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ON SETS FILLED BY ASYMPTOTIC SOLUTIONS OF DIFFERENTIAL EQUATIONS

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Consider an ordinary differential equation

$$(1) \qquad x' = f(t, x)$$
$$x = (x_1, \ldots, x_n), \qquad f = (f_1, \ldots, f_n).$$

Assumption I. Suppose that the domain D of $f(t, x)$ is open, $f(t, x)$ is continuous on D and through each point

$$A : t = a_0, \qquad x = a = (a_1, a_2, \ldots, a_n)$$

of D passes only one integral $x = x(t, A)$ of (1).

Denote by $(\alpha(A), \beta(A))$ the maximal interval on which there exists the integral passing through A. We shall denote

$$X(t, A) = (t, x(t, A)) \qquad \text{for} \qquad t \in (\alpha(A), \beta(A)).$$

Let E be an open subset of D. In the following we shall deal with the set Z(E) of such points A, that $X(t, A) \in E$ for $a_0 \leqslant t < \infty$. Obviously set Z(E) depends on both set E and system (1). It is evident that $E \subset F$ implies $Z(E) \subset Z(F)$. Let φ be a family of subsets F of D. We shall consider the following properties of equation (1).

PROPERTY I (of equation (1) in respect to E and φ). — *For every* $F \in \varphi$ $Z(E) \cap F$ *is empty or consists of one point.*

PROPERTY II. — *For every* $F \in \varphi$ $Z(E) \cap F$ *is not empty.*

Let $I^+(A)$ denote the set of all points $B = X(t, A)$ for $t \geqslant a_0$.

We say that the point $A \in P(G) \cap D$, where $P(G)$ denotes the boundary of an open set G, is the point of egress from G (with respect to equation (1) and set D) if there exists such an integral $x(t)$ of (1) and a positive number $\varepsilon > 0$ that

$$x(a_0) = a \qquad \text{and} \qquad (t, x(t)) \in G$$

for $a_0 - \varepsilon < t < a_0$ (under Assumption I, $X(t, A) \in G$ for $a_0 - \varepsilon < t < a_0$). If no point of $P(G)$ is a point of egress from G then $A \in G$ implies $I^+(A) \subset G$. If Property I is satisfied and $B \in Z(E) \cap F$ then $(F - B) \cap Z(E) = \emptyset$, where $F - B$ denotes the set of all points of the set F except the point B. It follows that for every $A \in F$, $A \neq B$ either $I^+(A) \sim\in E$ or $\beta(A) < \infty$. Let G be such a set that $\overline{G} \cap E$ has no common point with a plane $t = c > a_0$, where $\overline{G}$ denotes the closure of G, then $I^+(A) \subset \overline{G}$ implies $A \sim\in Z(E)$.

Lemma. — *Suppose Assumption I and the following conditions. For each set* $G_i (i = 1, \ldots) G_i \cap E$ *is contained in a halfspace* $t < c_i$. *No point of* $P(G_i)$ *is a point of egress. Set F satisfies inclusion* $F - O \subset \bigcup_{i=1}^{\infty} G_i$.

Then $(F - O) \cap Z(E) = \emptyset$.

Theorem 1. — *Suppose Assumption I and the following conditions. The intersection* $E(s)$ *of a given set E and the plane* $t = s$ *satisfies the inequality* $\operatorname{diam}(E(t)) < p(t)$, *where* $p(t)$ *is a positive function continuous on* $(-\infty, \infty)$. *No point of* $P(G_i)$ *is a point of egress in respect to the equation*

$$x' = f(t + a_0, x + a(t)) - f(t + a_0, a(t)),$$

where a_0 *is a real number and* $x = a(t)$ *is such a Lipchitzian function that the right side of the equation is defined. Set F satisfies inclusion* $F - O \subset \bigcup_{i=1}^{\infty} G_i$. *For any i and s there exists a constant* $c(i, s)$ *that* $\operatorname{dist}(G_i(t), 0) \geqslant p(t + s)$ *for* $t \geqslant c(i, s)$, *where* $G_i(s)$ *is the intersection of* G_i *and the plane* $t = s$.

Under these assumptions if $A \in Z(E)$, then

$$(F(A) - A) \cap Z(E) = \emptyset,$$

where $F(A)$ denotes set obtained from A by translation of R^{n+1} transforming O on A.

THEOREM 2. — If assumptions of Theorem 1 are satisfied and F is a plane then equation (1) possesses property I in respect to E and the family of planes parallel to F (and of the same dimension).

Suppose now that set F is a plane and in the coordinate system t, $x = (u, v)$, $u = (u_1, \ldots, u_k)$, $v = (v_1, \ldots, v_{n-k})$ it has the equation $t = 0$, $u = 0$. Now Property I (for the family of planes $t = c_0$, $u = (c_1, \ldots, c_k)$, c_i arbitrary) is necessary and sufficient for set Z(E) to be the graph of a single-valued function $v = q(t, u)$. Putting $g = (f_1, \ldots, f_k)$, $h = (f_{k+1}, \ldots, f_n)$ system (1) takes the form

$$(2) \qquad u' = g(t, u, v), \qquad v' = h(t, u, v).$$

The following result formulated in terms of inequalities can be obtain from Theorem 1 formulated in terms of sets [1]

THEOREM 3. — *Suppose that system* (2) *satisfies Assumption I and that the functions* $g(t, u, v)$, $h(t, u, v)$ *for*

$$(t, u, v) \in D, \qquad (t, \bar{u}, \bar{v}) \in D$$

satisfy inequalities

$$(3) \quad (g(t, u, v) - g(t, \bar{u}, \bar{v}))\,(u - \bar{u}) \leqslant \gamma(t)\,(u - \bar{u})^2$$

for $|v - \bar{v}| = |u - \bar{u}|$, *where* $|z|$ *denotes Euclidean distance of point* z *from* 0,

$$(4) \quad (h(t, u, v) - h(t, \bar{u}, \bar{v}))\,(v - \bar{v}) \geqslant \gamma(t)\,(v - \bar{v})^2,$$

for

$$|u - \bar{u}| \leqslant |v - \bar{v}|,$$

where $\gamma(t)$ *is a function summable in every finite interval, and such that*

$$\int_0^\infty \gamma(s)\,ds = \infty,$$

then set Z *of points* A *lying on the integrals of* (2) *(remaining in* D*) bounded for* $a_0 \leqslant t < \infty$ *is a graph of a single-valued function* $v = q(t, u)$ *defined in a certain set* $S(S \subset R^{k+1})$ *satisfying the Lipschitz condition with respect to all the variables*

(1) Such kind of formulation was suggested by T. Wazewski.

and in particular the condition

$$|q(t, u) - q(t, \bar{u})| \leqslant |u - \bar{u}|$$

in the set S *or the set* Z *is an empty set.*

Theorem 3 is a particular case of theorem 2 in [1].

Now for illustration of Property II we present a variant of an example from [2].

Let system (2) satisfy Assumption I on a neighbourhood D of the set H: $|u| \leqslant 1$, $|v| \leqslant 1$, $-\infty < t < \infty$. Moreover suppose that $g(t, u, v)u < 0$ for $|u| = 1$, $|v| \leqslant 1$ and arbitrary t, $h(t, u, v) > 0$ for $|v| = 1$, $|u| \leqslant 1$ and arbitrary t.

Under these assumptions for every $\bar{u}$, $|\bar{u}| < 1$ and arbitrary $\bar{t}$, there exists $\bar{v}$, that $I^+(\bar{t}, \bar{u}, \bar{v}) \in H$.

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