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TOPOLOGICAL STABILITY THEOREM FOR COMPOSITE MAPPINGS

by Isao NAKAI

CONTENTS

CHAPTER 0

INTRODUCTION

0.1. The main theorems and some other known results.

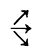
Let $G = (V.L.\Lambda)$ be an oriented graph where V is the set of vertices, L the set of edges and $\Lambda = (\alpha.\beta) : L \to V \times V$ is the orientation. Let $M = (M_v)_{v \in V}$ be a collection of C^∞ manifolds. *A diagram of smooth* (proper) *mappings on* (G,M) is a family $f = (f_\ell)_{\ell \in L}$ of (proper) mappings $f_\ell : M_{\alpha(\ell)} \to M_{\beta(\ell)}$. We denote the set of those diagrams by

$$C^\infty(G,M) = \prod_{\ell \in L} C^\infty(M_{\alpha(\ell)}, M_{\beta(\ell)}), \quad C^\infty_{\mathrm{pr}}(G,M) = \prod_{\ell \in L} C^\infty_{\mathrm{pr}}(M_{\alpha(\ell)}, M_{\beta(\ell)}).$$

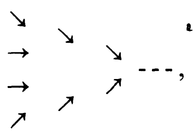
Two diagrams f, $g \in C^\infty(G,M)$ are C^r *equivalent* (*topologically equivalent*, if $r=0$) if there are C^r diffeomorphisms ϕ_v of M_v such that $\phi_{\beta(\ell)} \circ f_\ell = g_\ell \circ \phi_{\alpha(\ell)}$ for $\ell \in L$. The C^r equivalence class of f is denoted $\mathcal{O}^r(F)$ and f is C^r *stable* if $\mathcal{O}^r(f)$ is a neighbourhood of f in the Whitney topology.

Our first question is:

Is C^r stability a generic property?

It is easy to see that the answer to this question depends deeply on the combinatorial type of the underlying graph G and manifolds M_v, $v \in V$. For example if G is of the types either  (cycle) or  (divergent) it is known that topological stability does not hold in general by the study of discrete dynamics and Web geometry [Ca, Du 2-3]. We will touch on these counter examples later in this section and also in Appendix 2.

The graphs which we study in this paper are the (finite) convergent graphs:



defined below. We will establish a foundation for differential calculus of convergent diagrams of smooth mappings for such graphs.

The relation $\alpha(\ell) < \beta(\ell)$, $\ell \in L$ generates the partial order $<$ of vertices for an oriented tree. A finite oriented tree G is *convergent* if there is only one maximal vertex v_0: the *root* (*sink*) of G. If G is convergent, then for each vertex $v \neq v_0$, there is a unique edge ℓ_v with source $\alpha(\ell_v) = v$. We define $\beta(v)$ by $\beta(\ell_v)$ for $v \neq v_0$. The *height of* a vertex $v \in V$ is defined inductively $h(v_0) = 0$; otherwise $h(v) = 1 + h(\beta(v))$. Each vertex v of G defines a *branch* G_v, which is the subgraph consisting of all vertices $v' \leqslant v$ and edges ℓ with $\beta(\ell) \leqslant v$.

In this paper we call also a union of tress a *tree*.

Our goal in this paper is to prove

THEOREM. — *Let* $G = (V, L, \Lambda)$ *be a finite convergent tree and let* $M = (M_v)_{v \in V}$ *be a collection of smooth manifolds and* $P = (\dim M_v)_{v \in V}$. *If* P *satisfies the condition* G *defined in Section* 2.1, *then topologically stable mappings are dense in* $C^\infty_{\mathrm{pr}}(G, M)$ *with the Whitney topology.*

As a consequence of the above theorem and Theorem 2 in the paper [N2], we have

COROLLARY. — *Let* G, M, P *be as above.*

If p_v *satisfies one of the following conditions, for any* $v \in V$:

(1) $p_v \leqslant p_{\beta^n(v)}$ *for* $0 \leqslant n \leqslant h(v)$

(2) $p_{\beta(v)} \leqslant p_{\beta^n(v)}$ *for* $1 \leqslant n \leqslant h(v)$

(3) $p_v,\ p_{\beta(v)} \geqslant p_{\beta^2(v)} \leqslant p_{\beta^n(v)}$ *for* $2 \leqslant n \leqslant h(v)$

and the pair $(p_v - p_{\beta^2(v)}, p_{\beta(v)} - p_{\beta^2(v)})$ *is semi-nice*, i.e.

$$^2\sigma(p_v - p_{\beta^2(v)}, p_{\beta(v)} - p_{\beta^2(v)}) \geqslant p_v - p_{\beta^2(v)}.$$

Then topologically stable diagrams are dense in $C^\infty_{\mathrm{pr}}(G, W)$. *Here* $^2\sigma(n,p)$ *is the function defined by Mather* [M2] (*see also* [W2-3]).

The main theorem above generalizes well known topological stability theorem for single mappings due to Mather [M4] and also gives a partially affirmative answer to a conjecture by Baas and Mather [B1-3, L-T]: topologically stable diagrams are dense in $C^\infty_{\mathrm{pr}}(G,M)$ if G is a finite convergent tree.

We now recall some known results on the C^r stability problem respectively for various types of diagrams.

Case 1 : is an arrow $\rightarrow$. In this case our problem turns into the ordinary singularity theory of smooth mappings between two manifolds. We recall the main global results:

(1) C^∞ stable mappings are dense in $C^\infty_{\mathrm{pr}}(N^n, P^p)$ if the dimension pair (n,p) is nice, i.e. ${}^1\sigma(n,p) > n$ [M2].

(2) C^0 stable mappings are dense in $C^\infty_{\mathrm{pr}}(N,P)$ for any smooth manifolds $N.P$ [M4, Gi].

(3) The C^0 stability and the C^∞ stability are equivalent in $C^\infty(N,P)$ if (n, p) is a nice pair and N is compact (see e.g. [Da]).

(4) The complement of the union of equivalence classes $\mathcal{O}^\infty(f)$ with finite codimension in $C^\infty(N, P)$ has infinite codimension for compact N, if and only if (n, p) is a semi-nice pair [P, W2].

(5) C^1 stable mappings are dense in $C^\infty_{\mathrm{pr}}(N, P)$ if and only if (n, p) is a nice pair [W1].

A survey of these nice and semi-nice properties is available in the paper [W2] and the complete determination of those ranges is given by Mather [M3] and Wall [W3], respectively.

Case 2 : G is the composition $\rightarrow\rightarrow$. In this case C^1 stability does not hold generically even for some triples (M, N, P) of manifolds of small dimensions. In fact du Plessis showed that

(6) C^1 stable compositions are not dense in the space of proper composite mappings $C^\infty_{\mathrm{pr}}(M^3 \rightarrow N^4 \rightarrow P^2)$.

We will give a proof for this in Appendix 2. On the other hand, the triples (3.4.2) satisfy the condition (3) of the above corollary. Since the pair (1, 2) is nice and in particular semi-nice, C^0 stable mappings are dense in this space of compositions.

A technical reason for the restriction to the case of convergent diagrams is that the Malgrange-Mather division theorem does not hold for the other cases. In fact the nature of the space of diagrams C^∞ (G, M) presents a remarkable difference between the convergent and the other types. Some of these aspects will be found in the following two typical non convergent cases.

Case 3 : G is a cycle $\circlearrowleft$. In this case our problem corresponds to the theory of endomorphisms of manifolds, which have been long studied by many mathematicians. It is known that C^0 stability is not a generic property. This phenomenon is caused by the topological structure of orbits of endomorphisms $f\colon M \circlearrowleft$. The structure of compositions of their singularities along orbits is the same as that of their developments $\hat{f}\colon \cdots \overset{f}{\Rightarrow} \overset{f}{\Rightarrow} \cdots \Rightarrow f$: (covering f), for which it seems that the argument in this paper remains effective. So the topological structure of endomorphisms may be described by a certain combination of singularities and the orbit structure of f.

Case 4 : G is the divergent graph $\swarrow\!\nearrow$. In this case our problem is related to envelope theory in the papers [A, Ca, Du 2-3, Th]. The recent results by Carneiro [Ca] and Dufour [Du 2-3] present a new aspect of the topological classification problem for diagrams of this type by using a topological method in web geometry. Namely, Dufour proved that

(7) In the divergent mapping space $C^{\infty}(M^1 \leftarrow N^2 \rightarrow P^2)$, C^0 stable diagrams are not dense [Du2-3].

In the final section we will show that if P^2 is orientable and N^2 is not then all topological equivalence classes have infinite codimension. Surprisingly it was proved quite recently by Dufour [Du 4] that even in the space of pairs of functions $C^{\infty}(M^1 \leftarrow N^2 \rightarrow P^1)$, C^0 stability does not hold in general.

0.2. Sketch of the proof of the theorem.

First we begin by recalling the idea due to Thom and Mather for topological study of singularities of mappings, known as the theory of canonical stratification.

A stratification of a smooth mapping $f: N \rightarrow P$ is a pair $(\mathscr{S}_N, \mathscr{S}_P)$ of stratifications of manifolds N, P such that f restricts on each stratum $X \in \mathscr{S}_N$ to a submersion $f: X \rightarrow Y$ to some stratum $Y \in \mathscr{S}_P$. Thom's second isotopy lemma (Theorem 3.2.4) says that if a family of proper mappings $(f_t \times \mathrm{id}, \mathrm{Pr})$ $N \times \mathbb{R} \rightarrow P \times \mathbb{R} \rightarrow \mathbb{R}$ is simultaneously stratified by a triple $(\bar{\mathscr{S}}_N . \bar{\mathscr{S}}_P, \mathbb{R})$ of stratifications of $N \times \mathbb{R}$, $P \times \mathbb{R}$ and $\mathbb{R}$, and $\bar{\mathscr{S}}_N$ satisfies Thom's condition A_{f_t} then the family f_t is locally topologically trivial. This suggests that topological stability of mappings may be deduced from a certain stability of their A_f regular stratifications under small perturbations. A canonical stratification was explicitly constructed by Mather [M4] by using his highly systemized method in papers in a series, where the finite determinacy theorem and the unfolding theory played a crucial role.

Using the same basic idea as above, a fundamental part of the proof of the topological stability theorem for convergent diagrams will be a construction of their stratifications in a canonical way. For the simplest case of two-compositions $(f,g): M \to N \to P$, this may be done by refining a canonical stratification $\mathcal{S}_N(f)$ of N for f to a stratification $\mathcal{S}'_N(f)$ such that for some stratification $\mathcal{S}_P$ of P the pair $(\mathcal{S}'_N(f), \mathcal{S}_P)$ is A_g regular. Thus this problem is called the problem of the *second stratification* by Thom. In the following we will explain how the second stratifications of convergent diagrams are constructed in a canonical way.

Let $f_{v'v}: M_{v'} \to M_v$ denote the composition of f_ℓ along the oriented path from v' to v. The main technical problem in this paper is to give an intrinsic notion for the singularities of convergent diagrams f involving these compositions.

Given a diagram $f \in C^\infty(G,M)$ and another convergent graph Γ, a *diagram of f of type* Γ consists of

i) a morphism $\iota: (V_\Gamma, L_\Gamma) \to (V_G, L_G)$ with $\alpha_G \circ \iota = \iota \circ \alpha_\Gamma$, $\beta_G \circ \iota = \iota \circ \beta_\Gamma$,

ii) points $x_t \in M_{\iota(t)}$ for $t \in V_\Gamma$ such that $f_{\iota(\ell)}(x_{\alpha(\ell)}) = x_{\beta(\ell)}$ for $\ell \in L_\Gamma$.

We shall seek to understand the singularities of f in terms of its multigerms along such diagrams.

A diagram of f is determined by the set $X = \{x_{\alpha(\ell)} | \ell \in L_\ell\}$, so we denote it simply by f_X: the Collection of multi germs $f_{\ell_{X_\ell}}$, $\ell \in L_G$, where $X = \bigcup_{\ell \in L} X_\ell \subset \bigcup_{\ell \in L} M_{\alpha(\ell)}$. Note that $V_\Gamma \cong \bigcup_{\ell \in L} (X_\ell \cup f_\ell(X_\ell))$.

We first explain the role of trees for the case of a single mapping. Here a diagram f_X, $X \subset N$ of $f: N \to P$ is an oriented graph of height 1 consisting of $\# f(X)$ disjoint trees: forest. The germs of canonical stratifications $\mathcal{S}_N$, $\mathcal{S}_P$ of f at $f^{-1}(y)$, y are characterized by the multigerm f_{X_y} at $X_y = \Sigma(f) \cap f^{-1}(y)$ [M4].

For a general convergent tree G, our first problem is to describe the singularity type of convergent diagrams of smooth mapping, in other words to seek the smallest subset $X \subset \bigcup_{\ell \in L} M_{\alpha(\ell)}$ with $y \in f(X)$ for which the germ f_X characterizes the property of the germ of f along the fibres $f_{v'v}^{-1}(y)$, $v' < v$ on $y \in M_v$.

In Section 0.3, we define the critical point sets $C_{\alpha(\ell)}(f) \subset M_{\alpha(\ell)}$ and the critical values sets $D_v(f) = \bigcup_{\beta(\ell)=v} f_\ell(C_{\alpha(\ell)}(f))$ for convergent diagrams

$f \in C^{\infty}(G,M)$, using the notion of trees. The restriction $f_{\Sigma} = (f_{\ell} : C_{\alpha(\ell)}(f) \to D_{\beta(\ell)}(f))$ is considered as a skeleton of f. In fact f_{Σ} contains complete information about the singularities of f. The author would suggest these sets as good candidates for the notion of singularity for diagrams, in proving the C^{∞} stability theorem (Thm. 2.3.1) and in constructing a canonical stratification for diagrams in a certain class $f \in T_{\infty} \subset A_{\infty}$ (Theorems 2.1.2, 3.1.2).

The fundamental question for these critical sets is: whether the restrictions $f_{\ell} : C_{\alpha(\ell)}(f) \to D_{\beta(\ell)}(f)$, $\ell \in L$ are proper and finite-to-one. We say a convergent diagram $f \in C^{\infty}(G,M)$ is a *good representative* of a tree f_X if the restrictions $f : C_{\alpha(\ell)}(f) \to D_{\beta(\ell)}(f)$ are all proper and finite-to-one and also satisfy a certain additional condition on maximal trees (see Section 0.3 for the definition).

In the paper [N2] we proved that a (finite) tree f_X of a convergent diagram admits a good representative if the I_0 codimension of f_X is finite (Proposition 1.4.1 [N2]). Under the conditions $f \in A_{\infty} \cap C^{\infty}_{\mathrm{pr}}(G,M)$ that f_{ℓ} are proper and all trees have finite I_0 codimension, it is proved that the critical sets $C_{\alpha(\ell)}(f), D_{\beta(\ell)}(f)$ are closed and the restrictions $f_{\ell} : C_{\alpha(\ell)}(f) \to D_{\beta(\ell)}(f)$ are all proper and finite-to-one (Theorem 2.2.1).

Now we are ready to explain the construction of stratification of diagrams $f \in C^{\infty}_{\mathrm{pr}}(G,M)$. By Theorem 0.3.2, a tree f_X with finite I_0 codimension admits an (infinitesimally) stable unfolding $F_X = (F_{\ell x} : M_{\alpha(\ell)} \times \mathbb{R}^r, x \times 0 \to M_{\beta(\ell)} \times \mathbb{R}^r, f(x) \times 0)_{x \in X}$ of the form $F_{\ell x}(x,u) = (f_{\ell x u}(x), u)$, $f_{\ell x 0} = f_{\ell}$. By the finite determinacy of stable diagrams (Theorem 0.3.1), we may suppose F_X is a diagram of polynomial mappings. Then by a standard technique in the theory of semi-algebraic sets, we can construct a critical value stratification (CVS) $\mathscr{C}(F_X) = (\mathscr{C}_x(F_X))_{x \in F(X)}$, which yields immediately a Thom A_{F_X} regular stratification $\mathscr{S}(F_X) = (\mathscr{S}_x(F_X))_{x \in X \cup f(X)}$ of F_x. A tree f_X is *topologically transversal* if all inclusions $i_x : M_{vx} \hookrightarrow M_{vx} \times \mathbb{R}^r$ are transversal to $\mathscr{S}_x(F_X)$. Then the pullbacks $i_x^* \mathscr{S}_x(F_X)$ give the stratification of f_X denoted by $\mathscr{S}(f_X)$. A diagram $f \in A_{\infty}$ is *topologically transversal* if so are all trees in it (it is sufficient to consider topological transversality of maximal trees). The set of those diagrams is denoted by T_{∞}. By the naturality of $\mathscr{C}(f_X)$, $\mathscr{S}(f_X)$ with respect to coordinate transformations (Proposition 1.2.1) and the coherence of maximal trees and branches (Theorem 2.2.1), the CVS $\mathscr{C}_x(f_X)$, $x \in X \cup f(X)$ glue up to give a stratification of M_v denoted $\mathscr{C}(f) = (\mathscr{C}_v(f))$, from which we obtain immediately a canonical stratification $\mathscr{S}(f)$ of f.

By the argument outlined above, the genericity of topological stability is deduced from the openness and density of the sets A_∞ and T_∞. These properties are proved in Theorem 3.2.2 and Theorem 2.1.2 under Condition G defined below by arguments using transversality of jet sections. We explain these briefly.

A multi jet $z \in {}_mJ^\infty(G,M) = \prod_{\ell \in L} {}_mJ^\infty(M_{\alpha(\ell)}, M_{\beta(\ell)})$ is a collection of jets $Jf_{\ell i}(x_{\ell i}) \in J(M_{\alpha(\ell)}, M_{\beta(\ell)})$, $\ell \in L$, $i = 1, \ldots, m$. We regard again z as a combinatorial tree Γ_z (possibly a union of many connected components) consisting of the vertices x_{ℓ_i}, $f_{\ell_i}(x_{\ell_i})$ and edges $Jf_{\ell_i}(x_{\ell_i}) : x_{\ell_i} \to f_{\ell_i}(x_{\ell_i})$.

We say the dimension function $P = (\dim M_v)$ satisfies *Condition G* if, for any m, $z \in {}_mJ(G,M)$ off a subset of infinite codimension with any combinatorial type, Γ_z is finitely I_0 determined. Some range of such dimensions P is presented in the paper [N2].

The canonical stratification S of ${}_mJ(G,M)$ is roughly the partition by topological types of the stratification of stable unfoldings of those trees Γ_Z.

Proposition 1.3.3 and the argument in Section 3.2 say that topological transversality of $f \in A_\infty$ is equivalent to the transversality of the multi jet section ${}_mJf$ to the S for sufficiently large m. Therefore T_∞ is a countable intersection of open dense subsets by the transversality theorem (Theorem 0.3.5), hence it is a dense subset by the Baire property of $C^\infty(G,M)$. The openness of T_∞ is shown in Theorem 3.2.2 in the same way as that of A_∞ in Theorem 2.1.2.

0.3. Terminology and Preliminaries.

For a tuple of positive integers $P = (p_v)$, let

$$\mathscr{E}(G,P) = \bigoplus_{\ell \in L} m(p_{\alpha(\ell)})\, \mathscr{E}(p_{\alpha(\ell)}, p_{\beta(\ell)})$$

denote the set of diagrams of map germs $f_\ell : \mathbb{R}^{p_{\alpha(\ell)}}, 0 \to \mathbb{R}^{p_{\beta(\ell)}}, 0$. Here $\mathscr{E}(n)$ is the local ring of smooth function germs on $\mathbb{R}^n$ at 0 with maximal ideal $m(n)$ and $\mathscr{E}(n,p) = \bigoplus^p \mathscr{E}(n)$. Let $\theta(P) = \bigoplus_{v \in V} \theta(p_v)$, $\theta(f) = \bigoplus_{\ell \in L} \theta(f_\ell)$ and define the morphism $T(f) : \theta(P) \to \theta(f)$ by

$$T(f)\left(\bigoplus_{v \in V} \chi_v\right) = \bigoplus_{\ell \in L} \omega f_\ell(\chi_{\beta(\ell)}) - t f_\ell(\chi_{\alpha(\ell)}).$$

(For these notions, see [M1-4]). We say f is *infinitesimally stable* or simply *stable* if $T(f)$ is surjective, and f is *trivial* if

$$T(f)\left(\underset{v\neq v_0}{\oplus}\theta(p_v).\right) = \theta(f).$$

The I_0 codimension $C_{I_0}(f)$ of f is defined by

$$C_{I_0}(f) = \dim_{\mathbb{R}}\theta(f)/\mathrm{Im}\, T(f) + \underset{\ell\in L}{\oplus} f^*_{\alpha(\ell)v_0} m(p_{v_0})\theta(f_\ell),$$

where f_{vv_0} denotes the composition of f_ℓ along the oriented path from v to the root v_0.

A diagram $F\in\mathscr{E}(G,P+r)$, $P+r=(p_v+r)$ is called an r parameter unfolding of $f\in\mathscr{E}(G,P)$ if there are smooth imbeddings $\imath_v$: $\mathbb{R}^{p_v}\hookrightarrow\mathbb{R}^{p_v+r}$ such that $F_\ell\circ\imath_{\alpha(\ell)}=\imath_{\beta(\ell)}\circ f_\ell$ and $\imath_{\beta(\ell)}$ is transversal to f_ℓ. Two unfoldings F, G of f are *equivalent as unfoldings* if there are germs of diffeomorphisms ϕ_v of $\mathbb{R}^{p_v}$ such that $F_\ell\circ\phi_{\alpha(\ell)}=\phi_{\beta(\ell)}\circ G_\ell$ and $\phi_v\circ\imath_{vG}=\imath_{vF}$, where $\imath_{vF}$, $\imath_{vG}$ are the transversal inclusions of f to G, F.

The equivalence relation I_0 introduced in the paper [N1] is defined for diagrams f with $C_{I_0}(f)<\infty$. We say that f, g are I_0 equivalent if and only if they have unfoldings F, G which are equivalent as diagrams (see Theorem 4.2.1 [N1]). If F, G are unfoldings of the same dimension of f, g respectively then f and g are I_0 equivalent if and only if F and G are I_0 equivalent. The I_0 equivalence classes $\mathcal{O}^{I_0}(f)$ project to locally C^∞ trivial semialgebraic manifolds in the jet space $J^k(G,P)=\prod_{\ell\in L} J^k(p_{\alpha(\ell)},p_{\beta(\ell)})$ denoted $\mathcal{O}^{I_0k}(f)$ (Proposition 2.4.2-2 [N1]).

Let $F\in\mathscr{E}(G,P+r)$ be an unfolding of f of the normal form : $F_\ell(x,u)=(f_{\ell u}(x),u)$, $x\in\mathbb{R}^{p_{\alpha(\ell)}}$, $u\in\mathbb{R}^r$. Then the jet section $\bar{J}^kF:\Pi\mathbb{R}^{p_{\alpha(\ell)}}\times\mathbb{R}^r\to J^k(G,\mathbb{R}^p)=\prod_{\ell\in L}J^k(\mathbb{R}^{p_{\alpha(\ell)}},\mathbb{R}^{p_{\beta(\ell)}})$ is defined by $\bar{J}^kF((x_{\alpha(\ell)}),u)=(J^kf_{\ell u}(x_{\alpha(\ell)}))$. Let $\Delta_G\subset\prod_{\ell\in L}\mathbb{R}^{p_{\alpha(\ell)}}\times\mathbb{R}^{p_{\beta(\ell)}})$ denote the diagonal set $\{(x_\ell,y_\ell)_{\ell\in L}|x_\ell=y_{\ell'}\Leftrightarrow\alpha(\ell)=\beta(\ell')\}$.

THEOREM 0.3.1 (Theorem 3.1.1 [N1]). — *C^∞ stable diagrams $f\in E(G,P)$ are finitely determined : there is a function $e(G,P)$ such that any diagram g with the same $e(G,P)+1$ jet as f is equivalent to f.*

THEOREM 0.3.2 (Theorem 5.1.1 [N1]). – *The following conditions are equivalent :*

(1) *F is infinitesimally stable,*

(2) $\bar{J}^{e(G,P+r)}F$ is transversal to $\Delta_G \times \mathcal{O}^{I_0 e(G,P+r)}(f)$,

(3) $(\partial f_{\ell u}/\partial u_i(u=0))_{\ell \in L}$, $i = 1, \ldots, r$ *span* coker $T(f)$,

(4) $(\partial f_{\ell u}/\partial u_i(u=0))_{\ell \in L}$, $i = 1, \ldots, r$ *span*

$$\theta(f)/\mathrm{Im}\, T(f) + \bigoplus_{\ell \in L} f^*_{\alpha(\ell)v_0} m(p_{v_0})\, \theta(f_\ell).$$

THEOREM 0.3.3 (Theorem 5.2.1 [N1]). – *Two stable r-parameter unfoldings F, G of f are equivalent as unfoldings.*

THEOREM 0.3.4 (Proposition 2.1.1 [N2]). – *The condition $r < C_{I_0}(f)$ is an algebraic condition on the $e(r)\,(=e(G,P+r)+1)$ jet of f, which defines an algebraic set $\Sigma^{e(r)} \subset J^{e(r)}(G,P = \prod_{\ell \in L} J^{e(r)}(p_{\alpha(\ell)}, p_{\beta(\ell)})$ such that $\pi_{e(r)e(s)}(\Sigma^{e(r)}) \subset \Sigma^{e(s)}$ for any $s < r$. If $C_{I_0}(f) \leqslant r$, f is $e(r) - I_0$ determined, i.e., any g with the same $e(r)$ jet as f is I_0 equivalent to f. (This is a consequence of* Theorem 0.3.1 *and* 0.3.3.)

We say finite I_0 determinacy holds in general in $\mathscr{E}(G,P)$ if codim $\Sigma^{e(r)} \to \infty$ as $r \to \infty$.

Here we state our transversality theorem.

THEOREM 0.3.5. – *Let $G = (V,L,\Lambda)$ be a finite oriented graph, $M = (M_v)$ a collection of smooth manifolds and $S \subset J^k(G,M)$ a submanifold. Then the set $\mathscr{F}_S$ of diagrams $f \in C^\infty(G,M)$ for which the k jet sections $J^k f = (J^k f_\ell) : \prod_{\ell \in L} M_{\alpha(\ell)} \to J^k(G,M)$ are transversal to S is a countable intersection of open dense subsets.*

From now on we apply all concepts for convergent graphs and convergent diagrams of map germs to trees of diagrams $f \in C^\infty(G,M)$.

Let f_X, $X_\ell \in M_{\alpha(\ell)}$ be a (finite) tree of a diagram $f \in C^\infty(G,M)$. The *prolongation of* f_X is the tree f_{X^-} defined by the set $X^- = X \cup f(X) - M_{v_0}$.

The *critical point set* $C_{\alpha(\ell)}(f) \subset M_{\alpha(\ell)}$, $v \neq v_0$ of f is the set of roots x of (finite connected) trees f_X, for which the prolongation $f_{X^-} = f_X \cup (f_{\ell x} : x \to f_\ell(x))\ (X^- = X \cup f(X) = X \cup x)$ is not trivial. The *critical value set* $D_{\beta(\ell)}(f) \subset M_{\beta(\ell)}$ is the set of roots $f_\ell(x)$ of those prolongations : $D_{\beta(\ell)}(f) = \bigcup_{\beta(\ell')=\beta(\ell)} f_{\ell'}(C_{\alpha(\ell')}(f))$.

A tree f_X is *indecomposable* if $X_\ell \subset C_{\alpha(\ell)}(f)$ for all $\ell \in L$, and *maximal* if furthermore any tree $f_{X'}$ with $X \subsetneq X'$ is not indecomposable. We see easily that any point $x \in C_u(f) \cup D_v(f)$ is contained in unique maximal tree (possibly infinite) called the *maximal tree of* x and denoted f_{X_x}, and its branch on $x \in X_x \cup f(X_x)$ is called the *maximal branch on* x and denoted $f_{X_x^{\mathrm{br}}}$. Note that $X_x^{\mathrm{br}} = \bigcup_{v'<v} C_{v'}(f) \cap f_{v'v}^{-1}(x)$. Conventionally we define $X_x = x$ if $x \notin C_v(f) \cup D_v(f)$.

We call a diagram $f \in C^\infty(G,M)$ a *good representative of a tree* f_X if the following conditions are satisfied:

(1) f_X is maximal if f_X is indecomposable,

(2) The function $C_{I_0}(f_{X_x})$ is upper semi continuous with $x \in M_v$, $v \in V$.

(3) $C_v(f)$, $D_v(f) \subset M_v$ are closed and the restrictions $f_\ell : C_{\alpha(\ell)}(f) \to D_{\beta(\ell)}(f)$ are proper and locally uniformy finite-to-one.

(4) For any subgraph G', the restriction $f_{G'}$ is a good representative of the subtree $f_{X'}$, $X' = \bigcup_{\ell \in L'} X_\ell$.

PROPOSITION 0.3.6 (Proposition 1.6.1 [N2]). – *Any finitely I_0 determined convergent diagram of smooth map germs admits a good representative.*

CHAPTER 1

CRITICAL VALUE STRATIFICATION (CVS)

1.1. A canonical construction of CVS.

Let $G = (V,L,\Lambda)$ be a convergent diagram of height 1 with root v_0: $V = \{v_0, v_1, \ldots, v_k\}$, $L = \{\ell_1, \ldots, \ell_k\}$, ℓ_i: $v_i \to v_0$. Let $M = (M_0, \ldots, M_k)$ be a collection of smooth manifolds and

$$f = (f_i)_{i=1,\ldots,k}, \qquad f_i \in C^\infty(M_i, M_{v_0}).$$

We suppose Whitney (B) regular stratifications S_i of M_i are given. Let $\Sigma(f_i|A)$, $A \in S_i$ denote the set of points $x \in A$ where $f_i|A$ is not a C^∞ submersion. Let $\Sigma_{S_i}(f_i) = \bigcup_{A \in S_i} \Sigma(f_i|A)$, $D_{S_i}(f_i) = f_i(\Sigma_{S_i}(f_i))$ and $D_S(f) = \bigcup_{i=1}^{k} D_{S_i}(f_i)$, where S stands for the k-tuple $(S_i)_{i=1,\ldots,k}$. By the A regularity condition for S_i, $\Sigma_{S_i}(f_i)$ is closed in M_{v_i}.

A *critical value stratification* (CVS) S' *of* $D_S(f)$ is a Whitney regular stratification of $D_S(f)$ which possesses the following properties: for any i and strata A, $B \in S_i$, U, $V \in S'$ (where we allow $A=B$ and $U=V$),

(1) $\Sigma(f_i|A) \cap f_i^{-1}(U)$ is a smooth submanifold of A,

(2) f_i: $\Sigma(f_i|A) \cap f_i^{-1}(U) \to U$ is locally diffeomorphic,

(3) $\Sigma(f_i|B) \cap f_i^{-1}(V)$ is Whitney regular over $\Sigma(f_i|A) \cap f_i^{-1}(U)$,

(4) $B \cap f_i^{-1}(U) - \Sigma(f_i|B)$, $B \cap f_i^{-1}(M_{v_0} - D_S(f))$ are Whitney regular over $\Sigma(f_iA) \cap f_i^{-1}(U)$.

If $D_S(f)$ is a closed subset, the mapping f admits the Thom regular stratification (S_i', S') called the *stratification of* f *associated with the critical value stratification* S', defined by

$$S_i' = \{\Sigma(f_i|A) \cap f_i^{-1}(U), (A - \Sigma(f_i|A)) \cap f_i^{-1}(U), \\ A \cap f_i^{-1}(M_{v_0} - D_S(f)) \mid A \in S_i, U \in S'\}$$

for $i = 1, \ldots, k$; (Thom's A_{f_i} regularity and Whitney B regularity of S_i' follow immediately from the properties (2) and (3), (4) respectively. For the definitions of Whitney B regularity and Thom's A_f condition, see [Gi]).

In general let $G = (V, L, \Lambda)$ be a convergent graph with root v_0, $M = (M_v)$ a collection of smooth manifolds, and $f \in C^\infty(G, M)$. Let v^+ denote the set of vertices $v' \in V$ with $\beta(v') = v$ $(\ell_{v'}: v' \to v)$ and let $f_{v^+} = (f_\ell)_{\beta(\ell)=v}$. Now suppose that we have stratifications $\mathscr{C}_v(f)$ of M_v, for which the union of strata with positive codimension gives CVS of the critical value set $D_v(f)$ of f_{v^+}:

$$D_v(f) = D_{\mathscr{C}_{v^+}}(f_{v^+}), \qquad \mathscr{C}_{v^+} = (\mathscr{C}_{v'})_{v' \in v^+},$$

for any $v \in V$ ($D_v(f) = \varnothing$ for the source v of G).

Then f admits the *stratification* $\mathscr{S}(f) = (\mathscr{S}_v(f))_{v \in V}$ *associated with* $\mathscr{C}(f)$ defined by $\mathscr{S}_{v_0}(f) = \mathscr{C}_{v_0}(f)$ and

$$\mathscr{S}_{\alpha(\ell)}(f) = \{\Sigma(f_\ell|A) \cap f_\ell^{-1}(U), \\ (A - \Sigma(f_\ell|A) \cap f_\lambda^{-1}(U) \mid A \in \mathscr{C}_{\alpha(\ell)}(f), U \in \mathscr{S}_{\beta(\ell)}(f)\}.$$

Now we construct a CVS for the following mappings. Let $p_0, \ldots, p_k$ be positive integers and U_i a semialgebraic open neighbourhood of $0 \in \mathbb{R}^{p_i}$ for $i = 0, \ldots, k$. Let S_i, $i = 1, \ldots, k$ be Whitney regular semialgebraic stratifications of U_i and f_i: U_i, $0 \to U_0$, 0 polynomial mappings. If the $f_i|\Sigma_{S_i}(f_i)$ are finite to one, then by shrinking U_i if necessary, we may assume $f_i^{-1}(0) \cap \Sigma_{S_i}(f_i) = 0$ and $f_i|\Sigma_{S_i}(f_i)$ are proper hence $D_S(f)$, $S = (S_i)$ is closed in U_0. In this situation the germ of $D_S(f)$ at 0 is well defined by the germs of f_i at 0, and we have :

PROPOSITION 1.1.1. – *$D_S(f)$ admits the CVS $\mathscr{C}_S(f)$ so-called canonical critical value stratification, which possesses the following properties : The germ of $\mathscr{C}_S(f)$ at 0 is well defined by the germs of the f_i. Let $g_i : V_i \to V_0$ be polynomial mappings and S'_i stratifications of V_i satisfying the above conditions. If there are germs of diffeomorphisms ϕ_i of $\mathbb{R}^{p_i}$ such that $\phi_0 \circ f_i = g_i \circ \phi_i$ and $\phi_i(S_i) = S'_i$, then $\phi_0(\mathscr{C}_S(f)) = \mathscr{C}_{S'}(g)$.*

Proof. – We construct a filtration $Y_d \supset Y_{d-1} \supset \cdots \supset Y_0$ of the critical value set $Y_d = D_S(f)$ ($d = \dim D_S(f)$) by semialgebraic subsets Y_i of dimension $\leqslant i$ inductively, so that $M_i = Y_i - Y_{i-1}$ and $M_0 = Y_0$ are Whitney regular submanifolds, $\dim M_i = i$ and possess the properties (1) – (4) of CVS.

As induction hypothesis we assume that we have constructed a filtration $Y_d \supset \cdots \supset Y_i$ with the conditions (1) – (4) for A, $B \in S_j$, $j = 1, \ldots, k$ and $U = M_m$, $V = M_n$, $i + 1 \leqslant m$, $n \leqslant d$. Then we define Y_{i-1} in the following way : Let

(i) $Y_i^1 = Y_i - \Sigma(Y_i)$

(ii) $Y_i^2 = Y_i^1 - \overline{\bigcup_{\ell = i+1, \ldots, d} B(Y_i^1, M_\ell)}$

(iii) $X_A^1 = A \cap f_j^{-1}(Y_i^2) \cap \Sigma(f_j|A)$ for each $A \in S_j$

(iv) $X_A^2 = X_A^1 - \Sigma(X_A^1)$

(v)
$$X_A^3 = X_A^2 - \overline{\bigcup_{B \in S_j} B(X_A^2, X_B^2)} - \overline{\bigcup_{B \in S_j} B(X_A^2, f_j^{-1}(Y_i^2) \cap B - \Sigma(f_j|B))} - \overline{\bigcup_{B \in S_j, \ell = i+1, \ldots, d} B(X_A^2, \Sigma(f_j|B) \cap f_j^{-1}(M_\ell))} - \overline{\bigcup_{B \in S_j, \ell = i+1, \ldots, d} B(X_A^2, f_j^{-1}(M_\ell) \cap B - \Sigma(f_j|B))} - \overline{\bigcup_{B \in S_j} B(X_A^2, f_j^{-1}(U_0 - D_{S_j}(f_j)) \cap B)}$$

(vi) $X_A^4 = X_A^3 - \operatorname{Sing}(f_j|X_A^3)$.

Here $\Sigma(Y_i)$ denotes the set of points $x \in Y_i$ where Y_i is not a smooth manifold of dimension i, $B(X,Y)$ denotes the set of points $x \in X$ where Y is not Whitney regular over X and sing $(h : X \to Y)$ is the set of points $x \in X$ where $h : X \to Y$ is not of maximal rank. Then Y_i^1, Y_i^2, X_A^1, X_A^2, X_A^3 and X_A^4 are all semialgebraic, X_A^4 is open in X_A^1 and $\dim(X_A^1 - X_A^4) < i$ (for the properties of semialgebraic sets, see [Gi]), so we see that $\dim f_j(X_A^1 - X_A^4) < i$ and $f_j : X_A^4 \to f_j(X_A^4) \subset Y_i^2$ is locally isomorphic by the assumption that $f_i | \Sigma_{S_i}(f_i)$ is finite-to-one and the refining process (vi). Let

$$\text{(vii)} \qquad Y_i^3 = Y_i^2 - \overline{\bigcup_{A \in S_j, j=1,\dots,k} f_j(X_A^1 - X_A^4)}.$$

Then Y_i^3 is smooth of dimension i. Now we define $Y_{i-1} = Y_i - Y_i^3$. We claim that Y_{i-1} possesses the required property. The properties (1) – (4) of CVS for $M_i = Y_i - Y_{i-1}$ involving the other strata M_ℓ, $\ell = i+1, \dots, d$ and $A, B \in S_j$ follow respectively from the refining process (iv), (vi) and (vii), the 4-th term in RHS of (v), and the 5-th and 6-th terms in RHS of (v). If $0 \neq n = \dim Y_{i-1} < i - 1$, then we define $Y_{i-2} = Y_{i-2} = \cdots = Y_n$ and go on to the next step of refining Y_n to define Y_{n-1} so that $M_n = Y_n - Y_{n-1}$ is smooth of dimension n and satisfies the required properties. If $n = 0$, we complete the induction.

By construction the filtration $Y_d \supset \cdots \supset Y_0$ is determined by the germs of f_j at $\Sigma_{S_j}(f_j) \subset U_j$. Since $f_j | \Sigma_{S_j}(f_j)$ are proper and $f_j^{-1}(0) \cap \Sigma_{S_j}(f_j) = 0$, the germ of Y_i at $0 \in U_0$ is determined by the germs of f_j. The naturality of the germ of $\mathscr{C}_S(f)$ at 0 with respect to coordinate transformations is clear.

1.2. Some properties of CVS.

PROPOSITION 1.2.1. – *Any stable convergent diagram of smooth map germs $f \in \mathscr{E}(G,M)$ admits a representative $\hat{f} = (\hat{f}_\ell)$, $\hat{f}_\ell : U_{\alpha(\lambda)} \to U_{\beta(\ell)}$ defined on open neighbourhoods U_v of $0 \in \mathbb{R}^{p_v}$, with a CVS $\mathscr{C}(\hat{f}) = (\mathscr{C}_v(\hat{f}))$ such that the restrictions $f_\ell : \Sigma_{\mathscr{C}_{\alpha(\ell)}(f)}(\hat{f}_\ell) \to U_{\beta(\ell)}$ are proper and finite-to-one. The germs of $\mathscr{C}_v(\hat{f})$ at 0 are well defined by f, and called the canonical CVS and denoted $\mathscr{C}(f) = (\mathscr{C}_v(f))$. If $\hat{f}$ is a diagram of polynomial map germs, U_v, $\mathscr{C}_v(\hat{f})$ and $\mathscr{C}_v(f)$ are semialgebraic. Let $g \in \mathscr{E}(G,P)$ and assume there are germs of diffeomorphisms ϕ_v of $(\mathbb{R}^{p_v},0)$ with $\phi_{\alpha(\ell)}^{-1} \circ g_\lambda \circ \phi_{\beta(\ell)} = f_\ell$ for $\ell \in L$. Then $\phi_v(\mathscr{C}_v(f)) = \mathscr{C}_v(g)$ for $v \in V$.*

Proof. — By the determinacy theorem (Theorem 0.3.1), stable diagrams are equivalent to diagrams of polynomial map germs. So it suffices to prove the statement for polynomials f, g. We construct $\mathscr{C}_v(\hat{f})$ by descending induction on the height of vertices $v \in V$.

Let f_{v^+} denote the restriction of f to the edges $\ell_{v'} : v' \to v$. We assume that there are semialgebraic open neighbourhoods U_v of $0 \in \mathbb{R}^{p_v}$, $f_\ell(U_{\alpha(\ell)}) \subset U_{\beta(\ell)}$ such that the restrictions of f to the branches $G_{v'}$, on v', $\beta(v') = v$, admit the canonical CVS $\mathscr{C}_{v''}$ on $U_{v''}$, $v'' \leqslant v'$. We will construct a stratification $\mathscr{C}_v(f)$ of U_v so that the union of strata with positive codimension gives the CVS of the critical value set $D_{\mathscr{C}_{v^+}(f)}(f_{v^+})$ of f_{v^+} with respect to $\mathscr{C}_{v^+}(f) = (\mathscr{C}_{v'}(f))_{\beta(v')=v}$.

By Proposition 0.3.6, we may assume that $(f_\ell | U_{\alpha(\ell)})$ is a good representative of f. By the definition of the critical sets in Section 0.3, a tree f_X with its root in U_v and vertices of height 1 off the critical sets $\mathscr{C}_{v'}(f)$, $\beta(v') = v$ is trivial. Therefore we see $\Sigma_{\mathscr{C}_{\alpha(\ell)}(f)}(f_\ell) \subset \mathscr{C}_{\alpha(\ell)}(f)$ for ℓ with $\beta(\ell) = v$. By the properties of good representatives, the restrictions $f_\ell | \Sigma_{\mathscr{C}_{\alpha(\ell)}(f)}(f)$ are also proper and finite-to-one, for $\Sigma_{\mathscr{C}_{\alpha(\ell)}(f)}(f_\ell)$ is closed in $U_{\alpha(\ell)}$.

By Proposition 1.1.1, there are semialgebraic open neighbourhoods $U'_{v'}$ of $0 \in U_{v'}$ for v', $\beta(v') = v$ such that the restriction $(f_\ell | U'_{\alpha(\ell)})_{\beta(\ell)=v}$ admits the canonical CVS $\mathscr{C}_v(f)$ of the critical value set $\bigcup_{\beta(\ell)=v} f_\ell(\Sigma_{\mathscr{C}_{\alpha(\ell)}(f) \cap U'_{\alpha(\ell)}}(f_\ell))$. We put $U'_{v''} = U_{v''} \cap f^{-1}_{v''v'}(U'_{v'})$, $\beta(v') = v$ for $v'' < v$. Then the restrictions $\mathscr{C}_{v''}(f) \cap U'_{v''}$, $v'' \leqslant v$ gives the CVS of $(f_\ell | U'_{\alpha(\ell)})_{\alpha(\ell)<v}$. This completes the construction of the canonical CVS of f by induction. The final property of CVS in the proposition follows from the naturality of the canonical CVS in Proposition 1.1.1.

Now we state some properties of the above CVS.

PROPOSITION 1.2.2. — *Let $G = (V,L,\Lambda)$, $G' = (V',L',\Lambda')$ be convergent diagrams with a common root v_0, and let $P = (p_v)_{v \in V}$, $P' = (p'_v)_{v \in V'}$ be tuples of positive integers with $p_{v_0} = p'_{v_0}$. Let $f \vee f'$ denote the union of $f \in \mathscr{E}(G, P)$ and $f' \in \mathscr{E}(G', P')$. If $f \vee f'$ is stable, so are f and f', and $\mathscr{C}_{v_0}(f')$, $\mathscr{C}_{v_0}(f')$ meet in a general position at $0 \in \mathbb{R}^{p_{v_0}}$ and $\mathscr{C}_{v_0}(f \vee f') = \mathscr{C}_{v_0}(f) \cap \mathscr{C}_{v_0}(f')$.*

Proof. — This follows from the construction of the canonical CVS and its naturality with respect to coordinate transformations.

For a stable convergent diagram $f \in \mathscr{E}(G,P)$ we define codim $(f) = (\mathrm{codim}_v(f))_{v \in V}$ by $(\mathrm{codim}\, X_v)_{v \in V}$, where X_v is the stratum of the canonical CVS containing the origin $0 \in \mathbb{R}^{p_v}$ (we put codim $X_v = 0$ for the sources v of G). The uniqueness of stable unfolding (Theorem 0.3.3) enables us to define codim (f) for finitely I_0 determined $f \in \mathscr{E}(G,P)$ to be the codimension codim (F) of its stable unfolding F. By definition of the equivalence relation I_0 in this paper (section 0.3), codim (f) is determined by the I_0 equivalence class of f.

From now on, we say that a finitely I_0 determined diagram f is *topologically trivial* if codim $_{v_0}(f) = 0$ (it seems that f is topologically trivial if and only if C^∞ is trivial. For the definition, see Section 0.3), and we call *f is topologically indecomposable* if all prolongations $f_{G_v^-}$ of branches f_{G_v} on $v \in V$ are topologically non-trivial, in other words, for some stable unfoldings, F, $\Sigma_{\mathscr{C}_{\alpha(\ell)}(F)}(F) = \varnothing$ for all $\ell \in L$.

We call a sub tree f_X of the maximal tree f_{X_x} of x of $f \in C^\infty(G,M)$ with $C_{I_0}(f_{X_x}) < \infty$, the *topologically maximal tree of* x if f_X contains x as a vertex, f_X is topologically indecomposable and its complement $f_{X_x - X}$ is topologically trivial. We denote this tree by $f_{\mathrm{top}X_x}$ and its branch on x by $f_{\mathrm{top}X_v^{\mathrm{br}}}$, i.e.

$$^{\mathrm{top}}X_x^{\mathrm{br}} = \bigcup_{v' < v} (f_{v'v}^{-1}(x) \cap {}^{\mathrm{top}}X_x), \qquad x \in M_v .$$

The *topologically characterizing tree* $f_{\mathrm{top}X_x^{\mathrm{ch}}}$ *of* x is the union of the above $f_{\mathrm{top}X_x}$, *the sequence* $f : x \to f(x) \to f^2(x) \cdots \to f^h(x) \in M_{v_0}$ *and the tree*

$$f_{\mathrm{top}X_{f^h(x)}} : \ {}^{\mathrm{top}}X_x^{\mathrm{ch}} = {}^{\mathrm{top}}X_x \cup \{x, f(x), \ldots, f^{h-1}(x)\} \cup {}^{\mathrm{top}}X_{f^h(x)} .$$

The next proposition follows directly from the construction of the canonical CVS.

PROPOSITION 1.2.3. – *Let $f \in \mathscr{E}(G,P)$ be a stable diagram and let $\hat{f} = (\hat{f}_\ell : U_{\alpha(\ell)} \to U_{\beta(\ell)})$ be a good representative as in* Proposition 1.2.1. *Then the canonical* CVS $\mathscr{C}_v(\hat{f})$ *and the associated stratification* $\mathscr{S}_v(\hat{f})$ *coincide with the partitions of* U_v *by* codim $f_{\mathrm{top}X_x^{\mathrm{br}}}$, codim $f_{\mathrm{top}X_x^{\mathrm{ch}}}$, *respectively. And the germs $\mathscr{C}_v(\hat{f})_x$, $\mathscr{S}_v(\hat{f})_x$ at x coincide with $\mathscr{C}_x(\hat{f}_{\mathrm{top}X_x^{\mathrm{br}}})$, $\mathscr{S}_x(\hat{f}_{\mathrm{top}X_x^{\mathrm{ch}}})$ respectively, where x's are regarded as vertices of the underlying oriented trees $\hat{f}_{\mathrm{top}X_x^{\mathrm{br}}}$, $\hat{f}_{\mathrm{top}X_x^{\mathrm{ch}}}$.*

1.3. Construction of the stratification of the jet space of convergent diagrams.

Let $e(r)(= e(G,P+r)+1)$ be the increasing function in Theorem 0.3.4 and $\Sigma^{e(r)} \subset J^{e(r)}(G,P)$ be the well-defined set of $e(r)$ jets of diagrams $f \in \mathscr{E}(G,P)$ such that $C_{I_0}(f) > r$ (in other words, f does not admit a stable r-parameter unfolding). By Theorem 0.3.1, 0.3.3 and 0.3.4, a stable unfolding of f with $C_{I_0}(f) \leqslant r$ is uniquely determined up to the equivalence of diagrams by the $e(r)$ jet of f. So we can define the set $S_I^{e(r)}(G,P) \subset J^{e(r)}(G,P) - \Sigma^{e(r)}$ by

$$S_I^{e(r)}(G,P) = \{z \in J^{e(r)}(G,P) - \Sigma^{e(r)} \mid \operatorname{codim} f = I, \pi^{e(r)}(f) = z\},$$

for a tuple $I = (c_v)_{v \in V}$ of non-negative integers ($C_v = 0$ for the sources v of G). Then $S_I^{e(r)}(G,P)$ defines a partition of the complement of $\Sigma^{e(r)}$, denoted $S^{e(r)}(G,P)$. Again by the finite determinacy of stable diagrams, $S^{e(r)}(G,P)$, $r = 0, 1, \ldots$ defines a partition of $\mathscr{E}(G,P) - \Sigma$ by *pro-sets*, where Σ is the set of non finitely I_0 determined diagrams.

Proposition 1.3.1. — *Let $f \in \mathscr{E}(G,P)$ be a stable diagram and $\hat{f}$, $\mathscr{C}(\hat{f})$ and $\mathscr{S}(\hat{f})$ as in* Proposition 1.2.1. *Let X_v, X'_v ($X_v = U_v$ for sources $v \in X'_{\alpha(\ell)} = \Sigma(\hat{f}_\ell | X_{\alpha(\ell)}) \cap \hat{f}_\ell^{-1}(X'_{\beta(\ell)})$) be the strata of $\mathscr{C}_v(\hat{f})$, $\mathscr{S}_v(\hat{f})$ containing the origin in U_v, respectively. Let $I = \operatorname{codim} f = (\operatorname{codim} X_v)_{v \in V}$ and assume that* $\operatorname{codim} Y < \operatorname{codim} X_v$ *for all other strata $Y \in \mathscr{C}(\hat{f})$ and $v \in V$. Then $J^{e(0)}\hat{f}((x_{\alpha(\ell)})_{\ell \in L}) \in S_I^{e(0)}(G,P) \times \Delta_G$ if and only if $x_{\alpha(\ell)} \in X'_{\alpha(\ell)}$, $x_{\beta(\ell)} \in X'_{\beta(\ell)}$ and $\hat{f}_\ell(x_{\alpha(\ell)}) = x_{\beta(\ell)}$ for all $\ell \in L$, (where $e(0) = e(G,P) + 1$. See* Section 0.3).

Proof. — It suffices to prove the statement for indecomposable stable diagrams. First we prove the « if » part. Let $X = (x_{\alpha(\ell)})_{\ell \in L}$ be as above and let $\hat{f}_{\bar{X}}$ denote the maximal tree of the good representative $\hat{f}$ including the tree $\hat{f}_X$. By the properties (1), (2) of CVS, $\Sigma(\hat{f}_\ell | Y) \cap \hat{f}_\ell^{-1}(X'_{\beta(\ell)})$ is a smooth submanifold on which $\hat{f}_\ell$ restricts to a locally isomorphic covering map onto $X'_{\beta(\ell)}$ for any $Y \subset \mathscr{C}_{\alpha(\ell)}(\hat{f})$. Since $\Sigma_{\mathscr{C}_{\alpha(\ell)}(\hat{f})}(\hat{f}_\ell) \cap \hat{f}_\ell^{-1}(0) = 0$, we see

$$\Sigma_{\mathscr{C}_{\alpha(\ell)}(\hat{f})}(\hat{f}_\ell) \cap \hat{f}_\ell^{-1}(X'_{\beta(\ell)}) = \Sigma(\hat{f}_\ell | X_{\alpha(\ell)}) \cap \hat{f}_\ell^{-1}(X'_{\beta(\ell)}) = X'_{\alpha(\ell)}$$

and the restriction $\hat{f}_\ell : X'_{\alpha(\ell)}) \to X'_{\beta(\ell)}$ is isomorphic. So we see by Proposition 1.2.2, the CVS of $\hat{f}_{\bar{X}-X}$ is trivial at its roots $x_{\beta(\ell)}$ in those strata $X_{\beta(\ell)}$, in other words, $\hat{f}_{\bar{X}-X}$ is topologically trivial, thus $\mathscr{C}_{x_{\beta(\ell)}}(\hat{f}_{\bar{X}}) = \mathscr{C}_{x_{\beta(\ell)}}(\hat{f}_X)$ and the germ of $\mathscr{C}_{\beta(\ell)}(\hat{f})$ at the vertices $x_{\beta(\ell)}$ coincide with these germs. In particular we have

$$\begin{aligned}\operatorname{codim} \hat{f}_X &= (\text{codimensions of the strata of } \mathscr{C}_v(\hat{f}_{\bar{X}}) \text{ containing } x_v)\\ &= (\operatorname{codim} X_v)_{v \in V} = \operatorname{codim} f\,.\end{aligned}$$

Conversely we assume a connected tree f_X, $X = (x_{\alpha(\ell)})$ has a vertex $f_\ell(x_{\alpha(\ell)})$ off the stratum $X_{\beta(\ell)}$ for some $\ell \in L$. Let $\hat{f}_{\bar{X}}$ be the maximal tree with $X \subset \bar{X}$, and $x' \in U_{v'}$ one of the highest vertices of $\hat{f}_X$ where $\hat{f}_{\bar{X}}$ is branching off f_X. Let $\hat{f}_{X_1}$ denote the branch of $\hat{f}_X$ on x' and $\hat{f}_{X_2}$ the sub graph of $\hat{f}_{\bar{X}}$ branching off $\hat{f}_X$ at $x' : X_2 = X_{x'}^{\text{br}} - X_{x'}^{\text{br}}$ (a connected component of $\hat{f}_{\bar{X}} - f_X$). By Proposition 1.2.2, we have $\mathscr{C}_{v'}(\hat{f})_{x'} = \mathscr{C}_{x'}(\hat{f}_X) \cap \mathscr{C}_{x'}(\hat{f}_X)$. By the condition of the proposition, the stratum of $\mathscr{C}_{v'}(\hat{f})_{x'}$ containing x' has codimension smaller than $\operatorname{codim} X_{v'} = \operatorname{codim}_{v'}(f)$. So we have $\operatorname{codim}(\hat{f}_X) \neq \operatorname{codim}(f) = I$. This completes the proof.

Proposition 1.3.2. — *Let $I = (a_v)_{v \in V}$ be a tuple of positive integers ($a_v = 0$ for sources v). Then the set $S_I^{e(r)}(G,P) \subset J^{e(r)}(G,P)$ is a semialgebraic submanifold of codimension $\sum_{v \neq V_0} p_v + a_{v_0} - \sum_{v \in V} b_v . p_v$, where b_v denotes the number of edges $\ell \in L$ with $(\beta(\ell) = v$.*

Proof. — Let $z \in S_I^{e(r)}(G,P) \subset J^{e(r)}(G,P)$ and let

$$F = (F_\ell)_{\ell \in L} \in E(G, P+s), \qquad F_\ell : (\mathbb{R}^{p_{\alpha(\ell)}+s}, 0) \to (\mathbb{R}^{p_{\beta(\ell)}+s}, 0),$$

$F_\ell(x,u) = (f_{\ell,u}(x),u)$, $x \in \mathbb{R}^{p_{\alpha(\ell)}}$, $u \in \mathbb{R}^s$ be a stable sequence of polynomial map germs unfolding the polynomial map germ f, such that the r-jet section $\bar{J}^{e(r)}F : \prod_{v \neq v_0} \mathbb{R}^{p_v} \times \mathbb{R}^s \to J^{e(r)}(G, \mathbb{R}^P) \,(= \prod_{\ell \in L} J^{e(r)}(\mathbb{R}^{p_{\alpha(\ell)}}, \mathbb{R}^{p_{\beta(\ell)}}))$ is locally diffeomorphic at the origin, and by Theorem 0.3.2, F is stable. Let $\mathscr{S}(F) = (\mathscr{S}_v(f))_{v \in V}$ be the canonical stratification of F and let $\mathscr{S}_v$ be the strata of $\mathscr{S}_v(F)$ containing the origin in $\mathbb{R}^{p_v+s}$. By Proposition 1.2.1, each $\mathscr{S}_v$ is semialgebraic and $F_\ell | \mathscr{S}_{\alpha(\ell)} : \mathscr{S}_{\alpha(\ell)} \to \mathscr{S}_{\beta(\ell)}$ is isomorphic. Let

$$X = (x_{\alpha(\ell)})_{\ell \in L} \in \prod_{\ell \in L} \mathscr{S}_{\alpha(\ell)} \in \prod_{\ell \in L} \mathbb{R}^{p_{\alpha(\ell)}+s}, \qquad f_{\ell,u}(x_{\alpha(\ell)}) = x_{\beta(\ell)}$$

and let $\bar{X} = ((x_{\alpha(\ell)})_{\ell L}, u) \in \prod_{\ell \in L} \mathbb{R}^{p_{\alpha(\ell)}} \times \mathbb{R}^s$ with $(x_{\alpha(\ell)}, u) \in \mathscr{S}_{\alpha(\ell)}$ and $F_\ell(x_{\alpha(\ell)}, u) = (x_{\beta(\ell)}, u)$. Then both X and $\bar{X}$ are semialgebraic submanifolds. By Proposition 1.3.1, we have

$$X = J^r F^{-1}(S_I^r(G,P+r) \times \Delta_G), \qquad \Delta_G \subset \prod_{\ell \in L} \mathbb{R}^{p_{\alpha(\ell)}+r} \times \mathbb{R}^{p_{\beta(\ell)}+r},$$

and by the definitions of $S_I^r(G,P)$ and $S_I^r(G,P+r)$, we have

$$\bar{X} = \bar{J}^r F^{-1}(S_1^r(G,P) \times \Delta_G), \qquad \Delta_G \subset \prod_{\ell \in L} \mathbb{R}^{p_{\alpha(\ell)}} \times \mathbb{R}^{p_{\beta(\ell)}}.$$

Since F is a sequence of polynomial map germs, $\bar{J}^r F$ is also a polynomial map germ, and since $\bar{J}^r F$ is a diffeo-germ, the image $S_I^r(G,P) \times \Delta_G$ of $\bar{X}$ is a semialgebraic submanifold. Now we have the following equality,

$$\dim S_I^r(G,P) \times \Delta_G = \dim \bar{X} = \dim \mathscr{S}_{v_0},$$

from which we have

$$\begin{aligned}
\operatorname{codim} S_I^r(G,P) \times \Delta_G &= \sum_{v \neq v_0} p_v + r - \dim S_{v_0} \\
&= \sum_{v \neq v_0} p_v - p_{v_0} + \operatorname{codim} S_{v_0} \text{ in } \mathbb{R}^{p_{v_0}} \\
\operatorname{codim} S_I^r(G,P) &= \sum_{v \neq v_0} p_v - p_{v_0} + \operatorname{codim} S_{v_0} - \operatorname{codim} \Delta_G \\
&= \sum_{v \neq v_0} p_v + a_{v_0} - \sum_{v \in V} b_v \cdot p_v .
\end{aligned}$$

This completes the proof.

PROPOSITION 1.3.3. – *Let $f = (f_\ell)_{\ell \in L} \in \mathscr{E}(G,P)$ be a finitely I_0 determined diagram : $C_{I_0}(f) \leqslant r$. Let $F \in \mathscr{E}(G,P+s)$, $F_\ell : (\mathbb{R}^{p_{\alpha(\ell)}+s},0) \to (\mathbb{R}^{p_{\beta(\ell)}+s},0)$, $F_\ell(x,u) = (f_{\ell u}(x),u)$, $f_{\ell 0} = f_\ell$ be a stable unfolding of f, and let $\imath_v : (\mathbb{R}^{p_v},0) \to (\mathbb{R}^{p_v+s},0)$, $v \in V$ be the inclusions. Let $\mathscr{C}(F) = (\mathscr{C}_v(F))_{v \in V}$ be the canonical* CVS *of F ($\mathscr{C}_v(F)$ is trivial for sources v). Then $\imath_v$ is transversal to $\mathscr{C}_v$ for all $v \in V$, if and only if $J^{e(r)}f$ is transversal to $S_I^{e(r)}(G,P) \times \Delta_G$ at $(0)_{v=v_0} \in \prod_{v \neq v_0} \mathbb{R}^{p_v}$.*

Proof. – By Theorem 0.3.3, the transversality of $\imath_v$ to $\mathscr{S}_v$ is independent of the choice of the stable unfolding F. So we assume that the $e(r)$ jet section $\bar{J}^{e(r)}F : \prod_{v \neq v_0} \mathbb{R}^{p_v} \times \mathbb{R}^s \to J^{e(r)}(G,\mathbb{R}^p)$ defined by $\bar{J}^{e(r)}F((x_v)_{v \neq v_0}, u) = J^{e(r)}f_u((x_v)_{v \neq v_0})$ is the germ of a diffeomorphism, where $f_u = (f_{\ell u})_{\ell \in L} \in \mathscr{E}(G,P)$. Since $J^{e(r)}f = \bar{J}^{e(r)}F \circ \imath : \prod_{v \neq v_0} \mathbb{R}^{p_v} \to \prod_{v \neq v_0} \mathbb{R}^{p_v} \times \mathbb{R}^s \to J^{e(r)}(G,\mathbb{R}^p)$ we see $J^{e(r)}f \pitchfork S_I^{e(r)}(G,P) \times \Delta_G$ if and only if $\imath \pitchfork \bar{J}^{e(r)}F^{-1}(S_I^{e(r)}(G,P) \times \Delta_G)$, where $\imath$ is the natural inclusion. As we have seen in the proof of Proposition 1.3.2, $\bar{J}^{e(r)}F^{-1}(S_I^{e(r)}(G,P) \times \Delta_G)$ is the set of points $((x_v),u) \in \prod_{v \neq v_0} \mathbb{R}^{p_v} \times \mathbb{R}^s$ such that $\operatorname{codim} F_X = I$, $X = ((x_v,u))_{v \neq v_0}$, in other words, by Proposition 1.3.1,

$$(x_{\alpha(\ell)},u) \in \Sigma(F_\ell | X_{\alpha(\ell)}(F)) \cap F_\ell^{-1}(X'_{\beta(\ell)}(F)) = X'_{\alpha(\ell)}(F), \quad \ell \in L.$$

Let Pr: $\prod_{v \neq v_0} \mathbb{R}^{p_v} \times \mathbb{R}^s \to \mathbb{R}^s$ by the natural projection. The image of the inclusion $\imath$ is the fibre of Pr on $0 \in \mathbb{R}^s$, so the above transversality holds if and only if the restriction Pr: $\bar{J}^{e(r)}f^{-1}(S_I(G,P) \times \Delta_G) \to \mathbb{R}^s$ is a submersion. Since the $F_\ell : X'_{\alpha(\ell)}(F) \to X'_{\beta(\ell)}(F)$ are isomorphisms, this holds if and only if the second projections $P_2 : X'_v(F) \to \mathbb{R}^s$ are submersions, if and only if the inclusions $\imath_v : \mathbb{R}^{p_v} \to \mathbb{R}^{p_v+s}$ are transversal to $\mathscr{S}_v(F)$ for $v \in V$, and if and only if $\imath_{\beta(\ell)}$ is transversal to $C_{\beta(\ell)}(F)$ for $\ell \in L$.

From now we say a finitely I_0 determined diagram $f \in \mathscr{E}(G,P)$ is *topologically transversal* if the condition in the above proposition is satisfied.

COROLLARY 1.3.4. – *Let f, F be as above. Then there is a good representative $\hat{F}, \hat{F}_\ell : U_{\alpha(\ell)} \to U_{\beta(\ell)}$ of F defined on open neighbourhoods U_v of $C \in \mathbb{R}^{p_v+s}$ which admits the canonical* CVS $\mathscr{C}(F) = (\mathscr{C}_v(F))_{v \in V}$ *and the natural inclusions $\imath_v : \mathbb{R}^{p_v} \to \mathbb{R}^{p_v+s}$ are transversal to $\mathscr{C}_v(\hat{F})$ (for the definition of good representatives, see Section* 0.3). *In this situation, the restriction $\hat{f}(\hat{F}_\ell | U_{\alpha(\ell)} \times \mathbb{R}^{p_{\alpha(\ell)}} \times 0)$ is a good representative of f, which admits the canonical* CVS $\mathscr{C}(\hat{f}) = (\mathscr{C}_v(\hat{F}) | \mathbb{R}^{p_v} \times 0)$ *and all connected trees of $\hat{f}$ are topologically transversal. The germs of $\mathscr{C}_v(\hat{f})$ at* 0 *are independent of the choice of $\hat{F}$ and denoted $\mathscr{C}_v(f)$. Let $\mathscr{S}(\hat{f})$ denote the stratification of $\hat{f}$ associated whith $\mathscr{C}(\hat{f})$. Then $\mathscr{C}_v(\hat{f})$, $\mathscr{S}_v(\hat{f})$ coincide respectively with the partition of $U_v | \mathbb{R}^{p_v} \times 0$ by the numbers* $\operatorname{codim}_x \hat{f}_{\mathrm{top}\,X_x^{\mathrm{br}}} = \operatorname{codim}_x \hat{f}_{X_x^{\mathrm{br}}}$, $\operatorname{codim} \hat{f}_{\mathrm{top}\,X_x^{\mathrm{ch}}} = \operatorname{codim} \hat{f}_{X_x^{\mathrm{ch}}}$ *associated with those points $x \in U_v \cap \mathbb{R}^{p_v} \times 0$, where x is regarded as vertices of those trees of $\hat{f}$. Consequently the germs $\mathscr{C}_v(\hat{f})_x$, $\mathscr{S}_v(\hat{f})_x$ at x coincide with the germs $\mathscr{C}_x(\hat{f}_{\mathrm{top}\,X_x^{\mathrm{br}}}) = \mathscr{C}_x(\hat{f}_{X_x^{\mathrm{br}}})$, $\mathscr{S}_x(\hat{f}_{\mathrm{top}\,X_x^{\mathrm{ch}}}) = \mathscr{S}(\hat{f}_{X_x^{\mathrm{ch}}})$, respectively. (For the definition of the above trees, see Section* 1.2.)

CHAPTER 2

SOME PROPERTIES OF CRITICAL SETS, MAXIMAL TREES AND BRANCHES OF GENERIC CONVERGENT DIAGRAMS

2.1. Some generic properties.

Let $G = (V,L,\Lambda)$ be a convergent graph, $Q = (q_v)$ a tuple of integers $0 \leqslant q_v \leqslant \infty$. We call G a *Q graph* if each fibre $\beta^{-1}(v)$ of $\lambda : L \to V$ consists of at most q_v edges (finite if $q_v = \infty$). Let $\imath : \Gamma \to G$, $\imath = (\imath_V, \imath_L)$: $(V_\Gamma, L_\Gamma) \to (V_G, L_G)$ be a morphism of oriented graphs. We call $\imath$ a *Q morphism* if Γ is a $\imath_V^* Q$ - graph.

Let $P = (p_v)_{v \in V}$ be a tuple of integers $0 < p_v < \infty$. We say P satisfies *Condition* $G_Q(G)$ if finite I_0 determinacy holds in general in $\mathscr{E}(\Gamma, \imath^* P)$ for any Q-morphism (finite morphism) $\imath : \Gamma \to G$.

We call a diagram (tree) f_X of f of embedding type $\imath : \Gamma \to G$ (defined by the inclusions of germs) a Q-diagram (Q-tree), if $\imath$ is a Q-morphism.

Let $0 \leqslant r \leqslant \infty$ be an integer and $U \subset M_{V_0}$ a subset. We denote by A^r_{QU} the set of smooth diagrams $f \in C^\infty(G,M)$ such that for any Q tree f_X of the restriction $f_U = (f_\ell | f^{-1}_{\alpha(\ell)}(U))$, the I_0 codimension $C_{I_0}(f_X)$ is at most r (finite if $r = \infty$), and we denote $A^\infty_{QU} = A_{QU}$, $A^r_{QM_{V_0}} = A^r_Q$.

PROPOSITION 2.1.1. — *Let* $0 < r < \infty$, $P + r + 1 = (\dim M_v + r + 1)_{v \in V}$ *and let* $U \subset M_{v_0}$ *be a subset. Then*

$$A^r_{\infty,U} = A^r_{P+r+1,U}, \qquad \infty = (\infty)_{v \in V}.$$

Proof. — From the definition it follows immediately that $A^r_{\infty,U} \subset A^r_{P+r+1,U}$. Conversely let $f \in A^r_{P+r+1,U}$ and for simplicity of notations assume $U = M_{V_0}$. Then we prove that any connected and indecomposable tree of f with root in M_v is a $P + r$ tree, by the descending introduction on the height $h(v)$ of the vertices $v \in V$. It then follows that all finite trees of f admit stable unfoldings of dim $\leqslant r$ hence $f \in A^r_{\infty,U}$.

We may assume inductively all indecomposable trees of f with roots in M_v, $h(v) \geqslant h$ are $(P+r)$-trees of f. Let f_X be an arbitrary finite indecomposable tree of f with root $x \in M_v$ of height $h(v) = h$. Suppose that f_X is a union of the prolongations $f_{X\bar{x}_i}$ of the branches $f_{X_{x_i}}$ of f_X on $x_i \in X \cap M_{\alpha(\ell_i)}$, $\beta(\ell_i) = v$, $f_{\ell_i}(x_i) = x$ and that $q \geqslant p_v + r + 1$.

By the induction hypothesis these prolongations are all $(P+r)$-trees. Let f_Y be a $p_v + r + 1$ union of these prolongations. Then, by Corollary 1.1.4, at least one of these $p_v + r + 1$ branches must be trivial. This contradicts the assumption that f_X is indecomposable. Therefore f_X is a union of at most $p_v + r$ prolongations, and in particular is a $(P+r)$-tree. This completes the proof.

Our purpose in this chapter is to prove

THEOREM 2.1.2. — *Let $G = (V,L,\Lambda)$ be a convergent diagram with root v_0 and $M = (M_v)_{v \in V}$ a collection of smooth manifolds. Then the set $A^{\infty}_{\infty K} \cap C^{\infty}_{\mathrm{pr}}(G,M)$ is open in $C^{\infty}_{\mathrm{pr}}(G,M)$ with the Whitney topology for any closed subset $K \subset M_{v_0}$, and if K is compact the set is open in the weak C^{∞} topology. If $P = (\dim M_v)_{v \in V}$ satisfies the condition G_Q then $A^r_Q \cap C^{\infty}_{\mathrm{pr}}(G,M)$ is dense in $C^{\infty}_{\mathrm{pr}}(G,M)$ for any sufficiently large r and the complement of A^{∞}_Q has infinite codimension : any smooth family f_u, $u \in \mathbb{R}^s$ of arbitrary dimension s can be approximated by a smooth family f'_u in A^{∞}_Q.*

Remark. — It seems that if finite I_0 determinacy holds in general in $\mathscr{E}(G,P)$ then P satisfies Condition G_Q for any Q. So although we state everything for general Q in this and the next chapters, we will prove them only for the case $Q = \infty = (\infty)_{v \in V}$, and restrict ourselves to reminding here that the topological stability theorem in Section 0.1 can be proved under the Condition G_{P+1}. For the case of $Q = P + 1$, there is only one point of the proof that does not go the same in those proofs, that is, the maximal trees may not be finite. However, if we define topologically maximal trees by substituting C^0 triviality for C^{∞} triviality in the definition, then those trees are finite, and the rest of the proof remains valid.

2.2. Some properties of critical sets and maximal trees and branches.

To generalize the notions of $C_v(f)$ and $D_v(f)$ of diagrams $f \in C^{\infty}(G,M)$, let $Q = (q_v)_{v \in V}$ be a tuple of positive integers. The set $C_{vQ}(f)$ is defined to be the set of roots of Q trees of f in M_v, whose prolongation is not trivial and $D_{v,Q}(f) = \bigcup_{\beta(\ell)=v} f_{\ell}(C_{\alpha(\ell)Q}(f))$. Clearly, $C_{v\infty}(f)$, $D_{v\infty}(f)$, $\infty = (\infty)_{v \in V}$ coincide with the sets previously defined (in Section 0.3).

THEOREM 2.2.1. — *Let* $G = (V,L,\Lambda)$ *be a convergent diagram with root* v_0 *and* $M = (M_v)_{v \in V}$ *be a collection of smooth manifolds. Let* $K \subset M_{v_0}$ *be a subset,* $0 < r \leqslant \infty$ *an integer and* $f = (f_\ell) \in A^r_{\infty K} \cap C^\infty_{\mathrm{pr}}(G,M)$ *(resp.* $A^\infty_{QK} \cap C^\infty_{\mathrm{pr}}(G,M)$*),* $Q = (q_v)_{v \in V}$*,* $0 < q_v < \infty$ *integers). Then there is an open neighbourhood* U *of* K *in* M_{v_0} *such that the following properties are satisfied for any integers* $k = 0, 1, \ldots, h(G)$:

$(1)_k$ $C_v(f) \cap f^{-1}_{vv_0}(U)$ *(resp.* $C_{vQ}(f) \cap f^{-1}_{vv_0}(U)$*) is closed in* $f^{-1}_{vv_0}(U)$ *for any* $v \in V$, $h(v) = k$.

$(2)_k$ $D_v(f) \cap f^{-1}_{vv_0}(U)$ *(resp.* $D_{vQ}(f) \cap f^{-1}_{vv_0}(U)$*) is closed in* $f^{-1}_{vv_0}(U)$ *for any* $v \in V$, $h(v) = k$ *and the restriction* $f_\ell : C_{\alpha(\ell)}(f) \cap f^{-1}_{\alpha(\ell)v_0}(U) \to D_{\beta(\ell)}(f) \cap f^{-1}_{\beta(\ell)v_0}(U)$ *(resp.* $f_\ell : C_{\alpha(\ell)Q}(f) \cap f^{-1}_{\alpha(\ell)v_0}(U) \to D_{\beta(\ell)Q}(f) \cap f^{-1}_{\beta(\ell)v_0}(U)$*) is proper and locally uniformly finite-to-one for any* $\ell \in L$, $h(\beta(\rho)) = k$.

$(3)_k$ *For any* $v \in V$, $h(v) = k$*, the number of vertices of the maximal branch* $f_{X^{\mathrm{br}}_x}$ *on* x *is locally bounded at any point* $x \in f^{-1}_{vv_0}(U)$ *and if* $x_i \in f^{-1}_{vv_0}(U)$ *is convergent to a point* $x \in f^{-1}_{vv_0}(U)$ *as* $i \to \infty$ *then* $X^{\mathrm{br}}_{x_i} \to X^{\mathrm{br}}_x$ *i.e.,* X^{br}_x *is the cluster point set of* $\bigcup_{i=1}^{\infty} X^{\mathrm{br}}_{x_i}$. *(The coherence of maximal branches.)*

$(4)_k$ *For any connected tree (resp.* Q *tree)* f_X *of* f *with root in* $f^{-1}_{vv_0}(U)$, $h(v) = k$*, the* I_0 *codimension* $C_{I_0}(f_x) < r + 1$.

(5) $f \in A^r_{\infty U} \cap C^\infty_{\mathrm{pr}}(G,M)$ *(resp.* $A^r_{QU} \cap C^\infty_{\mathrm{pr}}(G,M)$*)*.

Proof. — We consider only the case $f \in A^r_{\infty k}$. The other case can be proved similarly.

We prove the statements by descending induction on the height k of vertices. The outline is as in the diagram:

$$\underbrace{(1)_{k+1} \quad (2)_{\ell \geqslant k+1} \quad (3)_{k+1}}_{\Downarrow} \qquad \searrow$$

$$(1)_k \quad (2)_k \Rightarrow (3)_k \;(\Rightarrow (1)_k) \qquad (4)_k, \qquad (4)_i,\ 0 \leqslant i \Rightarrow (5).$$

$(1)_{k+1}$, $(2)_{i>k}$, $(3)_{k+1} \Rightarrow (2)_k$. First we assume $(1)_{k+1}$: $C_v(f) \cap f_{vv_0}^{-1}(U)$ is closed for any $v \in V$, $h(v) = k+1$. Since f_ℓ are proper, the restrictions $f_\ell : C_{\alpha(\ell)}(f) \cap f_{\alpha(\ell)v_0}^{-1}(U) \to f_{\beta(\ell)v_0}^{-1}(U)$, $h(\alpha(\ell)) = k$ are also proper and the union of the images

$$D_v(f) \cap f_{\alpha(\ell)v_0}^{-1}(U) = \bigcup_{\lambda(\ell)=v} f_\ell((C_{\alpha(\ell)}(f) \cap f_{\alpha(\ell)v_0}^{-1}(U))$$

is closed in $f_{vv_0}^{-1}(U)$ for any $v \in V$, $h(v) = k$.

Next we assume $(2)_i$ for $k < i$ and $(3)_{k+1}$. Then for any point $x \in f_{\alpha(\ell)v_0}^{-1}(K)$ $h(\alpha(\ell)) = k$, the prolongation of the maximal branch $f_{X_x^{\mathrm{br}-}}$ is finite and $C_{I_0}(f_{X_x^{\mathrm{br}-}}) < r+1$, and by Proposition 0.3.6 there are disjoint open neighbourhoods U_y of the vertices $y \in M_v$, $v \in V$ of $f_{X_x^{\mathrm{br}-}}$ such that the restrictions $f_x = (f_\ell : U_y \to U_{f_\ell(y)})$, $y \in X_x^{\mathrm{br}^-} \cap M_{\alpha(\ell)}$, $\ell \in L$ is a good representative of the tree $f_{X_x^{\mathrm{br}-}}$. By $(2)_i$, $k < i$, we may assume that the maximal branch $f_{X_{x'}^{\mathrm{br}}}$ on $x' \in U_x$ is a tree of the restriction f_x for all $x \in f_{\alpha(\ell)v_0}^{-1}(K)$. Then by the properties of good representatives we see that $C_{\alpha(\ell)}(f) \cap U_x = C_x(f_x)$ and $f_\ell : C_{\alpha(\ell)}(f_x) \cap U_x \to U_{f_\ell(x)}$ are uniformly finite-to-one for any $x \in f_{\alpha(\ell)v_0}^{-1}(K)$, where x is regarded as a vertex of the underlying oriented graph of f_x. Since f_ℓ are proper, we may assume that, by shrinking the neighbourhood $U \supset K$,

$$f_{\alpha(\ell)v_0}^{-1}(U) \subset \bigcup_{x \in f_{\alpha(\ell)v_0}^{-1}(K)} U_x,$$

from which the statement $(2)_k$ follows.

$(2)_{i \geqslant k} \Rightarrow (3)_k$. We assume the statements $(2)_i$ for $k \leqslant i$:

$$f_\ell : C_{\alpha(\ell)} \cap f_{\alpha(\ell)v_0}^{-1}(U) \to D_{\beta(\ell)}(f) \cap f_{\alpha(\ell)v_0}^{-1}(U)$$

is proper and locally uniformly finite-to-one for any $\ell \in L$, $h(\beta(\ell)) > k$. Let $v \in V$ be a vertex of height k. Then $X_{xv'}^{\mathrm{br}} = C_{v'}(f) \cap f_{v'v}^{-1}(x)$ is a finite set of which the number of elements is locally bounded at any point $x \in f_{v'v}^{-1}(U)$ and $v' < v$ by the assumption above, and the union $X_x^{\mathrm{br}} = \bigcup_{v' < v} X_{xv'}^{\mathrm{br}}$ gives the maximal branch of f on x. The coherence of the maximal branches follows from the properness of $f_\ell | C_{\alpha(\ell)} \cap f_{\alpha(\ell)v_0}^{-1}(U)$, $\lambda(\ell) \leqslant v$.

$(3)_k \Rightarrow (1)_k$. We assume $(3)_k$. Let $v \in V$ be a vertex of height k and let $x_i \in C_v(f) \cap f_{vv_0}^{-1}(U)$ be a sequence convergent to a point $x \in f_{vv_0}^{-1}(U)$. By Proposition 0.3.6, there are open neighbourhoods U_y of vertices y of the prolongation $f_{X_x^{\mathrm{br}-}}$ of the tree $f_{X_x^{\mathrm{br}}}$ on x such that the restriction

$$f_x = (f_\ell : U_y \to U_{f_\ell(y)}), \; y \in X_x^{\mathrm{br}-} \cap M_{\alpha(\ell)}, \; \ell \in L$$

is a good representative of $f_{X_x^{\mathrm{br}-}}$. By the coherence of maximal branches, the prolongation $f_{X_{x_i}^{\mathrm{br}-}}$ is a tree of f_x hence $x_i \in C_x(f_x)$ for any sufficiently large i. By the property of good representatives, the critical point set $C_x(f_x) \subset U_x$ is closed so it follows that $x \in C_x(f_x) \subset C_v(f)$. Therefore $C_v(f) \cap f_{vv_0}^{-1}(U)$ is closed in $f_{vv_0}^{-1}(U)$.

$(3)_k \Rightarrow (4)_k$. Let $v \in V$ be a vertex of height k. By the same argument as the implication of $(2)_k$, any maximal branch $f_{X_{x'}^{\mathrm{br}}}$ on $x' \in f_{vv_0}^{-1}(U)$ is a tree of a good representative of some branch $f_{X_x^{\mathrm{br}}}$ on $x \in f_{vv_0}^{-1}(K)$. By the assumption, we have $C_{I_0}(f_{X_x^{\mathrm{br}}}) < r$ and by the property of good representatives, we have $C_{I_0}(f_{X_{x'}^{\mathrm{br}}}) \leqslant r$.

$(4)_i$, $0 \leqslant i \Rightarrow (5)$. Trivial.

This completes the proof of Theorem 2.2.1.

2.3. C^∞ stability and infinitesimal stability.

In this section, we prove a theorem on C^∞ stability of diagrams as an application of our theory of maximal trees and branches (Theorem 2.3.1). This theorem was proved already by Baas and Dufour [B1, Du], however the part of implication (3) $\Rightarrow$ (2) is not clear in their papers. The reader may appreciate our theory in proving this part.

Let $f \in C^\infty(G, M)$ be a convergent diagram of smooth mappings. Let $Q = (q_v)_{v \in V}$ be a tuple of integers $0 \leqslant q_v < \infty$. We say f is *multi* (resp. Q-) *infinitesimally stable* if any finite (resp. Q-) tree f_X of f is infinitesimally stable.

Our theorem is

THEOREM 2.3.1. – *Let* $G = (V,L,\Lambda)$ *be a finite convergent tree with root* v_0 *and let* $M = (M_v)_{v \in V}$ *be a collection of smooth manifolds and* $f = (f_\ell)_{\ell \in L} \in C^\infty_{\mathrm{pr}}(G,M)$. *Then the following conditions are equivalent :*

(1) f *is* C^∞ *stable,*

(2) f *is infinitesimally stable,*

(3) f *is multi infinitesimally stable,*

(4) f *is* $(P+1)$*-infinitesimally stable, where* $P+1 = (\dim M_v + 1)_{v \in V}$.

The part (2) $\Leftrightarrow$ (1) is a generalization of Mather's theory of adequate homomorphisms [M1], and can be found in the papers [Ba1, Bu, Du1]. The implication (2) $\Rightarrow$ (3) is obvious.

By Proposition 2.1.1, the conditions (3), (4) are equivalent.

Proof of the implication (3) $\Rightarrow$ (2). – We fix an element $v = \bigoplus_{\ell \in L} v_\ell \in \theta(f) = \bigoplus_{\ell \in L} \theta(f_\ell)$. In the remainder of this section, we construct a $u = \bigoplus_{v \in V} u_v \in \theta(M) = \bigoplus_{v \in V} \theta(M_v)$, such that $T(f)(u) = v$ by induction on the height of vertices $v \in V$.

Let $C^h_v(f) \subset M_v$ denote the set of points $x \in M_v$ whose maximal trees f_{X_x} have their roots in $M_{v'}$, $h(v') \leqslant h$, for any $v \in V$ and integer $h \geqslant h(v)$. It is easy to see $C^h_v(f) = \bigcap_{h(v) \geqslant h(v') > h} f^{-1}_{vv'}(C_{v'}(f))$.

By Theorem 2.2.1, the critical point sets $C^h_v(f)$, $h(v) > h$ are closed and the restrictions $f_\ell : C^h_{\alpha(\ell)}(f) \to C^h_{\beta(\ell)}(f)$, $h(\beta(\ell)) \leqslant h$ are proper and locally uniformly finite-to-one.

Let $0 \leqslant h < h(G)$ be an integer. We assume that for each $v \in V$, there is a vector field u^{h-1}_v defined on an open neighbourhood U^{h-1}_v of $C^{h-1}_v(f)$ in $M_v (U^{h-1}_v = M_v$ for $v \in V, h(v) \leqslant h-1)$ such that $f_\ell(U^{h-1}_{\alpha(\ell)}) \subset U^{h-1}_{\beta(\ell)}$, $\ell \in L$ and the restriction $f|U^{h-1}$ of f to the open neighbourhoods U^{h-1}_v, $v \in V$ satisfies

$$F(f|U^{h-1}_v)\left(\bigoplus_{v \in V} u^{h-1}_v\right) = \bigoplus_{\ell \in L} u_\ell | U^{h-1}_{\alpha(\ell)}.$$

We then extend u_v^{h-1} to vector fields u_v^h defined on open neighbourhoods U_v^h respectively for $v \in V$ so that the restriction $f|U^h$ satisfies the equality above. The final step $h = h(G)$ of the extension of vector fields completes the construction of a vector field $u = \bigoplus_{v \in V} u_v = \bigoplus_{v \in V} u_v^{h(G)}$ with the required property $T(f)(u) = v$.

Let $v \in V$ be a vertex of height h and $x \in M_v - C_v^{h-1}(f)$. Then the prolongation $f_{X_x^{br-}}$ of the maximal tree $f_{X_x^{br}}$ on x is trivial:

$$T(f_{X_x^{br-}}) = (\theta(M_{X_x^{br-}})) = \theta(f_{X_x^{br-}}),$$

where $\theta(M_{X_x^{br-}}) = \bigoplus_{\substack{x' \in X_x^{br-} \cap M_v \\ v' \in V}} \theta(M_{v'})_{x'}$. So there is a vector field $u^x = \bigoplus_{x'} u_{x'}^x \in \theta(M_{X_x^{br-}})$ such that

$$T(f_{X_x^{br-}})(u^x \oplus u_{\beta(v')f_\ell(x)}^{h-1}) = v_{X_x^{br-}} = \bigoplus_{\substack{x' \in X_x^{br} \cap M_{v'} \\ v' \in V}} u_{v'x'}.$$

Let $u_{x'}^x$ be representatives of $u_{x'}^x$ defined on disjoint open neighbourhoods $U_{x'}^x$ of vertices $x' \in X_x^{br-} \cap M_{v'}$, $v' \in V$ in $M_{v'} - C_{v'}^{h-1}(f)$ such that

(i) $f_\ell(U_{x'}^x) \subset U_{f_\ell(x')}^x$, for $x' \in X_x^{br} \cap M_{\alpha(\ell)}$, $\ell \in L$,

(ii) $C_{\alpha(\ell)}^h(f) \cap f_{\alpha(\ell)v}^{-1}(U_x^x) \subset \bigcap_{x' \in X_x^{br} \cap M_{\alpha(\ell)}} U_{x'}^x$ for $\ell \in L$,

and

(iii) $T(f|U^x)\left(\bigoplus_{x' \in X_x^{br-}} \tilde{u}_{x'}^x \oplus u_{\beta(\ell)}^{h-1}\right) = v|U^x$,

where $f|U^x$, $v|U^x$ denote respectively the sets of restrictions $f_\ell|U_{x'}^x$, $v_\ell|U_{x'}^x$, $x' \in X_x^{br-} \cap M_{\alpha(\ell)}$, $\ell \in L$ (the existence of such representatives is proved by Theorem 2.2.1). Let $x_i \in M_v - C_v^{h-1}(f)$, $i = 1, \ldots$ be a countable family of points such that $C_v^h(f) - U_v^{h-1} \subset \bigcup_i U_{x_i}^{x_i}$. Then by the property (ii), we have

$$C_{v'}^h f - U_{v'}^{h-1} \subset \bigcup_{\substack{x' \in X_{x_i}^{br} \\ i = 1, 2, \ldots}} U_{x'}^{x_i},$$

for all $v' \leqslant v$.

By shrinking the open neighbourhoods $U^{x_i}_{x'}$, $x' \in X^{\mathrm{br}^-}_{x_i}$, $i = 1, 2, \ldots$, we may assume that $\{U^{x_i}_{x_i}\}$ is locally finite. Then we can take a partition of unity $h_i\colon U^{x_i}_{x_i} \to \mathbb{R}$, $h_v\colon U^{h-1}_v(f) \to \mathbb{R}$ subordinate to the covering $\{U^{x_i}_{x_i},\ i = 1, 2, \ldots,\ U^{h-1}_v(f)\}$ of M_v. Now let

$$U^h_{v'} = U^{h-1}_{v'} \cup \bigcup_{\substack{x' \in X^{\mathrm{br}}_{x_i} \\ i=1,2,\ldots}} U^{x_i}_{x'}, \qquad v' \leqslant v$$

and define the vector field $\mathscr{u}^h_{v'}$ on $U^h_{v'}$ by $\mathscr{u}^h_v = \mathscr{u}^{h-1}_v$ for $v \in V$, $h(v) \leqslant h-1$, and

$$\mathscr{u}^h_{v'} = f^*_{v'v} h_v . \mathscr{u}^{h-1}_{v'} + \sum_{\substack{x' \in X^{\mathrm{br}^-}_{x_i} \\ i=1,2,\ldots}} f^*_{v'v} h_v . \mathscr{u}^{x_i}_{x'},$$

for $v' \leqslant v$. Then U^h_v and $\mathscr{u}^h_v$ have the required properties.

This completes the construction of the vector field $\mathscr{u}$ hence the proof of the implication (3) $\Rightarrow$ (2).

2.4. Proof of Theorem 2.1.2.

We prove the openness of $A^\infty_{\infty K}$. The openness of the other sets follows the same way.

First we prove that $A^{r+1}_{\infty K} \cap C^\infty_{\mathrm{pr}}(G,M)$ is a neighbourhood of $A^r_{\infty K} \cap C^\infty_{\mathrm{pr}}(G,M)$ in the weak C^∞ topology if K is compact. Since the weak C^∞ topology has countable open basis it suffices to prove that for any sequence $f_i \in C^\infty_{\mathrm{pr}}(G,M)$ convergent to an $f \in A^r_{\infty K} \cap C^\infty_{\mathrm{pr}}(G,M)$, $f_i \in A^\infty_{\infty K}$ for any sufficiently large i. Then f_i can be imbedded in a smooth one parameter famility $f_t \in C^\infty_{\mathrm{pr}}(G,M)$ so that $f_{t_i} = f_i$ with a sequence $t_i \in \mathbb{R}$ convergent to 0 (see the book [Gi], p. 146). Let $F \in C^\infty(G, M \times \mathbb{R})$, $F_\ell : M_{\alpha(\ell)} \times \mathbb{R} \to M_{\beta(\ell)} \times \mathbb{R}$, $F_\ell(x,t) = (f_{\ell t}(x), t)$. In general for an unfolding $H \in \mathscr{E}(G, P+s)$ of $h \in \mathscr{E}(G,P)$, we see $C_{I_0}(h) - s \leqslant C_{I_0}(H) \leqslant C_{I_0}(h)$ by definition of the I_0 codimension. So we see $F \in A^r_{\infty K} \cap C^\infty_{\mathrm{pr}}(G, M \times \mathbb{R})$, $K \subset M_{v_0} \times 0$ and then Theorem 2.2.1 applies to F and shows that there is an open neighbourhood U of $K \times 0$ in $M_{v_0} \times \mathbb{R}$, such that $F \in A^r_{\infty U} \cap C^\infty_{\mathrm{pr}}(G, M \times \mathbb{R})$. Since K is compact, $K \times t_i \subset U$ holds for any sufficiently large i, and for such i we see that $F \in A^r_{\infty K \times t_i}$, from which we have $f_{t_i} \in A^{r+1}_{\infty K} \subset A^{\infty}_{\infty K}$. (By a more detailed argument, we can prove that $f_{t_i} \in A^{r}_{\infty K}$.) This argument shows that $A^{\infty}_{\infty K} \cap C^{\infty}_{\mathrm{pr}}(G,M)$ is open.

Secondly, we prove the openness of $A^{\infty}_{\infty M_{v_0}} \cap C^{\infty}_{\mathrm{pr}}(G,M)$ in the Whitney topology. Let $K_i, i = 1, 2, \ldots$ be a locally finite covering of M_{v_0} by compact subsets and $f \in A^{r}_{\infty} \cap C^{\infty}_{\mathrm{pr}}(G,M)$, $r < \infty$. Naturally we then expect that the countable intersection $\bigcap_i A^{r+1}_{\infty K_i} \cap C^{\infty}_{\mathrm{pr}}(G,M) = A^{r+1}_{\infty} \cap C^{\infty}_{\mathrm{pr}}(G,M)$ of the open neighbourhood $A^{r+1}_{\infty K_i} \cap C^{\infty}_{\mathrm{pr}}(G,M)$ of f is again an open neighbourhood in the Whitney topology. This argument has already appeared in the book [Gi] to prove the topological stability theorem for single-mappings $f \in C^{\infty}_{\mathrm{pr}}(N,P)$. Unfortunately we cannot find a satisfactory reference for this argument in the generality needed here. So we present a sketch of a proof to cover this point.

Since $A^{r+1}_{\infty K_i} \cap C^{\infty}_{\mathrm{pr}}(G,M)$ is open in the weak C^{∞} topology there exist a positive integer r_i and an open neighbourhood $U_i \subset J^{r_i}(G,M)$ of $J^{r_i}f(M)$, $M = \prod_{\ell \in L} M_{\alpha(\ell)}$ with the property: if $J^{r_i}g(M) \subset U_i$ then $g \in A^{r+1}_{\infty K_i} \cap C^{\infty}_{\mathrm{pr}}(G,M)$ (for the definition of the weak C^{∞} and Whitney topologies, see [M2]). We claim that r_i can be chosen independently of i. Then the openness of the intersection $\bigcap_i A^{r+1}_{\infty K_i} \cap C^{\infty}_{\mathrm{pr}}(G,M)$ is easily seen.

By Proposition 2.1.1, $A^{r+1}_{\infty K_i} = A^{r+1}_{P+r+1 K_i}$ for $P = (\dim M_v)_{v \in V}$. By Theorem 0.3.1, there is a positive integer $e = e(G,P+r+2) + 1 < \infty$ with the following property: let f_X be a connected $(P+r+2)$-tree of a diagram $f \in C^{\infty}(G,M)$ of I_0 codimension $C_{I_0}(f_X) \leqslant r + 1$. If $g \in C^{\infty}(G,M)$, and g has the same e-jet as f at $X = \cup X_\ell$ then $C_{I_0}(g_X) \leqslant r + 1$.

Now we use the following lemma which is proved in Appendix 1.

LEMMA 2.4.1. – *Let $f : N \to P$ be a smooth mapping of manifolds N, P and U an open neighbourhood of $J^k f(N)$ in $J^k(N,P)$ and let $0 < s$, $q < \infty$ be integers. Then there is an open neighbourhood $U' \subset J^{q(s+1)}(N,P)$ of $J^{q(s+1)}f(N)$ with the following property: for any $g \in C^{\infty}(N,P)$ with $J^{q(s+1)}g(N) \subset U'$ and any q distinct points $x_1, \ldots, x_q \in N$, there is a g' such that $J^k g'(N) \subset U$ and $J^s g'(x_i) = J^s g(x_i)$, $i = 1, \ldots, q$.*

We apply the lemma to our problem in the setting: $U = U_i$, $k = r_i$, $s = e = e(G,P+r+2) + 1$ and $q = \prod_{\ell \in L} \dim M_{\beta(\ell)} + r + 2$. Then we get an open neighbourhood $U_i' \subset J^{q(s+1)}(G,M)$ of $J^{q(s+1)}f(M)$ with the following property: for any $g \in C^\infty(G,M)$ with $J^{q(s+1)}g(M) \subset U_i'$ and any $(P+r+2)$-tree g_X of g, there is a $g' \in C^\infty(G,M)$ such that

(i) $J^{r_i}g'(M) \subset U_i$

(ii) g_ℓ' has the same e-jet as g_ℓ at $X_\ell = X \cap M_{\alpha(\ell)}$, for all $\ell \in L$.

From (i) and the property of U_i, it follows $g' \in A^{r+1}_{\infty K_i} \cap C^\infty_{\mathrm{pr}}(G,M)$ and in particular $C_{I_0}(g_X') < r + 2$, and from (ii) and the property of the number e, it follows that $g \in A^{r+1}_{P+r+2K_i} = A^{r+1}_{\infty K_i}$.

This completes the proof of our claim.

Now to complete the proof of Theorem 2.1.2, we prove that if $P = (\dim M_v)_{v \in V}$ satisfies Condition G_Q, then

(1) A_Q^r is dense in $C^\infty(G,M)$ with the Whitney topology for any sufficiently large r,

(2) the complement of A_Q has infinite codimension.

Let $G' = (V',L',\Lambda')$ be a finite union of convergent trees and $(\imath_V,\imath_L): G' \to G(\imath_V: V' \to V. \imath_L: L \to L)$ be a morphism and assume G' is a union of $\imath_V^* Q$ trees: these are strictly less than $q_{\imath_V(v')} + 1$ edges $\ell' \in L'$ with $\beta'(\ell') = v'$ at each vertex $v' \in V'$. The set V' is naturally indexed by the set V as $v' \in V' \to \imath_V(v') \in V$. We denote by Γ_Q the set of these triplets $(G',\imath_V,\imath_L)$ as above.

Let $q = \prod_{\ell \in L} q_{\beta(\ell)}$, ${}_qJ^k(G,M) = J^k(G,M)^q$ and

$$\pi: {}_qJ^k(G,M) \to \prod_{\ell \in L} (M_{\alpha(\ell)} \times M_{\beta(\ell)})^q$$

the natural projection. Let $\Delta \subset \prod_{\ell \in L} (M_{\alpha(\ell)} \times M_{\beta(\ell)})^q$ denote the set of $(x_{\alpha\ell i}, y_{\beta\ell i})_{\ell \in L}$, $i = 1, \ldots, q$ such that $x_{\alpha\ell i} = x_{\alpha\ell j}$ for an $\ell \in L$, $i \neq j$. A point $(X,Y): X = (x_{\alpha\ell i})$, $Y = (y_{\beta\ell i})$ is naturally regarded as an oriented graph $G_{XY} = (V_{XY}, L_{XY}): V_{XY} = \{x_{\alpha\ell i}, y_{\alpha\ell i}\}$, $L_{XY} = \{\ell_i : x_{\alpha\ell i} \to y_{\beta\ell i}\}$. The set V_{XY} is naturally indexed by V: to $x_{\alpha\ell i}$, $y_{\beta\ell i} \in V_{XY}$ it assigns the vertices $\alpha(\ell)$, $\beta(\ell) \in V$, respectively. For each triple $(G',\imath_V,\imath_L) \in \Gamma_Q$, we denote by $\Delta_{G'}$ the set of points $(X,Y) \subset \prod_{\ell \in L} (M_{\alpha(\ell)} \times M_{\beta(\ell)})^q - \Delta$ for which the associated graph G_{XY} is equivalent to G' as oriented graphs indexed by the set V. It is easy to see that the $\Delta_{G'}$, $G' \in \Gamma_Q$ are smooth submanifolds and Whitney B regular over each other.

Let ${}_qJ^kf : \prod_{\ell \in L} M^q_{\alpha(\ell)} \to {}_qJ^k(G,M)$ be the multi k jet section of $f = (f_\ell) \in C^\infty(G,M)$. Then we see $\pi \circ {}_qJ^kf(X) \in \Delta_{G'}$, if and only if the tree f_X (regarded as an oriented graph with index $v \in V$ for each vertex $x \in (X \cup f(X)) \cap M_v$) is equivalent to G', where

$$X = \bigcup_{\ell \in L} X_\ell,\ X_\ell \in M_{\alpha(\ell)},\ f(X) = \bigcup_{\ell \in L} f_\ell(X_\ell).$$

We regard the fibre ${}_qJ^k(G,P)$ of the projection π on $(X,Y) \in \Delta_{G'}$ as the jet space $J^k(G', \iota_V^* P)$.

By theorem 0.3.4, the set $\Sigma \subset \mathscr{E}(G', \iota_V^* P)$ of f with finite codimension is a pro algebraic set defined by algebraic subsets

$$\Sigma^{e(r)} \subset J^{e(r)}(G', \iota_V^* P)$$

$(\pi^{e(r)}(f) \notin \Sigma^{e(r)} \Leftrightarrow C_{I_0}(f) \leqslant r)$, and by the Condition G_Q, codim $\Sigma^{e(r)} \to \infty$ as $r \to \infty$. Choose an r so that codim $\Sigma^{e(r)} > q \cdot \sum_{\ell \in L} P_{\alpha(\ell)}$ for all $G' \in \Gamma_Q$.

Let $S_{G'}$ be a stratification of $\Sigma^{e(r)}$ invariant under diffeomorphisms $J^{e(r)}(G', \iota_V^* P)$ induces from coordinate transformations of the germs $\mathbb{R}^{p_{\iota_V(v')}}, 0$ associated with vertices $v' \in V'$, and let $S'_{G'} \subset \pi^{-1}(\Delta_{G'})$ be the stratified set with fibre $S_{G'}$ over each $(X,Y) \in \Delta_{G'}$ and finally let S be the union of $S_{G'}$ for $G' \in \Gamma_Q$. Then the set S possesses the following property: if ${}_qJ^{e(r)}f$ is transversal to S at $X \in \prod_{\ell \in L} M^q_{\alpha(\ell)}$ then ${}_qJ^{e(r)}f(X) \notin S$ $\left(\text{for codim } S > \dim \prod_{\ell \in L} M^q_{\alpha(\ell)}\right)$ hence $C_{I_0}(f_X) \leqslant r$. Conversely any connected Q-tree of $f_{X'}$ of f is realised as a connected component of diagrams f_X defined above for $X \in \prod_{\ell \in L} M^q_{\alpha(\ell)}$. From this property of S and the transversality theorem (Theorem 0.3.5), the density of A^r_Q in $C^\infty(G,M)$ follows.

The infiniteness of codimension of A^∞_Q follows from the same argument using transversality and unboundedness of the codimension of $\Sigma^{e(r)}$ as $r \to \infty$.

Chapter 3

PROOF OF THE TOPOLOGICAL STABILITY THEOREM

3.1. Topological multi transversality, topologically maximal trees and branches.

Let $f \in C^\infty(G,M)$ be a convergent diagram of smooth mappings, $Q = (q_v)_{v \in V}$ a tuple of integers $0 \leqslant q_v < \infty$, S a Whitney regular stratification of M_{v_0} and $K \subset M_{v_0}$ a subset. We say f is *topologically Q-transversal relative to S on K* if : (i) any connected Q-tree f_X of f is topologically transversal and (ii) if f_X has root $x_0 \in M_{v_0}$ then the canonical CVS $C_{x_0}(f_X)$ of f_X at x_0 is transversal to S.

For the case $q_v = \infty$, $v \in V$, we say simply f is *topologically transversal relative to* S (for the definition of topological transversality of trees f_X, see Proposition 1.3.3).

Let $F = (F_\ell) \in C^\infty(G, M \times \mathbb{R}^r)$, $M \times \mathbb{R}^r = (M_v \times \mathbb{R}^r)_{v \in V}$, $F_\ell(x,t) = (f_{\ell t}(x), t)$, $x \in M_{\alpha(\ell)}$, $t \in \mathbb{R}^r$, $f_{\ell 0} = f_\ell$ be an unfolding of f such that the trees F_X, $X \subset \bigcup_{\ell \in L} M_{\alpha(\ell)} \times 0$ are infinitesimally stable unfoldings of f_X and let $\mathscr{C}(F_X) = (\mathscr{C}_x(F_X))_{x \in X \cup F(X)}$ be the canonical critical value stratification of F_X constructed in Chapter 1.1. By Proposition 1.3.3, the tree f_X is topologically transversal if and only if the inclusions $\imath_v : M_v \hookrightarrow M_v \times \mathbb{R}^r$, $v \in V$ are transversal to $\mathscr{C}_x(F_X)$ at the vertices $x \in X \cup F(X)$.

Let $\chi_x(F_X)$ denote the stratum of $\mathscr{C}_x(F_X)$ containing the vertex $x \in X \cup F(X)$. Let $x_1, \ldots, x_q \in X$ be the vertices of f_X such that $f(x_i) = x$, and let $f_{X_{x_i}}$, $F_{X_{x_i}}$ be the branches of f_X, F_X on x_i, and $f_{X^-_{x_i}}$, $F_{X^-_{x_i}}$ their prolongations (with root x). By proposition 1.2.2, the canonical CVS's $\mathscr{C}_x(F_{X^-_{x_i}})$ meet in general position at x and $\mathscr{C}_x(F_x) = \bigcap_{i=1,\ldots,q} \mathscr{C}_x(F_{X^-_{x_i}})$, and in particular $\chi_x(F_X) = \bigcap_{i=1,\ldots,q} \chi_x(F_{X^-_{x_i}})$. Therefore if $\imath_v : M_v \hookrightarrow M_v \times \mathbb{R}^r$ is transversal to $\mathscr{C}_x(F_X)$ at $x \in M_v$, the number of indices i for which $\mathscr{C}_x(F_{X^-_{x_i}})$ is topologically non-trivial is at most $\dim M_v$.

From the above fact and the same argument as in the proof of Proposition 2.1.1, we have

PROPOSITION 3.1.1. — *A convergent diagram $f \in A_\infty$ is topologically transversal if and only if f is topologically $(P+1)$-transversal. $P+1 = (\dim M_v + 1)_{v \in V}$. Any topologically maximal tree of f is a P tree.*

Using the same idea as Theorem 2.2.1, we prove

THEOREM 3.1.2. — *Let $f \in C^\infty_{pr}(G,M)$ be a convergent diagram, $P+1 = (\dim M_v + 1)_{v \in V}$, $K \subset M_{v_0}$ a closed subset and S a Whitney regular stratification of M_{v_0}. If $f \in T_{\infty KS} \subset A_{\infty K}$, then there is an open neighbourhood U of K such that $f \in T_{\infty US} \subset A_{\infty U}$ and the restriction $f_U = (f_\ell : f^{-1}_{\alpha(\ell)v_0}(U) \to f^{-1}_{\beta(\ell)v_0}(U))_{\ell \in L}$ admits a critical value stratification with the following properties : $f_\ell : \Sigma_{\mathscr{C}_{\alpha(\ell)}(f_U)}(f_U) \to f^{-1}_{\beta(\ell)v_0}(U)$ are proper, $\bigcup_{\beta(\ell)=v} f_\ell^{-1}(x) \cap \Sigma_{\mathscr{C}_{\alpha(\ell)}(f_U)}(f_\ell)$ consists of at most $\dim M_v$-points for each $x \in f^{-1}_{vv_0}(U)$, and $\mathscr{C}_{v_0}(f_U)$, S meet in general position in U. The topologically maximal branch $f_{\mathrm{top}X^{\mathrm{br}}_x}$ on $x \in f^{-1}_{vv_0}(U)$ is given by the set*

$$^{\mathrm{top}}X^{\mathrm{br}}_x = \bigcup_{\alpha(\ell)<v} \ \bigcap_{\alpha(\ell)\leqslant\alpha(\ell')<v} \Sigma_{\mathscr{C}_{\alpha(\ell')}(f_U)}(f_{\ell'}) \cap f^{-1}_{\alpha(\ell')v}(x)$$

and is topologically transversal (relative to S if $v = v_0$) and the germ of $\mathscr{C}_v(f_U)$ at x, $\mathscr{C}_v(f_U)_x$ coincides with the canonical CVS *$\mathscr{C}_x(f_{\mathrm{top}X^{\mathrm{br}}_x})$ of the branch $f_{\mathrm{top}X^{\mathrm{br}}_x}$ given in* Corollary 1.3.4.

Proof. — By Theorem 2.1.2, we may assume $f \in A_{\infty U}$. We construct the CVS with the properties in the theorem by descending induction on the height of vertices $v \in V$. So we assume that f_U admits a CVS $\mathscr{C}_{v'}(f_U)$ for $v' < v$ with the desired properties for the restriction of f_U to the branches $G_{v'}$ on v', $\beta(v') = v$, and then we construct $\mathscr{C}_v(f_U)$.

By definition the topologically maximal branch $f_{\mathrm{top}X^{\mathrm{br}}_x}$ on $x \in f^{-1}_{vv_0}(U)$ is a union of prolongations of topologically maximal branches on some points $x' \in f_\ell^{-1}(x)$, $\beta(\ell) = v$. Let $F_{\mathrm{top}X^{\mathrm{br}-}_{x'}}$ be an infinitesimally stable unfolding of the prolongation $f_{\mathrm{top}X^{\mathrm{br}-}_{x'}}$ of dim r. Since $f_{\mathrm{top}X^{\mathrm{br}}_{x'}}$ is topologically transversal by the induction hypothesis, the inclusion $i_{x'} : M_{\alpha(\ell)} \to M_{\alpha(\ell)} \times \mathbb{R}^r$ is transversal to the canonical CVS $\mathscr{C}_{x'}(F_{\mathrm{top}X^{\mathrm{br}}_{x'}})$ (Corollary 1.3.4). Hence

$$\Sigma_{\mathscr{C}_{x'}(f_{\mathrm{top}X^{\mathrm{br}}_{x'}})}(f_\ell) = i_{x'}^{-1}(\Sigma_{\mathscr{C}_{x'}(F_{\mathrm{top}X^{\mathrm{br}}_{x'}})}(F_{x'}))$$

and $f_{\mathrm{top}\,X_{x'}^{\mathrm{br}}}$ is topologically trivial if and only if $\Sigma_{\mathscr{C}_{x'}(F_{\mathrm{top}\,X_{x'}^{\mathrm{br}}})}(F_{x'}) = \varnothing$ if and only if $\Sigma_{\mathscr{C}_{x'}(f_{\mathrm{top}\,X_{x'}^{\mathrm{br}}})}(f_\ell) = \varnothing$. So we have

$$^{\mathrm{top}}X_x^{\mathrm{br}} = \bigcup_{\alpha(\ell)<v} \bigcap_{\alpha(\ell)\leqslant\alpha(\ell')<v} \Sigma_{\mathscr{C}_{\alpha(\ell')}(f_U)}(f_{\ell'}) \cap f_{\alpha(\ell')v}^{-1}(x).$$

Since the f_ℓ are proper and $\Sigma_{\mathscr{C}_{\alpha(\ell)}(f_U)}(f_\ell) \subset f_{\alpha(\ell)v_0}(U)$ are closed, $f_\ell : \Sigma_{\mathscr{C}_{\alpha(\ell)}(f_U)}(f_\ell) \to f_{\beta(\ell)v_0}^{-1}(U)$ are also proper and in particular the topologically maximal branches $f_{\mathrm{top}\,X_x^{\mathrm{br}}}$, $x \in f_{vv_0}^{-1}(U)$ are coherent in the sense of $(3)_k$ in Theorem 2.2.1.

Let $x \in f_{vv_0}^{-1}(K)$. Then the branch $f_{\mathrm{top}\,X_x^{\mathrm{br}}}$ is topologically transversal. Let $F^x = (F_\ell^x)_{\ell\in L}$, $F_\ell^x : M_{\alpha(\ell)} \times \mathbb{R}^r \to M_{\beta(\ell)} \times \mathbb{R}^r$, $F_\ell^x(y,u) = (f_{\ell u}^x(y),u)$ for $y \in M_{\alpha(\ell)}$, $u \in \mathbb{R}^r$, $f_{\ell 0}^x = f_\ell$ be a smooth unfolding of f of dim r such that the tree $F_{\mathrm{top}\,X_x^{\mathrm{br}}}^x$ on x is infinitesimally stable and its restriction $\hat{F}_{\mathrm{top}\,X_x^{\mathrm{br}}}^x = (F_\ell^x : U_{x'} \to U_{f(x')})$, $x' \in {}^{\mathrm{top}}X_x^{\mathrm{br}}$ to some open neighbourhoods $U_{x'}$ of the vertices $x' \in {}^{\mathrm{top}}X_x^{\mathrm{br}-} \cap M_{x'}$ in $M_{v'} \times \mathbb{R}^r$, $v' \leqslant v$ is a good representative of $F_{\mathrm{top}\,X_x^{\mathrm{br}}}^x$ with the properties in Corollary 1.3.4.

Using the same notation as in Corollary 1.3.4, the transversal intersections $\mathscr{C}_{x'}(\hat{F}_{\mathrm{top}\,X_x^{\mathrm{br}}}^x) \cap M_v \times 0$, $x' \in {}^{\mathrm{top}}X_x^{\mathrm{br}-} \cap M_{v'}$, $v' \leqslant v$ give the canonical CVS denoted $\mathscr{C}_{x'}(\hat{f}^x)$ of the restriction of f to open neighbourhoods $U_{x'} \cap M_{v'} \times 0$ of $x' \in M_{v'}$. Since $f_{\mathrm{top}\,X_x^{\mathrm{br}}}$, $x \in f_{vv_0}^{-1}(U)$ are coherent, we may assume, by shrinking $U_{x'}$ that $f_{\mathrm{top}\,X_{x''}^{\mathrm{br}}}$ is a tree of $\hat{f}^x$ if $x'' \in U_x \cap M_v \times 0$. Then by Corollary 1.3.4, the germ of $\mathscr{C}_x(\hat{f}^x)$ at x'' coincides with the canonical CVS $\mathscr{C}_{x''}(f_{\mathrm{top}\,X_{x''}^{\mathrm{br}}})$ of the maximal tree f on x''. Therefore the $\mathscr{C}_x(\hat{f}^x)$ $x \in f_{vv_0}^{-1}(K)$ glue up to give a stratification of $\bigcup_{x\in f_{vv_0}^{-1}(K)} U_x \cap M_v \times 0$.

Finally, by shrinking U so that $f_{vv_0}^{-1}(U) \subset \bigcup_{x\in f_{vv_0}^{-1}} U_x \cap M_v \times 0$, we complete the induction step.

In the same way as the implication of Theorem 2.2.1 to Theorem 2.1.2, the above theorem (Theorem 3.1.2) for topological transversality implies the following.

THEOREM 3.1.3. – *The set* $T_{\infty KS} \cap C^\infty_{\rm pr}(G,M) \subset A_{\infty K} \cap C^\infty_{\rm pr}(G,M)$ *of convergent diagrams* f *topologically multi transversal on* K *relative to a Whitney regular stratification* S *of* M_{v_0} *is an open subset in the Whitney topology if* $K \subset M_{v_0}$ *is closed.*

3.2. Proof of the theorem.

First we prove the following theorem.

THEOREM. 3.2.1. – *Let* $f \in A_\infty \cap C^\infty_{\rm pr}(G,M)$ *be a convergent diagram of proper smooth mappings and let* S *be a Whitney regular stratification of* M_{v_0} *by relatively compact submanifolds. If* f *is topologically* $P+1$ *(hence, multi)-transversal on* M_{v_0} *relative to* S, *then* f *is topologically stable.*

Proof. – By Theorem 3.1.3, there is an open neighbourhood U of f in $C^\infty_{\rm pr}(G,M)$ such that any $g \in U$ is topologically multi-transversal relative to S and joined to f by a smooth path $f_t \in U$, $t \in \mathbb{R}$ with $f_0 = f$, $f_1 = g$. Define the unfolding $F = (F_\ell)$, $F_\ell : M_{\alpha(\ell)} \times \mathbb{R} \to M_{\beta(\ell)} \times \mathbb{R}$ by $F_\ell(x,t) = (f_{t\ell}(x),t)$, $x \in M_{\alpha(\ell)}, t \in \mathbb{R}$. Let $f_{{\rm top}X^{\rm br}_x}$, $F_{{\rm top}X^{\rm br}_x}$ be the topologically maximal branch of f, F on $x \in M_v \subset M_v \times \mathbb{R}$, and let $\bar F_{{\rm top}X^{\rm br}_x}$ be an infinitesimally stable unfolding of $F_{{\rm top}X^{\rm br}_x}$. Then the canonical CVS of $f_{{\rm top}X^{\rm br}_x}$, $F_{{\rm top}X^{\rm br}_x}$ are given by the transversal intersections of the canonical CVS $\mathscr{C}_{x'}(\bar F_{{\rm top}X^{\rm br}_x})$, $x' \in {}^{\rm top}X^{\rm br-}_x$ with M_v, $M_v \times \mathbb{R}$ respectively as described in Corollary 1.3.4. Therefore $F_{{\rm top}X^{\rm br}_x}$ is also topologically transversal relative to $S \times \mathbb{R}$ and the inclusion $\imath_v : M_v \to M_v \times \mathbb{R}$ are transversal to the canonical CVS $\mathscr{C}_{x'}(F_{{\rm top}X^{\rm br}_x})$ at each vertex $x' \in {}^{\rm top}X^{\rm br-}_x \cap M_v$, $v \in V$ and $\mathscr{C}_{x'}(f_{{\rm top}X^{\rm br}_x}) = \imath_v^{-1}(F_{{\rm top}X^{\rm br}_x})$. By Theorem 3.1.2, F admits CVS $\mathscr{C}_v(F)$ of $M_v \times \mathbb{R}$ such that $\mathscr{C}_{v_0}(F)$ is transversal to $S \times \mathbb{R}$, and the germ $\mathscr{C}_v(F)_{(x,t)}$ coincides with $\mathscr{C}_{(x,t)}(F_{{\rm top}X^{\rm br}_{(x,t)}})$ for any $(x,t) \in M_v \times \mathbb{R}$, $v \in V$. The transversality of the inclusions $\imath_{(x,t)} : M_v \to M_v \times t \subset M_v \times \mathbb{R}$ to the CVS's shows that the second projections $M_v \times \mathbb{R} \to \mathbb{R}$ are stratified submersions, i.e., submersive restricted to each stratum of $\mathscr{C}_v(F)$. Let $\mathscr{S}_v(F) = \bigcap_{v<v'} F^{-1}_{vv_0}(\mathscr{C}_v(F)) \cap F^{-1}_{vv_0}(S)$, $v \in V$ be the canonical stratification associated to the CVS's $\mathscr{C}_v(F)$, $v \in V$. Clearly the second projection $M_v \times \mathbb{R} \to \mathbb{R}$ are still stratified submersions.

Since $f_t: M_{\alpha(\ell)} \to M_{\beta(\ell)}$, $t \in \mathbb{R}$, $\ell \in L$ are all proper and the strata of S are relatively compact, the strata of $S_v(F)$, $v \in V$ are also relatively compact. Now we apply Thom's second isotopy lemma to the prolongation (F,Pr) with the second projection $\mathrm{Pr}: M_{v_0} \times \mathbb{R} \to \mathbb{R}$. Then (F,Pr) is topologically locally trivial and in particular the sectional mappings $f = f_0$, $g = f_1$ are topologically equivalent. This completes the proof of Theorem 3.2.1.

Now we prove the main theorem.

THEOREM 3.2.2. – *The set* $T_{\infty S} \cap C^\infty_{\mathrm{pr}}(G,M)$ *is open dense in* $A_\infty \cap C^\infty_{\mathrm{pr}}(G,M)$ *with the Witney topology.*

Proof. – The openness is given by Theorem 3.1.3, and by Proposition 3.1.1, $T_{\infty S} = T_{P+1S}$ for $P+1 = (\dim M_v + 1)_{v \in V}$. It remains to show $T_{P+1S} \cap C^\infty_{\mathrm{pr}}(G,M)$ is dense. We use the same relation as in the proof of the density of $A_{P+1} \cap C^\infty_{\mathrm{pr}}(G,M)$ in Theorem 2.1.2.

Let $q = \prod_{\ell \in L} q_{\beta(\ell)}$ and let $\pi: {}_qJ^k(G,M) \to \prod_{\ell \in L} (M_{\alpha(\ell)} \times M_{\beta(\ell)})^q$ be the natural projection. Let $(G', \imath_V, \imath_L) \in \Gamma_Q$ be a morphism of an oriented graph G' to G and let $\Delta_{G'} \in \prod_{\ell \in L} (M_{\alpha(\ell)} \times M_{\beta(\ell)})^q - \Delta$ be the set of points $(x_{\alpha\ell i}, y_{\beta\ell i})$, $x_{\alpha\ell i}, i = 1, \dots, q$ all distinct, for which the associated graph G_{XY} is equivalent to G' as an oriented graph indexed by the set V.

We regard the fibre of π over $\Delta_{G'}$ as the k jet space of diagrams in $\mathscr{E}(G', \imath_V^* P)$. Let $\Sigma^{e(r)} \subset J^{e(r)}(G', i_V^* P)$ be the set in Theorem 0.3.4, which defines the pro-algebraic set $\Sigma \subset \mathscr{E}(G', \imath_V^* P)$ of non finitely I_0 determined diagrams, and let $\mathrm{S}^{e(r)}(G', i_V^* P)$ be the stratification of the complement of $\Sigma^{e(r)}$ defined in Section 1.3. Since these sets are invariant under coordinate transformations of spaces, these sets and stratifications define a locally trivial partition of the fibre bundle $\pi^{-1}(\Delta_{G'}) \to \Delta_{G'}$, denoted by $\Sigma^{e(r)}_{G'}$, $S^{e(r)}_{G'}$, respectively. The image of the projection to the roots $\pi': \Delta_{G'} \to M^q_{v_0}$ is the complement of the diagonal set of $M^{q'}_{v_0}$, where q' is the number of connected components of G'. Let $S^{e(r)}_{G'S}$ denote the refinement $S^{e(r)}_{G'} \cap (\pi' \circ \pi)^{-1}(S^{q'})$. By the transversality theorem (Theorem 0.3.5), the set $\mathscr{F}^{e(r)}_{G'}$ of diagrams $f \in C^\infty(G,M)$ for which the $e(r)$-jet section $J^{e(r)}f: \prod_{\ell \in L} M_{\alpha(\ell)} \to J^{e(r)}(G,M)$ is transversal to $\Sigma^{e(r)}_{G'}$, $S^{e(r)}_{G'S}$ is a countable intersection of open dense subsets. So the countable intersection $\mathscr{F}$ of those $\mathscr{F}^{e(r)}_{G'}$ for $r = 0, 1, 2, \dots$ and all morphisms

$G' = (G', i_V, i_L) \in \Gamma_G$ is still dense. By the definition of topological transversality, we see that if $f \in \mathscr{F}^{e(G', i_V^{P+r})}$ then all trees f_X of f equivalent to G' with $C_{I_0}(f_X) \leqslant r$ are topologically transversal relative to S (for the definition of the number $e(G', i_V^* P+r) + 1$ see Section 0.3).

Therefore the intersection $\mathscr{F} \cap A_\infty$ coincides with the set $T_{\infty S} \subset A_\infty$ and $T_{\infty S} \cap C^\infty_{\mathrm{pr}}(G,M)$ is open dense in the open subset $A_\infty \cap C^\infty_{\mathrm{pr}}(G,M)$. This completes the proof of Theorem 3.2.2.

COROLLARY 3.2.3. – *If $P = (\dim M_v)_{v \in V}$ satisfies the condition $G = G_\infty$, $\infty = (\infty)_{v \in V}$ in Section 2.1, then the set of topologically stable convergent diagrams of proper mappings $f \in C^\infty_{\mathrm{pr}}(G,M)$ is open dense in $C^\infty_{\mathrm{pr}}(G,M)$ with Whitney topology.*

Proof. – The statement follows immediately from Theorem 2.1.2 and Theorem 3.2.1-2.

THEOREM 3.2.4 (Thom's second isotopy lemma). – *Let $f \in C^\infty_{\mathrm{pr}}(G,M)$ be a convergent diagram of proper smooth mappings $f_\ell : M_{\alpha(\ell)} \to M_{\beta(\ell)}$. Assume that there are Whitney regular stratifications $S_v(f)$ of M_v, $v \in V$ and $S_{\alpha(\ell)}(f)$ are Thom A_{f_ℓ} regular $\ell \in L$, and S_{v_0} is trivial : $S_{v_0} = \{M_{v_0}\}$. Then f is locally topologically trivial over M_{v_0} : for any point $p \in M_{v_0}$, there is an open neighbourhood $U \subset M_{v_0}$ and homeomorphisms $\phi_v : f^{-1}_{vv_0}(U) \to f^{-1}_{vv_0}(p) \times U$, $v \in V$ such that the following diagram commutes*

$$\begin{array}{ccc} f_\ell : f^{-1}_{\alpha(\ell)v_0}(U) & \to & f^{-1}_{\alpha(\ell)v_0}(U) \\ \phi_{\alpha(\ell)} \downarrow & & \downarrow \phi_{\beta(\ell)} \\ f_{\ell p} \times 1 : f^{-1}_{\alpha(\ell)v_0}(p) \times U & \to & f^{-1}_{\beta(\ell)v_0}(p) \times U \end{array}$$

for $\ell \in L$. In particular the restriction $f_{p'} = (f_{\ell p'})_{\ell \in L}$ is topologically equivalent to f_p for any $p' \in U$.

Proof. – This is a natural generalization of Thom's second isotopy lemma. For the proof, see e.g. [Gi, M4].

APPENDIX 1.

PROOF OF LEMMA 2.4.1.

Let $\phi_i : O_i \xrightarrow{\sim} \mathbb{R}^n$, $\psi_i : O'_j \xrightarrow{\sim} \mathbb{R}^p$, $i, j = 1, 2, \ldots$ be coordinate systems of smooth manifolds N, P such that $f(O_i) \subset O'_j$ with some $j(i)$ for any i. For a smooth mapping $h : O_i \to \mathbb{R}^n$ we define $f + h : O_i \to P$ with the addition of coordinates on O'_j. Let $0 < r_{i0} < s_{i0} < r_{i1} < s_{i1} < \ldots < r_{iq} < s_{iq}$ be numbers such that $\{\phi_i^{-1}(D(r_{ij})) \mid i = 1, 2, \ldots\}$ is a locally finite covering of M for $j = 0, 1, \ldots, q$, where $D(r)$ denotes the open disk with radius r centred at $0 \in \mathbb{R}^n$. Let $\phi_{ij} : M \to \mathbb{R}$ be smooth functions with supports in $\phi_i^{-1}(D_i(r_{j+1}))$ and identically equal to 1 on $\phi_i^{-1}(D_i(s_j))$. Let Z_i be an open neighbourhood of 0 in the linear space A_n of polynomial functions on $\mathbb{R}^n$ of degree at most ℓ, $\ell = (s+1)^q$ such that

$$J^k(f + h \cdot \phi_{ij} \cdot \prod_{m=1,\ldots,p} (1 - \phi_{k\ell_m}))(N) \subset U$$

for any j, $\ell_m = 0, \ldots, q$, $p \leqslant q$ and $h \in Z_i$.

We apply the following Lemma A with $Z = Z_i \subset A_n^\ell$ and the compact neighbourhood $\bar{D}(s_{iq})$ of $0 \in \mathbb{R}^n$ and let $0 < \varepsilon_i < \infty$ be a number with the property in the lemma.

Let U' be the set-theoretical union of jet sections $J^{(s+1)}g(N)$ of g such that

$$\|(f - g) \circ \phi_i^{-1}\|_{q(s+1)}^{D(s_{iq})} < \varepsilon_i$$

for any $i = 1, 2, \ldots$, where $\| \quad \|_a^K$ denotes the sup. norm of derivatives of order $\leqslant a$ on the set $K \subset \mathbb{R}^n$. We claim that U' possesses the required property in Lemma 2.4.1.

Let $X = \{x_1, \ldots, x_q\} \subset N$. By an easy argument we see there is a function $j(i)$ such that

$$X \cap \phi_i^{-1}(D(r_{i\,j(i)+1}) - D(r_{i\,j(i)})) = \varnothing,$$

for any $i = 1, 2, \ldots$, and by renumbering the index i, we have

$$X \subset \bigcup_{i=1,\ldots,p} \phi_i^{-1}(D(r_{i\,j(i)})), \qquad p \leqslant q.$$

We define a partition of X into the disjoint p sets

$$X_i \subset \phi_i^{-1}(D(r_{i\,j(i)})) - \bigcup_{m=1,\ldots,i-1} \phi_m^{-1}(D(r_{i\,j(m)}))$$

$i = 1, \ldots, p$. By Lemma A, there are $h_i \in A_n^\ell$ such that the mappings $f_i' : N \to P$:

$$f_i' = f + h_i \circ \phi_i^{-1} \cdot \phi_{i\,j(i)} \cdot \prod_{m=1,\ldots,i-1} (1 - \phi_{n\,j(m)})$$

are well defined and satisfy

$$J^k f_i'(N) \subset U \quad \text{and} \quad J^r f_i'(x) = J^r(f + h_i)(x) = J^r g(x),$$

for $x \in X_i$, $i = 1, \ldots, p$. Now define $f' \in C^\infty(N,P)$ by $f' = f_i'$ on $\phi_i^{-1}(D(r_{i\,j(i)+1})) - \bigcup_{m=1,\ldots,i-1} \phi_m^{-1}(D(s_{m\,j(m)}))$ and $f' = f$ on the complement of these subsets above. Then f' possesses the required properties:

$$J^k f'(N) \subset U \quad \text{and} \quad J^r f'(x) = J^r f_i'(x) = J^r g(x)$$

for $x \in X_i$, $i = 1, \ldots, p$.

Lemma A (Golbitsky-Guillemin, Lemma 2.5 [GG]). – *Let* $0 \leqslant s, 0 < q$ *be integers,* $K \subset \mathbb{R}^n$ *a compact connected neighbourhood of the origin and let* $Z \subset A_n^\ell$, $\ell = (s+1)^q$ *be a neighbourhood of the constant mapping* $0 \in A_n^\ell$. *Then there is a positive number* $\varepsilon > 0$ *such that for any distinct* q *points* $p_1, \ldots, p_q \in K$ *and any smooth function* $g : \mathbb{R}^n \to \mathbb{R}$ *with* $\|g\|_{q(s+1)}^K < \varepsilon$ *there is a polynomial function* $V \in Z$ *for order* $\leqslant \ell$ *such that*

$$\frac{\partial^{(\alpha)} V}{\partial x^\alpha}(p_i) = \frac{\partial^{(\alpha)} g}{\partial x^\alpha}(p_i)$$

for $i = 1, \ldots, p$, $0 \leqslant |\alpha| \leqslant s$.

Appendix 2.

TWO EXAMPLES DUE TO DU PLESSIS AND DUFOUR

Example 1: due to du Plessis

C^1 stability is not generic in $C^\infty(M^3 \to N^4 \to P^2)$.

Proof. – Let (f,g): $M \to N \to P$ be a composition of proper mappings and assume the composition $g \circ f : M \to P$ is submersive at $x_i \in M - \Sigma(f)$, $i = 1, \ldots, 4$, $f(x_i) = y$ and the multi germ $f_{x_i} : (M,x_i) \to (N,y)$ is C^∞ stable, i.e., $\operatorname{Im} df_{x_i}$ are in general position. Let (f',g') be a perturbation of (f,g). Then by the stability of the multigerm above, there are again 4 points x_i' close to x_i respectively such that $f'(x_i') = y'$ and $g' \circ f'$ is submersive at x_i'. The cross ratio

C_y of $\mathrm{Im}\, df_{x_i} \cap \ker dg_y$ in $\ker dg_u$ is clearly C^1 invariant of diagrams of the type or $\substack{\searrow \\ \rightarrow \\ \nearrow} -$, $\substack{\searrow\searrow \\ \nearrow\nearrow} -$, etc., while the ratio C_y can vary by a perturbation of f, g. Hence (f,g) is not C^1 stable.

Example 2 : due to Dufour.

All topological equivalence classes in an open dense subset of $C^\infty(M^1 \leftarrow N^2 \rightarrow P^2)$ have infinite codimension, if $W_1(N) = W_1(P) = 0$, $W_2(N) \neq 0$ and $W_2(P) = 0$.

Proof. – Let $(f,g) : M \leftarrow N \rightarrow P$ be a divergent diagram of smooth mappings. The Thom polynomial for the singularity $\Sigma^{1,1}(g)$ is the polynomial $W_2(\gamma) - W_1(\gamma)^2$ of Stiefel-Whitney class of the difference bundle $\gamma = TN - g^*TP$. By the condition above we see the polynomial is not 0 in $H^2(N, \mathbb{Z}_2)$.

So generic mappings $g : N \rightarrow P$ have cusp singularities and in their neighbourhoods there are triples of points $x_i \notin \Sigma(f) \cup \Sigma(g)$ with $f(x_i) = y$. Dufour [D2] proved that the germs of (f,g) at x_1, x_2, x_3 are C^∞ equivalent if and only if they are topologically equivalent and C^∞ equivalence classes are all of infinite codimension in the jet space $J^\infty(2.1)^3 \times J^\infty(2.2)^3$. From this fact the statement follows.

Furthermore, Dufour [D3] proved that C^∞ classification and topological classification are the same for mappings in $C^\infty(M^1 \leftarrow N^2 \rightarrow P^2)$.

The two examples above are caused by the existence of « wild » diagrams of map germs imbedded in the global diagrams as multi germs. Now we denote them in terms of morphisms of oriented diagrams as follows :

$$\text{example 1} \qquad \begin{array}{cccccll} 3 & \searrow & & & & & \\ 3 & \searrow & & & & & \\ & & 4 & \rightarrow & 2 & \tilde{D}_5 & \text{wild} \\ 3 & \nearrow & & & & \Downarrow & \\ 3 & \nearrow & & & & & \\ 3 & \rightarrow & 4 & \rightarrow & 2 & A_3 & \end{array}$$

$$\text{example 2} \qquad \begin{array}{cccccll} 1 & \leftarrow & 2 & \searrow & & & \\ 1 & \leftarrow & 2 & \rightarrow & 2 & \tilde{E}_6 & \text{wild} \\ 1 & \leftarrow & 2 & \nearrow & & \Downarrow & \\ 1 & \leftarrow & 2 & \rightarrow & 2 & A_3 & \end{array}$$

This explanation suggests that the stability problem is closely related with morphisms of oriented graphs and their expanded diagrams.

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