hölder estimate and Poincaré inequality, for $x = (x', x_n) \in D(0,r)$, we have

$$\big|\nabla u(x)\big| \leq C\big(x_n - \psi(x')\big)^{\eta-1} \frac{1}{r^{\eta-1}} \Big\{ \frac{1}{r^n} \int_{D(0,\alpha_1 r)} |\nabla u|^2 \,\mathrm{d}y \Big\}^{1/2}, \tag{4.3}$$

where $\eta > \frac{1}{2}$ if $n = 2$, and $\eta > 0$ if $n \geq 3$. It follows that if $(p-2)(\eta-1) > -1$,

$$\begin{aligned}\int_{D(0,r)} |\nabla u|^p \, \mathrm{d}x &\leq \frac{C}{r^{(p-2)(\eta-1)}} \int_{D(0,r)} |\nabla u|^2 \big(x_n - \psi(x')\big)^{(p-2)(\eta-1)} \, \mathrm{d}x \\ &\qquad\qquad \times \Big\{ \frac{1}{r^n} \int_{D(0,\alpha_1 r)} |\nabla u|^2 \, \mathrm{d}x \Big\}^{p/2-1} \\ &\leq \frac{C}{r^{(p-2)(\eta-1)}} \int_{\Delta_r} \big|(\nabla u)^*\big|^2 \, \mathrm{d}\sigma \cdot \int_0^{cr} t^{(p-2)(\eta-1)} \, \mathrm{d}t \\ &\qquad\qquad \times \Big\{ \frac{1}{r^n} \int_{D(0,\alpha_1 r)} |\nabla u|^2 \, \mathrm{d}y \Big\}^{p/2-1} \\ &\leq C r \int_{\Delta_r} \big|(\nabla u)^*\big|^2 \, \mathrm{d}\sigma \cdot \Big\{ \frac{1}{r^n} \int_{D(0,\alpha_1 r)} |\nabla u|^2 \, \mathrm{d}x \Big\}^{p/2-1},\end{aligned}$$

where $\Delta_r = \{(x', \psi(x')) : |x'| < r\}$ and

$$(\nabla u)^*\big(x', \psi(x')\big) = \sup\big\{ |\nabla u(x', x_n)| : (x', x_n) \in D(0,r) \big\}.$$

This, together with the inequality

$$\int_{\Delta_r} \big|(\nabla u)^*\big|^2 \, \mathrm{d}\sigma \leq \frac{C}{r} \int_{D(0,\alpha_1 r)} |\nabla u|^2 \, \mathrm{d}x, \tag{4.4}$$

gives the desired estimate (1.5) for $2 < p < \bar{p}$, where $\bar{p} = 2 + 1/(1-\eta)$. Note that $\bar{p} > 3$ if $n \geq 3$, and $\bar{p} > 4$ if $n = 2$. Finally we point out that since $u = 0$ on $\Delta_{\alpha_2 r}$, (4.4) follows from the L^2 solvability of the regularity problem for Laplace's equation on Lipschitz domains, by an integration argument (see [8]). □

Remark 4.5. — If Ω is a C^1 domain, then Hölder estimate (4.3) holds for any $0 < \eta < 1$. It follows that operators with constant coefficients satisfy condition (i) in Theorem B for any $p > 2$. Consequently, the Riesz transform $\nabla(\mathcal{L})^{-1/2}$ is bounded on $L^p(\Omega)$ for all $1 < p < \infty$.

A function f in $\mathrm{BMO}(\mathbb{R}^n)$ is said to be in $\mathrm{VMO}(\mathbb{R}^n)$ if

$$\lim_{r \to 0} \sup_{x_0 \in \mathbb{R}^n} \frac{1}{r^n} \int_{B(x_0,r)} |f - f_{B(x_0,r)}| \, \mathrm{d}x = 0, \tag{4.6}$$

where $f_{B(x_0,r)} = \int_{B(x_0,r)} f \, \mathrm{d}x / |B(x_0,r)|$ is the average of f over $B(x_0,r)$.

Lemma 4.7. — *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$. Suppose that the coefficients of operator $\mathcal{L}$ in (1.1) are in* $\mathrm{VMO}(\mathbb{R}^n)$*. Then there exist a function $\phi(r)$ and some constants $C>0$, $\alpha>1$, $r_1>0$ and $p>3$ ($p>4$, if $n=2$) with the following properties*: 1) $\lim_{r\to 0}\phi(r)=0$, 2) *for any weak solution of $\mathcal{L}u=0$ in $D(x_0,\alpha r)$ and $u=0$ on $B(x_0,\alpha r)\cap\partial\Omega$ with $x_0\in\overline{\Omega}$ and $0<r<r_1$, there exists a function $v\in W^{1,p}(D(x_0,r))$ such that*

$$\left\{\frac{1}{r^n}\int_{D(x_0,r)}|\nabla v|^p\,\mathrm{d}x\right\}^{1/p}\leq C\left\{\frac{1}{r^n}\int_{D(x_0,\alpha r)}|\nabla u|^2\,\mathrm{d}x\right\}^{1/p}, \tag{4.8}$$

$$\left\{\frac{1}{r^n}\int_{D(x_0,r)}|\nabla u-\nabla v|^2\,\mathrm{d}x\right\}^{1/2}\leq \phi(r)\left\{\frac{1}{r^n}\int_{D(x_0,\alpha r)}|\nabla u|^2\,\mathrm{d}x\right\}^{1/2}. \tag{4.9}$$

Proof. — Let u be a weak solution of $\mathcal{L}u=0$ in $D(x_0,\alpha r)$ and $u=0$ on $B(x_0,\alpha r)\cap\partial\Omega$ with $x_0\in\overline{\Omega}$ and $0<r<r_1$. Consider the operator $\mathcal{L}_0=-\partial_j b_{jk}\partial_k$, where b_{jk} is a constant given by

$$b_{jk}=\frac{1}{|B(x_0,\alpha r)|}\int_{B(x_0,\alpha r)}a_{jk}(x)\,\mathrm{d}x. \tag{4.10}$$

Let v be a weak solution of $\mathcal{L}_0 v=0$ in $D(x_0,\beta r)$ such that $u-v$ belongs to $H_0^1(D(x_0,\beta r))$, where $\beta=\frac{1}{2}\alpha=\alpha_1$. We will show that v satisfies estimates (4.8)–(4.9). To this end, we first note that $v=0$ on $B(x_0,\beta r)\cap\partial\Omega$. Thus, by Lemma 4.1,

$$\begin{aligned}\left\{\frac{1}{r^n}\int_{D(x_0,r)}|\nabla v|^p\,\mathrm{d}x\right\}^{1/p}&\leq C\left\{\frac{1}{r^n}\int_{D(x_0,\beta r)}|\nabla v|^2\,\mathrm{d}x\right\}^{1/2}\\&\leq C\left\{\frac{1}{r^n}\int_{D(x_0,\beta r)}|\nabla u|^2\,\mathrm{d}x\right\}^{1/2},\end{aligned} \tag{4.11}$$

where $p>3$ for $n\geq 3$, and $p>4$ if $n=2$. This gives (4.8).

To see (4.9), we observe that

$$\mathcal{L}_0(u-v)=(\mathcal{L}_0-\mathcal{L})u=-\partial_j(b_{jk}-a_{jk})\partial_k u.$$

It follows from the energy estimate that

$$\begin{aligned}&\left\{\frac{1}{r^n}\int_{D(x_0,r)}|\nabla u-\nabla v|^2\,\mathrm{d}x\right\}^{1/2}\\&\qquad\leq C\sum_{j,k}\left\{\frac{1}{r^n}\int_{D(x_0,\beta r)}|b_{jk}-a_{jk}|^2|\nabla u|^2\,\mathrm{d}x\right\}^{1/2}\end{aligned}$$

$$\begin{aligned}&\le C\sum_{j,k}\Big\{\frac{1}{r^n}\int_{B(x_0,\beta r)}|b_{jk}-a_{jk}|^{2q'}\,\mathrm{d}x\Big\}^{1/(2q')}\\&\qquad\qquad\times\Big\{\frac{1}{r^n}\int_{D(x_0,\beta r)}|\nabla u|^{2q}\,\mathrm{d}x\Big\}^{1/(2q)}\\&\le \phi(r)\Big\{\frac{1}{r^n}\int_{D(x_0,\alpha r)}|\nabla u|^2\,\mathrm{d}x\Big\}^{1/2}\end{aligned}$$

where $q=1+\delta$ and $\delta>0$ is so small that the L^{2q} estimates hold for solutions of $\mathcal{L}u=0$ [15]. Also we have introduced the function $\phi(r)$ by

$$\phi(r)=C\sup_{x_0\in\overline{\Omega}}\sum_{j,k}\Big\{\frac{1}{r^n}\int_{B(x_0,\alpha r)}|a_{jk}-b_{jk}|^{2q'}\,\mathrm{d}x\Big\}^{1/(2q')}. \tag{4.12}$$

Finally we note that by the John-Nirenberg inequality, if $a_{jk}\in\mathrm{VMO}(\mathbb{R}^n)$, then $\phi(r)\to 0$ as $r\to 0$. This completes the proof. □

With Lemma 4.7 at our disposal, we may invoke the following approximation theorem to finish the proof of Theorem C.

THEOREM 4.13. — *Let $f\colon E\to\mathbb{R}^m$ be a locally square integrable function, where E is an open set of $\mathbb{R}^n$. Let $p>2$. Suppose that there exist three constants $\varepsilon>0$ and $\alpha,N>1$ such that for every ball $B=B(x_0,r)$ with $\alpha B=B(x_0,\alpha r)\subset E$, there exists a function $h=h_B\in L^p(B)$ with the properties*:

$$\Big\{\frac{1}{|B|}\int_B|f-h|^2\,\mathrm{d}x\Big\}^{1/2}\le\varepsilon\Big\{\frac{1}{|\alpha B|}\int_{\alpha B}|f|^2\,\mathrm{d}x\Big\}^{1/2}, \tag{4.14}$$

$$\Big\{\frac{1}{|B|}\int_B|h|^p\,\mathrm{d}x\Big\}^{1/p}\le N\Big\{\frac{1}{|\alpha B|}\int_{\alpha B}|f|^2\,\mathrm{d}x\Big\}^{1/2}. \tag{4.15}$$

Then, if $2<q<p$ and $0<\varepsilon<\varepsilon_0=\varepsilon_0(n,p,q,\alpha,N)$, we have

$$\Big\{\frac{1}{|B|}\int_B|f|^q\,\mathrm{d}x\Big\}^{1/q}\le C\Big\{\frac{1}{|\alpha B|}\int_{\alpha B}|f|^2\,\mathrm{d}x\Big\}^{1/2}, \tag{4.16}$$

for any ball B with $\alpha B\subset E$, where C depends only on n, p, q, α and N.

We remark that Theorem 4.13, whose proof is omitted here, is essentially proved in [6].

Proof of Theorem C. — It suffices to show that $\mathcal{L}$ satisfies condition (i) in Theorem B for some $p > 3$ if $n \geq 3$, and $p > 4$ if $n = 2$. This will be done by combining Lemma 4.7 with Theorem 4.13.

Let p and $\phi(r)$ be the same as in Lemma 4.7. Fix q so that $3 < q < p$ if $n \geq 3$, and $4 < q < p$ for $n = 2$. Choose $r_0 > 0$ so small that $\sup_{0<r<r_0} \phi(r) < \varepsilon_0$, where $\varepsilon_0 = \varepsilon_0(p, q, n, \alpha, C)$ is given in Theorem 4.13.

Now let u be a weak solution of $\mathcal{L}u = 0$ in $D(x_0, \alpha_2 r)$ and $u = 0$ on $B(x_0, \alpha_2 r) \cap \partial\Omega$, where $0 < r < r_0/\alpha_2$. If $B(x_0, \alpha_2 r) \subset \Omega$, we simply apply Theorem 4.13 to $f = \nabla u$ on $E = B(x_0, \alpha_1 r)$ with $h = \nabla v$ for each ball in E, given in Lemma 4.7.

In the case $x_0 \in \partial\Omega$, we also take $E = B(x_0, \alpha_1 r)$; but extend $f = \nabla u$ to be zero outside of Ω. Given any $B' = B(y_0, t)$ with $\alpha B' \subset E$. If $y_0 \in \overline{\Omega}$, we let $h_{B'} = \nabla v$ on $D(y_0, t)$ and zero otherwise. If $y_0 \notin \overline{\Omega}$ and $B' \cap \Omega \neq \emptyset$, we may find a ball $\widetilde{B} = B(z_0, 2t)$ centered on $\partial\Omega$, such that $B' \subset \widetilde{B}$. We let $h_{B'} = \nabla v$ on $D(z_0, 2t)$ and zero otherwise. Thus the desired estimates (4.14)–(4.15) for B' follow from (4.8)–(4.9). This completes the proof. □

5. Proof of Theorems 3.1 and 3.2.

In this section we give the proof of Theorems 3.1 and 3.3, using a line of argument similar to that in [6].

Proof of Theorem 3.1. — Let T be a bounded sublinear operator on $L^2(\mathbb{R}^n)$. Suppose that T satisfies assumption (3.2). We first note that with possibly different constants α_1, α_2, N, one may change balls B in (3.2) to cubes Q.

Fix $q \in (2, p)$. Let f be a bounded measurable function with compact support. For $\lambda > 0$, we consider the set

$$E(\lambda) = \{x \in \mathbb{R}^n : M(|Tf|^2)(x) > \lambda\}, \tag{5.1}$$

where M is the Hardy-Littlewood maximal operator defined by using cubes. Since $Tf \in L^2$, $|E(\lambda)| \leq C\|Tf\|_2^2/\lambda < \infty$. Let $A = 1/(2\delta^{2/q}) > 5^n$, where $\delta \in (0, 1)$ is a small constant to be determined. Applying the Calderón-Zygmund decomposition to $E(A\lambda)$, we obtain a collection of disjoint dyadic cubes $\{Q_k\}$ with the following properties: (a) $|E(A\lambda) \setminus \bigcup_k Q_k| = 0$, (b) $|E(A\lambda) \cap Q_k| > \delta|Q_k|$, (c) $|E(A\lambda) \cap \overline{Q}_k| \leq \delta|\overline{Q}_k|$, where $\overline{Q}_k$ denotes the dyadic "parent"of Q_k, i.e., Q_k is one of the 2^n cubes obtained by bisecting the sides of $\overline{Q}_k$. To see this, we first choose a large grid of dyadic cubes of $\mathbb{R}^n$ so that $|Q \cap E(A\lambda)| < \delta |Q|$ for each Q in the grid. We then proceed as in the proof of Lemma 1.1 in [CP] for each $Q \cap E(A\lambda) \subset Q$.

We claim that it is possible to choose constants $\delta, \gamma > 0$ so that

$$(5.2)\quad \big|E(A\lambda)\big| \leq \delta \big|E(\lambda)\big| + \Big|\big\{x \in \mathbb{R}^n : M(|f|^2)(x) > \gamma\lambda\big\}\Big| \text{ for any } \lambda > 0.$$

This would imply that for any $\lambda_0 > 0$,

$$(5.3)\quad \int_0^{A\lambda_0} \lambda^{q/2-1}\big|E(\lambda)\big|\,\mathrm{d}\lambda \leq \delta A^{q/2} \int_0^{\lambda_0} \lambda^{q/2-1}\big|E(\lambda)\big|\,\mathrm{d}\lambda + C(\delta,\gamma)\int_{\mathbb{R}^n} |f|^q\,\mathrm{d}x.$$

Using $\delta A^{q/2} = 1/2^{q/2} < 1$, $A > 1$ and $\sup_{\lambda>0} \lambda|E(\lambda)| < \infty$, we obtain

$$(5.4)\quad \int_0^{\lambda_0} \lambda^{q/2-1}\big|\big\{x \in \mathbb{R}^n : |Tf(x)|^2 > \lambda\big\}\big|\,\mathrm{d}\lambda \leq \int_0^{\lambda_0} \lambda^{q/2-1}\big|E(\lambda)\big|\,\mathrm{d}\lambda \leq C \int_{\mathbb{R}^n} |f|^q\,\mathrm{d}x.$$

Letting $\lambda_0 \to \infty$ in (5.4), we conclude that $\|Tf\|_q \leq C\|f\|_q$.

It remains to prove (5.2). To this end, it suffices to show that it is possible to choose $\delta, \gamma > 0$ such that if

$$\overline{Q}_k \cap \big\{x \in \mathbb{R}^n : M(|f|^2)(x) \leq \gamma\lambda\big\} \neq \emptyset,$$

then $\overline{Q}_k \subset E(\lambda)$. For this would imply that

$$(5.5)\quad \Big|E(A\lambda) \cap \big\{x \in \mathbb{R}^n : M(|f|^2)(x) \leq \gamma\lambda\big\}\Big| \leq \sum_{k'}\big|E(A\lambda) \cap \overline{Q}_{k'}\big| \leq \delta \sum_{k'} |\overline{Q}_{k'}| \leq \delta \big|E(\lambda)\big|,$$

where $\{\overline{Q}_{k'}\}$ is a disjoint subcover of $E(A\lambda) \cap \big\{x \in \mathbb{R}^n : M(|f|^2)(x) \leq \gamma\lambda\big\}$ with the property that $\overline{Q}_{k'} \cap \big\{x \in \mathbb{R}^n : M(|f|^2)(x) \leq \gamma\lambda\big\} \neq \emptyset$.

To finish the proof, we proceed by contradiction. Suppose that there exists $x_0 \in \overline{Q}_k \setminus E(\lambda)$ and $\{x \in \overline{Q}_k : M(|f|^2)(x) \leq \gamma\lambda\} \neq \emptyset$. Then, if Q contains $\overline{Q}_k$, we must have

$$(5.6)\qquad \frac{1}{|Q|}\int_Q |f|^2\,\mathrm{d}x \leq \gamma\lambda \quad\text{and}\quad \frac{1}{|Q|}\int_Q |Tf|^2\,\mathrm{d}x \leq \lambda.$$

It follows that for $x \in Q_k$,

$$M\big(|Tf|^2\big)(x) \le \max\big(M_{2\overline{Q}_k}\big(|Tf|^2\big)(x), 5^n\lambda\big), \tag{5.7}$$

where M_Q is a localized maximal function defined by

$$M_Q(g)(x) = \sup_{\substack{Q' \ni x \\ Q' \subset Q}} \frac{1}{|Q'|} \int_{Q'} \big|g(y)\big|\,\mathrm{d}y \quad \text{for } x \in Q. \tag{5.8}$$

Since $A = 1/(2\delta^{2/q}) \ge 5^n$, we have

$$\begin{aligned} \big|Q_k \cap E(A\lambda)\big| &\le \big|\big\{x \in Q_k : M_{2\overline{Q}_k}(|Tf|^2)(x) > A\lambda\big\}\big| \\ &\le \big|\big\{x \in Q_k : M_{2\overline{Q}_k}(|T(f\chi_{\alpha_2\overline{Q}_k})|^2)(x) > \tfrac{1}{4}A\lambda\big\}\big| \\ &\quad + \big|\big\{x \in Q_k : M_{2\overline{Q}_k}(|T(f\chi_{\mathbb{R}^n\setminus\alpha_2\overline{Q}_k})|^2)(x) > \tfrac{1}{4}A\lambda\big\}\big| \\ &\le \frac{C_n}{A\lambda} \int_{2\overline{Q}_k} \big|T(f\chi_{\alpha_2\overline{Q}_k})\big|^2\,\mathrm{d}x \\ &\qquad\qquad + \frac{C_{n,p}}{(A\lambda)^{p/2}} \int_{2\overline{Q}_k} \big|T(f\chi_{\mathbb{R}^n\setminus\alpha_2\overline{Q}_k})\big|^p\,\mathrm{d}x. \end{aligned} \tag{5.9}$$

It then follows from the L^2 boundedness of T, assumption (3.2) and (5.6) that for any $\lambda > 0$,

$$\begin{aligned} |Q_k \cap E(A\lambda)| &\le |Q_k| \Big\{ \frac{C\gamma}{A} + \frac{C}{A^{p/2}} \Big\} \\ &= \delta\,|Q_k| \big\{2C\gamma\delta^{2/q-1} + C2^{p/2}\delta^{p/q-1}\big\}, \end{aligned} \tag{5.10}$$

where C depends only on $n, p, \alpha_1, \alpha_2, N$ as well as the operator norm of T on $L^2(\mathbb{R}^n)$.

Finally we choose $\delta \in (0,1)$ so small that $C2^{p/2}\delta^{p/q-1} \le \frac{1}{2}$ and $A = 1/(2\delta^{2/q}) \ge 5^n$. This is possible since $q < p$. With δ fixed, we choose $\gamma > 0$ so small that $2C\gamma\delta^{2/q-1} \le \frac{1}{2}$. It follows from (5.10) that $|Q_k \cap E(A\lambda)| \le \delta\,|Q_k|$. This contradicts with the fact that $|Q_k \cap E(A\lambda)| > \delta\,|Q_k|$. Thus we must have $\overline{Q}_k \subset E(\lambda)$ whenever the set $\{x \in \overline{Q}_k : M(|f|^2)(x) \le \gamma\lambda\}$ is not empty. The proof is complete. □

Remark 5.11. — Let T be a linear operator with kernel $K(x,y)$ satisfying

$$\big|K(x,y) - K(x+h,y)\big| \le \frac{C|h|^\eta}{|x-y|^{n+\eta}}, \tag{5.12}$$

where $x, y, h \in \mathbb{R}^n$ and $|h| < \frac{1}{4}|x-y|$. Suppose $\operatorname{supp} f \subset \mathbb{R}^n \setminus 8B$. Then

$$|Tf(x) - Tf(y)| \leq C \sup_{Q' \supset Q} \frac{1}{|Q'|} \int_{Q'} |f(z)| \, \mathrm{d}z \tag{5.13}$$

for any $x, y \in Q$. It follows that

$$\|Tf\|_{L^\infty(Q)} \leq \frac{1}{|Q|} \int_Q |Tf| \, \mathrm{d}x + C \sup_{Q' \supset Q} \frac{1}{|Q'|} \int_{Q'} |f| \, \mathrm{d}x. \tag{5.14}$$

Thus T satisfies assumption (3.2) in Theorem 3.1 for any $p > 2$. Consequently, if T is bounded on L^2, then it is bounded on L^p for any $2 < p < \infty$. In this regard, Theorem 3.1 may be considered as an extension of the Calderón-Zygmund Lemma.

The following is a weighted version of Theorem 3.1. Its proof may be carried out by a careful inspection of the proof of Theorem 3.1. The key observation is that if $\mathrm{d}\mu = \omega^\delta \, \mathrm{d}x$ where $\omega \in A_1(\mathbb{R}^n)$ and $0 < \delta < 1$, then $\mu(E) \leq C(|E|/|Q|)^{1-\delta}\mu(Q)$ whenever $E \subset Q$. We leave the details to the reader.

Theorem 5.15. — *Under the same assumption as in Theorem* 3.1, T *is bounded on* $L^2(\mathbb{R}^n, \omega^\delta \, \mathrm{d}x)$ *where* $\omega \in A_1(\mathbb{R}^n)$ *and* $0 < \delta < 1 - 2/p$.

Proof of Theorem 3.3. — The proof is similar to that of Theorem 3.1. We first note that with possibly different constants $\alpha_1, \alpha_2, N, r_0$, inequality (3.4) holds for any ball $B(x_0, r)$ with the property that $0 < r < r_0$ and $B(x_0, r) \cap \Omega \neq \emptyset$. Also one may replace balls B in (3.4) by cubes Q of side length r.

Next we choose a cube Q_0 such that $\Omega \subset Q_0$. Fix $q \in (2, p)$. Let $\delta \in (0, 1)$ be a small constant to be determined. For $\lambda > 0$, we consider the set

$$E(\lambda) = \left\{x \in Q_0 : M_{2Q_0}(|Tf|^2 \chi_\Omega)(x) > \lambda\right\}. \tag{5.16}$$

Then $|E(\lambda)| \leq C\|f\|_2^2/\lambda \leq \delta|Q_0|$ if

$$\lambda \geq \lambda_1 = \frac{C}{\delta|Q_0|} \int_\Omega |f|^2 \, \mathrm{d}x. \tag{5.17}$$

Let $A = 1/(2\delta^{2/q})$. For $\lambda \geq \lambda_1$, we apply the Calderón-Zygmund decomposition to $E(A\lambda)$. This produces a collection of dyadic subcubes $\{Q_k\}$

of Q_0 satisfying the same properties (a), (b), (c) as in the proof of Theorem 3.1. Note that $\delta\,|Q_k| < |E(A\lambda)| \le C\|f\|_2^2/(A\lambda) \le \delta\,|Q_0|/A$. It follows that $|Q_k| < |Q_0|/A$. Thus we may choose δ so small that the side length of $2\overline{Q}_k$ is less than r_0. With this observation, we may use the same argument as in the proof of Theorem 3.1 to show that for $\lambda \ge \lambda_1$,

$$\big|E(A\lambda)\big| \le \delta\,\big|E(\lambda)\big| + \big|\big\{x \in \mathbb{R}^n : M(|f|^2\chi_\Omega)(x) > \gamma\lambda\big\}\big|. \tag{5.18}$$

By integration, this implies that

$$\int_\Omega |Tf|^q \,\mathrm{d}x \le C\lambda_1^{q/2}|Q_0| + C\int_\Omega |f|^q \,\mathrm{d}x \le C\int_\Omega |f|^q \,\mathrm{d}x. \tag{5.19}$$

The proof is finished. $\square$

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