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ON THE RATIONAL HOMOTOPY LIE ALGEBRA OF SPACES WITH FINITE DIMENSIONAL RATIONAL COHOMOLOGY AND HOMOTOPY

by Martin MARKL

Introduction.

A path connected topological space S is said to have type F , if

$$\dim(H^*(S;Q)) < \infty \quad \text{and} \quad \dim(\pi^*_\psi(S)) < \infty ,$$

where $\pi^*_\psi(S)$ denotes the ψ-homotopy of the space S [12; p. 61]. If S is simply connected, the previous condition is, of course, equivalent with

$$\dim(H^*(S;Q)) < \infty \quad \text{and} \quad \dim(\pi_*(S) \otimes Q) < \infty \qquad \text{(see [2]).}$$

Spaces of type F were studied by many authors, see for example [2], [3], [4] and [5]. J. Friedlander and S. Halperin gave in [2] the characterization of all rational graded vector spaces V_* , for which there exists a space S of type F with $V_* \cong \pi_*(S) \otimes Q$ in the category of graded spaces.

Suppose that S is simply connected and denote by ΩS the loop space of S . The Samelson product induces on $\pi_*(\Omega S) \otimes Q \cong \pi_{*+1}(S) \otimes Q$ the structure of a graded Lie algebra over rationals which is called the (rational) homotopy Lie algebra of the space S [8; p.210]. It is natural to ask how to characterize all graded rational Lie algebras Π_* for which there exists a simply connected space S of type F with $\Pi_* \cong \pi_*(\Omega S) \otimes Q$ in the category of graded Lie algebras. Unfortunately, this problem seems to have

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no reasonable solution (see [5; p.114]). On the other hand, this question leads to the study of the set $f\mathcal{L}(W)$ of all graded Lie algebra structures on a given graded vector space W , that are the homotopy Lie algebras of spaces of type F . This set forms a subset of the algebraic variety $\mathcal{L}(W)$ of all graded Lie algebra structures on W (see § 2). We prove, roughly speaking, that there are (under suitable assumptions) only three possibilities :

- $f\mathcal{L}(W) = \emptyset$, *i.e.* no graded Lie algebra structure on W can be realized by the homotopy Lie algebra of a simply connected space of type F ,
- $f\mathcal{L}(W)$ is a proper, nonempty and Zariski-open subset of $\mathcal{L}(W)$,
- $f\mathcal{L}(W) = \mathcal{L}(W)$, *i.e.* every graded Lie algebra structure on W can be realized by the homotopy Lie algebra of a simply connected space of type F .

We also show that these cases are characterized by the combinatorial condition, similar to the "strong arithmetic condition" of [2; p.117].

1. Preliminaries.

In this paper we adopt the terminology of [12] and [3]. A minimal algebra $(\Lambda M, D)$ is said to be pure, if $D(M^{\text{even}}) = 0$ and $D(M^{\text{odd}}) \subset \Lambda M^{\text{even}}$ [3; p.179]. For a minimal algebra $(\Lambda M, d)$ we define the differential d_p by

$d_p(M^{\text{even}}) = 0$, $d_p(M^{\text{odd}}) \subset \Lambda M^{\text{even}}$ and $(d-d_p)(M^{\text{odd}}) \subset \Lambda^+ M^{\text{odd}} \cdot \Lambda M$.

The differential d_p is called the pure modification of d . If the dimension of the vector space M is finite, then

$$(1.1) \qquad \dim(H^*(\Lambda M, d)) < \infty \text{ if and only if } \dim(H^*(\Lambda M, d_p)) < \infty$$

by [3; Proposition 1]. Let C^* be the cochain functor from the category of differential graded Lie algebras to the category of differential graded commutative algebras, $C^* : LDG \to ADGC$ [12; I.1]. It relates the minimal model $(\Lambda M, d)$ of a simply connected space S and its homotopy Lie algebra Π_* by :

$$(1.2) \qquad C^*((\Pi_*, \partial = 0)) \cong (\Lambda M, d_2) ,$$

where d_2 denotes the quadratic part of the differential d [12; p.88].

Let V be a (positively) graded finite dimensional rational vector space and let $x_1, \dots, x_r, y_1, \dots, y_q$ be a homogeneous basis, $\deg(x_i) = 2a_i$,

$\deg(y_j) = 2b_j - 1$, $1 \le i \le r$, $1 \le j \le q$. The integers $b_1, \dots, b_q; a_1, \dots, a_r$ will be called, according to [2], the exponents of the graded space V .

Let [;] be a graded Lie algebra product (bracket) on a graded vector space W [12; 0.4]. Denote by sW the suspension of W , *i.e.* the graded vector space defined by $(sW)_p = W_{p-1}$. If we write $C^*((W, [\ ,\ ], \partial = 0)) = (\Lambda V, d)$ then, by definition, the differential d is quadratic and

$$V = (sW)^* (= \mathrm{Hom}(sW, Q)) .$$

Choose a basis $x_1, \dots, x_r, y_1, \dots, y_q$ of V as above and let $b_1, \dots, b_q, a_1, \dots, a_r$ be the exponents of the space V . Clearly, the pure modification d_p of the differential d is characterized by a sequence $g_1, \dots, g_q$ of quadratic polynomials from $Q[x_1, \dots, x_r]$, $g_j = d_p(y_j) \in \Lambda(x_1, \dots, x_r) = Q[x_1, \dots, x_r]$, $1 \le j \le q$. Using [2; Theorem 3] we can easily deduce the following observation (the proof is given in § 4).

Observation. — Suppose that $(W, [\ ;\ ])$ is the homotopy Lie algebra of a simply connected space of type F . Then the following condition must be satisfied (compare with the definition before [2; Theorem 1]) :

for every subsequence A^* of $(a_1, \dots, a_r)$ of length s $(1 \le s \le r)$ there exist at least s elements b_j of $(b_1, \dots, b_q)$ of the form $b_j = \sum_{a_i \in A^*} \gamma_{ij} a_i$, where γ_{ij} are non-negative integers and

- either $\sum_{a_i \in A^*} \gamma_{ij} \ge 3$,
- or $\sum_{a_i \in A^*} \gamma_{ij} = 2$ and each quadratic monomial $\prod_{a_i \in A^*} (x_i)^{\gamma_{ij}}$ occurs

in the polynomial g_j .

2. Results.

Let V be a finite dimensional rational graded vector space and $b_1, \dots, b_q, a_1, \dots, a_r$ its exponents. We shall always assume that $a_i > 0$ and $b_j > 1$, $1 \le i \le r$, $1 \le j \le q$. Denote by W the desuspension $s^{-1}V^*$, *i.e.* the graded space defined by $(s^{-1}V^*)_p = V^*_{p+1}$. Clearly $2b_1 - 2, \dots, 2b_q - 2, 2a_1 - 1, \dots, 2a_r - 1$ are the degrees of a homogeneous basis of W .

Let $\mathcal{L}(W)$ be the system of all graded Lie algebra structures on W . Systems of such a type will be considered as (not necessarily irreducible) affine algebraic varieties (= closed algebraic sets) over Q in the same sense as, for example, in [7]. Similarly, let $\mathcal{L}_p(W)$ denote the variety of all graded Lie algebra products on W satisfying the following "purity" condition :

$$\text{if } x \text{ and } y \text{ are homogeneous and } [x;y] \neq 0 \text{ , then } \deg(x) \text{ and } \deg(y) \text{ are both odd.} \tag{2.1}$$

This condition means nothing else than the purity of $C^*((W,[;],\partial=0))$. Finally, denote by $f\mathcal{L}(W)$ (resp. $f\mathcal{L}_p(W)$) the system of all graded Lie algebra structures (resp. graded Lie algebra structures satisfying (2.1)) on W which can be realized by the homotopy Lie algebra of a simply connected space of type F .

Write for simplicity $B = (b_1, \ldots, b_q)$ and $A = (a_1, \ldots, a_r)$. In the situation described above we denote, for a positive integer k , by "AC_k" the following condition :

for every subsequence A^* of A of length s $(1 \leq s \leq r)$ there exist at least s elements b_j of B of the form

$$b_j = \sum_{a_i \in A^*} \gamma_{ij} a_i \ ,$$

where γ_{ij} are non-negative integers and $\sum_{a_i \in A^*} \gamma_{ij} \geq k$.

Remark. — The condition "AC_2" is precisely the "strong arithmetic condition" introduced in [2], hence the simply connected case of Theorem 1 in [2] reads in the terminology introduced above as follows :

the condition "AC_2" is satisfied if and only if $f\mathcal{L}(W) \neq \emptyset$.

Moreover, it easily follows from (1.1) that $f\mathcal{L}(W) \neq \emptyset$ if and only if $f\mathcal{L}_p(W) \neq \emptyset$ (see also the following paragraphs). Notice also that the Jacobi identity in graded Lie algebras satisfying (2.1) is trivial, hence $\mathcal{L}_p(W)$ is in fact isomorphic with the affine space Q^d for suitable d . Therefore each Zariski-open subset of $\mathcal{L}_p(W)$ is dense.

THEOREM 1. — *There are only three possibilities :*

- *First case : $f\mathcal{L}_p(W)$ is empty*
- *Second case : $f\mathcal{L}_p(W)$ is a nonempty, Zariski-open (and hence dense) subset of $\mathcal{L}_p(W)$, but $f\mathcal{L}_p(W) \neq \mathcal{L}_p(W)$*
- *Third case : $f\mathcal{L}_p(W) = \mathcal{L}_p(W)$.*

These cases are characterized as follows :

- *First case is equivalent with "non AC_2"*
- *Second case is equivalent with "AC_2 et non AC_3"*
- *Third case is equivalent with "AC_3" .*

This theorem is proved in § 4. Note that the conditions "AC_k" are easily verifiable. From the previous theorem and the Observation we easily obtain :

COROLLARY 2. — *If the condition "AC_3" is satisfied, then each pure (= satisfying (2.1)) Lie algebra product on W can be realized by the homotopy Lie algebra of a simply connected space of type F . If the condition "AC_3" is not satisfied, then no simply connected space of type F has the homotopy Lie algebra isomorphic with the algebra $(W, [\ ;\ ] = 0)$.*

Let us denote by $\mathcal{M}(V)$ (resp. $\mathcal{M}_p(V)$) the affine variety of all minimal (resp. pure minimal) algebras of the form $(\Lambda V, d)$. We can define the map $F : \mathcal{M}(V) \to \mathcal{L}(W)$ by $F((\Lambda V, d)) = (W, [\ ,\ ])$, where the algebra $(W, [\ ;\ ])$ is characterized by $C^*((W, [\ ;\ ], \partial = 0)) = (\Lambda V, d_2)$. The restriction gives the map $F_p : \mathcal{M}_p(V) \to \mathcal{L}_p(W)$. Define the map $p : \mathcal{L}(W) \to \mathcal{L}_p(W)$ by $p((W, [\ ;\ ])) = (W, [\ ;\ ]_p)$, where $[x; y]_p = [x; y]$ for deg(x) and deg(y) odd and $[x; y]_p = 0$ otherwise, $x, y \in W$ are homogeneous elements. Finally, we denote by $P : \mathcal{M}(V) \to \mathcal{M}_p(V)$ the map $P((\Lambda V, d)) = (\Lambda V, d_p)$ (d_p is defined in § 1). Our maps form the following commutative diagram :

$$\begin{array}{ccc} \mathcal{L}(W) & \xleftarrow{F} & \mathcal{M}(V) \\ {\scriptstyle p}\downarrow & & {\scriptstyle P}\downarrow \\ \mathcal{L}_p(W) & \xleftarrow{F_p} & \mathcal{M}_p(V) \end{array}$$

THEOREM 3. — *Let $4.\min\{2a_i, 2b_j - 1;\ 1 \le i \le r, 1 \le j \le q\} > \max\{2a_i, 2b_j - 1;\ 1 \le i \le r,\ 1 \le j \le q\} + 2$ or, more generally, let the canonical map from $\mathcal{M}(V)$ to the pullback of the diagram*

$$\begin{array}{ccc} \mathcal{L}(W) & & \\ {\scriptstyle p}\downarrow & & \\ \mathcal{L}_p(W) & \xleftarrow{F_p} & \mathcal{M}_p(V) \end{array}$$

be an epimorphism. Then the classification given in Theorem 1 is valid also for $f\mathcal{L}(W)$ in $\mathcal{L}(W)$.

The previous theorem contains the following interesting information.

COROLLARY 4. — *Suppose that the condition "AC_3" is satisfied and that* $4.\min\{\deg(v)\ ;\ v \in V$ *is homogeneous* $\} > \max\{\deg(v)\ ;\ v \in V$ *is homogeneous* $\}+2$. *Then each Lie algebra structure on the vector space W can be realized by the homotopy Lie algebra of a simply connected space of type F .*

THEOREM 5. — *Let the variety $\mathcal{M}(V)$ be irreducible. Then the condition "AC_2" is satisfied if and only if the set $f\mathcal{L}(W)$ is dense in $\mathcal{L}(W)$.*

Of course, if the condition "AC_2" is not satisfied, then the set $f\mathcal{L}(W)$ is empty (see the remark before Theorem 1). Our theorems are proved in § 4. We give the example showing the necessity of the irreducibility assumption in the last one.

Let V be the space homogeneously generated by the set $\{y_1, y_2, y_3, x\}$, $\deg(y_1) = 3$, $\deg(y_2) = 11$, $\deg(y_3) = 13$ and $\deg(x) = 4$. Then clearly $\mathcal{M}(V) \cong \{(a,b) \in Q^2; ab = 0\}$ and this set is reducible. It is easy to see that $\mathcal{L}(W) \cong Q$ and that $f\mathcal{L}(W) =$ Point, although the condition "AC_3" (and hence also "AC_2") is satisfied. It is interesting to compare this with the situation of Theorem 1, where "AC_3" implies $f\mathcal{L}_p(W) = \mathcal{L}_p(W)$. We see that the couples $(\mathcal{L}(W), f\mathcal{L}(W))$ and $(\mathcal{L}_p(W), f\mathcal{L}_p(W))$ have, in general, quite different properties.

On the other hand, there are interesting examples when Theorem 5 is applicable. For example, if V is the graded space based by the set $\{y_1, y_2, y_2', y_3, x\}$, $\deg(y_1) = 3$, $\deg(y_2) = \deg(y_2') = 11$, $\deg(y_3) = 13$ and $\deg(x) = 4$, then clearly $\mathcal{M}(V) \cong \{(a,b,c,d) \in Q^4; ac + bd = 0\}$ which can be shown to be irreducible. By Theorem 5, $f\mathcal{L}(W)$ is dense in $\mathcal{L}(W) = Q^2$ (it can be shown even that $f\mathcal{L}(W) = \mathcal{L}(W)$).

3. Main lemma.

In this paragraph we deduce the lemma, which forms the basis tool for proving our theorems. We adopt the usual terminology of [6], [9] and [10]. All objects are considered over an arbitrary (not necessary algebraically closed) field k of characteristic zero. Let $x_1, \ldots, x_r, a_1, \ldots, a_s$ be graded indeterminates, $\deg(x_i) > 0$, $\deg(a_j) = 0$ for $1 \leq i \leq r$, $1 \leq j \leq s$. We shall denote for brevity $x = (x_1, \ldots, x_r)$ and $a = (a_1, \ldots, a_s)$. For example, the graded polynomial ring $k[x_1, \ldots, x_r, a_1, \ldots, a_s]$ will be denoted simply by $k[x,a]$. Let A be the affine space with "coordinates" $a_1,\dots,a_s$:

$$A = \{(a_1,\dots,a_s);\ a_j \in k,\ 1 \le j \le s\} \cong k^s \ .$$

For a point $\alpha \in A$ and an ideal $I \subset k[x,a]$ let I_α be the ideal in $k[x]$ defined by

$$I_\alpha = \{f(x,\alpha); f(x,a) \in I\} \ .$$

Finally, for a subset $X \subset A$ write

$$X^I = \{\alpha \in X; \dim_k(k[x]/I_\alpha) < \infty\} \ .$$

The main result of this paragraph reads as follows :

MAIN LEMMA. — *Suppose that the ideal I is homogeneous (i.e. generated by a set of homogeneous elements, see [10; chap. VII]) in the graded ring $k[x,a]$. Then*

$$A^I = \{\alpha \in A; \dim_k(k[x]/I_\alpha) < \infty\}$$

is a (possibly empty) Zariski-open subset of A .

It can be easily shown that the lemma is not valid without the homogeneity assumption. Also the assumption $\deg(x_i) > 0$, $\deg(a_j) = 0$, $1 \le i \le r$, $1 \le j \le s$, is necessary.

Fix an algebraic closure $\overline{k}$ of the field k . The inclusion $k \subset \overline{k}$ defines the natural injection $k[x,a] \hookrightarrow \overline{k}[x,a]$ and we can clearly consider all objects over $\overline{k}$; I generates the ideal $\overline{I} \subset \overline{k}[x,a]$ and the "$\overline{k}$–version" of A is :

$$\overline{A} = \{(a_1,\dots,a_s); a_j \in \overline{k}, 1 \le j \le s\} \cong \overline{k}^s \ .$$

Then again $A \subset \overline{A}$. We can easily verify that for each $\alpha \in A$:

$$\dim_k(k[x]/I_\alpha) < \infty \text{ if and only if } \dim_{\overline{k}}(\overline{k}[x]/\overline{I}_\alpha) < \infty \ ,$$

hence $A^I = \overline{A}^{\overline{I}} \cap A$. Because $A \cap U$ is clearly Zariski-open (over k) in A for each Zariski-open (over $\overline{k}$) subset U of $\overline{A}$, it is sufficient to prove the lemma under the assumption that k *is algebraically closed.* First step towards the proof of Main Lemma is the following proposition.

PROPOSITION 1. — *For each Zariski-closed subset F of the affine space A either $F^I = \emptyset$ or F^I contains a nonempty subset, Zariski-open in F .*

Proof of the proposition. — Because clearly $(F_1 \cup F_2)^I = F_1^I \cup F_2^I$, we can always suppose that the set F is irreducible, hence the ideal

$$J = \{f \in k[a]; f(\alpha) = 0 \text{ for each } \alpha \in F\}$$

is prime. Denote by B the affine space

$$B = \{(x_1, \ldots, x_r, a_1, \ldots, a_s); x_i, a_j \in k, 1 \le i \le r, 1 \le j \le s\}$$

and let $P : B \to A$ be the natural projection. As usually, for an ideal K of a polynomial ring, denote by $Z(K)$ the zero set of K in the corresponding affine space [6; I.1]. We know that [2; Remark 1.9] :

(3.1) $\dim_k(k[x]/I_\alpha) < \infty$ if and only if the set $Z(I_\alpha)$ is finite.

Denote $M = Z(I) \cap P^{-1}(F)$. Because clearly $Z(I_\alpha) = Z(I) \cap P^{-1}(\alpha)$, we obtain easily from (3.1) that

(3.2) $$F^I = \{\alpha \in F; P^{-1}(\alpha) \cap M \text{ is finite}\}.$$

The ideal J can be considered as a subset of $k[x,a]$ and it makes sense to denote by D the ideal generated by I and J in $k[x,a]$. Note that $M = Z(D)$. If we decompose the algebraic set M into the union of irreducible components, $M = M_1 \cup \ldots \cup M_m$, then

$$Q_i = \{f \in k[x,a]; f(\xi) = 0 \text{ for each } \xi \in M_i\}$$

are the associated primes of the ideal D , $1 \le i \le m$. Similarly as above we obtain

(3.3) $$F^{Q_i} = \{\alpha \in F; P^{-1}(\alpha) \cap M_i \text{ is finite}\} \ , \ 1 \le i \le m \ ,$$

hence it is clear from the description (3.2) of the set F^I that

$$F^I = \bigcap_{1 \le i \le m} F^{Q_i} \ .$$

The set F is supposed to be irreducible, hence every nonempty Zariski-open subset of F is dense in F and it is clearly sufficient to prove that for each i , $1 \le i \le m$,

(3.4) either $F^{Q_i} = \emptyset$ or F^{Q_i} contains a nonempty subset, Zariski-open in F .

Fix i , $1 \le i \le m$. Because the ideals I and J are homogeneous, the ideal $D = (I, J)$ is homogeneous, too. By [10; p.154] each associated prime Q_i of D is also homogeneous, hence Q_i is generated by a system of the form

$$g_1(x,a), \ldots, g_u(x,a), h_1(a), \ldots, h_v(a) \ ,$$

where $g_t \in k[x,a]$ are homogeneous of positive degrees and $h_j \in k[a]$ are homogeneous of degree zero, $1 \le t \le u$, $1 \le j \le v$ (because $\deg(x_k) > 0$, no x_k can occur in a polynomial of degree zero, $1 \le k \le r$). This observation is the *key point* of our proof.

Denote by H the ideal generated in $k[a]$ by the polynomials $h_1, \ldots, h_v$. We claim that $P(M_i) = Z(H)$. Indeed, because the polynomials $g_1, \ldots, g_u$ have positive degrees, they are zero on elements of the form $(0, \alpha)$ for each $\alpha \in A$. Consequently, $(0, \alpha) \in Z(Q_i) = M_i$ provided $\alpha \in Z(H)$. Because $\alpha = P(0, \alpha)$, we see that $Z(H) \subset P(M_i)$. On the other hand, if $(\xi, \alpha) \in M_i = Z(Q_i)$ then clearly $h_j(\alpha) = 0$ for each j , $1 \leq j \leq v$, and $\alpha = P(\xi, \alpha) \in Z(H)$, which proves the inclusion $P(M_i) \subset Z(H)$.

By definition, $P(M_i) \subset F$ and we distinguish the following two cases :

A. $P(M_i) \subsetneqq F$. In this case, the set $U_i = F \setminus Z(H)$ is nonempty and Zariski-open in F . Because $P^{-1}(\alpha) \cap M_i = \emptyset$ for each $\alpha \in U_i$, $U_i \subset F^{Q_i}$ by (3.3) and the condition (3.4) is satisfied.

B. $P(M_i) = F$. Denote $F' = \{(0, \alpha); \alpha \in F\}$. Clearly $F' \subset M_i$, hence $\dim(F) = \dim(F') \leq \dim(M_i)$. The restriction $P|M_i$ defines the map $\pi : M_i \to F$, which is epic by our assumption. Again we distinguish two cases :

B.1. $\dim(M_i) > \dim(F)$. By the definition of the dimension, the set $\pi^{-1}(\alpha)$ is finite if and only if $\dim(\pi^{-1}(\alpha)) = 0$. The theorem [11; I.6. Theorem 7] (compare also [1; AG 10.1]) says that the set

$$F^{Q_i} = \{\alpha \in F; \dim(\pi^{-1}(\alpha)) = 0\}$$

is empty and (3.4) is valid.

B.2. $\dim(M_i) = \dim(F)$. Because $F' \subset M_i$ and $\dim(F') = \dim(M_i)$, from the irreducibility of the set M_i we see that $F' = M_i$, hence $\pi^{-1}(\alpha) = \{(0, \alpha)\}$. We have $F^{Q_i} = F$ and (3.4) is again satisfied. Our proposition is proved.

Proof of Main Lemma. — Suppose we have constructed a sequence $A_1 \supsetneqq A_2 \supsetneqq \cdots \supsetneqq A_k$, $k \geq 1$, of closed subsets of A with the property $(A \setminus A_k) \subset A^I$. If $A_k^I = \emptyset$ then $A^I = (A \setminus A_k)$ is open. In the opposite case there exists, by Proposition 1, a nonempty open subset $U_k \subset A_k$ with $U_k \subset A_k^I$. In this case we define $A_{k+1} = (A_k \setminus U_k)$. The set A_{k+1} is closed, $A_k \supsetneqq A_{k+1}$ and $(A \setminus A_{k+1}) \subset A^I$. Since the topological space A is Noetherian [6; 1.4.7], this procedure gives rise to a closed $A_m \subset A$ with $(A \setminus A_m) = A^I$. The lemma is proved.

4. Remaining proofs.

In this paragraph we prove the theorems of § 2. We adopt the notation introduced in previous paragraphs.

Let $f\mathcal{M}_p(V)$ denote the subset of $\mathcal{M}_p(V)$ consisting of all pure minimal algebras having finite dimensional cohomology. It is not hard to deduce from (1.1) that $f\mathcal{L}_p(W) = F_p(f\mathcal{M}_p(V))$. The algebras belonging to $\mathcal{M}_p(V)$ are of the form

$$(\Lambda(x_1,\dots,x_r,y_1,\dots,y_q),d)\ ,\quad \deg(x_i)=2a_i\ ,\quad \deg(y_j)=2b_j-1\ ,$$

with $d(x_i) = 0$ and $d(y_j) \in \Lambda(x_1,\dots,x_r) = Q[x_1,\dots,x_r]$ for $1 \le i \le r$, $1 \le j \le q$. Thus each element of $\mathcal{M}_p(V)$ is characterized by a sequence $f_1,\dots,f_q$ of polynomials, $f_j = d(y_j) \in Q[x_1,\dots,x_r]$, $1 \le j \le q$. Our minimal algebra clearly belongs to $f\mathcal{M}_p(V)$ if and only if

$$\dim_Q(Q[x_1,\dots,x_r]/(f_1,\dots,f_r)) < \infty\ , \quad \text{see also [2].}$$

PROPOSITION 2.

a) *"$f\mathcal{M}_p(V) = \emptyset$" is equivalent with "non AC_2",*

b) *"$f\mathcal{M}_p(V)$ is a nonempty subset, Zariski-open in $\mathcal{M}_p(V)$" is equivalent with "AC_2",*

c) *"$F_p(f\mathcal{M}_p(V)) = \mathcal{L}_p(W)$" is equivalent with "$AC_3$".*

Proof of a). — This equivalence is in fact the main result of [2]; see also the note before Theorem 1.

Proof of b). — For each j , $1 \le j \le q$, denote by Φ_j the family of all at least quadratic (*i.e.* of length ≥ 2) monomials $\sigma \in Q[x_1,\dots,x_r]$ with $\deg(\sigma) = 2b_j$. Write $\Phi_j = \{\sigma_1^j,\dots,\sigma_{k_j}^j\}$ and denote

$$f_j(x,a^j) = f_j(x_1,\dots,x_r,a_1^j,\dots,a_{k_j}^j) = \sum_{1\le s\le k_j} a_s^j\sigma_s^j\ ,\quad 1\le j\le q\ .$$

Then $\mathcal{M}_p(V)$ is isomorphic to the affine space A with the "coordinates" $a_1^1,\dots,a_{k_1}^1,\dots,a_1^q,\dots,a_{k_q}^q$ in the evident sense. If we put $\deg(a_s^j) = 0$ for $1 \le j \le q$, $1 \le s \le k_s$, then $I = (f_1,\dots,f_q)$ is a homogeneous ideal in the graded polynomial ring $Q[x_1,\dots,x_r,a_1^1,\dots,a_{k_1}^1,\dots,a_1^q,\dots,a_{k_q}^q]$. Applying Main Lemma to this situation we see that the set A^I , which is clearly isomorphic with $f\mathcal{M}_p(V)$, is Zariski-open in $A \cong \mathcal{M}_p(V)$. Combining this with a) we obtain the requisite equivalence.

Proof of c). — The set $\mathcal{L}_p(W)$ can be identified with the subset of $\mathcal{M}_p(V)$ consisting of all minimal algebras with pure quadratic differential in the natural way. Under this identification F_p acts as taking the quadratic part and "$F_p(f\mathcal{M}_p(V)) = \mathcal{L}_p(W)$" means that for each pure quadratic differential δ on ΛV there exists a pure minimal algebra $(\Lambda V, d) \in f\mathcal{M}_p(V)$ such that the quadratic part d_2 of the differential d is equal to δ . Especially the equation $F_p(f\mathcal{M}_p(V)) = \mathcal{L}_p(W)$ implies the existence of $(\Lambda V, d) \in f\mathcal{M}_p(V)$ with trivial quadratic part. Then "AC_3" must be satisfied by Observation in § 1.

On the other hand, let "AC_3" be satisfied and let ψ_j be, similarly as in the proof of b), the set of all at least cubic (= of length ≥ 3) monomials $\mu \in Q[x_1, \dots, x_r]$ with $\deg(\mu) = 2b_j$, $1 \leq j \leq q$. The families $\psi_1, \dots, \psi_q$ satisfy the condition P.C. of [2; p.119] and there is a sequence $f_1, \dots, f_q \in Q[x_1, \dots, x_r]$ of polynomials such that each f_j is a linear combination of monomials from ψ_j and

$$\dim_Q(Q[x_1, \dots, x_r]/(f_1, \dots, f_q)) < \infty \qquad [2\ ;\ \text{Theorem 3}].$$

By the definition of ψ_j all the polynomials $f_1, \dots, f_q$ have zero quadratic part.

Now, let $(\Lambda V, \delta)$ be a pure minimal algebra with quadratic differential and denote $g_j = \delta(y_j) \in Q[x_1, \dots, x_r]$, $1 \leq j \leq q$. Then the pure differential d , defined for each sequence $\alpha_1, \dots, \alpha_q$ of nonzero rationals by

$$d(y_j) = (\alpha_j)^{-1} \cdot f_j + g_j\ , \quad 1 \leq j \leq q\ ,$$

has the quadratic part equal to δ . By the following lemma we can find the rationals $\alpha_1, \dots, \alpha_q$ such that $(\Lambda V, d) \in f\mathcal{M}_p(V)$ which completes our proof.

LEMMA. — *Let $f_1, \dots, f_q, g_1, \dots, g_q \in Q[x_1, \dots, x_r]$ be homogeneous elements and let $\dim_Q(Q[x_1, \dots, x_r]/(f_1, \dots, f_q)) < \infty$. Then there exists a sequence $\alpha_1, \dots, \alpha_q$ of nonzero rational numbers such that*

$$\dim_Q(Q[x_1, \dots, x_r]/((\alpha_1)^{-1} f_1 + g_1, \dots, (\alpha_q)^{-1} f_q + g_q)) < \infty\ .$$

Proof of the lemma. — For $1 \leq i \leq q$ define $h_i(x, a) = f_i(x) + a_i g_i(x)$. If we define $\deg(a_i) = 0$ for $1 \leq i \leq q$, then $h_1, \dots, h_q$ are homogeneous elements of the polynomial ring $k[x_1, \dots, x_r, a_1, \dots, a_q]$; let us denote by I the ideal $(h_1, \dots, h_q)$. If we abbreviate by A the affine space $A = \{(a_1, \dots, a_q)\ ;\ a_i \in Q\ ,\ 1 \leq i \leq q\}$, the set A^I is Zariski-open in A by

Main Lemma. By our assumption, $\dim_Q(k[x_1,\dots,x_r]/(f_1,\dots,f_q)) < \infty$, hence $(0,\dots,0) \in A^I$ and A^I is nonempty. Clearly there exists a point $(\alpha_1,\dots,\alpha_q) \in A^I$ having all coordinates different from zero. Because

$$(f_1+\alpha_1 g_1,\dots,f_q+\alpha_q g_q) = ((\alpha_1)^{-1}f_1+g_1,\dots,(\alpha_q)^{-1}f_q+g_q) ,$$

our point $(\alpha_1,\dots,\alpha_q)$ has the requisite properties.

Proof of Theorem 1. — As we remarked in the proof of Proposition 2, the affine space $\mathcal{L}_p(W)$ can be identified with an affine subspace of the affine space $\mathcal{M}_p(V)$, under this identification $F_p : \mathcal{M}_p(V) \to \mathcal{L}_p(W)$ is simply the canonical projection, hence an open epimorphism. Theorem 1 now follows from the classification given in Proposition 2.

Proof of Theorem 3. — We easily deduce from (1.1) that $f\mathcal{L}(W) = FP^{-1}(f\mathcal{M}_p(V))$. Taking the space $\{(x,y) \in \mathcal{L}(W) \times \mathcal{M}_p(V); p(x) = F_p(y)\}$ as the pullback of the diagram we see that if the canonical map from $\mathcal{M}(V)$ to the pullback is epic, then $f\mathcal{L}(W) = p^{-1}(f\mathcal{L}_p(W))$. The theorem now follows from Theorem 1 and from the evident fact that $p : \mathcal{L}(W) \to \mathcal{L}_p(W)$ is a continuous epimorphism.

For $p > 0$ the set $\Lambda^p V = \{v_1 \wedge \dots \wedge v_p; v_1,\dots,v_p \in V\}$ forms a vector subspace of ΛV and $\bigoplus_{p\geq 0} \Lambda^p V \cong \Lambda V$ (we put $\Lambda^0 V = Q$). Let $q_p : \Lambda V \to \Lambda^p V$ be the projection. For a linear endomorphism G of ΛV and $i \geq 2$ denote by $G_i : \Lambda V \to \Lambda V$ the linear map defined by $G_i|\Lambda^p V = q_{p+i-1} \circ G$. Finally, for $j \geq 1$ denote $G_{>j} = \sum_{i>j} G_i$.

The canonical map from $\mathcal{M}(V)$ to the pullback is clearly epic if and only if for each pure minimal differential d on ΛV and for each quadratic differential D on ΛV whose pure modification D_p is equal to the quadratic part d_2 of d there exists a differential δ on ΛV whose pure modification is equal to d and whose quadratic part is equal to D .

Let D and d be as above. Define the derivation δ by $\delta = D + d_{>2}$.

Then clearly $\delta^2 = D^2 + (\delta^2)_{>3} = (\delta^2)_{>3}$ and it is not hard to verify that under the assumption

$$\begin{aligned}4.\ &\min\{\deg(v); v \in V \text{ is homogeneous}\}\\ &> \max\{\deg(v); v \in V \text{ is homogeneous}\} + 2\end{aligned}$$

is always $(\delta^2)_{>3} = 0$, consequently δ is a differential satisfying $\delta_p = d$ and $\delta_2 = D$.

Proof of Theorem 5. — Recall that $f\mathcal{L}(W) = FP^{-1}(f\mathcal{M}_p(V))$ (see the proof of Theorem 3). The map $P : \mathcal{M}(V) \to \mathcal{M}_p(V)$ is continuous and epic and the set $P^{-1}(U)$ is, because of the irreducibility of $\mathcal{M}(V)$, dense for each nonempty open subset $U \subset \mathcal{M}_p(V)$. The map $F : \mathcal{M}(V) \to \mathcal{L}(W)$ is also continuous and epic and the rest follows from Proposition 2.

Proof of Observation. — Let Ω_j be, for $1 \leq j \leq q$, the system of all monomials $\omega \in Q[x_1, \ldots, x_r]$ with $\deg(\omega) = 2b_j$, such that

- either ω is at least cubic (= of length ≥ 3),
- or ω is quadratic and it occurs in the polynomial g_j .

Suppose that there exists $(\Lambda V, D) \in f\mathcal{M}(V)$ with $C^*((W,[\ ;\ ],\partial=0)) = (\Lambda V, D_2)$. Then each polynomial $f_j = D_p(y_j)$ must be clearly a rational linear combination of elements of Ω_j , $1 \leq j \leq q$. Being $(W, [\ ;\ ])$ the homotopy Lie algebra of a space of type F , by [2; Theorem 3] the systems $\Omega_1, \ldots, \Omega_q$ must satisfy the condition P.C. of [2; p. 119]. But P.C. for $\Omega_1, \ldots, \Omega_q$ is clearly equivalent with the condition given in Observation.

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