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**DE RHAM THEOREMS
AND NEUMANN DECOMPOSITIONS ASSOCIATED
WITH LINEAR PARTIAL DIFFERENTIAL EQUATIONS**

by **D. C. SPENCER** (Stanford)

1. Introduction.

Our purpose is to associate, to a (homogeneous) system of linear partial differential equations, a resolution of its sheaf of germs of solutions defined in a canonical manner, the terms of which are connected by a linear differential operator of order 1. If the system is regular, i.e., if it has constant rank, the resolution is a sheaf of germs of differential forms with values in a vector bundle. A procedure is thus defined for extending the classical resolution of de Rham to arbitrary systems of equations and, if the system is regular, the exactness of the resolution has been established in the analytic case, in the elliptic case, and in various special cases which are simple enough to examine directly.

This procedure also provides a means of generalizing, to arbitrary elliptic systems of equations, the theory of harmonic differential forms. In particular, it enables us to generalize the Neumann problems recently solved by Kohn [5] and Ash [1] to arbitrary elliptic systems, and it provides a method for associating with each elliptic system a class of domains generalizing the pseudoconvex domains associated with the Cauchy-Riemann equations — namely the class of domains on which the Neumann problem for the elliptic system is solvable.

Finally, as an application of the generalized de Rham-Hodge theory, we obtain a theorem (see [9(b)]) for the existence of local coordinates compatible with structures on

manifolds defined by elliptic pseudogroups. An elliptic pseudogroup is a transitive, continuous pseudogroup whose Lie pseudoalgebra of infinitesimal transformations is defined by an elliptic system of equations. Thus we generalize the complex Frobenius theorem of A. Newlander and L. Nirenberg [7] (see also Nirenberg [8]) to structures defined by elliptic systems. We remark that any *complex* transitive, continuous pseudogroup is elliptic (see [9](a)). However, there is some reason to believe that this theorem remains valid for transitive, continuous pseudogroups without the assumption of ellipticity.

2. Jet forms.

Our principal task is to define the jet forms associated with a system of linear partial differential equations.

By « differentiable » we shall always mean « differentiable of class C^∞ ». Let $p = (p_1, \dots, p_n)$ denote an ordered set of n non-negative integers p_1, \dots, p_n , and write

$$|p| = p_1 + p_2 + \dots + p_n.$$

Moreover, if $x = (x^1, \dots, x^j, \dots, x^n)$ is a coordinate, we define $\partial_p = (\partial/\partial x^1)^{p_1} (\partial/\partial x^2)^{p_2} \dots (\partial/\partial x^n)^{p_n}$. Finally let

$$O^\nu(\mathbf{R}^n), \quad \nu \geq 0,$$

be the ν -tuple symmetric product of (real) n -space \mathbf{R}^n , and write

$$F^\mu = \text{Hom} \left(\bigoplus_{0 \leq \nu \leq \mu} O^\nu(\mathbf{R}^n), \mathbf{R}^m \right) = \bigoplus_{0 \leq \nu \leq \mu} F_{\nu-1}^\nu$$

where

$$F_{\nu-1}^\nu = \text{Hom}(O^\nu(\mathbf{R}^n), \mathbf{R}^m)$$

and $F_{-1}^0 = F^0 = \mathbf{R}^m$. Let $\sigma^\mu = \{\sigma_j^\mu | 1 \leq j \leq m, 0 \leq |p| \leq \mu\}$ be the coordinate of F^μ where $\sigma = \sigma^0 = (\sigma^1, \dots, \sigma^j, \sigma^m)$ is the coordinate of \mathbf{R}^m .

Now let M be a differentiable manifold of dimension n , and let Q be a differentiable vector bundle over M with fibre \mathbf{R}^m . For each non-negative integer μ , we denote by $S_d^\mu = S_d^\mu(Q)$ the differentiable vector bundle over M , with fibre F^μ , of all jets of order μ of differentiable sections of Q over M . Let U be

a neighborhood of M , covered by a differentiable coordinate $x = (x^1, \dots, x^j, \dots, x^n)$, such that $Q|U \cong U \times \mathbb{R}^m$. Then $S_d^\mu|U \cong U \times \mathbb{F}^\mu$ is covered by the coordinate $(x, \sigma) = (x, \sigma^\mu)$.

A differentiable section $s: M \rightarrow Q$ induces, for each μ , a differentiable section $\iota^\mu(s): M \rightarrow S_d^\mu$ which, expressed in terms of a local coordinate (x, σ) , sends x into

$$\iota^\mu(s)(x) = (x, (\partial s)(x))$$

where

$$\partial s = \partial^\mu s = \{ \partial_p s^j | 1 \leq j \leq m, \quad 0 \leq |p| \leq \mu \}.$$

DEFINITION 2.1. — *A linear partial differential equation on M of virtual order ν_0 is the kernel $E^{\nu_0} = E^{\nu_0}(Q, R)$ of a differentiable map $a^{\nu_0}: S_d^{\nu_0}(Q) \rightarrow R$, where Q, R are differentiable vector bundles over M , and a^{ν_0} maps each fibre of*

$$S_d^{\nu_0} = S_d^{\nu_0}(Q) \rightarrow M$$

linearly onto a fibre of $R \rightarrow M$ and induces the identity map on the base space M . A solution of the equation $E^{\nu_0} = E^{\nu_0}(Q, R)$ is a differentiable section $s: M \rightarrow Q$ which induces a section $\iota^{\nu_0}(s): M \rightarrow E^{\nu_0}$.

A linear partial differential equation $E^{\nu_0} = E^{\nu_0}(Q, R)$ is defined locally, in terms of a local coordinate (x, σ) for $S_d^{\nu_0}$, by a finite number of equations

$$f^k(x, \sigma) = 0, \quad k = 1, 2, \dots, h,$$

each of which is linear in σ . Therefore, a solution s of E^{ν_0} , expressed locally in terms of a coordinate (x, σ) , satisfies the equations

$$f^k(x, (\partial s)(x)) = 0, \quad k = 1, 2, \dots, h.$$

Let $E^{\nu_0} = E^{\nu_0}(Q, R)$ be a partial differential equation (of virtual order ν_0). Then E^{ν_0} can be prolonged, by differentiation, to a linear partial differential equation

$$E^{\nu_0+1} = \iota^1(E^{\nu_0})$$

of virtual order $\nu_0 + 1$ where E^{ν_0+1} is defined locally, in terms of a coordinate, by the equations

$$\begin{cases} f^k(x, \sigma^\mu) = 0, \\ \partial f^k(x, \sigma^{\mu+1}) = 0, \end{cases} \quad k = 1, 2, \dots, h,$$

where

$$\delta f^k(x, \sigma^{\mu+1}) = \delta^1 f^k(x, \sigma^{\mu+1}) = \{ \delta_j f^k(x, \sigma^{\mu+1}) \mid j = 1, 2, \dots, n \}$$

$$\text{and} \quad \delta_j f^k = \frac{\delta f^k}{\delta x^j} + \sum_{l=1}^m \sum_{0 \leq |p| \leq \mu} \frac{\delta f^k}{\delta \sigma_p^l} \cdot \sigma_{p+1_j}^l$$

where 1_j is the set of n integers all of which are zero except the j -th and it is equal to 1. For each non-negative integer μ , we define $E^\mu = E^\mu(E^{\nu_0})$ as follows:

$$E^\mu = \begin{cases} E^{\nu_0} = \iota^0(E^{\nu_0}), & \mu = \nu_0, \\ \iota^{\mu-\nu_0}(E^{\nu_0}) = \iota^1(\iota^{\mu-\nu_0-1}(E^{\nu_0})), & \mu > \nu_0, \\ pr^\mu(E^{\nu_0}), & 0 \leq \mu < \nu_0, \end{cases}$$

where $pr^\nu(E^{\nu_0})$ denotes the projection of E^{ν_0} in S_a^ν . Set

$$S^\mu = \bigcap_{\nu \geq \mu} pr^\nu(E^{\nu_0}).$$

For each non-negative integer μ , we then have the projection $S^{\mu+1} \rightarrow S^\mu$ whose kernel we denote by $S_{\mu+1}^\mu$, i.e.,

$$(2.1) \quad 0 \rightarrow S_{\mu+1}^\mu \rightarrow S^{\mu+1} \rightarrow S^\mu \rightarrow 0$$

where

$$(2.2) \quad S^\mu \subseteq E^\mu \subseteq S_a^\mu.$$

We denote by $S = S(E^{\nu_0}) = pr \lim S^\mu$ the projective limit of the system $\{\{\mu\}, S^\mu\}$ over the (directed) set $\{\mu\}$ of non-negative integers (with the natural order).

DEFINITION 2.2. — *We call $S = S(E^{\nu_0})$ the (canonical) prolongation of the linear partial differential equation*

$$E^{\nu_0} = E^{\nu_0}(Q, R).$$

For each p , $|p| = \mu - \nu \leq \mu$, we have the map

$$\delta_p: F_{\mu-1}^\mu \rightarrow F_{\nu-1}^\nu$$

sending the element σ of $F_{\mu-1}^\mu$, with components

$$\sigma_r = \{ \sigma_r^j \mid 1 \leq j \leq m, |r| = \mu \},$$

into the element $\delta_p \sigma$ of $F_{\nu-1}^\nu$, with components $(\delta_p \sigma)_q = \sigma_{p+q}$. For each p , $|p| = \mu - \nu \leq \mu$, we therefore have the *local* map

$$\delta_p: S_{a, \mu-1}^\mu \rightarrow S_{a, \nu-1}^\nu.$$

DEFINITION 2.3. — We say that the linear partial differential equation $E^{\nu_0} = E^{\nu_0}(Q, R)$ is regular if the following conditions hold:

- (i) S^{ν_0} is a (differentiable) vector sub-bundle of $S_a^{\nu_0}$.
- (ii) For $\mu > \nu_0$, we have locally

$$S_{\mu-1}^{\mu} = \bigcap_{|p|=\mu-\nu_0} \delta_p^{-1}(S_{\nu_0-1}^{\nu_0}),$$

i.e., the element σ of $S_{a, \mu-1}^{\mu}$, $\mu > \nu_0$, belongs to $S_{\mu-1}^{\mu}$ if and only if, for each p , $|p| = \mu - \nu_0$, $\delta_p \sigma$ belongs to $S_{\nu_0-1}^{\nu_0}$.

Suppose that condition (i) of Definition 2.3 is satisfied. Then for $0 \leq \mu \leq \nu_0$, S^{μ} and $S_{\mu-1}^{\mu}$ are (differentiable) vector bundles. If, in addition, condition (ii) of Definition 2.3 holds, i.e., if E^{ν_0} is regular, the argument used to establish Lemma 5.6 of [9(a)] shows that, for $\mu > \nu_0$, $S_{\mu-1}^{\mu}$ is a (differentiable) vector bundle and we then infer from (2.1), by recursion, that S^{μ} is a (differentiable) vector bundle for $\mu > \nu_0$ and hence for all μ . Thus we have:

PROPOSITION 2.1. — If E^{ν_0} is regular then, for $\mu \geq 0$, S^{μ} and $S_{\mu-1}^{\mu}$ are (differentiable) vector bundles.

DEFINITION 2.4. — Suppose that $E^{\nu_0} = E^{\nu_0}(Q, R)$ is regular. We then denote by μ_0 the smallest positive integer ν_0 for which condition (ii) of Definition 2.3 holds, we set $\mathcal{G}^{\mu_0} = S^{\mu_0}$, and we call \mathcal{G}^{μ_0} a regular partial differential equation of order μ_0 with prolongation $S = \text{pr lim } S^{\mu}$. We say that the (differentiable) manifold M is an \mathcal{G}^{μ_0} -manifold if a regular partial differential equation \mathcal{G}^{μ_0} of order μ_0 is defined over it.

Let M be an \mathcal{G}^{μ_0} -manifold, denote by $T^*(M)$ the dual of the tangent bundle $T(M)$ of M , and let $\Lambda^i T^*(M)$ be the i -tuple exterior product of $T^*(M)$. We let

$$S_a^{\mu, i} = S_a^{\mu} \otimes \Lambda^i T^*(M)$$

for $\mu \geq 0$, and we define $S_a^{\mu, i} = 0$ for $\mu < 0$. Moreover, we let $S_{a, \mu-1}^{\mu, i}$ be the kernel of the projection $S_a^{\mu, i} \rightarrow S_a^{\mu-1, i}$. Let U be a sufficiently small neighborhood of an arbitrary point of M ; then

$$S_{a, \mu-1}^{\mu, i}|U \cong U \times \{F_{\mu-1}^{\mu} \otimes \text{Hom}(\Lambda^i(\mathbb{R}^n), \mathbb{R})\}$$

and $S_{d, \mu-1}^{\mu, i}$ is covered by the coordinate (x, σ) where

$$\sigma = \{\sigma_q \mid |q| = \mu\}, \quad \sigma_q = \{\sigma_q^j \mid 1 \leq j \leq m\}$$

and

$$\sigma_q^j = \sum \sigma_{q, k_1 \dots k_i}^j dx^{k_1} \wedge \dots \wedge dx^{k_i},$$

where the summation is over k_1, \dots, k_i satisfying

$$1 \leq k_1 < \dots < k_i \leq n.$$

We have the map

$$(2.3) \quad \delta: S_{d, \mu}^{\mu+1, i} \rightarrow S_{d, \mu-1}^{\mu, i+1}$$

sending σ into $S\sigma$ where $S\sigma$ has the components

$$\{(\delta\sigma)_p \mid |p| = \mu\}$$

and

$$(2.4) \quad (\delta\sigma)_p = \sum_{j=1}^n dx^j \wedge \sigma_{p+1, j}$$

where $p + 1_j = (p_1, \dots, p_j + 1, \dots, p_n)$. Clearly we have $\delta^2 = \delta \circ \delta = 0$.

Define $S^{\mu, i}$ in the same way as $S_d^{\mu, i}$ with S^μ replacing S_d^μ . Then $S^{\mu, i} \subset S_d^{\mu, i}$ and, in particular, $S^{\mu, i} = 0$ for $\mu < 0$. If $\mu \geq \mu_0$, we infer from (ii), Definition 2.3, that the restriction of δ to $S_{\mu}^{\mu+1, i}$, the kernel of the projection $S^{\mu+1, i} \rightarrow S^{\mu, i}$, defines a map

$$(2.5) \quad \delta: S_{\mu}^{\mu+1, i} \rightarrow S_{\mu-1}^{\mu, i+1},$$

and we denote the kernel of (2.5) by $L_{\mu}^{\mu+1, i}$.

The proof of the following theorem is the same as that of Theorem 5.1 of [9(a)]:

THEOREM 2.1. — *If M is an \mathcal{G}^{μ_0} -manifold, there is an integer $\mu_1 = \mu_1(\mu_0, n)$, depending only on the order μ_0 of \mathcal{G}^{μ_0} and the dimension n of M , where $\mu_1 > \mu_0$, such that the sequence*

$$(2.6) \quad 0 \rightarrow L_{\mu}^{\mu+1, i} \rightarrow S_{\mu}^{\mu+1, i} \xrightarrow{\delta} L_{\mu-1}^{\mu, i+1} \rightarrow 0$$

is exact for $\mu \geq \mu_1$ (and all i , $0 \leq i \leq n$).

Suppose that M is an \mathcal{G}^{μ_0} -manifold, and let $\Sigma^\mu = \oplus_i \Sigma^{\mu, i}$ where $\Sigma^{\mu, i}$ is the sheaf over M of germs of (differentiable) sections of $S^{\mu, i}$. Moreover, let $J^{\mu, i}$ be the sheaf of pairs $u = (\sigma, \xi)$ where, for some element $\sigma^{\mu+1}$ of $\Sigma^{\mu+1, i}$, $\sigma = \sigma^\mu$ is the projec-

tion of $\sigma^{\mu+1}$ in $\Sigma^{\mu,i}$ and $\xi = \delta\sigma^{\mu+1}$ has components defined by (2.4) for $0 \leq |p| \leq \mu$. For each i , $0 \leq i \leq n$, we have the surjective map

$$\#^\mu : \Sigma^{\mu+1,i} \rightarrow J^{\mu,i}$$

sending $\sigma^{\mu+1}$ into $u = (\sigma, \xi)$, and its kernel is the sheaf $\Lambda_{\mu}^{\mu+1,i}$ of germs of sections of $L_{\mu}^{\mu+1,i}$, i.e.,

$$0 \rightarrow \Lambda_{\mu}^{\mu+1,i} \rightarrow \Sigma^{\mu+1,i} \xrightarrow{\#^\mu} J^{\mu,i} \rightarrow 0.$$

We have the map (see [9(a)])

$$D : J^{\mu,i} \rightarrow J^{\mu,i+1}$$

sending $u = (\sigma, \xi)$ into $Du = D(\sigma, \xi) = (d\sigma - \xi, -d\xi)$. Clearly we have $D^2 = D \cdot D = 0$.

DEFINITION 2.5. — We call $J^\mu = \oplus J^{\mu,i}$ the sheaf (over M) of jet forms of order μ belonging to \mathcal{G}^{μ} .

The regularity implies that J^μ is a sheaf of germs of differentiable sections of a (differentiable) vector bundle over M and therefore, in particular, J^μ is a fine sheaf.

Finally, let $D^\mu : \Sigma^{\mu+1,i} \rightarrow \Sigma^{\mu,i+1}$ be the map sending $\sigma^{\mu+1}$ into $D^\mu(\sigma^{\mu+1}) = d\sigma^\mu - \delta\sigma^{\mu+1}$. For $\nu \leq \mu$, let $\Sigma_{\nu}^{\mu+1,i}$ be the kernel of the projection $\Sigma^{\mu+1,i} \rightarrow \Sigma^{\nu,i}$. The restriction of D^μ to $\Sigma_{\nu}^{\mu+1}$ maps $\Sigma_{\nu}^{\mu+1}$ into $\Sigma_{\nu-1}^{\mu}$, where $\Sigma_{-1}^{\mu} = \Sigma^{\mu}$, and we denote by $\Xi_{\nu}^{\mu+1} = \oplus \Xi_{\nu}^{\mu+1,i}$ the maximal subsheaf of $\Sigma_{\nu}^{\mu+1}$ which is mapped by D^μ into Ξ_{ν}^{μ} . Then (see [9(a)], page 383)

$$\Xi_{\mu-1}^{\mu} = \Lambda_{\mu-1}^{\mu},$$

where $\Lambda_{\mu-1}^{\mu} = \oplus \Lambda_{\mu-1}^{\mu,i}$, and we have the exact commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & \Sigma_{\mu}^{\mu+1} & \rightarrow & \Xi_{\mu-1}^{\mu+1} & \rightarrow & \Lambda_{\mu-1}^{\mu} \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \rightarrow & \Sigma_{\mu}^{\mu+1} & \rightarrow & \Sigma^{\mu+1} & \rightarrow & \Sigma^{\mu} \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & J^{\mu-1} & = & J^{\mu-1} \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Moreover, denoting by $J_{\mu-1}^\mu = \oplus J_{\mu-1}^{\mu,i}$ the kernel of the projection $J^\mu \rightarrow J^{\mu-1}$, we have the exact commutative diagram

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 0 & \rightarrow & \Lambda_\mu^{\mu+1} & \rightarrow & \Xi_{\mu-1}^{\mu+1} & \xrightarrow{\neq \iota} & J_{\mu-1}^\mu \rightarrow 0 \\
 & & \parallel & & \downarrow & & \downarrow \\
 0 & \rightarrow & \Lambda_\mu^{\mu+1} & \rightarrow & \Sigma^{\mu+1} & \xrightarrow{\neq \iota} & J^\mu \rightarrow 0 \\
 & & & & \downarrow & & \downarrow \\
 & & & & J^{\mu-1} & = & J^{\mu-1} \\
 & & & & \downarrow & & \downarrow \\
 & & & & 0 & & 0
 \end{array}$$

Now let M be an \mathcal{G}^{μ_0} -manifold of dimension n , and let Θ be the sheaf over M of germs of solutions of the regular partial differential equation of order μ_0 . Moreover, let

$$i = i^\mu : \Theta \rightarrow J^{\mu,0}$$

be the injection sending θ into

$$i(\theta) = i^\mu(\theta) = (\iota^\mu(\theta), \delta \iota^{\mu+1}(\theta)) = (\iota^\mu(\theta), d\iota^\mu(\theta)).$$

If $\mu \geq \mu_0 - 1$, it is easily seen that the sequence

$$(2.7) \quad 0 \rightarrow \Theta \xrightarrow{i} J^{\mu,0} \xrightarrow{D} J^{\mu,1} \xrightarrow{D} J^{\mu,2} \xrightarrow{D} \dots \xrightarrow{D} J^{\mu,n} \rightarrow 0$$

is exact at $J^{\mu,0}$.

DEFINITION 2.6. — We call (2.7) the resolution (by jet forms) of order μ of the sheaf Θ of solutions of \mathcal{G}^{μ_0} .

Examples. 1) (de Rham's theorem). — Let M be an \mathcal{G}^1 -manifold, where \mathcal{G}^1 is the equation $df = 0$ for the real-valued function f . Then $\Theta = \mathbb{R}$ (real numbers) and $J^{\mu,i} = A^i$ for $\mu \geq 0$ and $0 \leq i \leq n$, where A^i is the sheaf of germs of (real-valued) differential forms of degree i . In this case (2.7) coincides (for arbitrary $\mu \geq 0$) with the classical (exact) resolution of de Rham, namely

$$0 \rightarrow \mathbb{R} \rightarrow A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \dots \xrightarrow{d} A^n \rightarrow 0.$$

2) Let M be a differentiable manifold with a foliate structure whose sheets are real m -dimensional manifolds. This

structure is represented by a covering $\mathcal{U} = \{U_\alpha\}$, where U_α is a domain of the local coordinates

$$(x_\alpha, y_\alpha) = (x_\alpha^1, \dots, x_\alpha^m, y_\alpha^1, \dots, y_\alpha^{n-m}), \quad n > m,$$

and the transition functions for these coordinates have the form

$$(2.8) \quad \begin{cases} x_\alpha = f_{\alpha\beta}(x_\beta, y_\beta) \\ y_\alpha = g_{\alpha\beta}(y_\beta), \end{cases}$$

where $f_{\alpha\beta}$ is differentiable in x_β, y_β , the jacobian matrix

$$\partial(x_\alpha)/\partial(x_\beta)$$

is non-singular, and $g_{\alpha\beta}$ is differentiable in y_β and the jacobian matrix $\partial(y_\alpha)/\partial(y_\beta)$ is non-singular. Then M is an \mathcal{G}^1 -manifold, where \mathcal{G}^1 is represented, in terms of the coordinates (x_α, y_α) , by the equations

$$(2.9) \quad \frac{\partial f}{\partial x_\alpha^j} = 0, \quad j = 1, 2, \dots, m,$$

for the real-valued function f . It is now convenient to write $x^{m+1} = y^1, \dots, x^n = y^{n-m}$. The equations (2.9) (which remain unchanged) imply that an element $\sigma^{\mu+1}$ of $\Sigma^{\mu+1} = \oplus_i \Sigma^{\mu+1, i}$ has the components σ_p , where $\sigma_p = 0$ unless $p = (p_1, \dots, p_n)$ where $p_1 = 0, \dots, p_m = 0$. In this case $J^{0, i}$ is composed of the pairs $u = (\sigma, \xi)$, where σ is a (real-valued) differential form of degree i and ξ is locally equal to a (real-valued) differential form of degree $i + 1$ which belongs to the ideal generated by dx^{m+1}, \dots, dx^n . The sequence (2.7) is exact for $\mu \geq 0$ (see [4], [9(a)]).

3) (Cauchy-Riemann equations). Let M be a complex analytic manifold of (complex) dimension m , and let

$$z = (z^1, \dots, z^j, \dots, z^m)$$

be a local holomorphic coordinate on M . Write

$$z^j = x^{2j-1} + \sqrt{-1} x^{2j},$$

$j = 1, 2, \dots, m$, where $x = (x^1, \dots, x^j, \dots, x^n)$, $n = 2m$, and define

$$\frac{\partial}{\partial z^j} = \frac{\partial}{\partial x^j} = \frac{1}{2} \left(\frac{\partial}{\partial x^{2j-1}} + \sqrt{-1} \frac{\partial}{\partial x^{2j}} \right), \quad j = 1, 2, \dots, m.$$

The equations

$$(2.10) \quad \frac{\partial f}{\partial z^j} = 0, \quad j = 1, 2, \dots, m,$$

for the complex-valued function f , have as solutions the functions holomorphic in $z = (z^1, \dots, z^j, \dots, z^n)$. Introduce the self-conjugate coordinate

$$(z, \bar{z}) = (z^1, \dots, z^j, \dots, z^m, z^{\bar{j}}, \dots, z^{\bar{j}}, \dots, z^{\bar{m}})$$

where $z^{\bar{j}} = \bar{z}^j$, and write

$$r = p + \bar{p} = (p^1, \dots, p^j, \dots, p^n, p^{\bar{1}}, \dots, p^{\bar{j}}, \dots, p^{\bar{n}})$$

where p^j and $p^{\bar{j}}$ are non-negative integers. An element $\sigma^{\mu+1}$ of $\Sigma^{\mu+1} = \bigoplus_i \Sigma^{\mu+1, i}$ has the components $\sigma_{p+\bar{p}}$, where $\sigma_{p+\bar{p}} = 0$ unless $\bar{p} = 0$. In this case $J^{0, i}$ is the sheaf of germs of pairs $u = (\sigma, \xi)$, where σ is a (complex-valued) differential form of degree i and ξ is a (complex-valued) differential form of degree $i+1$ which belongs to the ideal generated over the (differentiable) functions by $dz^1, \dots, dz^j, \dots, dz^n$. The sequence (2.7) is exact for $\mu \geq 0$ (see [4], [9(a)]). Finally, let $A^{0, i}$ denote the sheaf over M of germs of (complex-valued) differential forms of type $(0, i)$, and let $\pi: J^{0, i} \rightarrow A^{0, i}$ be the projection sending $u = (\sigma, \xi)$ into the component of σ of type $(0, i)$. The differential operator D on J^0 splits into the sum of two operators D', D'' , where

$$D'(\sigma, \xi) = (\partial\sigma - \xi, -\partial\xi), \quad D''(\sigma, \xi) = (\bar{\partial}\sigma, -\bar{\partial}\xi)$$

and $d = \bar{\partial} + \partial$ is the usual splitting of the exterior differential operator d into operators $\partial, \bar{\partial}$ of types $(1,0), (0,1)$, respectively. The following diagram is exact and commutative:

$$\begin{array}{ccccccccccc} 0 & \rightarrow & \Theta & \rightarrow & J^{0,0} & \xrightarrow{D} & J^{0,1} & \xrightarrow{D} & J^{0,2} & \xrightarrow{D} & \dots & \rightarrow & J^{0,m} & \xrightarrow{D} & \dots & \xrightarrow{D} & J^{0,n} & \rightarrow & 0 \\ & & \parallel & & \downarrow \pi & & \downarrow \pi & & \downarrow \pi & & & & \downarrow \pi & & & & & & & \\ 0 & \rightarrow & \Theta & \rightarrow & A^{0,0} & \xrightarrow{\bar{\partial}} & A^{0,1} & \xrightarrow{\bar{\partial}} & A^{0,2} & \xrightarrow{\bar{\partial}} & \dots & \xrightarrow{\bar{\partial}} & A^{0,m} & \rightarrow & 0. \end{array}$$

The second line of this diagram is the classical Dolbeault resolution of the sheaf Θ of germs of holomorphic functions on M .

3. Elliptic systems of equations.

Let M be an \mathcal{G}^{μ_0} -manifold, choose a metric, let $S^*(M)$ be the corresponding unit cotangent sphere bundle and let $\pi: S^*(M) \rightarrow M$ be the projection. Denote by $\pi^*S_{\nu-1}^{\nu,i}$ the bundle over $S^*(M)$ which is induced from the bundle $S_{\nu-1}^{\nu,i}$ over M by the map π .

If $\mu \geq \mu_0$, we have the map

$$(3.1) \quad d\delta: \Sigma_{\mu}^{\mu+1,0} \rightarrow \Sigma_{\mu-1}^{\mu,2}$$

where $d\delta = d \cdot \delta$ is the composition of formal and actual exterior differentiation and $d\delta = -\delta d$. The symbol $s(d\delta)$ of the differential operator $d\delta$ then defines a homomorphism of vector bundles, namely

$$(3.2) \quad s(d\delta): \pi^*S_{\mu}^{\mu+1,0} \rightarrow \pi^*S_{\mu-1}^{\mu,2}.$$

The map $s(d\delta)$ is described in terms of a local coordinate as follows. Let σ be a vector belonging to the fibre of $\pi^*S_{\mu}^{\mu+1,0}$, and let the point of $S^*(M)$ over which σ lies be (x, ξ) , where $x = (x^1, \dots, x^k, \dots, x^n)$ and

$$\xi = \sum_{k=1}^n \xi_k dx^k.$$

Denote by $\delta_k \sigma$ the vector of $\pi^*S_{\mu-1}^{\mu,0}$ lying over the same point of $S^*(M)$, which has the components $(\delta_k \sigma)_p = \{\sigma_{p+1k} | 1 < j < m\}$. Then the map (3.2) sends σ into $s(d\delta)\sigma$ where

$$(3.3) \quad s(d\delta)\sigma = \sqrt{-1} \sum_{j < k} (\xi_j \delta_k \sigma - \xi_k \delta_j \sigma) dx^j \wedge dx^k.$$

The following definition of ellipticity is a natural one in the present context:

DEFINITION 3.1. — *We say that \mathcal{G}^{μ_0} is elliptic if and only if the map (3.2) is injective for $\mu \geq \mu_1 = \mu_1(\mu_0, n)$.*

Examples. (1) (Cauchy-Riemann equations). — Let M be a complex analytic manifold of (complex) dimension m , and let \mathcal{G}^1 be the system of Cauchy-Riemann equations (2.10). We

suppose that $\mu \geq 0$. Let σ be a vector of $\pi^*S_{0,\mu}^{\mu+1,0}$ lying over the point (z, ξ) of the real unit cotangent sphere, where

$$z = (z^1, \dots, z^k, \dots, z^m),$$

$$\xi = \sum_{k=1}^n (\xi_k dz^k + \xi_{\bar{k}} dz^{\bar{k}}),$$

and $\xi_{\bar{k}} = \bar{\xi}_k$, $dz^{\bar{k}} = d\bar{z}^k$. Then (see Example (3), § 2) we have $\delta_{\bar{k}}\sigma = 0$, and hence

$$s(d\delta)\sigma = \sqrt{-1} \left\{ \sum_{j < k} (\xi_j \delta_k \sigma - \xi_k \delta_j \sigma) dz^j \wedge dz^k - \sum_{j,k} \xi_{\bar{k}} \delta_j \sigma dz^j \wedge dz^{\bar{k}} \right\}.$$

Therefore, the vanishing of $s(d\delta)\sigma$ implies that

$$\xi_{\bar{k}} \delta_j \sigma = 0, \quad j, k = 1, 2, \dots, m.$$

Since $\xi_{\bar{k}} \neq 0$ for one value of k , at least, we conclude that $\delta_j \sigma = 0$, $j = 1, 2, \dots, m$, i.e., $\sigma = 0$. Thus the map (3.2) is injective and hence the Cauchy-Riemann equations are elliptic.

(2) Let M be a subdomain of euclidean n -space with coordinates $x = (x^1, \dots, x^k, \dots, x^n)$, and let \mathcal{G}^2 be represented by the usual laplacian

$$\sum_{k=1}^n \left(\frac{\partial}{\partial x^k} \right)^2 f = 0.$$

We suppose that $\mu \geq 1$, and let σ be a vector of $\pi^*S_{0,\mu}^{\mu+1,0}$ over (x, ξ) . Let $\delta_k^2 \sigma = \delta_k(\delta_k \sigma)$ be the element with the components $(\delta_k^2 \sigma)_p = \sigma_{p+2k}$, where $2_k = 1_k + 1_k$. Then

$$(3.4) \quad \sum_{k=1}^n \delta_k^2 \sigma = 0.$$

Now suppose that $s(d\delta)\sigma = 0$, i.e.,

$$(3.5) \quad \xi_j \delta_k \sigma = \xi_k \delta_j \sigma, \quad j, k = 1, 2, \dots, n.$$

Operating on (3.5) with $\xi_k \delta_j$, we obtain

$$\xi_j \xi_k \delta_j \delta_k \sigma = \xi_k^2 \delta_j^2 \sigma$$

and hence, by symmetry,

$$\xi_j^2 \delta_k^2 \sigma = \xi_k^2 \delta_j^2 \sigma.$$

Summing on k from 1 to n , we have by (3.4)

$$(\sum_k \xi_k^2) \cdot \delta_j^2 \sigma = \xi_j^2 \cdot \sum_k \delta_k^2 \sigma = 0$$

and, since $\sum_k \xi_k^2 \neq 0$, we conclude that $\delta_j^2 \sigma = 0$, $j = 1, 2, \dots, n$. Applying δ_j to (3.5), we therefore have

$$(3.6) \quad \xi_j \delta_j \delta_k \sigma = \xi_k \cdot \delta_j^2 \sigma = 0.$$

Choose j such that $\xi_j \neq 0$. Then we infer from (3.5) that $\delta_k \sigma = 0$ if $\xi_k = 0$, and hence $\delta_j \delta_k \sigma = 0$ if either ξ_j or ξ_k is equal to zero. If $\xi_j \neq 0$, we infer from (3.6) that $\delta_j \delta_k \sigma = 0$. Thus $\delta_j \delta_k \sigma = 0$ for all j, k , i.e., $\sigma = 0$ and the map (3.2) is injective. We have thus verified that the laplacian is elliptic!

Let D^* be the (formal) adjoint operator defined in terms of a metric, and let $\square = DD^* + D^*D$ be the corresponding laplacian. We have the following theorem (see [9(b)]), which justifies Definition 4.1.

THEOREM 3.1. — *The system \mathcal{G}^{μ_0} is elliptic (in the sense of Definition 4.1) if and only if the laplacian $\square = DD^* + D^*D$ is an elliptic operator (in the « interior » sense) on the sections of $J^\mu = \oplus_i J^{\mu_i}$ for $\mu \geq \mu_1$.*

4. Neumann decompositions.

We say that a manifold M is *finite* if it is a subdomain of a differentiable manifold M' where M has compact closure in M' and a boundary M which is a regularly imbedded differentiable submanifold of M' of codimension 1. We say that M is a finite \mathcal{G}^{μ_0} -manifold if it is a finite subdomain of an \mathcal{G}^{μ_0} -manifold M' .

Suppose that \mathcal{G}^{μ_0} is elliptic, and let M be a finite \mathcal{G}^{μ_0} -manifold, i.e., M is a finite subdomain of an \mathcal{G}^{μ_0} -manifold M' . Let μ be a fixed integer and suppose that $\mu \geq \mu_1$, where $\mu_1 = \mu_1(\mu_0, n)$. We have over M' the sheaf $J^\mu = \oplus_i J^{\mu_i}$, and we denote by $A = \oplus_i A^i$ the restriction to M of the space of sections over M' of $J^\mu = \oplus_i J^{\mu_i}$. Thus A is the space of sections of J^μ over M which are differentiable up to and including the boundary of M .

Choose a metric on M' , which fits the structure as closely as possible, denote by (u, v) the scalar product, defined in

terms of the metric, of the elements u, ν of \mathbf{A} , and let D^* be the formal adjoint of the differential operator D , i.e., if u has compact support on M , D^* is the operator satisfying $(Du, \nu) = (u, D^*\nu)$ for all elements ν of \mathbf{A} . Let $\mathbf{N} = \oplus_i \mathbf{N}_i$, (Neumann space) be the (graded) subspace of \mathbf{A} composed of the forms u which satisfy the following pair of boundary conditions :

$$(4.1) \quad \begin{cases} (D^*u, \nu) = (u, D\nu), \\ (D^*Du, \nu) = (Du, D\nu), \end{cases}$$

for all ν of \mathbf{A} . Denote by $\mathbf{H} = \oplus_i \mathbf{H}_i$ the (graded) subspace of \mathbf{N} composed of the forms which are annihilated by the laplacian $DD^* + D^*D$ or, equivalently (in view of (4.1)), \mathbf{H} is the subspace of \mathbf{N} composed of the elements u satisfying $Du = 0, D^*u = 0$. If \mathbf{H} is finite dimensional, we denote by $H: \mathbf{A} \rightarrow \mathbf{H}$ the orthogonal projection of \mathbf{A} onto \mathbf{H} . If \mathbf{H} is infinite dimensional, let $\overline{\mathbf{A}}, \overline{\mathbf{H}}$ be the completions of \mathbf{A}, \mathbf{H} , respectively, and let $\overline{H}: \overline{\mathbf{A}} \rightarrow \overline{\mathbf{H}}$ be the orthogonal projection of $\overline{\mathbf{A}}$ onto $\overline{\mathbf{H}}$.

DEFINITION 4.1. — *We say that the Neumann problem is solvable for a finite \mathcal{G}^k -manifold M if the following assertions are true.*

I) The restriction of \overline{H} to \mathbf{A} is a projection

$$(4.2) \quad H: \mathbf{A} \rightarrow \mathbf{H}$$

of \mathbf{A} onto \mathbf{H} .

II) The Neumann operator N exists, i.e., there is the surjective map, of degree 0,

$$(4.3) \quad N: \mathbf{A} \rightarrow \mathbf{N}$$

which is characterized by the following conditions :

i) $HN = NH = 0$.

ii) $DN = ND$.

iii) (Neumann decomposition). For $u \in \mathbf{A}$, we have the orthogonal decomposition

$$(4.4) \quad u = DD^*Nu + D^*DNu + Hu,$$

which, in view of (ii), can be written in the form

$$(4.5) \quad u = D(D^*N)u + (D^*N)Du + Hu.$$

The Neumann decomposition therefore has the form of a cochain homotopy. In fact, let $Z(\mathbf{A}) = \oplus_i Z(\mathbf{A}^i)$ be the kernel of the map $D : \mathbf{A} \rightarrow \mathbf{A}$; then

$$(4.6) \quad Z(\mathbf{A})/D(\mathbf{A}) = Z(\mathbf{A}^0) \oplus Z(\mathbf{A}^1)/D(\mathbf{A}^0) \oplus \dots \oplus Z(\mathbf{A}^n)/D(\mathbf{A}^{n-1})$$

is the D-cohomology of \mathbf{A} , where $Z(\mathbf{A}^0)$ is the space of sections of Θ over M which are differentiable up to and including the boundary of M . The Neumann decomposition (if it exists) provides a representation of the D-cohomology of \mathbf{A} by the space $\mathbf{H} = \oplus_i \mathbf{H}^i$ of harmonic forms, i.e., it gives a linear isomorphism (of graded vector spaces)

$$(4.7) \quad \mathbf{H} \cong Z(\mathbf{A})/D(\mathbf{A}).$$

The solvability of the Neumann problem for a given finite manifold M depends only on \mathcal{G}^{ν_0} , i.e., it is independent of the choice of metric. We denote by $\mathcal{C}(\mathcal{G}^{\nu_0})$ the set of finite \mathcal{G}^{ν_0} -manifolds for which the Neumann problem is solvable. Our program is to solve the following problem:

Problem. — Determine $\mathcal{C} = \mathcal{C}(\mathcal{G}^{\nu_0})$ for each elliptic \mathcal{G}^{ν_0} .

Examples 1. — \mathcal{G}^1 is the system of equations $df = 0$ (see Example (1), § 2). Then $\mathcal{C}(\mathcal{G}^1)$ is the set of all finite manifolds (see Duff and Spencer [3], Conner [2], Morrey [6]).

2) \mathcal{G}^1 is the class of the Cauchy-Riemann equations in m variables, i.e., the system of equations in Example (3), § 2. Then $\mathcal{C}(\mathcal{G}^1)$ is the class of all strongly pseudoconvex (finite) manifolds (see Kohn [5]).

3) Let $x = (x^1, \dots, x^j, \dots, x^m)$, $z = (z^1, \dots, z^k, \dots, z^n)$, where x^j is real, z^k complex, and write $\bar{z} = (\bar{z}^1, \dots, \bar{z}^k, \dots, \bar{z}^n)$, where $\bar{z}^k = \overline{z^k}$ is the complex conjugate of z^k . Let \mathcal{G}^1 be the system of equations

$$(4.8) \quad \begin{cases} \frac{\partial f}{\partial x^j} = 0, & j = 1, 2, \dots, m, \\ \frac{\partial f}{\partial z^k} = 0, & k = 1, 2, \dots, n, \end{cases}$$

for the complex-valued function f (compare (2.10)).

Now let M' be a differentiable manifold with a foliate structure whose sheets are real m -dimensional manifolds

with a complex analytic structure transverse to them. This mixed structure is represented by a locally finite covering $\mathcal{V} = \{V_\alpha\}$, where V_α is an open set covered by the coordinates (x_α, z_α) , and the transition functions have the form

$$\begin{cases} x_\alpha = f_{\alpha\beta}(x_\beta, z_\beta, \bar{z}_\beta), \\ z_\alpha = g_{\alpha\beta}(z_\beta), \end{cases}$$

where $f_{\alpha\beta}$ is differentiable in $x_\beta, z_\beta, \bar{z}_\beta$, the jacobian matrix $\partial(x_\alpha)/\partial(x_\beta) = \partial(f_{\alpha\beta})/\partial(x_\beta)$ is non-singular, and $g_{\alpha\beta}$ is a biholomorphic transformation. Through each point of M' there passes a real m -dimensional sheet, which is defined in V_α by setting the z_α^k equal to (complex) constants, and the local differentiable coordinate along this sheet is

$$x_\alpha = (x_\alpha^1, \dots, x_\alpha^j, \dots, x_\alpha^m).$$

Let $T(M')$ be the tangent bundle of M' , and denote by $T_s(M')$ the sub-bundle of $T(M')$ of tangent vectors along the sheets. Then the restriction $T_s(V_\alpha)$ of $T_s(M')$ to V_α is covered by the coordinates $(x_\alpha, z_\alpha, \partial/\partial x_\alpha)$, where

$$\partial/\partial x_\alpha = (\partial/\partial x_\alpha^1, \dots, \partial/\partial x_\alpha^j, \dots, \partial/\partial x_\alpha^m).$$

The equations (4.8) are defined on M' , i.e., M' is an \mathcal{G}^1 -manifold where \mathcal{G}^1 is represented, in terms of local coordinates (x_α, z_α) , by a system of the form (4.8). Let M be a finite subdomain of M' , and denote by ∂M the boundary of M . The boundary ∂M is tangent to a sheet, at the point x_0 , if $T_s(M')|_{x_0}$ is contained in the tangent space of ∂M at x_0 . We denote by $\partial_s M$ the (closed) set of boundary points of M at which ∂M is tangent to the sheets. The works of Ash [1] and Kohn [5], together with an observation of L. Nirenberg, yield the following result:

$\mathcal{C}(\mathcal{G}^1)$ is the class of finite \mathcal{G}^1 -manifolds M such that, at each point of $\partial_s M$, the boundary is strongly pseudoconvex in the sense of the complex structure transverse to the sheet and strongly convex along the sheet through the point.

The method of Kohn [5], in the form applied by Ash [1], can be generalized to establish the following result:

PROPOSITION 4.1. — *Suppose that \mathcal{G}^s is elliptic. Then $\mathcal{C}(\mathcal{G}^{s_0})$ contains all sufficiently small, sufficiently convex (finite)*

subdomains of euclidean n -space and, for these domains, $H^i = 0$ for $i > 0$.

Suppose that \mathcal{G}^{μ_0} is elliptic, and let M be an \mathcal{G}^{μ_0} -manifold of dimension n . The exactness of the sequence (2.7), for $\mu \geq \mu_1$, follows at once from Proposition 4.1. In fact, suppose that μ is a fixed integer, $\mu \geq \mu_1$, and let \tilde{u} be a germ of $J^{\mu, i}$, where $i > 0$, which satisfies $D\tilde{u} = 0$. Then \tilde{u} is represented by a section u of $J^{\mu, i}$, which is defined over a neighborhood containing the closure of a sufficiently small coordinate ball and satisfies $Du = 0$. By Proposition 4.1, the Neumann problem is solvable on the coordinate ball and $Hu = 0$. Hence, by formula (4.5), $u = D\varpi$ where $\varpi = D^*Nu$, i.e., the Poincaré lemma for D is valid and the sequence (2.7) is exact.

Now let $L(J^\mu) = \oplus_i L(J^{\mu, i})$ be the graded vector space of sections of J^μ over M , and let $Z(J^\mu) = \oplus_i Z(J^{\mu, i})$ be the kernel of the map $D: L(J^\mu) \rightarrow L(J^\mu)$. Then

$$(4.9) \quad \begin{aligned} Z(J^\mu)/DL(J^\mu) \\ = Z(J^{\mu, 0}) \oplus Z(J^{\mu, 1})/DL(J^{\mu, 0}) \oplus \dots \oplus Z(J^{\mu, n})/DL(J^{\mu, n-1}) \end{aligned}$$

is the (graded) D-cohomology of sections of J^μ over M . Moreover, let

$$(4.10) \quad H^*(M, \Theta) = H^0(M, \Theta) \oplus H^1(M, \Theta) \oplus \dots \oplus H^n(M, \Theta)$$

be the (graded) cohomology of M with values in the sheaf Θ of germs of solutions of the system \mathcal{G}^{μ_0} of linear partial differential equations on M .

We denote by $\mu_2 = \mu_2(\mathcal{G}^{\mu_0}, n)$ the smallest positive integer for which the sequence (2.6) is exact for $\mu \geq \mu_2$ and

$$0 \leq i \leq n - 1,$$

and we denote by $\nu_1 = \nu_1(\mathcal{G}^{\mu_0}, n)$ the larger of the two integers μ_0, μ_2 .

The following theorem is an immediate consequence of Proposition 4.1.

THEOREM 4.1. (Theorem of de Rham for elliptic systems). — *Let M be an \mathcal{G}^{μ_0} -manifold of dimension n , and suppose that*

\mathcal{G}^{μ} is elliptic. Then, for $\mu \geq \nu_1 - 1$, a fortiori for $\mu \geq \mu_1 - 1$, the sequence

$$0 \rightarrow \Theta \xrightarrow{i} J^{\mu,0} \xrightarrow{D} J^{\mu,1} \xrightarrow{D} \dots \xrightarrow{D} J^{\mu,n} \rightarrow 0$$

is an exact sequence of fine sheaves, and we have the isomorphism of graded vector spaces

$$(4.11) \quad H^*(M, \Theta) \cong Z(J^{\mu})/DL(J^{\mu})$$

which is derived from the exact sequence of sheaves in a canonical manner.

In fact, suppose that $\mu \geq \nu_1 - 1$, $i > 0$, and let u be a local section of $J^{\mu,i}$ satisfying $Du = 0$. If $\mu < \mu_1$, then u can be lifted up to a section ν of $J^{\nu,i}$ satisfying $D\nu = 0$ (see [9(a)], § 5). By Proposition 4.1, $\nu = D(D^*N\nu)$, and it follows that $u = D\omega$, where ω is the projection of $D^*N\nu$ into a (local) section of $J^{\mu,i-1}$.

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