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ON THE AXIOMATIC OF HARMONIC FUNCTIONS I

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The aim of the present paper is to present some remarks on Brelot's axiomatic of harmonic functions [2] and to show that any space, which locally has a countable basis and on which there exists a positive superharmonic function, possesses a countable basis.

1. Let X be a locally compact connected space and \mathcal{H} a sheaf on X , of real vector spaces of continuous functions ⁽¹⁾ called harmonic functions. An open set $U \subset X$ is called regular if it is non-empty, relatively compact and, if for any continuous function f on the boundary ∂U of U , there exists a unique function on \bar{U} equal to f on ∂U and harmonic on U , non-negative if f is non-negative. The restriction of this function on U will be denoted by H_f^U . For any $x \in U$ the functional

$$f \rightarrow H_f^U(x)$$

is linear and non-negative on the real vector space of continuous functions on ∂U . There exists therefore a measure $\omega_x^U = \omega_x$ on ∂U , called harmonic measure, such that

$$H_f^U(x) = \int f d\omega_x^U$$

for any continuous function f on ∂U .

We assume that \mathcal{H} satisfies the following axioms.

A₁. *The regular domains form a basis of X .*

A₂. *The limit of any increasing sequence of harmonic functions on a domain is either harmonic or identically infinite.*

⁽¹⁾ The term « function » means, in this paper, « real finite function ».

If u is a non-negative harmonic function on a domain U , then it follows from A_2 , considering the sequence $\{nu\}$, that u is either positive or identically zero.

THEOREM 1. — *Let \mathcal{U} be an increasingly directed set of harmonic functions on a domain U ; the least upper bound of \mathcal{U} is either harmonic or identically infinite.*

We shall prove the assertion using an idea from R. NEVAN-LINNA (*Uniformisierung*, Springer Verlag, 1953).

Let us suppose the least upper bound of \mathcal{U} is not identically infinite and let x be a point of U at which it is finite. There exists an increasing sequence $\{u_n\}$ of \mathcal{U} such that

$$\lim_{n \rightarrow \infty} u_n(x) = \sup_{u \in \mathcal{U}} u(x).$$

We denote

$$u_0 = \lim_{n \rightarrow \infty} u_n.$$

Then u_0 is harmonic by A_2 . Let y be a point of U different from x and $\{\nu_n\}$ be an increasing sequence of \mathcal{U} such that $u_n \leq \nu_n$ and

$$\lim_{n \rightarrow \infty} \nu_n(y) = \sup_{u \in \mathcal{U}} u(y).$$

We denote

$$\nu_0 = \lim_{n \rightarrow \infty} \nu_n.$$

Since ν_0 is finite at x it is harmonic. Obviously $u_0 \leq \nu_0$. Since u_0 and ν_0 are equal at x they coincide everywhere. It follows that

$$u_0(y) = \sup_{u \in \mathcal{U}} u(y).$$

y being arbitrary, u_0 is the least upper bound of \mathcal{U} .

The theorem shows that axiom A_2 is equivalent to axiom 3 [2].

2. A lower semi-continuous numerical function ⁽²⁾ s on an open set V , which does not take the value $-\infty$, is called hyperharmonic if for any regular domain U , $\bar{U} \subset V$, and $x \in U$

$$s(x) \geq \int s d\omega_x^U.$$

⁽²⁾ « Numerical function » will mean a function whose values are real numbers or $\pm \infty$.

A hyperharmonic function on the open set V is called superharmonic if it is not identically infinite on any component of V . A function s is called hypoharmonic (resp. subharmonic) if $-s$ is hyperharmonic (resp. superharmonic). A non-negative hyperharmonic function is called a potential if its greatest harmonic minorant is zero. A set A is called polar if for any $x \in X$ there exists a positive superharmonic function on a neighbourhood U of x infinite on $U \cap A$.

THEOREM 2. — *Let $U \subset X$ be an open non-compact ⁽³⁾ set on which there exists a positive superharmonic function. Then any superharmonic function on U non-negative outside a compact subset of U is non-negative.*

Let s_0 be a positive superharmonic function on U and s be a superharmonic function on U non-negative outside a compact set. For any positive real number α we denote

$$K_\alpha = \{x \in U \mid s(x) + \alpha s_0(x) \leq 0\}.$$

K_α is a compact set and we have $K_\alpha \subset K_\beta$ for $\alpha \geq \beta$ and

$$K_\alpha = \bigcap_{\alpha > \beta} K_\beta.$$

Suppose s negative at a point. Then since $\inf s > -\infty$ there exists a real number $\alpha > 0$ such that $K_\alpha \neq \emptyset$ and $K_\beta = \emptyset$ for $\beta > \alpha$. The function $s + \alpha s_0$ is superharmonic and non-negative. Let x be a point of K_α and V the component of U which contains x . Since $s + \alpha s_0$ vanishes at x it vanishes on the whole V ([2] Theorem 3 (i)). It follows $V \subset K_\alpha$ which is a contradiction since V is non-compact. Hence s is non-negative.

COROLLARY 1. — *Let $U \subset X$ be an open non-compact set and s_0 a superharmonic function on U such that*

$$\inf s_0 > 0.$$

Then any superharmonic function s on U for which

$$\liminf_{x \rightarrow a_U} s(x) \geq 0,$$

where a_U is the Alexandroff point of U , is non-negative.

⁽³⁾ This means either X non-compact or $U \neq X$.

Indeed for any $\varepsilon > 0$ the function $s + \varepsilon s_0$ is non-negative outside a compact set and therefore non-negative on U .

THEOREM 3. — *Let f be a lower semi-continuous numerical function on X , which does not take the value $-\infty$. The greatest lower bound of the set of hyperharmonic functions which dominate f is hyperharmonic, continuous at any point x where f is continuous; if, moreover, it is superharmonic and different at x from $f(x)$ then it is harmonic on a neighbourhood of x ⁽⁴⁾.*

Let \mathcal{G} denote the set of hyperharmonic functions which dominate f , s_0 be its greatest lower bound, U a regular domain and $y \in U$. We have

$$s_0(y) = \inf_{s \in \mathcal{G}} s(y) \geq \inf_{s \in \mathcal{G}} \int s d\omega_y^U \geq \overline{\int s_0 d\omega_y^U}.$$

From this relation it follows that the regularised function \hat{s}_0 is hyperharmonic ([2] Theorem 7). Since f is lower semi-continuous $\hat{s}_0 \in \mathcal{G}$, $\hat{s}_0 = s_0$.

Let x be a point at which f is continuous and $s \in \mathcal{G}$ such that

$$s(x) \neq f(x).$$

There exists a harmonic functions u , defined on a neighbourhood of x , for which

$$f(x) < u(x) < s(x).$$

Let U be a regular neighbourhood of x , where these inequalities still hold. For any $y \in U$ we have

$$\int s d\omega_y^U \geq \int u d\omega_y^U = u(y) > f(y)$$

and therefore the balayaged function of s relative to

$$X - U, \quad \hat{R}_s^{X-U},$$

belongs to \mathcal{G} . Herefrom it follows that if s_0 is superharmonic and $s_0(x) \neq f(x)$, then s_0 is harmonic on a neighbourhood of x . Further we get

$$\limsup_{y \rightarrow x} s_0(y) \leq \limsup_{y \rightarrow x} \hat{R}_s^{X-U}(y) = \hat{R}_s^{X-U}(y) \leq s(x).$$

⁽⁴⁾ This theorem was proved in the classical case by M. BRELOT, *Journ. de Math. Pures et Appl.*, 24, 1945, 1-32.

Let U' be a domain, K a compact set in U' and f' the function defined on U' equal to f on K and equal to s_0 on $U' - K$. We denote by \mathcal{G}' the set of hyperharmonic functions on U' which dominate f' and by s'_0 its greatest lower bound. Obviously $s'_0 \leq s_0$ on U' and $s'_0 = s_0$ on $U' - K$. The function on X equal to s_0 on $X - K$ and equal to s'_0 on U' is hyperharmonic ([2] Theorem 4) and dominates f . Hence $s_0 = s'_0$ on U' .

We take U' as being a regular neighbourhood of x and K a compact neighbourhood of x . For any $\epsilon > 0$ we have

$$s'_0 + \epsilon H_1^{U'} \in \mathcal{G}'$$

and

$$s'_0(x) + \epsilon H_1^{U'}(x) \neq f'(x).$$

From the preceding considerations we have, since f' is continuous at x ,

$$\limsup_{y \rightarrow x} s_0(y) = \limsup_{y \rightarrow x} s'_0(y) \leq s'_0(x) + \epsilon H_1^{U'}(x) = s_0(x) + \epsilon H_1^{U'}(x).$$

ϵ being arbitrary s is continuous at x .

COROLLARY 2. — *A superharmonic function which dominates a continuous function is equal to the least upper bound of the set of its continuous finite superharmonic minorants.*

COROLLARY 3([2] Proposition 12). — *Let F be a closed set with a non-empty interior. If there exists a potential on X then there exists a continuous positive potential on X harmonic on $X - F$.*

It is sufficient to take f as being a continuous non-negative function, $f \not\equiv 0$, whose carrier lies in F .

3. We shall denote by \mathfrak{P} (resp. \mathfrak{H}) the class of spaces (X, \mathcal{H}) for which there exists at least a positive potential (resp. a positive harmonic function) on X . The type \mathfrak{P} (resp. \mathfrak{H}) of X is not altered by the multiplication of all the functions of \mathcal{H} by a positive continuous function. An open set $U \subset X$ is said to be of type \mathfrak{P} (resp. \mathfrak{H}) if any component of U belongs to \mathfrak{P} (resp. \mathfrak{H}). The spaces of type $\mathfrak{P} \cup \mathfrak{H}$ (resp. \mathfrak{P}) are exactly those on which there exists a positive superharmonic (resp. positive superharmonic non-harmonic) function. On a space of type $\mathfrak{H} - \mathfrak{P}$ any two positive superharmonic functions are

proportional. If $X \in \mathfrak{P}$ (resp. \mathfrak{H}) and U is a domain in X , then $U \in \mathfrak{P}$ (resp. \mathfrak{H}). This is trivial for \mathfrak{H} and for \mathfrak{P} it results from the fact that there exists a positive superharmonic function on X which is not harmonic on U . If $U \subset X$, $U \in \mathfrak{P}$ and $X - U$ is polar, then X also belongs to \mathfrak{P} since any potential can be extended to a superharmonic function on X , ([2] page 125) which is obviously a potential. This result does not hold if we take \mathfrak{H} instead of \mathfrak{P} .

Let X_1 denote the real axis and X_2 the unit circumference in the complex plane and let \mathcal{H}_1 (resp. \mathcal{H}_2) be the sheaf of solutions of the equation $u'' + \alpha u = 0$ on X_1 (resp. X_2), where α is a real number. If α is positive, no positive superharmonic function exists on the spaces (X_1, \mathcal{H}_1) , (X_2, \mathcal{H}_2) . If α is positive and irrational, the only harmonic function on X_2 is identically zero. For $\alpha = 0$ we obtain examples of spaces of the type $\mathfrak{H} - \mathfrak{P}$. For $\alpha < 0$, X_1 belongs to $\mathfrak{P} \cap \mathfrak{H}$ and X_2 belongs to $\mathfrak{P} - \mathfrak{H}$. We do not know if there exists non-compact spaces of the type $\mathfrak{P} - \mathfrak{H}$.

LEMMA 1 ⁽⁵⁾. — *If \mathcal{H} satisfies axiom A_1 the axiom A_2 is equivalent to the following assertion. Let U be a domain, $V \subset U$ an open set, $K \subset V$ a compact set and $x \in U$. There exists a positive number $\alpha = \alpha(U, V, K, x)$ such that for any non-negative superharmonic function s on U harmonic on V*

$$\sup_{y \in K} s(y) \leq \alpha s(x).$$

Suppose A_2 fulfilled. If α does not exist there exists for any natural number n a non-negative superharmonic function s_n on U , harmonic on V and such that

$$\sup_{y \in K} s_n(y) > n, \quad s_n(x) < \frac{1}{n^2}.$$

This leads to a contradiction since the function

$$\sum_{n=1}^{\infty} s_n$$

is superharmonic on U and infinite on a component of V .

⁽⁵⁾ This lemma was inspired by a similar result of R.-M. HERVÉ ([3] n° 2; propriété 7).

Suppose now the existence of an α asserted in the lemma, and let $\{u_n\}$ be an increasing sequence of harmonic functions on U . If there exists a point $x \in U$ for which $\{u_n(x)\}$ is convergent, then $\{u_n(x) - u_{n-1}(x)\}$ converges to zero. Hence

$$\{u_n - u_{n-1}\}$$

converges to zero uniformly on any compact set of U . It follows immediately that $\lim_{n \rightarrow \infty} u_n$ is harmonic.

The axiom A_2 can be strengthened by requiring

$$\lim_{K \rightarrow x} \alpha(U, U, K, x) = 1.$$

This assertion was called axiom 3' ([2], page 147). From this axiom it follows that the positive harmonic functions on U equal to 1 at a point of U form an equicontinuous set of functions.

LEMMA 2. — Let $x \in X$, I be an increasingly directed ordered set and for any $\iota \in I$ let U_ι be a domain on X containing x , V_ι be an open subset of U_ι and s_ι a non-negative superharmonic function on U_ι harmonic on V_ι equal to 1 at x . We suppose $U_\iota \subset U_\kappa$, $V_\iota \subset V_\kappa$ for any $\iota \leq \kappa$. Let \mathfrak{U} be an ultrafilter on I finer than the filter of sections of I ⁽⁶⁾. If \mathcal{H} satisfies the axiom 3' then s_ι converges uniformly along \mathfrak{U} on any compact subset of $\bigcup_{\iota \in I} V_\iota$ to a harmonic function.

Let K be a compact set in $\bigcup_{\iota \in I} V_\iota$ and $\kappa \in I$ such that $K \subset V_\kappa$. Since $\{s_\iota | \iota \geq \kappa\}$ is an equicontinuous family of functions on K , s_ι converges uniformly along \mathfrak{U} on K . Its limit is therefore harmonic on $\bigcup_{\iota \in I} V_\iota$.

THEOREM 4. — Let X be non-compact. If \mathcal{H} satisfies the axiom 3' and any relatively compact domain of X belongs to \mathfrak{S} then X belongs to \mathfrak{S} .

Let $x \in X$, I be the set of relatively compact domains containing x ordered by the inclusion relation and for any $\iota \in I$ denote $U_\iota = V_\iota = \iota$ and let s_ι be a positive harmonic

⁽⁶⁾ This is filter generated by the family of sets $\{\iota \in I | \iota \geq \kappa\}_{\kappa \in I}$.

function on U_i equal to 1 at x . By means of lemma 2 one can construct a positive harmonic function on X .

COROLLARY 4 [3] ⁽⁷⁾. — *If \mathcal{H} satisfies the axiom 3' and X belongs to \mathfrak{B} and is non-compact, then X belongs to \mathfrak{S} .*

THEOREM 5 [3] ⁽⁷⁾. — *If \mathcal{H} satisfies the axiom 3' and X belongs to \mathfrak{B} then there exists for any point $x \in X$ a positive potential on X harmonic on $X - \{x\}$.*

Let p be a positive potential on X and I be the set of compact neighbourhoods of x ordered by the inverse inclusion relation. For any $\iota \in I$ we denote

$$U_\iota = X, \quad V_\iota = X - \iota, \\ s_\iota = \frac{\hat{R}_p^\iota}{\hat{R}_p^\iota(x_0)},$$

where x_0 is a fixed point different from x . Let \mathfrak{U} be an ultrafilter on I finer than the filter of sections of I and for any $y \in X$

$$s(y) = \lim_{\iota, \mathfrak{U}} s_\iota(y).$$

By lemma 2 s is harmonic on $X - \{x\}$.

Let U be a regular neighbourhood of x and $y \in U$. Since by lemma 2 s_ι converges uniformly along \mathfrak{U} to u on ∂U we have

$$s(y) = \lim_{\iota, \mathfrak{U}} s_\iota(y) \geq \lim_{\iota, \mathfrak{U}} \int s_\iota d\omega_y^\iota = \int s d\omega_y^U.$$

The regularised function \hat{s} of s is therefore superharmonic. From the above uniform convergence we deduce the existence of a positive number α and a $\kappa \in I$ such that

$$s_\iota \leq \alpha p$$

on ∂U for any $\iota \geq \kappa$. It follows ([3] Lemma 3.1)

$$s_\iota \leq \alpha p, \quad s \leq \alpha p$$

on $X - U$. Hence \hat{s} is a potential. It cannot be harmonic on a neighbourhood of x since then it would be harmonic on X and therefore zero.

⁽⁷⁾ Théorème 16-1.

THEOREM 6. — *On a compact space of the type \mathfrak{B} any superharmonic function is a potential. Particularly any superharmonic function is non-negative.*

If s is a superharmonic function then $-\min(s, 0)$ is a subharmonic function. Since the space is compact and of type \mathfrak{B} it is dominated by a potential. Hence it vanishes and s is non-negative. The greatest harmonic minorant of s vanishes being dominated by a potential. s is therefore a potential.

REMARK. — *A space of the type $\mathfrak{B} \cap \mathfrak{H}$ is non-compact.*

THEOREM 7. — *Let $X \in \mathfrak{B} \cup \mathfrak{H}$ and U be a domain on X . If $X - U$ is non-polar then $U \in \mathfrak{B} \cap \mathfrak{H}$.*

Let s be a positive superharmonic function on X . Suppose its restriction on U is a potential. There exists then a positive superharmonic function s' on U such that

$$\lim_{U \ni x \rightarrow \partial U} s'(x) = \infty$$

([1] Lemma 1). If we extend s' to a function on X equal to $+\infty$ on $X - U$ we obtain a superharmonic function. This is a contradiction since $X - U$ is non-polar. Hence the restriction of s on U is not a potential and $U \in \mathfrak{H}$.

∂U is non-polar. This is obvious if $\partial U = X - U$. If

$$\partial U \neq X - U,$$

∂U is non-polar since $X - \partial U$ is non-connected. There exists therefore a point $x \in \partial U$ such that the intersection of any neighbourhood V of x with ∂U is non-polar in V . Let V be a regular domain which contains x and K be a compact non-polar set, $K \subset V \cap \partial U$. The reduced function $(R_K^x)_V$ of s relative to K , where the operation is made on V , does not vanish, it converges to zero at the boundary of V and is harmonic on $V - K$. The function s' on U equal to s on $U - V$ and equal to $s - (R_K^x)_V$ on $V \cap U$ is superharmonic and non-proportional to s . Hence $U \in \mathfrak{B}$.

COROLLARY 5. — *If $X \in \mathfrak{B} \cup \mathfrak{H}$ and U is an open non-connected set, then U is of the type $\mathfrak{B} \cap \mathfrak{H}$.*

Let V be a component of U . Since $X - V$ has interior points it is non-polar. V is therefore of the type $\mathfrak{B} \cap \mathfrak{H}$.

4. LEMMA 3. — *The sum of a sequence of potentials convergent at a point is a potential.*

Let $\{p_n\}$ be a sequence of potentials such that

$$s = \sum_{n=1}^{\infty} p_n$$

be finite at a point. Let u be a harmonic minorant of s . We shall prove inductively that

$$u \leq \sum_{n=m}^{\infty} p_n.$$

Suppose

$$u \leq \sum_{n=m}^{\infty} p_n.$$

Then $u - \sum_{n=m+1}^{\infty} p_n$ is a subharmonic minorant of p_m and therefore non-positive.

A sequence $\{U_n\}$ of relatively compact domains on X is called a pseudo-exhaustion of X if

$$\bar{U}_n \subset U_{n+1}$$

for any n and

$$X - \bigcup_{n=1}^{\infty} U_n$$

is polar.

THEOREM 8. — *Any space of the type $\mathfrak{B} \cup \mathfrak{S}$ possesses a pseudo-exhaustion.*

Let $K \subset X$ be a compact non-polar set such that $X - K$ contains only a finite number of components, let p be a positive potential on $X - K$ and \mathcal{G} be the set of functions $(\hat{R}_p^{X-U})_{X-K}$, where U is a relatively compact domain which contains K . The greatest lower bound of \mathcal{G} , being a non-negative harmonic minorant of p , is equal to zero. There exists therefore a sequence $\{U_n\}$ of relatively compact domains containing K , such that for any n $U_n \subset U_{n+1}$ and

$$\sum_{n=1}^{\infty} (\hat{R}_p^{X-U_n})_{X-K}$$

is a superharmonic function on $X - K$, infinite on $X - \bigcup_{n=1}^{\infty} U_n$.

THEOREM 9. — *If $X \in \mathfrak{P}$ and $\{U_n\}$ is a pseudo-exhaustion of X there exists a continuous potential on X , which is infinite exactly on $X - \bigcup_{n=1}^{\infty} U_n$.*

Let p be a continuous finite potential on X and, for any n , let f_n denote a continuous non-negative function on X equal to 0 on U_n , equal at most to p on $U_{n+1} - U_n$ and equal to p on $X - U_{n+1}$. The function

$$p_n = R_{f_n}^X$$

is a continuous finite potential, harmonic on U_n (Theorem 3) and $p_n \geq p_{n+1}$. Let u denote the limit of the sequence $\{p_n\}$.

The function u is locally bounded and harmonic on $\bigcup_{n=1}^{\infty} U_n$. Since $X - \bigcup_{n=1}^{\infty} U_n$ is polar there exists a harmonic function on X equal to u on $\bigcup_{n=1}^{\infty} U_n$. Being a harmonic minorant of p it vanishes. Hence u is equal to zero on $\bigcup_{n=1}^{\infty} U_n$. We may therefore assume that the function

$$p_0 = \sum_{n=1}^{\infty} p_n$$

is finite at a certain point. Since p_n is harmonic on U_n , p_0 is continuous and finite on $\bigcup_{n=1}^{\infty} U_n$. According to lemma 3 p is a potential and it is equal to infinite on $X - \bigcup_{n=1}^{\infty} U_n$.

COROLLARY 6. — *Let f be a finite non-negative upper semi-continuous function on $X \in \mathfrak{P}$. R_f^X is the greatest lower bound of the set of continuous hyperharmonic majorants of f . If \hat{R}_f^X is a potential, R_f^X is the greatest lower bound of the set of continuous potentials which dominate f .*

Let $x \in X$ and s be a superharmonic majorant of f . Let further $\{U_n\}$ be a pseudo-exhaustion of X , $U_1 \ni x$, $U = \bigcup_{n=1}^{\infty} U_n$ and let p be a continuous potential on X finite at x and equal to ∞ on $X - U$. Since U is a normal space there exists a

continuous finite function g on U , $f \leq g \leq s$. Let g_0 be the lower semi-continuous function on X equal to g on U and equal to 0 on $X - U$. The function $R_{g_0}^x$ is superharmonic and continuous on U according to theorem 3. Hence the function $s_0 = R_{g_0}^x + \varepsilon p$ is a continuous superharmonic majorant of f for any $\varepsilon > 0$ and we have

$$s_0(x) \leq s(x) + \varepsilon p(x).$$

In order to prove the last assertion it is sufficient to show that there exists a potential which dominates f . The function

$$u = \lim_{n \rightarrow \infty} \hat{R}_f^{x-U_n}$$

is a harmonic function on U . Since \hat{R}_f^x is locally bounded u is locally bounded. There exists therefore a harmonic function on X equal to u on U . This function is a minorant of \hat{R}_f^x . Hence u vanishes on U . We may therefore assume that

$$\sum_{n=1}^{\infty} R_f^{x-U_n}(x)$$

is convergent. Let us denote

$$U_0 = \emptyset, \quad G_n = U_{n+1} - \bar{U}_{n-1}.$$

We have

$$\sum_{n=1}^{\infty} \hat{R}_f^{G_n}(x) < \infty.$$

The function

$$p + \sum_{n=1}^{\infty} \hat{R}_f^{G_n}$$

is a potential which dominates f .

5. LEMMA 4. — *Let X be a locally compact locally connected space, F a closed nowhere disconnecting set in X , and \aleph a cardinal number. If $X - F$ possesses a basis whose cardinal is at most equal to \aleph and if there exists a set of continuous functions on X whose cardinal is at most equal to \aleph and which separates the points of F , then X possesses a basis whose cardinal is at most equal to \aleph .*

Let U be an open set on X and $\{U_i\}_{i \in I}$ the family of components of U . Since F is nowhere disconnecting, $\{U_i - F\}_{i \in I}$

are exactly the family of components of $U - F$. Since $X - F$ possesses a basis whose cardinal is most equal to \aleph , the cardinal of I is at most equal to \aleph .

There exists a set \mathcal{F} whose cardinal is at most equal to \aleph of continuous functions on X which separates the points of X . For any $f \in \mathcal{F}$ and any two rational numbers α, β we denote

$$U(f; \alpha, \beta) = \{x \in X | \alpha < f(x) < \beta\}.$$

Let \mathcal{B}' denote the family

$$\mathcal{B}' = \{U(f; \alpha, \beta) | f \in \mathcal{F}, \alpha, \beta \text{ rational numbers}\}.$$

The cardinal number of \mathcal{B}' is at most \aleph . Let us denote by \mathcal{B}

the system of components of the sets of the form $\bigcap_{i=1}^n U_i$, $U_i \in \mathcal{B}'$. According to the above remark the cardinal number of \mathcal{B} is at most equal to \aleph .

We want to prove that \mathcal{B} is a basis of X . Let x be a point of X and U be a relatively compact neighbourhood of x . For any $y \in \partial U$ let f_y be a function of \mathcal{F} such that

$$f_y(x) \neq f_y(y).$$

There exist two rational numbers α_y, β_y such that

$$x \in U(f_y; \alpha_y, \beta_y), \quad y \notin \overline{U(f_y; \alpha_y, \beta_y)}.$$

Since ∂U is compact we may find a finite number of points $\{y_i | i = 1, \dots, n\}$ on ∂U such that

$$x \in \bigcap_{i=1}^n U(f_{y_i}; \alpha_{y_i}, \beta_{y_i}), \quad \left(\bigcap_{i=1}^n U(f_{y_i}; \alpha_{y_i}, \beta_{y_i}) \right) \cap \partial U = \emptyset.$$

Let V denote the component of $\bigcap_{i=1}^n U(f_{y_i}; \alpha_{y_i}, \beta_{y_i})$ which contains x . Since $V \cap \partial U = \emptyset$ we have $V \subset U$. \mathcal{B} is hence a basis, since $V \in \mathcal{B}$.

THEOREM 10. — *Let $X \in \mathfrak{B} \cup \mathfrak{S}$, F be a closed polar set on X and let \aleph be a cardinal number. If for any point of $X - F$ there exists a neighbourhood which possesses a basis whose cardinal is at most equal to \aleph , then X possesses a basis whose cardinal is at most equal to \aleph .*

Since any space of the type $\mathfrak{B} \cup \mathfrak{H}$ can be covered with a finite system of domains of the type \mathfrak{B} , it is sufficient to prove the theorem for the case $X \in \mathfrak{B}$.

Let $\{U_n\}$ be a pseudo-exhaustion on $X - F$. Since F is polar $\{U_n\}$ is a pseudo-exhaustion of X . We denote $U = \bigcup_{n=1}^{\infty} U_n$ and assume $F = X - \bigcup_{n=1}^{\infty} U_n$. Let p be a continuous potential on X such that p is infinite exactly on F . Since any U_n possesses a basis whose cardinal is at most equal to \aleph , U possesses a basis \mathfrak{B} whose cardinal is at most equal to \aleph . For any two relatively compact sets $V, W \in \mathfrak{B}$, $\bar{V} \subset W$, let $f_{V,W}$ denote a continuous function on X , $0 \leq f_{V,W} \leq 1$, equal to 1 on V and equal to 0 on $X - W$. We denote by \mathcal{F} the set of functions of the form

$$\max_{1 \leq i \leq n} f_{V_i, W_i}.$$

The cardinal number of \mathcal{F} is at most equal to \aleph . We denote further for any $f \in \mathcal{F}$

$$s_f = R_p^x$$

$$\mathcal{S} = \{s_f | f \in \mathcal{F}\}.$$

and

The cardinal number of \mathcal{S} is at most equal to \aleph . Hence, according to the preceding lemma, it remains only to prove that \mathcal{S} separates the points of F .

Let $x, y \in F$, $x \neq y$ and V be a neighbourhood of x , $y \notin \bar{V}$. We denote by \mathcal{S}_V the family of functions $s_f \in \mathcal{S}$ for which the carrier of f is contained in V . \mathcal{S}_V is an upper directed family of superharmonic functions. Its least upper bound s is therefore superharmonic. We have

$$s \leq R_p^y, \quad s(y) \leq R_p^y(y) < \infty$$

and $s = p$ on $V - F$. Since F is polar we have $s = p$ on V and therefore $s(x) = \infty$. There exists therefore an $s_f \in \mathcal{S}_V$ such that

$$s_f(x) > s_f(y).$$

COROLLARY 7. — *If $X \in \mathfrak{B} \cup \mathfrak{H}$ and any point of X possesses a neighbourhood with a countable basis, X possesses a countable basis. Particularly if X is a manifold, and $X \in \mathfrak{B} \cup \mathfrak{H}$, X possesses a countable basis.*

There exist for any cardinal number \aleph examples of spaces on which the constants are harmonic and which possess points for which the cardinal number of any fundamental system of neighbourhoods is at least equal to \aleph . Let M be a set whose cardinal is \aleph and Γ the set of points of the complex plane $\{e^{i\theta} \mid \theta \text{ real number}\}$. For any finite set $I \subset M$ we denote by X_I the topological space obtained from the topological space $\Gamma \times I$, where I is considered with the discrete topology, identifying the points $(1, \iota)$ with $\iota \in I$. We denote by a_I this point of X_I . The harmonic functions on $X_I - \{a_I\}$ will be the functions which are linear in θ_i . A continuous function u defined on a neighbourhood of a_I is harmonic if it is harmonic outside $\{a_I\}$ and for sufficiently small $\varepsilon > 0$

$$u(a_I) = \frac{1}{2n} \sum_{\iota \in I} [u(e^{i\varepsilon}, \iota) + u(e^{-i\varepsilon}, \iota)],$$

where n is the cardinal number of I . It is easy to verify that the harmonic functions satisfy the axioms A_1, A_2 .

For any $I \subset J$ we denote by φ_{IJ} the map $X_J \rightarrow X_I$ defined by

$$\varphi_{IJ}(z, \iota) = \begin{cases} (z, \iota) & \text{if } \iota \in I \\ a_I & \text{if } \iota \notin I \end{cases}; \quad \varphi_{IJ}(a_J) = a_I.$$

The system $\{X_I, \varphi_{IJ}\}$ is a projective system of topological spaces. Let $\{X, \varphi_I\}$ be its projective limit and a the point of X corresponding to the points a_I . X is compact and the cardinal number of any fundamental system of neighbourhoods of a is at least equal to \aleph . The harmonic functions on X will be the functions of the form $u \circ \varphi_I$, where u is a harmonic function on X_I . It can be verified that the sheaf of harmonic functions on X satisfies the required axioms (and even the axiom 3').

THEOREM 11. — *The set of non-relatively compact components of an open set on $X \in \mathfrak{B} \cup \mathfrak{H}$ is at most countable.*

Let $\{G_i\}_{i \in I}$ be a family of pairwise disjoint domains on X and U be a relatively compact domain on X . We denote by I_U the set of $i \in I$ for which

$$G_i \cap U \neq \emptyset, \quad G_i - \bar{U} \neq \emptyset.$$

For any $i \in I_U$ we denote by f_i the function on ∂U equal to 1 on $G_i \cap \partial U$ and equal to 0 on $\partial U - G_i$. This function is resolutive with respect to U [1] and let $H_{f_i}^U$ denote its solution. This function doesn't vanish since in the contrary case there would exist a non-negative superharmonic function s on U converging to infinite at any point of $G_i \cap \partial U$. The function on $U \cup G_i$ equal to s on U and equal to infinite on $G_i - U$ would be a superharmonic function infinite on an open set. This is a contradiction. From

$$\sum_{i \in I_U} H_{f_i}^U \leq H_1^U$$

it follows that I_U is at most countable.

Let G be an open set $\{G_i\}_{i \in I}$ be the family of its non-relatively compact components and $\{U_n\}$ be a pseudo-exhaustion of X . From the above proof it follows that $I = \bigcup_{n=1}^{\infty} I_{U_n}$ is at most countable.

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