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MEANS ON $CV_p(G)$-SUBSPACES OF $CV_p(G)$ WITH RNP AND SCHUR PROPERTY

by Françoise LUST-PIQUARD

Introduction.

Let G be a lca group and $1 \leqslant p \leqslant 2$. We generalize to the space $CV_p(G)$ of bounded convolution operators: $L^p(G) \to L^p(G)$ $(1<p<2)$ some results which are obvious for $p = 1$ and were obtained for $p = 2$ by L. H. Loomis, G. S. Woodward, P. Glowacki and the author. We also generalize some results of N. Lohoué on convolution operators. Our motivation was a question raised by E. Granirer: is there a generalization of Loomis theorem [Loo] for convolution operators? A positive answer is given in theorem 2.8: Let $E \subset G$ be compact and scattered. Then $CV_p(E)$, the space of convolution operators on $L^p(G)$ which are supported on E, is the norm closure of finitely supported measures on E, and this space has Radon-Nikodym property. We also prove (theorem 2.14) that under the same assumptions $CV_p(E)$ has the Schur property.

The natural predual of $CV_p(G)$ is $A_p(G)$, which by C. Herz fundamental result is an algebra for pointwise multiplication and has some properties similar to those of $A_2(G)$ (we recall that $A_2(G)$ is isometric to $L^1(\hat{G})$ and $CV_2(G)$ is isometric to $L^\infty(\hat{G})$). But the proofs of Loomis theorem for $p = 2$ actually use the fact that every $\chi \in \hat{G}$ defines an isometric multiplier : $CV_2(G) \to CV_2(G)$ and that if $S \subset CV_2(G)$ has a compact support

$$\|S\|_{CV_2(G)} = \sup_{\chi \in \hat{G}} |\langle S, \chi \rangle|$$

where $\hat{G}$ is a group (the dual group of G).

Key-words : Invariant means - Convolution operators - Schur property - Radon-Nikodym property.

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One of the ingredients in this paper is to provide $CV_p(G)$ with an equivalent norm such that

$$|||S|||_p = \sup_{f \in \mathcal{S}_p(G)} |\langle S, f\rangle| .$$

where $\mathcal{S}_p(G)$ is a semi-group of functions of $A_p(G)$. This is done by using numerical ranges. We can thus adapt to $CV_p(G)$ a theory of means which is the usual one on $CV_2(G)$ or rather on $L^\infty(\hat{G})$[Gr], and which fits Eberlein's theory ([Eb1] Part. I). Topological means on $CV_p(G)$ were already defined in [G]. This is done in part 1 where we also give notation, definitions and recall the properties of $CV_p(G)$ and $A_p(G)$ that we need.

In part 2 we prove our main results theorems 2.8 and 2.14. The crucial lemma 2.2 allows to adapt the techniques of [Loo] [W1] [W2] [L-P1] [L-P2] [Gl]. In part 3 we show how theorems 2.8 and 2.14 also imply results on some $CV_p(\Lambda)$ where Λ is discrete. The main result is theorem 3.3, which is a generalization of a result of [L-P1] and [L-P3].

In part 3, 4 we give some transfer theorems between $CV_p(G)$ and $CV_p(G_d)$ (G_d is G provided with the discrete topology) and we prove an Eberlein decomposition (theorem 4.2) for elements of $CV_p(G)$ which are totally topologically p-ergodic (see definition 1.7) and we precise it for (weak) p-almost periodic elements of $CV_p(G)$ (see definition 4.5). This generalizes results of [Eb2] [W2] [L-P2] [Gra] [Loh1].

We take this opportunity to thank Ed. Granirer for nice and useful discussions.

1. Notation, definitions, states and means on $CV_p(G)$.

We consider Banach spaces over the field $\mathbb{C}$ of complex numbers. We denote by X^* the dual space of a Banach space X.

For $\varepsilon > 0$ D_ε is the open disc in $\mathbb{C}$ centered at $\{0\}$ with radius ε.

G denotes a lca group, G_d is the same group provided with the discrete topology, $\hat{G}$ is the dual group of G.

For $1 \leqslant p < \infty$ $L^p(G)$ is the space of equivalence classes of p-integrable functions with respect to the Haar measure on G; $L^\infty(G)$ is the dual space of $L^1(G)$. For $1 \leqslant p \leqslant 2$ p' is defined by $\frac{1}{p}+\frac{1}{p'}=1$;

the duality between $L^p(G)$ and $L^{p'}(G)$ is defined by

$$\langle f, g\rangle = \int_G f(x)\, g(x)\, dx\,.$$

$C_0(G)$ is the space of continuous functions on G which tends to 0 at infinity. $M(G)$ is the space of bounded Borel measures on G, i.e. the dual space of $C_0(G)$. For $1 \leqslant p \leqslant 2$ $CV_p(G)$ denotes the space of bounded convolution operators: $L^p(G) \to L^p(G)$, i.e. operators which commute with translation by elements of G, provided with the operator norm. We recall that $CV_1(G) = M(G)$ and $CV_2(G)$ is the space of Fourier transforms of the functions in $L^\infty(\hat{G})$.

$CV_p(G)$ is also the space of bounded convolution operators: $L^{p'}(G) \to L^{p'}(G)$ $(1 < p \leqslant 2)$ hence, by Riesz interpolation theorem, identity is continuous with norm 1

$$CV_{p_1}(G) \to CV_{p_2}(G), \qquad 1 \leqslant p_1 \leqslant p_2 \leqslant 2\,.$$

For $1 \leqslant p \leqslant \infty$ and $f \in L^p(G)$ we denote $\check{f}(x) = f(-x)$.

For $1 < p \leqslant 2$ $A_p(G)$ denotes the space of functions f on G which can be represented as

$$f = \sum_{n \geqslant 1} u_n * \check{v}_n$$

where $\sum_{n \geqslant 1} \|u_n\|_{L^p(G)} \|v_n\|_{L^{p'}(G)} < +\infty$ and the norm of f is the infimum of these sums over all such representations of f.

Hence $A_2(G)$ is the space of Fourier transforms of the elements of $L^1(\hat{G})$.

For $p = 1$ we replace $L^{p'}(G)$ by $C_0(G)$ in the definition above, hence $A_1(G) = C_0(G)$.

The duality between $CV_p(G)$ and $A_p(G)$ is defined by

$$\langle S, u * \check{v}\rangle = \langle S(u), v\rangle\,.$$

$CV_p(G)$ is clearly the dual space of $A_p(G)$. In particular

$$A_{p_1}(G) \leftarrow A_{p_2}(G), \qquad 1 \leqslant p_1 \leqslant p_2 \leqslant 2\,.$$

As functions which are continuous on G with a compact support are dense in $L^p(G)$ $(1 \leqslant p \leqslant 2)$ $A_{p_2}(G)$ is dense in $A_{p_1}(G)$, hence identity: $CV_{p_1}(G) \to CV_{p_2}(G)$ is one to one.

For $x \in G$ and $f \in L^p(G)\,(1 \leqslant p < \infty)$ or $A_p(G)\,(1 \leqslant p \leqslant 2)$ we denote by f_x the translate of f by x i.e. $f_x(t) = f(t-x)$. For $S \in CV_p(G)\,(1 \leqslant p \leqslant 2)$ the translate S_x is defined by $S_x(f) = (S(f))_x$ for $f \in L^p(G)$. Translation in $A_p^{**}(G)$ is defined by duality, i.e. $\langle S, F_x \rangle = \langle S_x, F \rangle$ for $F \in A_p^{**}(G)$; when restricted to $A_p(G)$ this definition coincides with the first one. The support of $S \in CV_p(G)$ is the (closed) set of x's $\in G$ such that for every neighborhood $V(x)$ there exists $f \in A_p(G)$ such that f is supported on $V(x)$ and $\langle S, f \rangle \neq 0$.

Let $E \subset G$ be a closed subset; we denote by $CV_p(E)$ the closed subspace of $CV_p(G)$ whose elements are supported on a subset of E. We denote by $\overline{\ell^1(E)}^{\|\ \|_{W_p}}$ the closed subspace of $CV_p(E) \subset CV_p(G)$ spanned by measures whose support is finite and lies in E. We denote by $CV_p(E_d)$ the closed subspace of $CV_p(G_d)$ whose elements are supported on a subset of E. We recall Herz's fundamental results ([P] proposition 10.2, 19.8): $A_p(G)$ is a Banach algebra for pointwise multiplication $(1 \leqslant p \leqslant 2)$. Let $B_p(G)$ denote the algebra of pointwise multipliers of $A_p(G)$. Then for $f \in A_p(G)$

$$\|f\|_{A_p(G)} = \|f\|_{B_p(G)}.$$

More generally let H be a lca group such that G_d is a subgroup of H_d, the embedding $G \to H$ is continuous and G is dense in H (hence H continuously embeds in $\bar{G}$ the Bohr compactification of G i.e. the dual group of $\hat{G}_d$). Then ([Ey] théorème 1, [Loh1] chap. IV, théorème IV.1, p. 108)

$$\forall f \in B_p(H), \quad \|f\|_{B_p(G)} = \|f\|_{B_p(H)}.$$

In the sequel we will write only $G \to H$ and this will mean that the above assumptions on G and H are satisfied. Actually we will only use the particular cases $G \to G$, $G_d \to G$, $G \to \bar{G}$.

Let $\varphi \in B_p(G)$; we will consider the pointwise multiplication operator associated to φ and the adjoint operators

$$\begin{aligned} A_p(G) &\to A_p(G) \\ f &\rightsquigarrow \varphi f \\ CV_p(G) &\leftarrow CV_p(G) \\ \varphi S &\leftarrow S \\ A_p^{**}(G) &\to A_p^{**}(G) \\ F &\rightsquigarrow \varphi F. \end{aligned}$$

Let $E \subset G$ be a closed subset; $I_p(E)$ is the closed ideal of functions of $A_p(G)$ which are zero on E. We denote the quotient algebra $\frac{A_p(G)}{I_p(E)}$ by $A_p(E)$. We recall that every $x \in G$ is a set of synthesis for $A_p(G)$ ([H1] theorem B, [P] proposition 19.19) which means that if $f \in A_p(G)$ and $f(x) = 0$, f is the norm limit of a sequence of functions in $A_p(G)$ which are zero on a neighborhood of x in G.

Let $W \subset G$ be a set of positive finite Haar measure. We denote

$$\varphi_W = |W|^{-1} 1_W * \check{1}_W.$$

$\|\varphi_W\|_{A_p(G)} = 1 = \varphi(0)\,(1 \leqslant p \leqslant 2)$.

The group G satisfies Fölner-condition ([Gre] theorem 3.6.2): for every $\varepsilon > 0$ and every compact $K \subset G$ there is a compact set $W = W(K) \subset G$ with finite positive Haar measure such that

$$\forall x \in K, \quad \frac{1}{|W|} |W_x \Delta W| \leqslant \varepsilon.$$

Hence

$$\forall x \in K, \quad \left\| \frac{(1_W)_x}{|W|^{1/p}} - \frac{1_W}{|W|^{1/p}} \right\|_{L^p(G)} \leqslant \varepsilon^{\frac{1}{p}}.$$

By [H2] 9. lemma 5, the family $(\varphi_{W(K)})_K$ is an approximate identity for $A_p(G)$ i.e. for every $\varepsilon > 0$ and $f \in A_p(G)$ there exists a compact set $K \subset G$ such that $\|f - f\varphi_{W(K)}\|_{A_p G)} \leqslant \varepsilon$. Obviously every $\varphi_{W(K)}$ has a compact support.

If G is provided with its discrete topology and if $F \subset G$ is a finite set (i.e. F is a compact set in G_d) we denote $P_F = |F|^{-1} 1_F * \check{1}_F$ (convolution is taken in G_d) instead of φ_F. Let $\mathscr{F}$ be the net of finite subsets of G. For every $x \in G$ $P_F(x) \xrightarrow{\mathscr{F}} 1$.

We recall that a Banach space X has *the Schur property* if every sequence $(x_n)_{n \geqslant 1}$ in X such that $x_n \to 0$ $\sigma(X, X^*)$ is norm convergent. A Banach space X has *the Radon-Nikodym property* (RNP in short) if every bounded linear operator $T\colon L^1[0\ 1] \to X$ is representable i.e. there exists a bounded strongly measurable function $F\colon [0\ 1] \to X$ s.t.

$$\forall \varphi \in L^1[0\ 1], \quad T(\varphi) = \int_{[0\ 1]} F(t)\varphi(t)\, dt.$$

We recall that if every separable subspace of X has RNP so has X and that every separable dual space has RNP.

States on $CV_p(G)$.

$CV_p(G)$ $(1 \leqslant p \leqslant 2)$ is a convolution algebra with unit δ_0.

Following the theory of numerical ranges [BD], we denote by $\mathscr{S}_p(G)$ the following set of states on $CV_p(G)$:

$$\mathscr{S}_p(G) = \{f \in A_p(G) \mid \|f\|_{A_p} = 1 = f(0)\}.$$

Let

$$\pi_p(G) = \{f \in A_p(G) \mid f = g * \check{h}, \|g\|_{L^p(G)} = \|h\|_{L^{p'}(G)} = \int g(x)h(x)dx = 1\}.$$

Obviously $\pi_p \subset \mathscr{S}_p$.

LEMMA 1.1. — (i) *$\mathscr{S}_p(G)$ is the norm closure of the convex hull of $\pi_p(G)$.*

(ii) $\mathscr{S}_p^{00}(G) = \{F \in A_p^{**}(G) \mid \|F\|_{A_p^{**}(G)} = 1 = \langle F, \delta_0 \rangle\}$.

Proof. — Let us denote the last set by $\mathscr{D}_p$.

Obviously $\mathscr{D}_p$ is norm closed and convex, and

$$\bar{C}_0\, \pi_p \subset \mathscr{S}_p \subset \mathscr{S}_p^{00} \subset \mathscr{D}_p.$$

By [BD] chap. 1, § 2, definition 1 and chap. 3, § 9, theorem 3:

$$\forall S \in CV_p(G), \quad \bar{C}_0 \{\langle S, f \rangle\}_{f \in \pi_p} = \{\langle S, F \rangle\}_{F \in \mathscr{D}_p} \subset \mathbb{C}.$$

As

$$\bar{C}_0 \{\langle S, f \rangle\}_{f \in \pi_p} \subset \overline{\{\langle S, f \rangle\}_{f \in \mathscr{S}_p}} = \{\langle S, F \rangle\}_{F \in \mathscr{S}_p^{00}} \subset \{\langle S, F \rangle\}_{F \in \mathscr{D}_p}$$

these sets are the same and Hahn-Banach theorem implies (i) and (ii). □

By the fundamental theorem on numerical ranges [BD] chap. 1, § 4, theorem 1,

$$\|S\|_{CV_p(G)} \geqslant \sup_{F \in \mathscr{D}_p} |\langle S, F \rangle| \geqslant e^{-1} \|S\|_{CV_p(G)}$$

hence by lemma 1.1

$$(1) \quad \forall S \in CV_p(G)\ \|S\|_{CV_p(G)} \geqslant \sup_{f \in \mathscr{S}_p(G)} |\langle S, f \rangle| \geqslant e^{-1} \|S\|_{CV_p(G)}.$$

As we are investigating geometric properties of subspaces of $CV_p(G)$ we can as well provide $CV_p(G)$ with the equivalent norm $\sup_{f \in \mathscr{S}_p(G)} |\langle S,f \rangle|$. The set $\mathscr{S}_2(G)$ is the set of functions in the unit sphere of $A_2(G)$ such that $\hat{f} \geqslant 0$ on $\hat{G}$. Hence $\mathscr{S}_p(G)\,(1 \leqslant p < 2)$ will replace the face of positive elements in the unit sphere of $L^1(\hat{G})$.

Remark 1.2. – Let us mention ([BD] chap. 6, § 31, theorem 1) that the mappings

$$S \rightsquigarrow (\langle S,f \rangle)$$
$$CV_p(G) \to C(\mathscr{S}_p) \quad \text{or} \quad CV_p(G) \to C(\mathscr{S}_p^{00})$$

are isometries of $CV_p(G)$ provided with its new norm into a closed subspace of the continuous functions on $\mathscr{S}_p$ or $\mathscr{S}_p^{00}$ provided with the $(A_p(G)^{**}, CV_p(G))$ topology. $\mathscr{S}_p^{00}$ is compact for this topology and the closure of $\mathscr{S}_p$. Every $F \in A_p^{**}(G)$ can be written as

$$F = \alpha_1 F_1 - \alpha_2 F_2 + i\alpha_3 F_3 - i\alpha_4 F_4$$

where $F_i \in \mathscr{S}_p^{00}(G)$, $\alpha_i \geqslant 0$ $(1 \leqslant i \leqslant 4)$ and $\sum_{i=1}^{4} \alpha_i \leqslant \sqrt{2} \sup |\langle S,F \rangle|$ where the supremum is taken on

$$\{S \in CV_p(G) | \, \forall f \in \mathscr{S}_p(G) \quad |\langle S,f \rangle| \leqslant 1\}. \qquad \square$$

As $A_p(G)$ is an algebra for pointwise multiplication $\mathscr{S}_p(G)$ is an abelian semi-group. Multiplication by $f \in \mathscr{S}_p(G)$ is continuous on $\mathscr{S}_p(G)$ provided with $\sigma(A_p(G)^{**}, CV_p(G))$, i.e. $\mathscr{S}_p(G)$ is a semi-topological semi-group. In this setting the measures $\alpha\delta_0 (\alpha \in \mathbb{C})$ are constant functions on $\mathscr{S}_p(G)$ and if $S \in CV_p(G)$, $f \in \mathscr{S}_p(G)$ fS is the translate of S (considered as a function on $\mathscr{S}_p(G)$) by f. The set $\{fS\}_{f \in \mathscr{S}_p(G)}$ is the orbit of S under the action of $\mathscr{S}_p(G)$. We denote by K_S its pointwise closure (for pointwise convergence on $\mathscr{S}_p(G)$); by remark 1.2 K_S can be also identified with the closure of $\{fS\}_{f \in \mathscr{S}_p(G)}$ for $\sigma(CV_p(G), A_p(G))$. $\mathscr{S}_p(G)$ is convex (as a subset of functions on G) and S defines an affine function on $\mathscr{S}_p(G)$.

Means on $CV_p(G)$.

DEFINITION 1.3. – *Let G be a* lca *group and let $G \to H$. Let $1 \leqslant p \leqslant 2$. A H-mean on $CV_p(G)$ is an element $\hat{m} \in \mathscr{S}_p^{00}(G)$ such that*

$$\forall \varphi \in \mathscr{S}_p(H), \quad \varphi \hat{m} = \hat{m}.$$

This definition is consistent because $\mathcal{S}_p(H) \subset B_p(G)$. The set of H-means is compact for $\sigma(A_p(G)^{**}, CV_p(G))$.

If $H = G$ a H-mean is called a topological mean [Gra].

If $H = \bar{G}$ a H-mean is called a mean. If $p = 2$ means and topological means on $CV_2(G)$ are Fourier transforms of usual means and topological means on $L_\infty(\hat{G})$. If G is discrete the only topological mean on $CV_p(G)$ is $1_{\{0\}}$ $(1 \leqslant p \leqslant 2)$. If $p = 1$ and G is any lca group the only mean on $CV_1(G) = M(G)$ is $1_{\{0\}}$.

LEMMA 1.4. – *Let G be a* lca *group, $G \to H$, $1 \leqslant p \leqslant 2$.*

(i) *Let $\hat{m}$ be a H-mean. Let $\varphi \in B_p(H)$ be such that $\|\varphi\|_{B_p(H)} = 1 = \varphi(0)$. Then $\varphi\hat{m} = \hat{m}$.*

(ii) *A topological mean on $CV_p(G)$ is a H-mean.*

Proof. – (i) Let $\varphi_0 \in \mathcal{S}_p(H)$. By definition $\varphi_0\hat{m} = \hat{m}$ hence $\varphi\varphi_0\hat{m} = \varphi\hat{m}$. As $\varphi\varphi_0 \in \mathcal{S}_p(H)$ $\varphi\varphi_0\hat{m} = \hat{m}$.

(ii) Let $\hat{m}$ be a topological mean and $\varphi \in \mathcal{S}_p(H)$. As $\varphi \in B_p(G)$ $\varphi\hat{m} = \hat{m}$ by (i). □

This proof is similar to [Gre] proposition 2.1.3.

LEMMA 1.5. – *Let G be a* lca *group, $G \to H$, $1 \leqslant p \leqslant 2$.*

(i) *Let $(W_\alpha)_{\alpha \in A}$ be a basis of open neighborhoods of $\{0\}$ in H. Let $(f_\alpha)_{\alpha \in A}$ be a net in $\mathcal{S}_p(G)$ such that f_α is supported on W_α for every α. Then every cluster point of $(f_\alpha)_{\alpha \in A}$ for $\sigma(A_p^{**}(G), CV_p(G))$ is a H-mean.*

(ii) *Conversely let $\hat{m}$ be a H-mean on $CV_p(G)$. There exists a net $(f_\alpha)_{\alpha \in A}$ in $\mathcal{S}_p(G)$ such that* (a) : *$f_\alpha \to \hat{m}$, $\sigma(A_p^{**}(G), CV_p(G))$;* (b) *for every open neighborhood W of $\{0\}$ in H there exists $\alpha_0 \in A$ such that for every $\alpha > \alpha_0$ f_α is supported on $W \cap G$.*

Proof. – (i) Let $F \in \mathcal{S}_p^{00}(G)$ be a cluster point of $(f_\alpha)_{\alpha \in A}$. Let $\varphi \in \mathcal{S}_p(H)$. As $\{0\}$ is a set of synthesis for $A_p(H)$, for every $\varepsilon > 0$ there exists φ_ε such that $\|\varphi - \varphi_\varepsilon\|_{A_p(H)} \leqslant \varepsilon$ and $\varphi = 1$ in a neighborhood W of $\{0\}$ in H. As soon as $W_\alpha \subset W$ $\varphi_\varepsilon f_\alpha = f_\alpha$ hence $\varphi_\varepsilon F = F$ and $\|\varphi F - \varphi_\varepsilon F\|_{A_p^{**}(G)} \leqslant \|\varphi - \varphi_\varepsilon\|_{B_p(G)} \leqslant \varepsilon$. This implies $F = \varphi F$.

(ii) Let $\hat{m}$ be a H-mean on $CV_p(G)$. For every neighborhood W of $\{0\}$ in H let W' be a neighborhood of $\{0\}$ in H such that $W' - W' \subset W$.

As $\varphi_{W'}$ is a multiplier of $\mathscr{S}_p(G)$ $\hat{m} = \varphi_{W'}\hat{m}$ lies in $\{\mathscr{S}_p(G)\cap I_p(W^c\cap G)\}^{00}$. Hence

$$\hat{m} \in \bigcap_W \{\mathscr{S}_p(G)\cap I_p(W^c\cap G)\}^{00}$$

where W runs through a basis of neighborhoods of $\{0\}$ in H, and this proves the claim. □

Let G be a lca group and $G \to H$. For $1 \leqslant p \leqslant 2$ and $S \in CV_p(G)$ let us define

$$M_p^H(S) = \{\langle S,\hat{m}\rangle \mid \hat{m} \text{ is a } H\text{-mean on } CV_p(G)\}.$$

If $H = G$ we will write $M_p^G(S) = M_p(S)$.

$M_p^H(S)$ is a compact subset of $\mathbb{C}$ and $M_p^H(S) \supset M_2^H(S)\ (1 \leqslant p \leqslant 2)$.

If $\varphi \in \mathscr{S}_p(G)$ $M_p^H(\varphi S) = M_p^H(S)$.

LEMMA 1.6. — *Let G be a* lca *group and $G \to H$. Let $S \in CV_p(G)\ (1 \leqslant p \leqslant 2)$. Then for every $\varepsilon > 0$ there exists an open neighborhood $W(0)$ in H such that $M_p^H(S) \subset \{\langle S,f\rangle \mid f \in \mathscr{S}_p(G),\ f$ is supported on $W\cap G\} \subset M_p^H(S) + D_\varepsilon$.*

Proof. — The left inclusion is obvious by lemma 1.5 (ii). If the right one does not hold there exists $\varepsilon > 0$ such that for every $W(0)$ in H there exists $f_{(W)} \in \mathscr{S}_p(G)$, supported on $W(0)$ such that $d(\langle S,f_{(W)}\rangle, M_p^H(S)) \geqslant \varepsilon$. By lemma 1.5 (i) any cluster point of $(f_{(W)})$ for $\sigma(A_p^{**}(G), CV_p(G))$ (when W runs through a basis of neighborhoods of $\{0\}$ in H) is a H-mean $\hat{m}$, and the distance from $\langle S,\hat{m}\rangle$ to $M_p^H(S)$ would be greater than ε, which is a contradiction.

DEFINITION 1.7. — *Let G be a* lca *group, $G \to H$, $1 \leqslant p \leqslant 2$. An element $S \in CV_p(G)$ is H-p-ergodic at 0 if $M_p^H(S)$ is a point. S is H-p-ergodic at $x \in G$ if S_x is H-p-ergodic at 0 and S is H-p-totally ergodic if it is H-p-ergodic at every point $x \in G$. If $H = G$ we say that S is topologically p-ergodic at x instead of G-p-ergodic at x.*

This definition is apparently weaker than [Eb1] definition 3.1. Hence our next lemma is stronger than [Eb1] theorem 3.1 applied to this setting.

For $p = 2$ it was proved in [W1] corollary 3, under the assumption that $\hat{S}$ is uniformly continuous and in full generality in [L-P2] proposition 1.

LEMMA 1.8. – *Let G be a* lca *group,* $G \to H$, $1 \leqslant p \leqslant 2$. *The following assertions on* $S \in CV_p(G)$ *are equivalent :*

(i) *S is H-p-ergodic at* 0.

(ii) *There exists $M \in \mathbb{C}$ such that*

$$\forall \varepsilon > 0, \quad \exists \varphi \in \mathcal{S}_p(H), \quad \|\varphi S - M\delta_0\|_{CV_p(G)} \leqslant \varepsilon .$$

(iii) *There exists $M \in \mathbb{C}$ such that for every $\varepsilon > 0$ there exists $\psi \in A_p(H)$ whose support is disjoint from* $\{0\}$ *and*

$$\|S - M\delta_0 - \psi S\|_{CV_p(G)} \leqslant \varepsilon .$$

Proof. – (iii) $\to$ (i) by lemma 1.5 (ii), and $M_p^H(S) = \{M\}$.

(i) $\Rightarrow$ (ii) : let us put $\{M\} = M_p^H(S)$ hence $M_p^H(S - M\delta_0) = \{0\}$. For every $\varepsilon > 0$ we choose W as in lemma 1.6. Hence if $W' - W' \subset W$ and W' is an open neighborhood of $\{0\}$ in H

$$\forall f \in \mathcal{S}_p(G), \quad |\langle S - M\delta_0, f\varphi_{W'} \rangle| \leqslant \varepsilon$$

which implies by (1)

$$\|\varphi_{W'} S - M\delta_0\|_{CV_p(G)} \leqslant e\varepsilon$$

(ii) $\Rightarrow$ (iii) For every $\varepsilon > 0$ let φ be as in (ii). As $\{0\}$ is a set of synthesis for $A_p(H)$ there exists $\varphi_\varepsilon \in A_p(H)$ such that $\|\varphi - \varphi_\varepsilon\|_{A_p(H)} \leqslant \varepsilon$ and $\varphi_\varepsilon = 1$ in a neighborhood of $\{0\}$ in H. For $\psi = 1 - \varphi_\varepsilon$

$$\|S - M\delta_0 - \psi S\|_{CV_p(G)} = \|\varphi_\varepsilon S - M\delta_0\|_{CV_p(G)} \leqslant \varepsilon + \varepsilon\|S\|_{CV_p(G)} . \quad \square$$

DEFINITION 1.9. – *Let G be a* lca *group,* $1 \leqslant p \leqslant 2$. $UC_p(G)$ *is the closed subspace of $CV_p(G)$ spanned by compactly supported elements.*

Obviously $UC_p(G)$ is the norm closure in $CV_p(G)$ of

$$\{fS \mid f \in A_p(G), S \in CV_p(G)\} .$$

It is a norm closed unitary subalgebra of $CV_p(G)$ ([Gra], proposition 12). $UC_2(G)$ is the space of Fourier transforms of uniformly continuous functions on $\hat{G}$. $B_p(G)$ can be identified with a subspace of $UC_p(G)^*$ in the following way : let $(\varphi_\alpha)_{\alpha \in A} \in \mathcal{S}_p(G)$ be an approximate identity for $A_p(G)$ and $F \in B_p(G)$. For every $S \in CV_p(G)$ and $f \in A_p(G)$

$$\langle fS, F\varphi_\alpha \rangle = \langle S, fF\varphi_\alpha \rangle \to \langle S, fF \rangle$$

hence the net $(F\varphi_\alpha)_{\alpha\in A}$ which is bounded in $A_p(G)$ (hence in $UC_p^*(G)$) converges for $\sigma(UC_p(G)^*, UC_p(G))$, its limit can be identified with F.

LEMMA 1.10. – *Let G be a* lca *group, $G \to H$, $1 \leqslant p \leqslant 2$.*

(i) *Let $\hat{m}$ be a H-mean on $CV_p(G)$. For every $\varphi \in \mathcal{S}_p(G)$ $\varphi\hat{m}$ is a topological mean.*

(ii) *A topological mean is uniquely determined by its restriction to $UC_p(G)$.*

(iii) *Let $S \in UC_p(G)$. Then $M_p^H(S) = M_p(S)$.*

Proof. – Let $K \subset G$ be a compact set. The topologies on K induced by G and H are the same. For every neighborhood V of $\{0\}$ in G there exists a neighborhood W of $\{0\}$ in H such that $V \cap K \supset W \cap K$.

(i) Let $(f_\alpha)_{\alpha\in A} \in \mathcal{S}_p(G)$, $f_\alpha \to \hat{m}$ as in lemma 1.5 (ii). Hence if $\varphi \in \mathcal{S}_p(G)$ $\varphi f_\alpha \to \varphi\hat{m}$, $\sigma(A_p^{**}(G), CV_p(G))$ and if φ has a compact support K the above remark and lemma 1.5 (i) imply that $\varphi\hat{m}$ is a topological mean. Every $\varphi \in \mathcal{S}_p(G)$ is the norm limit in $A_p(G)$ of $(\varphi_n)_{n\geqslant 1} \in \mathcal{S}_p(G)$ where φ_n has a compact support $(n \geqslant 1)$. Hence $\varphi_n\hat{m} (n\geqslant 1)$ and $\varphi\hat{m}$ are topological means.

(ii) Let $\hat{m}$ be a topological mean on $CV_p(G)$. Then

$$\forall S \in CV_p(G), \quad \forall \varphi \in \mathcal{S}_p(G), \quad \langle S,\hat{m}\rangle = \langle S,\varphi\hat{m}\rangle = \langle \varphi S,\hat{m}\rangle$$

hence if $\hat{m}$ and $\hat{m}'$ are topological means which coïncide on $UC_p(G)$ they coïncide on $CV_p(G)$.

(iii) Let us first assume that S has a compact support and let $K \subset G$ be a compact set whose interior contains the support of S. Let $\varphi \in \mathcal{S}_p(G)$. As $\{0\}$ is a set of synthesis for $A_p(G)$, for every $\varepsilon > 0$ there exists φ_ε such that $\|\varphi - \varphi_\varepsilon\|_{A_p(G)} \leqslant \varepsilon$ and $\varphi_\varepsilon = 1$ in a neighborhood of $\{0\}$ in G which we denote by V. Let $W \subset H$ be such that $W \cap K \subset V \cap K$. Hence for every $f \in A_p(G)$ which is supported on $W (1-\varphi)f \in \mathrm{I}_p(K)$ and $\langle S,(1-\varphi_\varepsilon)f\rangle = 0$. For every H-mean $\hat{m}$ lemma 1.5 (ii) now implies $\langle S,\hat{m}\rangle = \langle S,\varphi_\varepsilon\hat{m}\rangle$ hence $\langle S,\hat{m}\rangle = \langle S,\varphi\hat{m}\rangle$. The same is true if S is a norm limit of S_n's with compact supports. By (i) $\varphi\hat{m}$ is a topological mean, hence $M_p(S) = M_p^H(S)$.

Lemma 1.10 (iii) generalizes the fact that there is no need to distinguish means and topological means on uniformly continuous functions of $\hat{G}$ ([Gre], lemma 2.2.2).

Though we won't use the next results in the next parts of this paper we think they are worth being noticed.

LEMMA 1.11. – *Let G be a* lca *group, $G \to H$, $1 \leqslant p \leqslant 2$. Let $(V_\beta)_{\beta \in B}$ be a basis of neighborhoods of $\{0\}$ in H and $S \in CV_p(G)$. The following assertions are equivalent :*

(i) *S is H-p-ergodic.*

(ii) *For every net $(f_\alpha)_{\alpha \in A}$ in $\mathcal{S}_p(G)$ such that for every V_β there exists $\alpha(\beta)$ such that f_α is supported on V_β for every $\alpha > \alpha(\beta)$, the net $(\langle S, f_\alpha \rangle)_{\alpha \in A}$ converges.*

(iii) *For every net $(f_\alpha)_{\alpha \in A}$ as in* (ii) *$(f_\alpha S)_{\alpha \in A}$ is norm converging in $CV_p(G)$.*

Proof. – (i) $\Rightarrow$ (ii) : by lemma 1.5 (i) every cluster point of $(f_\alpha)_{\alpha \in A}$ for $\sigma(A_p^{**}(G), CV_p(G))$ is a H-mean.

(ii) $\Rightarrow$ (iii) : if $(f_\alpha)_{\alpha \in A}$ is a net as in (ii) such that $(f_\alpha S)_{\alpha \in A}$ is not a Cauchy filter for the norm there exists $\varepsilon > 0$ such that for every $\alpha \in A$ there exist $\alpha'' > \alpha' > \alpha$ and

$$\|f_{\alpha''}S - f_{\alpha'}S\|_{CV_p(G)} \geqslant \varepsilon,$$

hence by (1) there exists $g_\alpha \in \mathcal{S}_p(G)$ such that

$$|\langle f_{\alpha''}S, g_\alpha \rangle - \langle f_{\alpha'}S, g_\alpha \rangle| \geqslant \varepsilon e^{-1}.$$

The net $(h_\gamma)_{\gamma \in C}$ defined by $h_{\alpha,1} = f_{\alpha'}g_\alpha \quad h_{\alpha,2} = f_{\alpha''}g_\alpha$ i.e. $C = (A, \{1,2\})$ satisfies the assumptions of (ii), yet $(\langle S, h_\gamma \rangle)_{\gamma \in C}$ does not converge.

(iii) $\Rightarrow$ (i) : let $(f_\alpha)_{\alpha \in A}$ be a net as in (ii). The norm limit of $(f_\alpha S)_{\alpha \in A}$ must be $M\delta_0$ where $M \in \mathbb{C}$ might depend on $(f_\alpha)_{\alpha \in A}$. Hence $M\delta_0$ belongs to the norm closure of $\mathcal{S}_p(G)S$. Let $\hat{m}$ be a topological mean on $CV_p(G)$. Then $\langle S, \hat{m} \rangle = \langle M\delta_0, \hat{m} \rangle = M$ hence M does not depend on the net $(f_\alpha)_{\alpha \in A}$. In particular for every net $(f_\alpha)_{\alpha \in A}$ as in (ii)

$$f_\alpha S \to M\delta_0, \qquad \sigma(CV_p(G), \ A_p^{**}(G))$$

hence

$$f_\alpha S \to M\delta_0, \qquad \sigma(UC_p(G), \ UC_p^*(G)).$$

As the constant function 1 belongs to $B_p(G)$ hence to $UC_p^*(G)$

$$\langle S, f_\alpha \rangle = \langle f_\alpha S, 1 \rangle \to M.$$

By lemma 1.5 (ii) this implies $\langle S, \hat{m} \rangle = M$ for every H-mean $\hat{m}$ on $CV_p(G)$. □

Lemma 1.11 generalizes [L-P2], theorem 1.

Actually $(f_\alpha)_{\alpha\in A}$ in lemma 1.11 can be taken in $\mathscr{S}_2(G)$; hence if $S\in CV_p(G)$ is H-p-ergodic there is a scalar multiple of δ_0 in the norm closure of $\mathscr{S}_2(G)S$ in $CV_p(G)$.

Let $S\in CV_p(G)$. We recall that K_S is the closure of the convex set $\mathscr{S}_p(G)S$ for $\sigma(CV_p(G),A_p(G))$. K_S is compact for this topology. For every $F\in B_p(G)$ such that $\|F\|_{B_p(G)} = F(0) = 1$ FS belongs to K_S as a limit of $(\varphi_\alpha FS)_{\alpha\in A}$ where $(\varphi_\alpha)_{\alpha\in A}\in\mathscr{S}_p(G)$ is an approximate identity for $A_p(G)$. Bus this does not give the whole of K_S in general (especially if G is compact). Let $\varphi''\in\mathscr{S}_p(G)^{00}$. We define $\varphi''S$ as an element of $CV_p(G)$ as follows: let $(\varphi_\alpha)_{\alpha\in A}$ be a bounded net in $\mathscr{S}_p(G)$ converging to φ'' for $\sigma(A_p^{**}(G),CV_p(G))$; $\varphi''S$ is the limit of $(\varphi_\alpha S)_{\alpha\in A}$ for $\sigma(CV_p(G),A_p(G))$. Clearly

$$K_S = \{\varphi''S\,|\,\varphi''\in\mathscr{S}_p^{00}(G)\}$$

and actually we only have to consider the restriction of φ'''s to $UC_p(G)$. If G is discrete $UC_p(G)$ is the norm closure in $CV_p(G)$ of finitely supported measures. In this case $UC_p(G)^* = B_p(G)$ by [Loh], chap. IV, theorem 1, p. 79, [H2], theorem 2, [P], proposition 19.11.

We now consider the following questions: when is a H-mean constant on K_S? when is it a Baire $-$ 1 function on K_S (provided with its $\sigma(CV_p(G),A_p(G))$ topology)?

LEMMA 1.12. – *Let G be a* lca *group, $G\to H$, $1\leqslant p\leqslant 2$. Let $S\in CV_p(G)$. Let $\hat{m}$ be a H-mean which is constant on K_S. Then $\hat{m}$ coïncide on K_S with a topological mean and S is topologically p-ergodic.*

Proof. – By assumption for every $\varphi''\in\mathscr{S}_p^{00}(G)$ $\langle\varphi''S,\hat{m}\rangle = M$. For every $\varphi\in\mathscr{S}_p(G)$ $\varphi\varphi''S\in K_S$ hence

$$\langle\varphi''S,\varphi\hat{m}\rangle = \langle\varphi\varphi''S,\hat{m}\rangle = M$$

and $\varphi\hat{m}$ is a topological mean by lemma 1.10.

Let $(f_\alpha)_{\alpha\in A}$ be a net in $\mathscr{S}_p(G)$ converging to $\hat{m}$ for $\sigma(A_p^{**}(G),CV_p(G))$:

$$\forall\varphi''\in\mathscr{S}_p(G)^{00},\ \langle f_\alpha S,\varphi''\rangle = \langle\varphi''S,f_\alpha\rangle \to \langle\varphi''S,\hat{m}\rangle = M = \langle M\delta_0,\varphi''\rangle.$$

By Remark 1.2 it implies that $M\delta_0$ belongs to the weak closure of $\mathscr{S}_p(G)S$, hence to the norm closure of $\mathscr{S}_p(G)S$ which implies the claim by lemma 1.5.

LEMMA 1.13. — *Let G be a* lca *group, $G \to H$, $1 \leqslant p \leqslant 2$. Let $S \in CV_p(G)$. The following assertions are equivalent :*

(i) *S is H-p-ergodic*

(ii) *every H-mean on $CV_p(G)$ is constant on K_S*

(iii) *all H-means on $CV_p(G)$ are constant and equal on K_S.*

If $H = G$ these assertions are equivalent to

(iv) *there exists a topological mean which is constant on K_S.*

Proof. — (i) $\Rightarrow$ (ii) : By lemma 1.8 there exists $M \in \mathbb{C}$ such that for every $\varepsilon > 0$ there exists $\psi \in \mathscr{S}_p(H)$ with $\|\psi S - M\delta_0\| \leqslant \varepsilon$ hence for every H-mean $\hat{m}$ and $\varphi'' \in \mathscr{S}_p^{00}(G)$

$$\langle \varphi'' S, \hat{m} \rangle = \langle \psi\varphi'' S, \hat{m} \rangle \qquad \text{and} \qquad \|\psi\varphi'' S - M\delta_0\| \leqslant \varepsilon$$

which implies $\langle \varphi'' S, \hat{m} \rangle = M$.

(ii) $\Rightarrow$ (iii) by lemma 1.12.

(iii) $\Rightarrow$ (i) : we saw that $S \in K_S$ hence the claim is obvious.

If $H = G$ (iii) $\Rightarrow$ (iv) is obvious and (iv) $\Rightarrow$ (i) by lemma 1.12. $\square$

$S \in CV_p(G)$ may be topologically p-ergodic without K_S being the norm closure of $\mathscr{S}_p(G)S$: for example if G is discrete, if S does not belong to the norm closure of finitely supported measures, S belongs to K_S and not to $UC_p(G)$ hence not to $\overline{\mathscr{S}_p(G)S}^{\|\ \|}$, though S is topologically p-ergodic.

LEMMA 1.14. — *Let G be a* lca *group, $G \to H$, $1 \leqslant p \leqslant 2$. Let $S \in CV_p(G)$. Then $\mathscr{S}_p(H)S$ is dense in K_S for $\sigma(CV_p(G), A_p(G))$.*

Proof. — As $\mathscr{S}_p(H)$ lies in $B_p(G)$ we saw that $\mathscr{S}_p(H)S$ lies in K_S.

By [Loh1], chap. II, theorem 1.2 or [Loh2], theorem 1, if $T \in CV_p(G)$ has a compact support it determines $\tilde{T} \in CV_p(H)$ such that $\|T\|_{CV_p(G)} = \|\tilde{T}\|_{CV_p(H)}$ and

$$\forall F \in A_p(H), \quad \langle \tilde{T}, F \rangle = \lim_{\alpha} \langle FT, \varphi_\alpha \rangle$$

where $(\varphi_\alpha)_{\alpha \in A}$ is an approximate identity (in $\mathscr{S}_p(G)$) for $A_p(G)$.

Hence there is a canonical isometry from $UC_p(G)$ to a closed unitary subalgebra E_p of $UC_p(H) \subset CV_p(H)$.

Every $\varphi \in \mathscr{S}_p(G)$ defines a state on $UC_p(G)$ hence it can be identified with the restriction to E_p of an element $\tilde{\varphi} \in \mathscr{S}_p^{00}(H)$. Hence there exists a net $(\varphi_\beta)_{\beta \in B}$ in $\mathscr{S}_p(H)$ such that

$$\forall f \in A_p(G), \quad \langle \varphi_\beta S, f \rangle = \langle fS, \varphi_\beta \rangle \underset{\beta}{\rightarrow} \langle \tilde{f}\tilde{S}, \tilde{\varphi} \rangle = \langle \varphi S, f \rangle$$

which proves the claim.

Lohoué's theorem is obvious if $p = 2$ and easy if G is discrete (see lemma 13.2 below).

Lemma 1.14 implies that a H-mean which is continuous on K_S is constant on K_S.

PROPOSITION 1.15. – *Let G be a metric* lca *group, $G \rightarrow H$, $1 \leqslant p \leqslant 2$. Let $S \in CV_p(G)$ and let $\hat{m}$ be a H-mean on $CV_p(G)$. If $\langle S, \hat{m} \rangle \notin M_p(S)$ $\hat{m}$ is not a Baire 1-function on K_S.*

Proof. – If $\hat{m}$ is a Baire 1-function on K_S there is an open set $0 \subset K_S$ such that

$$\operatorname{diam} \{\langle 0, \hat{m} \rangle\} \leqslant \frac{1}{2} d(\langle S, \hat{m} \rangle, M_p(S)).$$

As $\mathscr{S}_p(G)S$ and $\mathscr{S}_p(H)S$ are dense in K_S by definition and lemma 1.14 there exist $\psi \in \mathscr{S}_p(G)$ and $\varphi \in \mathscr{S}_p(H)$ such that

$$\operatorname{diam} \{\langle 0, \hat{m} \rangle\} \geqslant |\langle \psi S, \hat{m} \rangle - \langle \varphi S, \hat{m} \rangle| = |\langle \psi S, \hat{m} \rangle - \langle S, \hat{m} \rangle|.$$

By lemma 1.10 $\psi \hat{m}$ is a topological mean, hence

$$|\langle \psi S, \hat{m} \rangle - \langle S, \hat{m} \rangle| \geqslant d(\langle S, \hat{m} \rangle, M_p(S))$$

which is a contradiction.

If G is discrete every $S \in CV_p(G)$ has a countable support hence K_S is metrizable and the conclusion of proposition 1.15 holds true:

If $\hat{m}$ is a H-mean and if $\langle S, \hat{m} \rangle \neq \langle S, 1_{\{0\}} \rangle$ $\hat{m}$ is not a Baire 1-function on K_S.

For general lca group G we do not know if there exist H-means on $CV_p(G)$ which are Baire 1-functions on K_S without being constant on K_S.

2. Some subspaces of $CV_p(G)$ with Radon-Nikodym and Schur property. A generalization of Loomis theorem.

We first prove a lemma (lemma 2.2 (b) below) which will be a key for this paper. It is obvious when $p = 2$ and is implicitly used in [W1], [W2] for $p = 2$, in [Loh1] for $1 \leqslant p \leqslant 2$. Neither in [W1] nor in [Loh] its whole strengh is used.

LEMMA 2.1. — *Let G be a* lca *group, $G \to H$, $1 \leqslant p \leqslant 2$. Let $F \subset G$ be a finite set. There exists a neighborhood W of $\{0\}$ in H such that, for every $(k,k') \in \pi_p(G)$ supported on $W \times W$, $(k * \check{k}') * P_F$ lies in $\mathcal{S}_p(G)$, where $(k * \check{k}') * P_F$ is defined by*

$$(k * \check{k}') * P_F = \sum_{P_F(x_i) \neq 0} P_F(x_i)(k * \check{k}')_{x_i}.$$

Proof. — We choose W a neighborhood of $\{0\}$ in H such that the sets $x_i + W$ $(x_i \in F)$ are pairwise disjoint. Let $(k,k') \in \pi_p(G)$ be supported on $W \times W$. Hence

(i) $1 = \left\| |F|^{-1/p} \sum_{x_i \in F} k_{x_i} \right\|_{L^p(G)} = \left\| |F|^{-1/p'} \sum_{x_j \in F} k'_{x_j} \right\|_{L^{p'}(G)}$

(ii) $1 \geqslant \left\| \left(|F|^{-1} \left(\sum_{x_i \in F} k_{x_i} \right) * \left(\sum_{x_j \in F} (\check{k}'_{x_j}) \right) \right\|_{A_p(G)}$

(iii) $|F|^{-1} \left(\sum_{x_i \in F} k_{x_i} \right) * \left(\sum_{x_j \in F} (\check{k}'_{x_j}) \right)$

$$= |F|^{-1} \sum\sum_{F \times F} (k * \check{k}')_{x_i - x_j} = (k * \check{k}') * P_F$$

(iv) $(k * \check{k}') * P_F(0) = k * \check{k}'(0) = 1$.

LEMMA 2.2. — *Let G be a* lca *group, $G \to H$, $1 \leqslant p \leqslant 2$.* a) *Let W be a neighborhood of $\{0\}$ in H.*

For every $f \in \mathcal{S}_p(G)$ $\varphi_W f$ lies in the norm closed convex hull of

$$\{k * \check{k}' \mid (k, k') \in \pi_p(G), (k,k') \text{ is supported on } W \times W\}.$$

b) *Let $F \subset G$ be a finite set and $\hat{m}$ be a H-mean on $CV_p(G)$. Then $\hat{m} * P_F$ lies in $\mathcal{S}_p^{00}(G)$, where $\hat{m} * P_F$ is defined by*

$$\hat{m} * P_F = \sum_{P_F(x_i) \neq 0} P_F(x_i) \hat{m}_{x_i}.$$

Proof. — a) The claim is proved for $f \in \mathcal{S}_p(G)$ as soon as it is proved for $f = g * \check{g}'$ where $(g,g') \in \pi_p(G)$ owing to lemma 1.1.

By the proof of [Ey] theorem 1, $(g * \check{g}')\varphi_W$ belongs to the norm closed convex hull of

$$\frac{g|W|^{-1/p}(1_W)_x}{\|g|W|^{-1/p}(1_W)_x\|_{L^p(G)}} * \frac{g'|W|^{-1/p'}(1_W)_x}{\|g'|W|^{-1/p'}(1_W)_x\|_{L^{p'}(G)}} = k * \check{k}'$$

where $x \in G$, and

$$k = \frac{g_{-x}|W|^{-1/p}1_W}{\|g_{-x}|W|^{-1/p}1_W\|_{L^p(G)}}, \qquad k' = \frac{g'_x|W|^{-1/p'}1_W}{\|g'_x|W|^{-1/p'}1_W\|_{L^{p'}(G)}}.$$

b) Let $(f_\alpha)_{\alpha \in A} \in \mathcal{S}_p(G)$ be such that $f_\alpha \to \hat{m}$, $\sigma(A_p^{**}(G), CV_p(G))$. Let W be chosen as in lemma 2.1. By lemmas 2.1 and 2.2 (a) $(f_\alpha \varphi_W) * P_F \in \mathcal{S}_p(G)$. Obviously

$$(f_\alpha \varphi_W) * P_F \underset{\alpha \in A}{\longrightarrow} \hat{m} * P_F, \quad \sigma(A_p^{**}(G), CV_p(G). \qquad \square$$

The proof of lemma 2.2 b is much simpler for $p = 2$: let $(f_\alpha)_{\alpha \in A}$ be a net as in lemma 1.5 b. Then $\hat{f}_\alpha \geqslant 0$ hence $\hat{f}_\alpha \hat{P}_F \geqslant 0$, $\|f_\alpha * P_F\|_{A_2(G)} = f_\alpha * P_F(0)$; moreover $f_\alpha * P_F(0) = f_\alpha(0)P_F(0) = 1$ as soon as the $x_i + W(x_i \in F)$ are disjoint and f_α is supported on W.

Lemma 2.2 will be the main ingredient in the definition of the mappings $A_{\hat{m}}$ in part 4. It is also an ingredient in the proof of proposition 2.3 below, and it will be revisited in the proof of lemma 2.10 below. Proposition 2.3 is a generalization of [W1] theorem 9 (ii). We keep some arguments of his proof but his crucial use of properties of almost periodic functions is replaced by lemma 2.2.

PROPOSITION 2.3. — *Let G be a* lca *group, $G \to H$, $1 \leqslant p \leqslant 2$. Let us assume that $S \in CV_p(G)$ is H-p-ergodic at every $x \neq 0$, $x \in G$. Then for every $\varepsilon > 0$ there exists $\varphi \in \mathcal{S}_p(H)$ such that for every finite set $F \subset G$*

$$\Bigg\| \sum_{\substack{x_i \neq 0 \\ P_F(x_i) \neq 0}} P_F(x_i)\varphi(x_i)\, M_p^H(S_{x_i})\, \delta_{x_i} \Bigg\|_{CV_p(G)} \leqslant \varepsilon.$$

Let us write it in another way: let $\hat{m}$ be a H-mean on $CV_p(G)$. Let

$$\varphi'' = \sum_{x_i \neq 0} P_F(x_i)\hat{m}_{x_i} = \hat{m} * (P_F - 1_{\{0\}}) \in A_p^{**}(G).$$

Then $\varphi''(\varphi S)$ defined as an element of $CV_p(G)$ as in part 1 (description of K_S) satisfies

$$\varphi''(\varphi S) = \sum_{x_i \neq 0} P_F(x_i)\varphi(x_i)M_p^H(S_{x_i})\delta_{x_i}.$$

Proposition 2.3 does not imply that S is H-p-ergodic at 0 in general. But if G is not discrete and if we apply it for $G = G_d$ and $H = G$ we get that for every $\varepsilon > 0$ there exists $\varphi \in \mathcal{S}_p(G)$ such that

$$\forall F \text{ finite } F \subset G \, \|P_F(\varphi S - \langle S, 1_{\{0\}} \rangle \delta_0)\|_{CV_p(G_d)} \leqslant \varepsilon$$

hence

$$\|\varphi S - \langle S, 1_{\{0\}} \rangle \, \delta_0\|_{CV_p(G_d)} \leqslant \varepsilon$$

which means by lemma 1.8 that $S \in CV_p(G_d)$ is G-p-ergodic at 0. For $p = 2$ this was noticed in [Gl].

Thus Proposition 2.3 easily implies the following corollary whose proof is the same as in [Gl] Corollary 2, where $p = 2$:

Corollary 2.4. – *Let G be a* lca *group,* $1 \leqslant p \leqslant 2$. *Let $E \subset G$ be closed and scattered. Then every $S \in CV_p(E_d) \subset CV_p(G_d)$ is G-totally p-ergodic.*

Proof. – Let $N = \{x \in G \,|\, S$ is not G-p-ergodic at $x\}$. By lemma 1.8 $N \subset E$ because E is closed in G. Let $\bar{N}$ be the closure of N in E. If N is not empty there exists $x \in \bar{N}$ which is an isolated point of $\bar{N}$ hence $x \in N$. But there exists $\varphi \in \mathcal{S}_p(G)$ such that the support of $\varphi_x S$ meets $\bar{N}$ only at $\{x\}$. By Proposition 2.3 and the remark above $\varphi_x S$ is G-p-ergodic at x hence so is S and this is a contradiction.

Proof of proposition 2.3. – For every $\varepsilon > 0$ we choose $W(0) \subset H$ as in lemma 1.6 and $\varphi = \varphi_{W'} \in \mathcal{S}_p(H)$ such that W' is an open neighborhood of $\{0\}$ in H and $W' - W' \subset W$. For every finite set $F \subset G$, every H-mean $\hat{m}$ on $CV_p(G)$ and every $g \in \mathcal{S}_p(G)$ lemma 1.6 and lemma 2.2 (b) imply

$$\langle g\varphi S, \sum_{P_F(x_i) \neq 0} P_F(x_i)\hat{m}_{x_i} \rangle \in M_p^H(S) + D_\varepsilon.$$

On the other hand

$$\langle g\varphi S, \sum_{P_F(x_i) \neq 0} P_F(x_i)\hat{m}_{x_i} \rangle = \langle S, \hat{m} \rangle + \sum_{\substack{P_F(x_i) \neq 0 \\ x_i \neq 0}} P_F(x_i) g(x_i) \varphi(x_i) \langle S, \hat{m}_{x_i} \rangle.$$

Hence for every $g \in \mathcal{S}_p(G)$, as S is H-p-ergodic at every $x \neq 0$

$$M_p^H(S) + \langle \sum_{\substack{P_F(x_i) \neq 0 \\ x_i \neq 0}} P_F(x_i)\varphi(x_i)M_p^H(S_{x_i})\delta_{x_i}, g\rangle \subset M_p^H(S) + D_\varepsilon .$$

Hence

$$\sup_{g \in \mathcal{S}_p(G)} |\langle \sum_{\substack{P_F(x_i) \neq 0 \\ x_i \neq 0}} P_F(x_i)\varphi(x_i)M_p^H(S_{x_i})\delta_{x_i}, g\rangle| \leqslant \varepsilon$$

which implies by (1)

$$\left\| \sum_{\substack{P_F(x_i) \neq 0 \\ x_i \neq 0}} P_F(x_i)\varphi(x_i)M_p^H(S_{x_i})\delta_{x_i} \right\|_{CV_p(G)} \leqslant e\varepsilon . \qquad \square$$

In order to prove our generalization of Loomis theorem (theorem 2.8 below) we now state the obvious generalization of a part of the original proof.

DEFINITION 2.5. – *Let G be a* lca *group, and* $1 \leqslant p \leqslant 2$. *An element* $S \in CV_p(G)$ *if p-almost periodic if* $S \in \overline{\ell^1(G)}^{\|\ \|_{CV_p(G)}}$ *i.e. if S lies in the norm closure in* $CV_p(G)$ *of finitely supported measures. S is said to be p-almost periodic at* $x \in G$ *if there exists* $f \in A_2(G)$ *such that* $f(x) \neq 0$ *and fS is p-almost periodic.*

Equivalent definitions of p-almost periodic elements of $CV_p(G)$ are given in theorem 4.8 below.

LEMMA 2.6. – *Let G be a* lca *group,* $G \to H$, $1 \leqslant p \leqslant 2$.

a) *If* $S \in CV_p(G)$ *is p-almost periodic, S is totally H-p-ergodic and for every* $\varepsilon > 0$ *there exists a finite set* $F \subset G$ *such that for every H-mean* $\hat{m}$

$$\|S - (\hat{m} * P_F)S\|_{CV_p(G)} \leqslant \varepsilon \quad \text{and} \quad S - (\hat{m} * P_F)S \in \overline{\ell^1(G)}^{\ CV_p(G)} .$$

b) *If* $S \in CV_p(G)$ *has a compact support K and is p-almost periodic at every point of K, S* is *p-almost periodic.*

c) *If* $S \in CV_p(G)$ *has a compact support K, such that* $0 \in K$, *is p-almost periodic at every* $x \in K$, $x \neq 0$, *and topologically p-ergodic at* 0, *S is p-almost periodic.*

Proof. — a) $(\hat{m} * P_F)S$ is defined as in part 1 (see also proposition 2.3) as a finitely supported measure. Moreover for every $S' \in CV_p(G)$

$$\|(\hat{m} * P_F)S'\|_{CV_p(G)} \leqslant \|S'\|_{CV_p(G)}$$

by definition and lemma 2.2. Both assertions of (a) are obvious if S is a finitely supported measure and verified by norm density if $S \in \overline{\ell^1(G)}^{CV_p(G)}$. (These facts will be used again in lemma 3.2 and theorem 4.1.)

The proof of b) is analogue to [Loo] theorem 1: there exist $(f_j)_{1 \leqslant j \leqslant n} \in A_2(G)$ such that $f_j S$ is p-almost periodic and $\sum_{1 \leqslant j \leqslant n} f_j > 0$ on K, there exists $f \in A_2(G)$ such that $f\left(\sum_{1 \leqslant j \leqslant n} f_j\right) = 1$ in a neighborhood of K hence $S = \sum_{1 \leqslant j \leqslant n} f f_j S$ is p-almost periodic.

c) Every ψS defined as in lemma 1.8 (iii) satisfies the assumptions of (b), hence ψS is p-almost periodic and so is S by lemma 1.8. □

We now prove a generalization of [Loo] theorem 2.3, but with a different proof: it will be a consequence of proposition 2.3.

PROPOSITION 2.7. — *Let G be a* lca *group,* $1 \leqslant p \leqslant 2$. *Let $S \in CV_p(G)$ with a compact support K such that $0 \in K$. If S is p-almost periodic at every $x \in k$ except $\{0\}$ then S is p-almost periodic.*

Proof. — By lemma 2.6 it is enough to show that S is topologically p-ergodic at $\{0\}$. S verifies the assumptions of Proposition 2.3 for $H = G$. For every $\varepsilon > 0$ we choose $\varphi \in \mathcal{S}_p(G)$ as in proposition 2.3 and we choose $f, g \in \mathcal{S}_p(G)$ such that

$$\operatorname{diam} M_p(S) - \varepsilon = \operatorname{diam} M_p(\varphi S) - \varepsilon = |\langle \varphi S, f - g \rangle| .$$

As $\{0\}$ is a set of synthesis for $A_p(G)$ our assumption on S implies that $(f-g)\varphi S$ is p-almost periodic at every $x \in G$ hence p-almost periodic by lemma 2.6 b). By lemma 2.6 a), for every $\varepsilon > 0$ there exists a finite set $F \subset G$ such that for any mean $\hat{m}$ on $CV_p(G)$

$$\|(f-g)\varphi S - (\hat{m} * P_F)(f-g)\varphi S \|_{\overline{\ell^1(G)}\|} \quad \|_{CV_p(G)} \leqslant \varepsilon .$$

Let $W \subset G$ be a compact set such that

$$\|(f-g) - (f-g)\varphi_W\|_{A_p(G)} \leqslant \varepsilon \|S\|^{-1}_{CV_p(G)} .$$

Hence by our choice of φ

$$|\langle\varphi S, f-g\rangle| \leqslant \varepsilon + |\langle(f-g)\varphi S, \varphi_W\rangle| \leqslant 2\varepsilon + |\langle(\hat{m} * P_F)(f-g)\varphi S, \varphi_W\rangle|$$
$$= 2\varepsilon + |\langle(\hat{m} * (P_F - 1_{\{0\}}))\varphi S, \varphi_W(f-g)\rangle| \leqslant 4\varepsilon .$$

Hence diam $M_p(S) \leqslant 5\varepsilon$ and S is topologically p-ergodic at $\{0\}$.

THEOREM 2.8. – *Let G be a* lca *group,* $1 \leqslant p \leqslant 2$.

a) *Let $E \subset G$ be compact and scattered. Then $CV_p(E) = \overline{\ell^1(E)}^{\|\ \|_{CV_p(G)}}$ and $CV_p(E)$ has Radon-Nikodym property.*

b) *If $E \subset G$ is compact and not scattered $CV_p(E)$ does not have Radon-Nikodym property nor Schur property.*

Theorem 2.8 is obvious for $p = 1$. For $p = 2$ theorem 2.8 (a) is Loomis theorem [Loo].

Proof. – a) Proposition 2.7 implies that every $S \in CV_p(E)$ is p-almost periodic at every $x \in G$ exactly as in [Loo] proof of theorem 4, or as in the proof of corollary 2.4 above. Lemma 2.6 finishes the proof of the first assertion. Every separable subspace of $CV_p(E)$ is a subspace of $CV_p(E')$ where E' is a separable closed subset of E. Hence E' is compact and countable. By the first assertion $CV_p(E')$ is separable, and it is a dual space. Hence $CV_p(E')$ and $CV_p(E)$ have RNP.

b) The proof is the same as for $p = 2$ [L-P1] proposition 3 : By [V] chap. 4.3, E has a closed perfect subset E' such that

$$M(E') = CV_2(E') = CV_p(E')$$

and $M(E')$ does not have RNP nor the Schur property. □

Theorem 2.8 (a) implies the following corollary exactly as Loomis theorem implies [Gl] Proposition 4 :

COROLLARY 2.9. – *Let G be a* lca *group and let $F \subset G$ be closed and scattered. Then every $S \in CV_p(E)(1 \leqslant p \leqslant 2)$ is totally topologically p-ergodic.*

Proof. – We prove that S is topologically p-ergodic at $\{0\}$. Let $f \in \mathcal{S}_p(G)$ with a compact support. The support of fS is compact and scattered hence by theorem 2.8 (a) and lemma 2.6 fS is topologically p-ergodic at $\{0\}$ hence so is S.

Our aim now is to prove (theorem 2.14 below) that under the assumptions of theorem 2.8 (a) $CV_p(E)$ has the Schur property. Exactly as in the case $p = 2$ [L-P1] theorem 1, we begin with the case where E is a convergent sequence. The following lemma is crucial. It is a generalization of [W1], proof of theorem 9 (ii), and the proof uses the same ideas as lemma 2.2, proposition 2.3 above.

LEMMA 2.10. – *Let G be a* lca *group and $E = (e_k)_{k\geq 1} \subset G$ be a sequence such that $e_k \to 0(k \to +\infty)$ and $e_k \neq 0(k \geq 1)$. Let $1 \leqslant p \leqslant 2$.*

a) *For every $N \geqslant 1$ and $\varepsilon > 0$ there exists $W_{N,\varepsilon}$ a neighborhood of $\{0\}$ in G such that for every f, $g \in \mathcal{S}_p(G)$ there exists $h \in \mathcal{S}_p(G)$ such that*

(i) $\|g-h\|_{A_p(E_N)} \leqslant 2\varepsilon$

(ii) $\|f-g\|_{A_p(\bar{W}_{N,\varepsilon})} \leqslant 2\varepsilon$

where $E_N = \{e_1, \ldots, e_N\}$.

b) *Let 0 be an open subset of the compact metric topological space $\mathcal{S}_p^{00}(G)$ provided with $\sigma(A_p^{**}(G), CV_p(E))$. There exists W a neighborhood of $\{0\}$ in G such that for every $S \in CV_p(E)$ which is supported on W and every topological mean $\hat{m}$ on $CV_p(G)$*

(iii) $\sup_{f \in \mathcal{S}_p(G)} |\langle S - \langle S,\hat{m}\rangle \delta_0, f\rangle| \leqslant 2 \sup_{h \in 0} |\langle S - \langle S,\hat{m}\rangle\delta_0, h\rangle|$,

(iv) $\|S - \langle S,\hat{m}\rangle\delta_0\|_{CV_p(G)} \leqslant 2e \operatorname{diam}\{\langle S,0\rangle\}$.

Proof. – a) Let $(g_i)_{i \in I_{N,\varepsilon}}$ be a finite family in $\mathcal{S}_p(G)$ such that

(v) $\forall g \in \mathcal{S}_p(G)$, $\exists i \in I_{N,\varepsilon}$, $\|g - g_i\|_{A_p(E_N)} \leqslant \varepsilon$.

As $\{0\}$ is a set of synthesis for $A_p(G)$ there exists $V_{N,\varepsilon}$ a neighborhood of $\{0\}$ in G such that

(vi) $\forall i \in I_{N,\varepsilon}$, $\|g_i - 1\|_{A_p(\bar{V}_{N,\varepsilon})} \leqslant \varepsilon$,

where $\bar{V}_{N,\varepsilon}$ is the closure of $V_{N,\varepsilon}$ in G.

There exists a finite set $F_{N,\varepsilon} \subset G$ such that

(vii) $\|1 - P_{F_{N,\varepsilon}}\|_{A_2(E_N)} \leqslant \sum_{k=1}^{N} |1 - P_{F_{N,\varepsilon}}(e_k)| \leqslant \varepsilon$.

There exists $V'_{N,\varepsilon}$ a neighborhood of $\{0\}$ in G such that $V'_{N,\varepsilon} - V'_{N,\varepsilon} \subset V_{N,\varepsilon}$, and the $x_i + V'_{N,\varepsilon} - V'_{N,\varepsilon}(x_i \in F_{N,\varepsilon} \cup \{F_{N,\varepsilon} - F_{N,\varepsilon}\})$ are pairwise disjoint. There exists $W_{N,\varepsilon}$ a neighborhood of $\{0\}$ in G such that

(viii) $\|1-\varphi_{V'_{N,\varepsilon}}\|_{A_p(\bar{W}_{N,\varepsilon})} \leqslant \varepsilon$.

For every $f \in \mathscr{S}_p(G)$ $(\varphi_{V'_{N,\varepsilon}} f) * P_{F_{N,\varepsilon}} \in \mathscr{S}_p(G)$ by lemmas 2.1, 2.2 (a). Hence by (vii)

(ix) $\forall i \in I_{N,\varepsilon} ((\varphi_{V'_{N,\varepsilon}} f) * P_{F_{N,\varepsilon}}) g_i \in \mathscr{S}_p(G)$.

(x) $\|g_i - (\varphi_{V'_{N,\varepsilon}} f * P_{F_{N,\varepsilon}}) g_i\|_{A_p(E_N)} = \|g_i \sum_{k=1}^{k} (1 - P_{F_{N,\varepsilon}}(e_k))\|_{A_p(E_N)} \leqslant \varepsilon$.

For every $g \in \mathscr{S}_p(G)$ we choose $i_0 \in I_{N,\varepsilon}$ such that $\|g - g_{i_0}\|_{A_p(E_N)} \leqslant \varepsilon$. Let $h = ((\varphi_{V'_{N,\varepsilon}} f) * P_{F_{N,\varepsilon}}) g_{i_0}$.

Then $h \in \mathscr{S}_p(G)$ by (ix) and satisfies (i) by our choice of g_{i_0} and (x). Moreover by our choice of $V'_{N,\varepsilon}$, (viii) and (vi)

$$\|f-h\|_{A_p(\bar{W}_{N,E})} \leqslant \|f - \varphi_{V'_{N,\varepsilon}} f\|_{A_p(\bar{W}_{N,\varepsilon})} + \|(\varphi_{V'_{N,\varepsilon}} f) * P_{F_{N,\varepsilon}} - h\|_{A_p(\bar{W}_{N,\varepsilon})} \leqslant 2\varepsilon$$

which proves (ii).

b) Let 0 be as in the statement. By theorem 2.8 (a) there exist $h_0 \in \mathscr{S}_p(G)$, N and $0 < \varepsilon < (6e)^{-1}$ such that

$$0 \supset \{h \in \mathscr{S}_p^{00}(G) \mid \forall 1 \leqslant k \leqslant N \quad |h(e_k) - h_0(e_k)| < 2\varepsilon\}.$$

Let $W = W_{N,\varepsilon}$ be chosen as in (a). Let $S \in CV_p(E)$ which is supported on W and let $f \in \mathscr{S}_p(G)$ be such that

(xi) $(1-\varepsilon) \sup_{f' \in \mathscr{S}_p(G)} |\langle S - \langle S, \hat{m}\rangle \delta_0, f'\rangle| \leqslant |\langle S - \langle S, \hat{m}\rangle \delta_0, f\rangle|$.

Let us define h as in (a) for this f and $g = h_0$. By (i) $h \in 0$ and (ii), (xi) imply (iii) via (1).

We now prove (iv): let $P_{F_{N,\varepsilon}}$ be defined as in (a) and let

$$h' = (\hat{m} * P_{F_{N,\varepsilon}}) h_0.$$

By lemma 2.2 (b) $h' \in \mathscr{S}_p^{00}(G)$; for $k \geqslant 1$ $\langle \delta_{e_k}, h'\rangle = P_{F_{N,\varepsilon}}(e_k) h_0(e_k)$ hence $h' \in 0$ by (vii). By (ii) and our choice of W

(xii) $|\langle S - \langle S, \hat{m}\rangle \delta_0, f\rangle| = |\langle S, f\rangle - \langle S, \hat{m}\rangle| = |\langle S, f\rangle - \langle S, h'\rangle|$
$\leqslant 2\varepsilon \|S - \langle S, \hat{m}\rangle \delta_0\|_{CV_p(G)} + |\langle S, h\rangle - \langle S, h'\rangle|$.

Hence (xi) and (xii) imply (iv) via (1).

PROPOSITION 2.11. — *Let G be a* lca *group and $E \subset G$ be a compact countable set with only one cluster point. Then $CV_p(E)$ $(1 \leqslant p \leqslant 2)$ has the Schur property.*

Proof. — We assume that $E = (e_k)_{k \geqslant 1}$ as in lemma 2.10. Let $(S_n)_{n \geqslant 1}$ be a sequence in $CV_p(E)$ such that $S_n \to 0$ $\sigma(CV_p(E), A_p^{**}(G))$. By theorem 2.8 (a) and by eventually extracting a subsequence we may assume that there exists a sequence $(S'_n)_{n \geqslant 1}$ of measures whose finite support lies in $E \backslash \{0\}$, such that $\|S_n - S'_n\|_{CV_p(E)} \leqslant 2^{-n} (n \geqslant 1)$ and the S'_n are supported on disjoint blocks $\{e_{k_n}, e_{k_n+1}, \ldots, e_{k_{n+1}-1}\}$ where $(k_n)_{n \geqslant 1}$ is a strictly increasing sequence of positive integers. In order to prove the claim we assume that

$$\exists \delta > 0, \quad \forall n \geqslant 1, \quad \|S'_n\|_{CV_p(G)} > \delta$$

and we will show that this is impossible.

Let $C = \sup_n \|S_n\|_{CV_p(G)}$; we may assume that $\|S'_n\|_{CV_p(G)} \leqslant 2C$. Let $\varepsilon = \delta(8eC)^{-1}$. We define a subsequence $(S'_{n(j)})_{j \geqslant 1}$ and a decreasing sequence $(0_j)_{j \geqslant 1}$ in $\mathscr{S}_p^{00}(G)$ in the following way: $0_1 = \mathscr{S}_p^{00}(G)$; assume that 0_j and $S'_{n(j-1)}$ have been defined; by lemma 2.10 define a neighborhood W_j of $\{0\}$ in E such that assertion (iii) is satisfied for 0_j and ε; choose $n(j) > n(j-1)$ such that $S'_{n(j)}$ is supported on W_j, and 0_{j+1} such that

$$0_{j+1} = \{h \in 0_j \,|\, |\langle S_{n(j)}, h \rangle| \geqslant \sup_{h' \in 0_j} \langle S'_{n(j)}, h' \rangle| - \varepsilon \|S'_{n(j)}\|\}.$$

Take h_j in the closure of 0_j for $\sigma(A_p^{**}(G), CV_p(E))$ such that

$$|\langle S'_{n(j)}, h_j \rangle| = \sup_{h' \in 0_j} |\langle S'_{n(j)}, h'_j \rangle|, \qquad j \geqslant 1.$$

Let $h_0 \in \mathscr{S}_p^{00}(G)$ be a cluster point of $(h_j)_{j \geqslant 1}$ for $\sigma(\mathscr{S}_p^{00}(G), CV_p(E))$.

Then

$$\forall j \geqslant 1, \quad |\langle S'_{n(j)}, h_0 \rangle| \geqslant \frac{1}{2} \sup_{f \in \mathscr{S}_p^{00}(G)} |\langle S'_{n(j)}, f \rangle| - 2\varepsilon C \geqslant \delta/4e$$

by (1). Hence $(S'_{n(j)})_{j \geqslant 1}$ does not converge weakly to zero, which is a contradiction. □

This proof is similar to [L-P1] lemma 2. It is sufficient in order to prove theorem 2.14 below. But proposition 2.11 can be improved as follows :

DEFINITION 2.12. — *A Banach space X has the strong Schur property if there exists $C > 0$ such that for every $0 < \delta < 2$ and every sequence $(x_n)_{n\geqslant 1}$ in X such that*

(i) $\|x_n\| \leqslant 1$ $(n\geqslant 1)$

(ii) $\|x_n - x_k\| \geqslant \delta$ $(n\neq k)$

there exists a subsequence $(x_{n_k})_{k\geqslant 1}$ such that

(iii) $\forall \alpha_1, \ldots, \alpha_N \in \mathbb{C}$, $\left\| \sum_{k=1}^{N} \alpha_k x\ x_{n_k} \right\| \geqslant \delta C \sum_{k=1}^{N} |\alpha_k|$.

PROPOSITION 2.13. — *Let G be a* lca *group and $E \subset G$ be a compact countable set with only one cluster point. Then $CV_p(E)$ $(1\leqslant p\leqslant 2)$ has the strong Schur property.*

Proof. — By (1) we can consider $CV_p(E)$ as a closed subspace of the continuous functions on the compact space $\mathcal{S}_p^{00}(G)$ provided with the $\sigma(A_p^{**}(G), CV_p(E))$ topology. As $CV_p(E)$ is separable by theorem 2.8 (a) this topology is metrizable. Proposition 2.13 is thus implied by theorem B of [S], if we replace lemma 1 of [S] by lemma 2.10 (b).

We do not know whether $CV_p(E)$ still has the strong Schur property when E is compact countable with an infinite number of cluster points.

THEOREM 2.14. — *Let G be a* lca *group, let $E \subset G$ be compact and scattered. Then $CV_p(E)$ $(1\leqslant p\leqslant 2)$ has the Schur property.*

Proof. — As we deal with sequences of elements in $CV_p(E)$ theorem 2.8 (a) shows that we actually work in $CV_p(E_1)$ where $E_1 \subset E$ is compact and countable. We can now use the proof of [L-P1] theorem 1, writing « $CV_p(E_1)$ » instead of « $PM(E)$ ». The proof uses transfinite induction and deduces the general case from the particular case where E_1 has only one cluster point i.e. from proposition 2.11. □

3. A consequence of theorems 2.8 and 2.14.

Let G be a lca group, $1 \leqslant p \leqslant 2$.

We denote by $X_p(G)$ *the closed subspace of $CV_p(G_d)$ of those elements which are totally G-p-ergodic, and by $Y_p(G)$ the closed subspace of $CV_p(G)$ of those elements which are totally topologically p-ergodic.*

We first show the existence of bounded linear mappings $B_\omega : CV_p(G_d) \to CV_p(G)$ $(1 \leqslant p \leqslant 2)$ which are identity on finitely supported measures on G. They were already defined in [L-P2] for $p = 2$.

THEOREM 3.1. – *Let G be a* lca *group,* $1 \leqslant p \leqslant 2$. *Let* $(P_F)_{F \in \mathscr{F}}$ *be an approximate identity in* $A_2(G_d)$. *Let* ω *be a cluster point of* $(P_F)_{F \in \mathscr{F}}$ *for* $\sigma(A_2^{**}(G_d), CV_2(G_d))$. *Let us define* $B_\omega : CV_p(G_d) \to CV_p(G)$ *by*

$$\forall f \in A_p(G),\ \forall S \in CV_p(G_d),\quad \langle B_\omega(S), f\rangle = \langle fS, \omega\rangle .$$

This mapping has the following properties :

(i) $\|B_\omega\|_{CV_p(G_d) \to CV_p(G)} \leqslant 1$.

(ii) B_ω *restricted to finitely supported measures is identity.*

(iii) B_ω *commutes with multiplication by elements of* $B_p(G)$.

(iv) *If* $\Lambda \subset G$ *and* $\bar{\Lambda}$ *is the closure of* Λ *in* G, B_ω *maps* $CV_p(\Lambda_d)$ *into* $CV_p(\bar{\Lambda})$.

(v) B_ω *is one to one on* $X_p(G)$ *and sends* $X_p(G)$ *into* $Y_p(G)$.

Proof. – (i) By definition $\omega \in \mathscr{S}_2^{00}(G_d) \subset \mathscr{S}_p^{00}(G_d)$. By [Ey] theorem 1 $A_p(G)$ is a subspace of $B_p(G_d)$ hence $\langle fS, \omega\rangle$ is well defined and

$$|\langle fS, \omega\rangle| \leqslant \|fS\|_{CV_p(G_d)} \leqslant \|S\|\,\|f\|_{A_p(G)} .$$

(ii) As $P_F(x) \to 1 (F \in \mathscr{F})$ for every $x \in G$,

$$\langle f\delta_x, \omega\rangle = f(x) = \langle \delta_x, f\rangle$$

for every $f \in A_p(G)$ hence $B_\omega(\delta_x) = \delta_x$.

(iii) By [Ey] theorem 1 $B_p(G)$ is a subspace of $B_p(G_d)$ hence (iii) holds by the definition of B_ω.

(iv) is obvious from the definitions.

(v) Let $S \subset CV_p(G_d)$, $S \neq 0$. Hence there exists $x_0 \in G$ such that $\langle S, 1_{\{x_0\}}\rangle \neq 0$. If moreovoer $S \in X_p(G)$, $M_p^G(S_x) = \langle S, 1_{\{x\}}\rangle$ for every $x \in G$. By Lemma 1.8 for every $\varepsilon > 0$ and $x \in G$ there exists $\varphi \in \mathscr{S}_p(G)$ such that $\|\varphi_x S - \langle S, 1_{\{x\}}\rangle \delta_x\|_{CV_p(G_d)} \leqslant \varepsilon$. By (i), (ii), (iii) $\|\varphi_x B_\omega(S) - \langle S, 1_{\{x\}}\rangle \delta_x\|_{CV_p(G)} \leqslant \varepsilon$ which implies by lemma 1.8 again that $B_\omega(S) \in Y_p(G)$ and that $\varphi_{x_0} B_\omega(S)$ is not zero for a suitable φ. □

The following lemma is proved in [Loh1] chap. 2, theorem 1.1, proprosition 3.2.0. Actually a more general result is proved there and we recall a short proof for this particular case.

LEMMA 3.2. — *Let G be a* lca *group, $1 \leqslant p \leqslant 2$. Let μ be a finitely supported measure on G. Then $\|\mu\|_{CV_p(G)} = \|\mu\|_{CV_p(G_d)}$.*

Proof. — The inequality $\|\mu\|_{CV_p(G_d)} \leqslant \|\mu\|_{CV_p(G)}$ is proved by a computation similar to the proof of lemma 2.1 : Let k, k' be finitely supported functions in the unit sphere of $L^p(G_d)$ and $L^{p'}(G_d)$ respectively. Let W be an open neighborhood of $\{0\}$ in G such that the $x_i + W - W$ are pairwise disjoint for x_i lying in the union of the supports of k, k', μ. Hence

(i) $\langle \mu, k * \check{k}' \rangle = \langle \mu, (k * \check{k}') * \varphi_W \rangle$

(ii) $$(k * \check{k}') * \varphi_W = \left(|W|^{-1/p} \sum_{k(x_i) \neq 0} k(x_i)(1_W)_{x_i} \right) * \left(|W|^{-1/p} \sum_{k'(x_j) \neq 0} \check{k}'(x_j)(\check{1}_W)_{x_j} \right)$$

(iii) $$1 = \left\| |W|^{-1/p} \sum_{k(x_i) \neq 0} k(x_i)(1_W)_{x_i} \right\|_{L^p(G)} = \left\| |W|^{-1/p'} \sum_{k'(x_j) \neq 0} \check{k}'(x_j)(\check{1}_W)_{x_j} \right\|_{L^{p'}(G)}$$

hence $(\check{k} * \check{k}') * \varphi_W$ belongs to the unit ball of $A_p(G)$.

The converse inequality $\|\mu\|_{CV_p(G)} \leqslant \|\mu\|_{CV_p(G_d)}$ comes from theorem 3.1 (i) and (ii). □

We can now prove a consequence of theorem 2.8 and 2.14 ; for $p = 2$ it was proved in [L-P1] theorem 3 and partly in [L-P3] theorem 2.2, by two different methods.

THEOREM 3.3. — *Let G be a discrete abelian group and $\Lambda \subset G$. We assume that there exists a* lca *group H such that $G \to H$ (as it was defined in part* 1) *and the closure $\bar{\Lambda}$ of Λ in H is compact and scattered. Then $CV_p(\Lambda)$ is the norm closure in $CV_p(G)$ of finitely supported measures on Λ ; it has the Radon-Nikodym and the Schur property.*

We give a first proof which is similar to [L-P1] proposition 2, theorem 3, but simpler, owing to corollary 2.4.

Proof. — By assumption G is a closed subgroup of H_d hence by [H1] theorem A, $CV_p(G)$ is a closed subspace of $CV_p(H_d)$ and $CV_p(\Lambda)$ is a closed subspace of $CV_p((\bar{\Lambda})_d) \subset CV_p(H_d)$. By theorem 3.1 (iv) and theorem 2.8, $B_\omega : CV_p((\bar{\Lambda})_d) \to \overline{\ell^1(\bar{\Lambda})}^{\|\,\|_{CV_p(H)}}$. By lemma 3.2 there exists an isometry which we denote by A : $\overline{\ell^1(\bar{\Lambda})}^{\|\,\|_{CV_p(H)}} \to \overline{\ell^1(\bar{\Lambda})}^{\|\,\|_{CV_p(H_d)}}$ which is identity when restricted to finitely supported measures.

By corollary 2.4 $CV_p((\bar{\Lambda})_d)$ lies in $X_p(H)$, hence with the notations of the proof of theorem 3.1 (v) for every $S \subset CV_p((\bar{\Lambda})_d)$ and $x \in G$

$$|\langle A \circ B_\omega(S), 1_{\{x\}}\rangle - \langle S, 1_{\{x\}}\rangle| = |\langle \varphi_x A \circ B_\omega(S), 1_{\{x\}}\rangle - \langle S, 1_{\{x\}}\rangle$$
$$\leqslant \|A\| \, \|\varphi_x B_\omega(S) - \langle S, 1_{\{x\}}\rangle \, \delta_x\|_{CV_p(H)} \leqslant \varepsilon$$

which implies that $A \circ B_\omega$ is identity on $CV_p((\bar{\Lambda})_d)$. This proves that $CV_p((\bar{\Lambda})_d) = \overline{\ell^1(\bar{\Lambda})}^{\|\ \|_{CV_p(H_d)}}$; as $\|B_\omega\| \leqslant 1$ this proves also that B_ω is an isometry : $CV_p((\bar{\Lambda})_d) \to CV_p(\bar{\Lambda})$. Hence theorem 2.8 and 2.14 imply that $CV_p((\bar{\Lambda})_d)$ and its subspace $CV_p(\Lambda)$ have RNP and the Schur property. □

Alternatively theorem 3.3 has another proof which is similar to [L-P3] theorem 2.2 : We keep the previous notations. By lemma 3.2 the spaces $\overline{\ell^1(\bar{\Lambda})_d)}^{\|\ \|_{CV_p(H_d)}}$ and $\overline{\ell^1(\bar{\Lambda})}^{\|\ \|_{CV_p(H)}}$ are isometric, hence by theorem 2.8 and 2.14 the first one has RNP and the Schur property. It remains to prove that this space is the same as $CV_p((\bar{\Lambda})_d)$ which is a consequence of the following lemma, a generalization of [L-P3] theorem 2.1 :

Lemma 3.4. — *Let G be a discrete abelian group, $\Lambda \subset G$, $1 \leqslant p \leqslant 2$. Then $\overline{\ell^1(\Lambda)}^{\|\ \|_{CV_p(G)}}$ has* RNP *iff it coincides with $CV_p(\Lambda)$.*

Proof. — Let S $CV_p(\Lambda)$. It defines a bounded multiplier: $A_2(G) \to CV_p(\Lambda)$, $f \rightsquigarrow fS$. As functions with finite support are dense in $A_2(G)$ the range of this multiplier lies in $\overline{\ell^1(\Lambda)}^{\|\ \|_{CV_p(\Lambda)}}$. If this space has RNP there exists a bounded strongly measurable function F: $\hat{G} \to \overline{\ell^1(\Lambda)}^{\|\ \|_{CV_p(G)}}$ such that

$$\forall f \in A_2(G), \quad fS = \int_{\hat{G}} \hat{f}(\gamma) F(\gamma)\, d\gamma .$$

In particular for every $\gamma' \in \hat{G}$

$$\int_{\hat{G}} \hat{f}(\gamma) \hat{S}(\gamma' - \gamma)\, d\gamma = fS(\gamma') = \int_{\hat{G}} \hat{f}(\gamma) \widehat{F(\gamma)}(\gamma')\, d\gamma$$

hence for almost all $\gamma \in \hat{G}$, $\widehat{F(\gamma)}(\gamma') = \widehat{(\gamma S)}(\gamma')$ and $F(\gamma) = \gamma S$. In particular $S \in \overline{\ell^1(\Lambda)}^{\|\ \|_{CV_p(G)}}$.

Conversely if $\overline{\ell^1(\Lambda)}^{\|\ \|_{CV_p(G)}} = CV_p(\Lambda)$ the same equality is true for every countable subset $\Lambda' \subset \Lambda$: hence $CV_p(\Lambda')$ is a separable dual and has RNP. This implies that every separable subspace of $CV_p(\Lambda)$ (which is a subspace of a $CV_p(\Lambda')$ where Λ' is countable) has RNP, hence $CV_p(\Lambda)$ has RNP.

DEFINITION 3.5. – *Let G be a discrete group,* $\Lambda \subset G$, $1 \leqslant p \leqslant 2$. *If* $\overline{\ell^1(\Lambda)}^{\|\ \|_{CV_p(G)}} = CV_p(\Lambda)$ *we call* Λ *a p-Rosenthal set.*

Obviously every Λ is a 1-Rosenthal set and a 2-Rosenthal set is usually called a Rosenthal set. Theorem 3.3 gives examples of sets Λ which are p-Rosenthal for every $1 \leqslant p \leqslant 2$. We do not know whether « Λ is p-Rosenthal » implies « Λ is q-Rosenthal » for $1 < q < p$, but we have the following result :

LEMMA 3.6. – *Let G be a countable discrete abelian group and* $\Lambda \subset G$. *Let* $1 < q < p \leqslant 2$. *Let* Λ *be a p-Rosenthal set.*

a) *Every bounded sequence in* $A_p(\Lambda)$ *has a weak Cauchy subsequence.*

b) *If* $\overline{\ell^1(\Lambda)}^{\|\ \|_{CV_p(G)}}$ *is weakly complete* Λ *is q-Rosenthal.*

Proof. – a) By assumption $CV_p(\Lambda)$ is a separable dual. Hence its predual $A_p(\Lambda)$ has no ℓ^1-sequence. Rosenthal's theorem [R] implies the claim.

b) Let $(P_{F_n})_{n\geqslant 1}$ be an approximate identity in $A_2(G)$. By (a) the sequence $(R(P_{F_n}))_{n\geqslant 1}$ of restrictions to Λ has a weak-Cauchy subsequence in $A_p(\Lambda)$. As identity : $CV_q(\Lambda) \to CV_p(\Lambda)$ is continuous, so is : $A_p(\Lambda) \to A_q(\Lambda)$. Hence $(R(P_{F_n}))_{n\geqslant 1}$ has a weak Cauchy subsequence in $A_q(\Lambda)$. For every $S \in CV_q(\Lambda)$, $n \geqslant 1$, $P_{F_n}S = R(P_{F_n})S \in \overline{\ell^1(\Lambda)}^{\|\ \|_{CV_q(G)}}$ and $P_{F_n}S \to S$, $\sigma(CV_q(\Lambda), A_q(\Lambda))$. It also has a weak Cauchy subsequence in $\overline{\ell^1(\Lambda)}^{\|\ \|_{CV_q(G)}}$ hence it converges weakly to S and S lies in $\overline{\ell^1(\Lambda)}^{\|\ \|_{CV_p(G)}}$. □

If $\Lambda \subset G$ is a Sidon set identity is continuous (by definition) :

$$\ell^1(\Lambda) \to CV_p(\Lambda) \to CV_2(\Lambda) \to \ell^1(\Lambda).$$

If $\Lambda_2 \subset G_1$ and $\Lambda_2 \subset G_2$ are two Sidon sets we have

$$\ell^1(\Lambda_1 \times \Lambda_2) \to CV_p(\Lambda_1 \times \Lambda_2) \to CV_2(\Lambda_1 \times \Lambda_2) = \ell^1 \hat{\hat{\otimes}} \ell^1.$$

Is $\Lambda_1 \times \Lambda_2$ a p-Rosenthal set for $1 < p < 2$? (This is true if Λ_1 and Λ_2 satisfy the assumptions of theorem 3.3 because $\Lambda_1 \times \Lambda_2$ also satisfy them.) We can also define p-Riesz sets as follows :

DEFINITION 3.7. — *Let G be a discrete abelian group and $\Lambda \subset G$, $1 < p \leqslant 2$. Λ is a p-Riesz set if every $f \in B_p(G)$ which is supported on Λ lies in $A_p(G)$.*

A 2-Riesz set is usually called a Riesz set. We do not define 1-Riesz sets because $A_1(G) = C_0(G)$, $B_1(G) = \ell^\infty(G)$ hence no infinite set is 1-Riesz. In order to generalize results on Riesz sets for p-Riesz sets $(1<p<2)$ it is necessary to know whether $A_p(G)$ is weakly complete or not when G is discrete, which the author does not know.

(This is true if G is compact by [L-P4] theorem 4.)

If there exists $f \in B_2(G)$ which is supported on Λ and such that $f \notin C_0(G)$ Λ is not a p-Riesz set for any $1 < p \leqslant 2$ because f is not in $A_p(G)$. This is the case if Λ contains the spectrum of a Riesz product.

4. Transfer theorems.

We have already proved one transfer theorem, namely theorem 3.1. We now prove a « converse » one, by defining mappings $A_{\hat{m}} : CV_p(G) \to CV_p(G_d)$. Actually all these mappings will coincide on $\overline{\ell^1(G)}^{\| \|_{CV_p(G)}}$ and their common restriction is the mapping A which we already used in the proof of theorem 3.3. Mappings A and B_ω were already used implicitly in [Loh1], [Loh2]. For $p = 2$ $A_{\hat{m}}$ was defined in [W2], p. 104 and [W1], p. 292, on $UC_2(G)$ and it was defined in full generality in [L-P2]. The proof below is different.

THEOREM 4.1. — *Let G be a* lca *group, $1 \leqslant p \leqslant 2$. Let $(P_F)_{F\in\mathcal{F}}$ be an approximate identity in $A_2(G_d)$. Let $\hat{m}$ be a topological mean on $CV_p(G)$. The linear mapping $A_{\hat{m}} : CV_p(G) \to CV_p(G_d)$ is defined by*

$$\forall S \in CV_p(G),\ \forall f \in A_p(G_d),\quad \langle A_{\hat{m}}(S), f\rangle = \lim_{\mathcal{F}} \langle(\hat{m} * P_F)S, f\rangle$$

$A_{\hat{m}}$ has the following properties :

(i) $\|A_{\hat{m}}\|_{CV_p(G)\to CV_p(G_d)} \leqslant 1$.

(ii) *$A_{\hat{m}}$ restricted to finitely supported measures on G is identity.*

(iii) *$A_{\hat{m}}$ commutes with multiplication by functions of $B_p(G)$.*

(iv) *If $E \subset G$ is a closed subset*

$$A_{\hat{m}} : CV_p(E) \to CV_p(E_d).$$

(v) *$A_{\hat{m}}$ maps $Y_p(G)$ into $X_p(G)$.*

Proof. – We first explain the definition of $A_{\hat{m}}$. $(\hat{m} * P_F)S$ is defined as in proposition 2.3, lemma 2.6 and part 1 by

(vi) $$\forall f \in A_p(G),\quad \langle(\hat{m} * P_F)S, f\rangle = \langle fS, \hat{m} * P_F\rangle = \sum_{P_F(x_i)\neq 0} P_F(x_i)\langle S, \hat{m}_{x_i}\rangle\langle\delta_{x_i}, f\rangle .$$

It is a finitely supported measure on G. By lemmas 2.2 and 3.2

(vii) $$\|S\|_{CV_p(G)} \geqslant \|(\hat{m} * P_F)S\|_{CV_p(G)} = \|(\hat{m} * P_F)S\|_{CV_p(G_d)} .$$

Let $f \in A_p(G_d)$ with a finite support. Then

(viii) $$\langle(\hat{m} * P_F)S, f\rangle = \sum_{f(x_i)\neq 0} P_F(x_i) f(x_i)\langle S, \hat{m}_{x_i}\rangle \underset{\mathscr{F}}{\to} \sum_{f(x_i)\neq 0} f(x_i)\langle S, \hat{m}_{x_i}\rangle .$$

Hence $(\hat{m} * P_F)S = \sum_{P_F(x_i)\neq 0} P_F(x_i)\langle S, \hat{m}_{x_i}\rangle\delta_{x_i}$ $(F \in \mathscr{F})$ is a bounded net in $CV_p(G_d)$ which converges for $\sigma(CV_p(G_d), A_p(G_d))$ to a limit which we denote by $A_{\hat{m}}(S)$. $A_{\hat{m}}$ is clearly a linear mapping.

(i) is implied by (vii) and (viii); (ii) is implied by (viii) because $\langle\mu, \hat{m}_{x_i}\rangle = \mu(x_i)$ if μ is finitely supported.

(iii) Let $F \in B_p(G)$ and $\varphi \in \mathscr{S}_p(G)$. For every $x \in G$, $\varphi_x F \in A_p(G)$. As x is a point of synthesis for $A_p(G)$ lemma 1.5 (b) implies $\langle\varphi_x FS, \hat{m}_x\rangle = F(x)\langle S, \hat{m}_x\rangle$. As $\langle\varphi_x FS, \hat{m}_x\rangle = \langle FS, \hat{m}_x\rangle$ (viii) implies (iii).

(iv) By lemma 1.5 (b) $\langle S, \hat{m}_x\rangle = 0$ if x lies outside the support of S. Hence (viii) implies (iv).

(v) If we write $M_p(S_x)$ instead of $\langle S, 1_{\{x\}}\rangle$ the proof of (v) is similar to the proof of (v) in theorem 3.1. □

Let us notice however that $A_{\hat{m}}$ is not one to one on $Y_p(G)$: e.g. if $\mu \in M(G)$ is a diffuse measure $A_{\hat{m}}(\mu) = 0$. This will be precised in theorem 4.2 below.

Theorem 4.2 provides an Eberlein p-decomposition for elements of $Y_p(G)$.

THEOREM 4.2. – *Let G be a* lca *group,* $1 \leqslant p \leqslant 2$. *Let $\hat{m}$ be any topological mean on $CV_p(G)$, let $A_{\hat{m}}$ and B be as in theorems* 3.1, 4.1.

a) *$A_{\hat{m}} \circ B_\omega$ is identity on $X_p(G)$; B_ω is an isometry on $X_p(G)$, $A_{\hat{m}}$ is an isometry on $B_\omega(X_p(G))$.*

b) *For every* $S \in Y_p(G)$, $S = B_\omega \circ A_{\hat{m}}(S) + S'_\omega$ *where*

$$B_\omega \circ A_{\hat{m}}(S) \in Y_p(G)$$

and does not depend on $\hat{m}$, *and* $A_{\hat{m}}(S'_\omega) = 0$.

c) *If* $\hat{m}$ *is a topological mean on* $CV_2(G)$, $X_p(G)$ *and* $Y_p(G)$ *can be replaced by* $X_2(G) \cap CV_p(G_d)$ *and* $Y_2(G) \cap CV_p(G)$ *in the assertions above.*

For $p = 2$ this result was partly proved in [W2] corollary 2, and proved in [L-P2] theorem 7.

Proof. – a) Let $S \in X_p(G)$. By the proof of theorem 3.1 (v), for every $\varepsilon > 0$ and $x \in G$ there exists $\varphi \in \mathcal{S}_p(G)$ such that

$$\|\varphi_x B_\omega(S) - \langle S, 1_{\{x\}}\rangle \delta_x\|_{CV_p(G)} \leqslant \varepsilon$$

hence by theorem 4.1

$$\|\varphi_x A_{\hat{m}} \circ B_\omega(S) - \langle S, 1_{\{x\}}\rangle \delta_x\|_{CV_p(G_d)} \leqslant \varepsilon$$

hence

$$\forall x \in G, \quad \langle A_{\hat{m}} \circ B_\omega(S), 1_{\{x\}}\rangle = \langle S, 1_{\{x\}}\rangle .$$

As $\|B_\omega\|$, $\|A_{\hat{m}}\| \leqslant 1$ the rest of the claim is now obvious.

b) Let $S \in Y_p(G)$. By theorems 4.1 (v) and 3.1 (v) $A_{\hat{m}}(S) \in X_p(G)$ and $B_\omega \circ A_{\hat{m}}(S) \in Y_p(G)$. By (a) $(A_{\hat{m}} \circ B_\omega) \circ A_{\hat{m}}(S) = A_{\hat{m}}(S)$ hence $S - B_\omega \circ A_{\hat{m}}(S) \in \ker A_{\hat{m}}$. On the other hand all $A_{\hat{m}}$ coïncide on $Y_p(G)$ for topological means $\hat{m}$ on $CV_p(G)$.

c) By (a) $A_{\hat{m}} \circ B_\omega$ is identity on $X_2(G)$ hence on $X_2(G) \cap CV_p(G_d)$. The rest of the proof is similar to the proof of (a), (b). □

Theorem 4.2 (c) implies [Loh1] chap. 2, corollaire de la proposition III. 2.0, p. 56, where $\overline{\ell^1(G)}^{\|\ \|_{CV_2(G)}} \cap CV_p(G)$ is shown to be isometric to $\overline{\ell^1(G)}^{\|\ \|_{CV_2(G_d)}} \cap CV_q(G_d)$. We do not know whether $X_2(G) \cap CV_p(G_d)$ is strictly larger than $X_p(G)$ or not (and the same question for $Y_2(G) \cap CV_p(G)$ and $Y_p(G)$). However let $1 \leqslant q < p$ and let $S \in CV_p(G_d)$. Lemma 1.8 and the interpolation inequality $\left(\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{2}\right)$

$$\|S\|_{CV_p(G_d)} \leqslant \|S\|^{\theta}_{CV_q(G_d)} \|S\|^{1-\theta}_{CV_2(G_d)}$$

imply that if $S \in X_2(G)$ then $S \in X_p(G)$. In the same way $CV_q(G) \cap Y_2(G) \subset Y_p(G)$. The following result is a generalization of [Gl] theorem 4, where $p = 2$.

THEOREM 4.3. — *Let G be a* lca *group, let $E \subset G$ be closed and scattered. Let $1 \leqslant p \leqslant 2$. Then $CV_p(E)$ and $CV_p(E_d)$ are isometric.*

Proof. — By corollary 2.4, $CV_p(E)$ is a closed subspace of $X_p(G)$. By theorem 4.2 (a) (b) B_ω is an isometry : $CV_p(E_d) \to CV_p(E)$ and $A_{\bar{m}}$ is an isometry : $CV_p(E) \to CV_p(E_d)$ if $CV_p(E) \subset B_\omega(X_p(G))$ hence if $A_{\bar{m}}$ is one to one on $CV_p(E)$. Let now $S \in CV_p(E)$, $S \neq 0$. Hence the support E' of S is a closed non empty subset of E. As E is scattered let x be an isolated point of E'. Let V be a neighborhood of x in G such that $V \cap E' = \{x\}$. By assumption there exists $\varphi_V \in A_p(G)$ which is supported on V and such that $\langle S, \varphi_V \rangle$ is not zero. The support of $\varphi_V S$ is $\{x\}$ hence $\varphi_V S = \langle S, \varphi_V \rangle \, \delta_x$ and $A_{\bar{m}}(\varphi_V S)$ is not zero. By theorem 4.1 (iii) $A_{\bar{m}}(\varphi_V S) = \varphi_V A_{\bar{m}}(S)$ which proves the claim.

Alternatively we could have used Glowacki's result (whose proof is the same as above, for $p = 2$) and theorem 4.2 (c). □

Theorem 3.3 is an obvious consequence of theorems 4.3, 2.8, 2.14. But we prefered to give a direct simpler proof.

Our next aim is to precise the Eberlein decomposition of $S \in CV_p(G)$ when S is p-weak almost periodic. We first establish a general lemma :

LEMMA 4.4. — *Let G be a* lca *group and $1 \leqslant p \leqslant 2$. $CV_p(G)$ is isometric to the space of multipliers : $A_p(G) \to CV_2(G)$ and to the space of multipliers : $A_2(G) \to CV_p(G)$ provided with operator norm.*

Proof. — (i) For every $f \in A_p(G)$, $g \in A_2(G)$, $S \in CV_p(G)$

$$\langle fS, g \rangle = \langle gS, f \rangle = \langle S, gf \rangle$$

hence

$$\|S\|_{A_p \to CV_2} = \|S\|_{A_2 \to CV_p} \leqslant \|S\|_{A_p \to CV_p} \leqslant \|S\|_{CV_p}.$$

(ii) Conversely let S be a multiplier : $A_2(G) \to CV_p(G)$. Let $(\varphi_\alpha)_{\alpha \in A}$ be an approximate identity with compact support in $\mathcal{S}_2(G)$. Hence $\|S(\varphi_\alpha)\|_{CV_p(G)} \leqslant \|S\|_{A_2 \to CV_p}$. For every $f \in A_p(G)$ with a compact support K there exists $g_K \in A_2(G)$ such that $g_K = 1$ on K. Hence as $\|\varphi_\alpha g_K - g_K\|_{A_2(G)} \xrightarrow{\alpha} 0$

$$\langle S(\varphi_\alpha), f \rangle = \langle S(\varphi_\alpha), g_K f \rangle = \langle S(\varphi_\alpha g_K), f \rangle \to \langle S(g_K), f \rangle .$$

It implies that $(S(\varphi_\alpha))_{\alpha \in A}$ converges for $\sigma(CV_p(G), A_p(G))$; let $s \in CV_p(G)$, $\|s\|_{CV_p(G)} \leqslant \|S\|_{A_2 \to CV_p}$ be the limit. In particular for f as above $\langle S(g_K), f\rangle = \langle g_K s, f\rangle$. We now verify that $hs = S(h)$ in $CV_p(G)$ when $h \in A_2(G)$. It is sufficient to prove it when h has a compact support K. Then for every $f \in A_p(G)$, as $g_K h = h$

$$\langle hs - S(h), f\rangle = \langle g_K s - S(g_K), hf\rangle = 0.$$

It implies the above claim hence $\|S\|_{A_2 \to CV_p} \leqslant \|s\|_{CV_p(G)}$.

The assertion of the lemma is now obvious. □

Let us recall the definition of p-WAP(G), the weak p-almost periodic elements of $CV_p(G)$:

DEFINITION 4.5 [Gra]. – *Let G be a* lca *group, $1 \leqslant p \leqslant 2$. p-WAP(G) is the subspace of $CV_p(G)$ of elements S which define weakly compact multipliers : $A_p(G) \to CV_p(G)$.*

Let $S \in CV_p(G)$. By remark 1.2 it is easy to see that $S \in p$-WAP(G) iff $\{fS\}_{f \in \mathcal{S}_p(G)}$ is relatively compact for $\sigma(CV_p(G), A_p^{**}(G))$ hence iff $\{fS\}_{f \in \mathcal{S}_p(G)}$ is relatively weakly compact in $C(\mathcal{S}_p(G))$, which means by [BJM] chapter 3, definition 8.1, that S is a weak almost periodic function on the semi-group $\mathcal{S}_p(G)$.

In the same way S is a compact multiplier: $A_p(G) \to CV_p(G)$ iff S is an almost periodic function on the semi-group $\mathcal{S}_p(G)$ [BJM] 3, definition 9.1.

By [Gra], proposition 9, p-WAP(G) is a closed subspace of $Y_p(G)$.

By [Gra] proposition 7, $M(G)$ is a subspace of p-WAP(G).

Assertion (c) $\Leftrightarrow$ (d) in the next theorem is Eberlein's decomposition of WAP function on $\hat{G}$ [Eb2] when $p = 2$. (b) $\Leftrightarrow$ (d) is a particular case of [BJM] chapter 3, corollary 16.14.

THEOREM 4.6. – *Let G be a* lca *group, $G \to H$, $1 \leqslant p \leqslant 2$. Let $S \in CV_p(G)$. The following assertions are equivalent :*

a) $S \in p$-WAP(G).

b) *$\mathcal{S}_p(G)S$ is relatively weakly compact in $CV_p(G)$.*

c) *$\mathcal{S}_p(H)S$ is relatively weakly compact in $CV_p(G)$.*

d) *$S = B_\omega \circ A_{\hat{m}}(S) + S'$ where $\hat{m}$ is a topological mean on $CV_p(G)$, B_ω, $A_{\hat{m}}$ are defined as in theorems* 3.1, 4.1, *$B_\omega \circ A_{\hat{m}}(S)$ belongs to $\overline{\ell^1(G)}^{\|\ \|_{CV_p(G)}}$ and does not depend on ω nor on $\hat{m}$, $S' \in p$-*WAP(G) *and* $A_{\hat{m}}(S') = 0$.

Proof. – (a) $\Rightarrow$ (b) is obvious.

(a) $\Leftarrow$ (b) is easy by remark 1.2 as we already told above.

(b) $\Rightarrow$ (c) : When we studied K_S in part 1 we saw that $\mathscr{S}_p(H)S$ lies in K_S.

If (b) holds K_S is the norm closure of $\mathscr{S}_p(G)S$ and K_S is weakly compact in $CV_p(G)$.

(c) $\Rightarrow$ (b) : By lemma 1.14, $\mathscr{S}_p(H)S$ is dense in K_S for $\sigma(CV_p(G), A_p(G))$.

If (c) holds K_S is the norm closure of $\mathscr{S}_p(H)S$ and K_S is weakly compact.

(b) $\Rightarrow$ (d) : the assumption implies that $S \in Y_p(G)$ hence theorem 4.2 (b) holds. We claim that $A_{\hat{m}}(S)$ lies in $\overline{\ell^1(G)}^{\|\ \|_{CV_p(G_d)}}$: by definition and lemma 2.2 $\{(\hat{m} * P_F)S \mid F \subset G,\ F \text{ finite}\}$ lies in K_S and in $\overline{\ell^1(G)}^{\|\ \|_{CV_p(G)}}$ (see the proof of theorem 4.1 (a)). By assumption it is relatively weakly compact in $CV_p(G)$ hence in $\overline{\ell^1(G)}^{\|\ \|_{CV_p(G)}}$ hence in $\overline{\ell^1(G)}^{\|\ \|_{CV_p(G_d)}}$ by lemma 3.2. The definition of $A_{\hat{m}}$ (see the proof of theorem 4.1) now proves the claim. As B_ω is identity on $\ell^1(G)$

$$B_\omega \circ A_{\hat{m}}(S) \in \overline{\ell^1(G)}^{\|\ \|_{CV_p(G)}},$$

it does not depend on ω, nor on $\hat{m}$ by theorem 4.2 (b), it lies obviously in p-WAP(G).

$d \Rightarrow a$ is obvious. □

Motivated by lemma 4.4 and a result of Lohoué on compact multipliers : $A_p(G) \to CV_p(G)$ [Loh1] chap. 2, theorem III.1, p. 50, we also consider elements of $CV_p(G)$ which are weakly compact multipliers : $A_2(G) \to CV_p(G)$. We do not know if they are weakly compact multipliers : $A_p(G) \to CV_p(G)$, but they have analogous properties. In particular they lie in $Y_p(G)$: let W be a decreasing basis of neighborhoods of $\{0\}$ in G. If $S \in CV_p(G)$ and if $\mathscr{S}_2(G)S$ is relatively weakly compact in $CV_p(G)$ $(\varphi_W S)_{W \in W}$ has a weak cluster point which must be a scalar multiple of δ_0 and which belongs to the norm closure of $\mathscr{S}_2(G)S$. Lemma 1.8 finishes the proof.

THEOREM 4.7. — *Theorem* 4.6 *holds true if we replace* $\mathscr{S}_p(G)$ *by* $\mathscr{S}_2(G)$ *and p*-WAP(G) *by the set of weakly compact multipliers* : $A_2(G) \to CV_p(G)$.

Proof. — By lemma 4.4 such a multiplier is given by an element $S \in CV_p(G)$. The proof then follows the same lines as the proof of theorem 4.6. It is even simpler: for example lemma 1.14 is obvious for $p = 2$, it implies that $\mathscr{S}_2(H)S$ and $\mathscr{S}_2(G)S$ have the same closure for $\sigma(CV_p(G), A_p(G))$. If $\hat{m}$ is a topological mean on $CV_2(G)$ and if $\mathscr{S}_2(G)S$ is relatively weakly compact in $CV_p(G)$ $(\hat{m} * P_F)S$ lies in the norm closure of $\mathscr{S}_2(G)S$ hence $A_{\hat{m}}(S) \in \overline{\ell^1(G)}^{\| \|_{CV_p(G_d)}}$ by the same proof as in theorem 4.6. As $S \in Y_p(G)$ $A_{\hat{m}}(S)$ does not depend on $\hat{m}$ when $\hat{m}$ is a topological mean on $CV_p(G)$. □

Theorem 4.7 implies the following improvement of [Loh1] chap. 2, theorem III.1 :

THEOREM 4.8. — *Let* G *be a* lca *group,* $G \to H$, $1 \leqslant p \leqslant 2$, *let* $S \in CV_p(G)$.

The following assertions are equivalent :

(a) $S \in \overline{\ell^1(G)}^{\| \|_{CV_p(G)}}$.

(b) S *is a compact multiplier* : $A_p(G) \to CV_p(G)$.

(c) $\mathscr{S}_p(G)S$ *is relatively compact in* $CV_p(G)$.

(d) $\mathscr{S}_2(G)S$ *is relatively compact in* $CV_p(G)$.

(e) $\mathscr{S}_2(H)S$ *is relatively compact in* $CV_p(G)$.

(f) $\mathscr{S}_2(G)S$ *is relatively weakly compact in* $CV_p(G)$ *and relatively compact in* $CV_2(G)$.

Proof. — (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (f) are obvious.

(e) $\Leftrightarrow$ (d) by the proof of theorem 4.7.

(f) $\Rightarrow$ (a) : By theorem 4.7

$$S = B_\omega \circ A_{\hat{m}}(S) + S' \quad \text{and} \quad B_\omega \circ A_{\hat{m}}(S) \in \overline{\ell^1(G)}^{\| \|_{CV_p(G)}}.$$

We only have to prove that $S' = 0$ in $CV_p(G)$ or that $S' = 0$ in $CV_2(G)$. We know that $A_{\hat{m}}(S') = 0$ and that $\mathscr{S}_2(\bar{G})S'$ is relatively compact in $CV_2(G)$ because $\mathscr{S}_2(\bar{G})S'$ is $\sigma(CV_2(G), A_2(G))$ dense in the $\sigma(CV_2(G), A_2(G))$ closure of $\mathscr{S}_2(G)S'$. Hence $\hat{S}'$ is an almost periodic function on $\hat{G}$ in the usual sense and $\langle \chi S', m \rangle = 0$ for every character χ on $\hat{G}$ and every mean m on $L^\infty(\hat{G})$. Hence $S' = 0$ by classical results. □

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