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# ISOSPECTRAL RIEMANN SURFACES

**by Peter BUSER**

## 1. Introduction.

Two compact Riemannian manifolds are called *isospectral* if they have the same spectrum of the Laplace-Beltrami operator. The first examples of isospectral non isometric manifolds were tori of dimension 16 given by Milnor in 1964 [8]. If the manifold is a compact Riemann surface of genus $g \geqslant 2$ endowed with the Poincaré metric of constant curvature $-1$, then Gel'fand conjectured that the spectrum of the Laplacian determines the geometry of the surface completely [4]. Early results in the affirmative direction were that Riemann surfaces do not admit isospectral continuous deformations [4], [14], and later by McKean that for a given Riemann surface the number of isospectral non isometric surfaces is at most finite [7]. From general results ([2]) it is also known that the spectrum determines the genus, and that a surface of constant curvature can never have the spectrum of a surface of non constant curvature.

In 1977/78 two results appeared almost simultaneously. Wolpert [19] [20] proved the Gel'fand conjecture in the following generic sense: Let $T_g$ be the Teichmüller space of compact Riemann surfaces of genus $g \geqslant 2$, and define

$$V_g = \{S \in T_g \mid \textit{there exists a surface } F \in T_g \textit{ which is isospectral but not isometric to } S\}, \tag{1.1}$$

then $V_g$ is a local real analytic subvariety of $T_g$ of *lower dimension*. Hence almost all Riemann surfaces of genus $g \geqslant 2$ are determined by their spectrum. However, at about the same time M. F. Vignéras [17] [18] found the first isospectral examples for, in fact, infinitely many sufficiently large

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$g$. Vignéras' examples are based on quaternion algebras over a suitable number field and are difficult to understand geometrically. In 1983 Sunada [12] found a new and much more general approach to isospectral manifolds. Sunada's construction is based on covering arguments which have been long known in algebraic number theory, and which he was the first to apply to Riemannian geometry. Among the various results, Sunada proves that $\dim V_g > 0$ for all $g = 17 + 8n$, $n = 0, 1, 2, \ldots$

In this paper we shall improve this result as follows:

1.2. THEOREM. — $\dim V_g > 0$ *for* $g = 5$ *and for all* $g \geqslant 7$.

The examples presented here have been found while investigating the combinatorial aspect of Sunada's construction. This aspect seems to be rather important, and we shall construct all surfaces in a purely combinatorial manner by pasting together building blocks according to a certain pasting scheme which bears the phenomenon of isospectrality.

The proof that the two surfaces thus obtained are isospectral will then be in the same spirit. We shall show that the eigenfunctions on the first surface can be suitably « transplanted » to yield eigenfunctions with the same eigenvalue on the second surface and vice-versa. One of the advantages of this procedure is that it works equally well for surfaces in space. In section 7 we shall therefore leave the realm of Riemann surfaces and prove.

1.3. THEOREM. — *There exist isospectral non isometric surfaces which are isometrically embedded in* $\mathbf{R}^3$.

In the same section we shall also give examples of isospectral flat two dimensional bordered domains which are isometrically embedded in $\mathbf{R}^3$. These domains which improve earlier results of Urakawa [16] have smooth boundary and are isospectral for either Dirichlet or Neumann boundary conditions. Since they are embedded as ruled surfaces, these examples may be easily realized by paper models.

The pasting scheme is described in section 2. Sections 3, 6, 7 give the examples. The transplantation of eigenfunctions takes place in section 4. In section 5 we carry out a similar transplantation technique for closed geodesics to prove that the examples are also also isospectral with respect to the length spectrum.

## 2. Pasting.

Two surfaces $S_1$, $S_2$ will be obtained by pasting together a certain set of *building blocks* in two combinatorially different ways. In order to simplify the description we shall often refer to the particular example of surfaces of genus 5 (section 3) which are illustrated in figure 1. There the building block, B, is a rectangular geodesic octagon in the hyperbolic plane H with sides $a'$, $b'$, $a^*$, $c'$, $a^{**}$, $b''$, $a''$, $c''$ (fig. 2). In section 6, the building blocks are more general geodesic polygons of the hyperbolic plane; in section 7 the building blocks are bordered surfaces in $\mathbf{R}^3$.

In all the examples, eight identical copies $B_1, \ldots, B_8$ of the same block B with corresponding sides $a_i'$, $b_i'$ etc $(i=1,\ldots,8)$ are pasted together along sides of equal length, where the combining of sides is given by the pasting scheme.

*Locally* the pasting is the usual one as used e.g. in [1] or [15], namely : If $u$ on $B_i$ and $v$ on $B_j$ are associated sides ($i=j$ is possible, but $u$ and $v$ are not allowed to coincide) we parametrize them in the form $t \mapsto u(t)$, $t \mapsto v(t)$; $t \in [0, 1]$, with constant speed and positive boundary orientation (with respect to some fixed orientation on B). With the identification

$$v(t) = u(1-t); \qquad t \in [0, 1], \tag{2.1}$$

we obtain the connected sum $B_i + B_j$ (mod. 2.1). Since the hyperbolic plane H has a twofold transitive isometry group, the identification – or *pasting* – (2.1) can be carried out with an orientation preserving isometry $\Phi : H \mapsto H$ satisfying

$$\Phi(v(t)) = u(1-t); \qquad t \in [0,1].$$

It follows that the hyperbolic structures (= metrics of constant curvature $-1$) of $B_i$ and $B_j$ extend to a smooth hyperbolic structure on the connected sum.

In the same way, we may paste together several pairs of sides simultaneously. However we then have the following additional *condition :* If $p_1, \ldots, p_n$ are vertices which together define an *inner* point $p$ of the connected sum, then the hyperbolic structures extend smoothly into $p$ if and only if

$$\alpha_1 + \cdots + \alpha_n = 2\pi, \tag{2.2}$$

where $\alpha_i$ is the inner angle at vertex $p_i$ of the building block, $(i=1,\ldots,n)$.

In the case of the octagons of section 3 e.g. the inner angles are $\pi/2$ and the number of meeting vertices will always be 4.

*Globally* the two pasting schemes may best be read out of figure 1 where the identifications are indicated by arrows. The formal description is as follows.

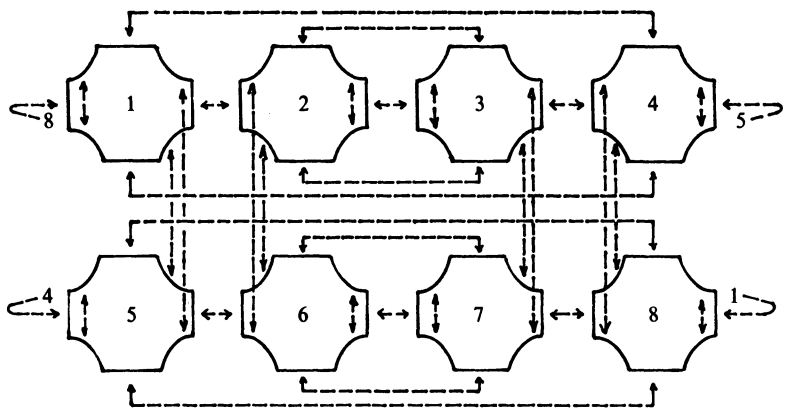


Fig. 1*a*.

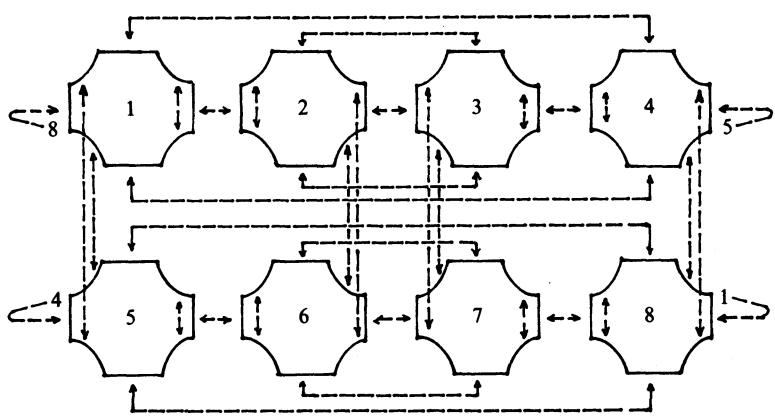


Fig. 1*b*.

The sides of the building block B split into four groups : sides of type $a, b, c$ and $d$ (with coherent labelling).

Type $d$ may be absent. In order to say that side $u$ of block $B_i$ is identified with side $v$ of block $B_j$ via (2.1) we shall write

$$v_j = \pi(u_i), \quad \text{or equivalently} \quad u_i = \pi(v_j).$$

The sides of type $a$ are $a'$, $a''$, $a^*$, $a^{**}$; the last two may be absent. For $S_1$ the pasting scheme of these sides is

$$(2.3)_1 \quad \begin{cases} \pi(a_i') = a_i'', & \pi(a_i^*) = a_{i+4}^{**}; \quad i = 1, 3, 5, 7, \\ \pi(a_i') = a_{i+4}'', & \pi(a_i^*) = a_i^{**}; \quad i = 2, 4, 6, 8. \end{cases}$$

For $S_2$ the scheme is

$$(2.3)_2 \quad \begin{cases} \pi(a_i') = a_{i+4}'', & \pi(a_i^*) = a_i^{**}; \quad i = 1, 3, 5, 7, \\ \pi(a_i') = a_i'', & \pi(a_i^*) = a_{i+4}^{**}; \quad i = 2, 4, 6, 8. \end{cases}$$

(All indices mod 8).

For the remaining types, $S_1$ and $S_2$ will have the same scheme. The sides of type $b$ are $b^1, \ldots, b^\beta$, the scheme is

$$(2.4) \quad \begin{cases} \pi(b_1^\nu) = b_4^{\sigma(\nu)}, & \pi(b_2^\nu) = b_3^{\sigma(\nu)} \\ \pi(b_5^\nu) = b_8^{\sigma(\nu)}, & \pi(b_6^\nu) = b_7^{\sigma(\nu)}; \end{cases} \quad \nu = 1, \ldots, \beta,$$

where $\sigma$ is an *involutive* permutation of $\{1, \ldots, \beta\}$. For the octagons in section 3 e.g. we only have $b^1 = b'$, $b^2 = b''$ and $\sigma$ is the identity.

The sides of type $c$ are $c'$, $c''$, $c^*$, $c^{**}$; the last two may be absent. The identifications are

$$(2.5) \quad \begin{cases} \pi(c_i') = c_{i+1}'' \\ \pi(c_i^*) = c_{i+1}^{**}; \end{cases} \quad i = 1, \ldots, 8 \pmod 8.$$

The sides of type $d$ are $d^1, \ldots, d^\delta$. The identifications are

$$(2.6) \quad \pi(d_i^\nu) = d_{i+4}^{\tau(\nu)}; \; i = i, \ldots, 8 \pmod 8, \; \nu = i, \ldots, \delta,$$

where $\tau$ is an involutive permutation of $\{1, \ldots, \delta\}$.

2.7. *Remark.* — The above pasting scheme is modeled on Gerst's example [5]

$$G = (\mathbf{Z}/8\mathbf{Z})^* \cdot (\mathbf{Z}/8\mathbf{Z})$$

(semi direct product) of a group $G$ which has the two subgroups

$$H_1 = \{(1,0),\ (3,0),\ (5,0),\ (7,0)\}$$
$$H_2 = \{(1,0),\ (3,0),\ (5,4),\ (7,4)\}$$

which are not conjugate, but *almost conjugate* in the sense that each element $g \in G$ has as many conjugates in $H_1$ as it has in $H_2$.

Sunada proves in [12] that any group $G$ with in this sense almost conjugate but not conjugate subgroups gives rise to isospectral Riemannian manifolds. Numerous further examples of such groups are found in [9]. The first example is by Gassmann [3] and was used to give a counter example to a conjecture of Kronecker.

## 3. Examples of genus five.

Before proving that $S_1$, $S_2$ are isospectral, we give the examples of compact Riemann surfaces of genus five. Here it is particularly easy to see that $S_1$ and $S_2$ are not isometric in general.

The building block in this section is a rectangular geodesic octagon in the hyperbolic plane which is obtained

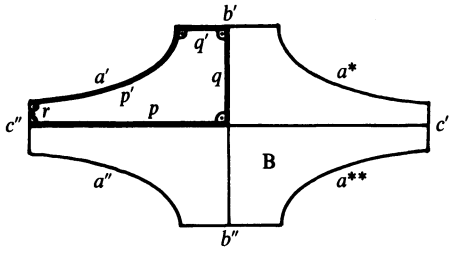


Fig. 2.

by pasting together four identical rectangular pentagons (fig. 2). For the pentagon with sides $r$, $p$, $q$, $q'$, $p'$ the following trigonometric formulae are known (eg [10])

$$\sinh r = \coth p . \coth p' \tag{3.1}$$
$$\cosh r = \sinh q . \sinh q' . \tag{3.2}$$

Moreover it is known that *such pentagons exist for any positive values of* $q'$ *and* $r$. Consequently one can construct such octagons for any given

lengths of sides $b'$ and $c'$. By construction we have

$$a' = a'' = a^* = a^{**}, \qquad c' = c''$$

so that the pasting of section 2 is possible. The following simple lemma helps distinguishing $S_1$ from $S_2$: ($\ell$ denotes arc length).

3.3. LEMMA. — *Let* $0 < c' < b' < 1$. *Then any curve* $\delta$ *on* B *which connects two non adjacent sides of* B *has length* $\ell(\delta) \geqslant c'$. *Equality holds only for* $\delta = c'$ *and* $\delta = c''$.

*Proof.* — Recall from the negative curvature that the unique shortest connecting curve $\delta$ between two non intersecting geodesics is the common perpendicular between these geodesics. Now $\sinh q . \sinh q' = \cosh r > 1$ and $q' = 1/2\, b' < 1$ together imply $q > 1$. Similarly (3.1), (3.2) with cyclic permutation imply $p' > 1$, $p > 1$. It follows that any $\delta$ different from $c'$ or $c''$ has length $\ell(\delta) > c' = c''$ as claimed.

We now take B such that the hypothesis of Lemma 3.3 is satisfied and glue together eight copies of B according to the pasting scheme of figure 1 to obtain Riemann surfaces $S_1$, $S_2$. Check that the angle condition 2.2 is satisfied. We have 8 faces, 32 edges and 16 vertices so that the Euler characteristic is $-8$ and the genus is 5.

3.4. PROPOSITION. — *Under the hypothesis of Lemma* 3.3 $S_1$ *and* $S_2$ *are non isometric.*

*Proof.* — On either surface we let, for $i = 1, \ldots, 8$, $\gamma_i$ be the simple closed geodesic which contains side $c_i'$ of block $B_i$. Observe that $B_i$ is rectangular so that either $c_i'$ itself or $c_i'$ together with $c_{i+4}'$ (indices mod 8) yields a closed geodesic. On $S_1$ we have $\ell(\gamma_i) = c'$ for $i = 2, 4, 6, 8$ and $\ell(\gamma_i) = 2c'$ for $i = 1, 3, 5, 7$. On $S_2$ it is the other way round.

We claim that *any further closed geodesic* $\eta$ *on* $S_1$ *resp.* $S_2$ *has length* $\ell(\eta) > c'$. In fact, since $\eta$ is not contractible, $\eta$ is composed of arcs $\delta$ which connect sides on building blocks.

If some $\delta$ connects two non adjacent sides, the claim follows from Lemma 3.3. Since by the negative curvature $\delta$ cannot return to the same side, it remains to consider the case that all $\delta$ connect adjacent sides. Let $\delta$, $\delta'$ be two consecutive segments of $\eta$ which cross side $u$, say. As indicated in figure 3, $\delta \cup \delta'$ connects the two perpendiculars at the

endpoints of $u$ since again $\delta \cup \delta'$ cannot return to the same geodesic. Hence $\ell(\eta) \geqslant \ell(\delta) + \ell(\delta') > u \geqslant c'$ and the claim is proved.

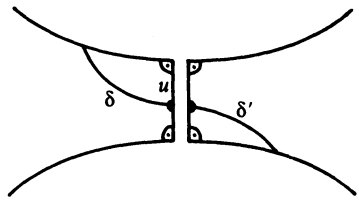


Fig. 3.

Thus it is proved that the $\gamma_i$ are the unique shortest closed geodesics on $S_1$ for $i = 2, 4, 6, 8$ and on $S_2$ for $i = 1, 3, 5, 7$. Now if $S_1$ and $S_2$ were isometric then cutting open $S_1$ and $S_2$ along all shortest closed geodesics would have the same effect.

However, $S_1$ remains connected and $S_2$ does not. This proves Proposition 3.4.

3.5. *Remark.* — Since the parameters $c', b'$ can be chosen freely in an open neighbourhood, the above examples together with Proposition 4.1 in the next section show that the variety $V_5$ of surfaces in $T_5$ which are not determined by their spectrum (1.1) has dimension $\dim V_5 \geqslant 2$. This proves Theorem 1.2 for $g = 5$. (Observe however that varying the parameters $c', b'$ is not an isospectral deformation).

## 4. Transplantation of eigenfunctions.

4.1. PROPOSITION. — *Let* $S_1$, $S_2$ *be the two dimensional Riemannian manifolds obtained by the pasting scheme of section* 2. *Then* $S_1$ *and* $S_2$ *have the same spectrum of the Laplacian.*

*Proof.* — The idea is to copy on $S_2$ each eigenfunction of $S_1$ in the most obvious way and see under what conditions the such « transplanted » function remains a smooth function. It would also be possible to use covering arguments. However, transplantation shows more directly how the two surfaces come to have the same spectrum.

Let $S_1^* = S_2^* = S^*$ be the interior of the surface which is obtained by pasting together the building blocks $B_1, \ldots, B_8$ only along the sides of type $c$ (2.5). The surface $S$ is a sort of bracelet on which an isometry $\varphi$ of

order eight acts in a natural way : If $p \in S^*$ lies on block $B_i$, then $\varphi(p)$ is the corresponding point on block $B_{i+1}$ (indices mod 8). $S^*$ is considered as a subset of $S_1$ resp. $S_2$ in a natural way. Note that $\varphi^4$ *extends to an isometry of* $S_1$ *resp.* $S_2$.

Since $S^*$ is open and dense in $S_i$ $(i=1,2)$, $\varphi$ induces a unitary transformation

$$\hat{\varphi} : L^2(S_i) \to L^2(S_i)$$

of the $L^2$-functions defined by

$$\hat{\varphi} f = f \circ \varphi, \quad f \in L^2(S_i).$$

Roughly speaking, $\hat{\varphi}$ shifts every function one block back. Observe that for continuous functions the image under $\hat{\varphi}$ is discontinuous in general.

If a function $f \in L^2(S_i)$ which originally has been defined only on $S_i^*$ proves to have a $C^\infty$-extension on $S_i$, we shall, by abuse of notation, say that $f \in C^\infty(S_i)$.

To define transplantation we let

$$J : S_2^* \to S_1^*$$

be the natural identification of $S^* = S_2^* \subset S_2$ with $S^* = S_1^* \subset S_1$. For every $f \in L^2(S_1)$ its transplanted image $f^t \in L^2(S_2)$ is then defined by

$$f^t = f \circ J,$$

and similarly we define $g^t = g \circ J^{-1} \in L^2(S_1)$ for $g \in L^2(S_2)$. Observe that this definition makes sense in $L^2$. However, continuous functions become discontinuous in general.

Now let $\lambda$ be a positive real number and let $E_i$ be the eigenspace of $\lambda$ on $S_i$, $(i=1,2)$, i.e. $E_i$ is the space of all $C^\infty$-functions $f$ on $S_i$ satisfying $\Delta f = \lambda f$ where $\Delta$ is the Laplace-Beltrami operator on $S_i$. We want to prove that

$$\dim E_1 = \dim E_2.$$

Certainly, if $f \in E_1$, then $\Delta f^t = \lambda f^t$ on $S_2^*$. However, since $f^t$ is not smooth in general, $f^t$ does not necessarily belong to $E_2$. This will be compensated by transplanting certain shifted functions $\hat{\varphi}^n f$ which in turn do not belong to $E_1$.

4.2. LEMMA. — $E_i \subset L^2(S_i)$ *is the orthogonal sum of* $E_i^+$ *and* $E_i^-$ $(i=1,2)$ *where*

$$E_i^+ = \{f \in E_i | \hat{\varphi}^4 f = f\}$$
$$E_i^- = \{f \in E_i | \hat{\varphi}^4 f = -f\}.$$

*Proof.* — This is immediate from the standard decomposition $f = 1/2\,(f+\hat{\varphi}^4 f) + 1/2(f-\hat{\varphi}^4 f)$. A priori, this decomposition takes place in $L^2(S_i)$. But since $\varphi^4$ is an isometry of $S_i$ (and not only of $S_i^*$), $\hat{\varphi}^4 f$ is again a $C^\infty$-eigenfunction so that the two components belong to $E_i$.

4.3. LEMMA. —

(*a*) *If* $f \in E_i^+$ *then* $f' \in E_{i+1}^+$.

(*b*) *If* $f \in E_i^-$ *then* $(\hat{\varphi} f - \hat{\varphi}^3 f)' \in E_{i+1}^-$.

(*Indices* mod 2).

*Proof.* — We let $i = 1$, for $i = 2$ the arguments are the same. We have to show that the transplanted functions in (*a*) and (*b*) have $C^\infty$-extensions on $S_2$. The reader may first check with figure 1 that the transplanted functions match at least continuously at the differently pasted sides in $S_2$. A closer look then shows that the matching is in fact smooth. The arguments are as follows. ($U_p$ denotes an open neighbourhood of $p$).

4.3.1. *For* $p \in S_2$ *there exists* $p' \in S_1$ *and a local isometry* $j : U_p \to U_{p'}$ *such that for all* $q \in U_p \cap S_2^*$ *we have*

$$j(q) = J(q) \qquad or \qquad j(q) = \varphi^4 \circ J(q).$$

*Proof.* — This is clear if $p \in S_2^*$. Now suppose $p \in S_2 \backslash S_2^*$. Then $p$ lies on the boundary of some building block. These blocks have the following property :

(i) *If* $B_m$ *and* $B_n$ *meet along a side of type* $a$ *in* $S_2$, *then* $B_m$ *and* $B_{n+4}$ *meet along the same side in* $S_1$,

and

(ii) *If* $B_m$ *and* $B_n$ *meet along a side of type* $b, c$ *or* $d$, *in* $S_2$ *then* $B_m$ *and* $B_n$ *meet along the same side in* $S_1$.

(Indices mod 8). To define the local isometry $j : U_p \to U_{p'}$ we let $U_p$ be a circular neighbourhood of $p$ with sufficiently small radius such that the open connected components $U^1, \ldots, U^s$ of $U_p \cap S_2^*$ are circle sectors, where any two circle sectors have at most one side of matching

building blocks in common (t.g. if $p$ lies in the interior of a side, we have $s=2$). The components $U^1, \ldots, U^s$ can be labelled such that for suitable $\sigma$, $1 \leqslant \sigma \leqslant s$, the images $J(U^1), \ldots, J(U^\sigma)$ and $\varphi^4 \circ J(U^{\sigma+1}), \ldots, \varphi^4 \circ J(U^s)$ become the components of $U_{p'} \cap S_1^*$ for a circular neighbourhood $U_{p'}$ in $S_1$. This follows immediately from (i), (ii) and proves 4.3.1.

4.3.2. *For* $p \in S_2$ *there exist* $p_1$, $p_3 \in S_1$ *and local isometries* $j_1 : U_p \to U_{p_1}$, $j_3 : U_p \to U_{p_3}$ *such that for all* $q \in U_p \cap S_2^*$ *we have either*

$$j_1(q) = \varphi \circ J(q) \text{ and at the same time } j_3(q) = \varphi^3 \circ J(q),$$

*or*

$$j_1(q) = \varphi^{-1} \circ J(q) \text{ and at the same time } j_3(q) = \varphi^{-3} \circ J(q).$$

*Proof.* — This is trivial for $p \in S_2^*$. If $p \in S_2 \backslash S_2^*$, let again $U_p$ be a small circular neighbourhood of $p$ such that the components $U^1, \ldots, U^s$ are circle sectors. Here we use the following property of the pasting scheme, (indices mod 8):

(iii) *If* $B_m$ *and* $B_n$ *meet along a side of type* $b$ *in* $S_2$, *then* $B_{m+1}$ *and* $B_{n-1}$ *as well as* $B_{m+3}$ *and* $B_{n-3}$ *meet along the same side in* $S_1$.

(iv) *If* $B_m$ *and* $B_n$ *meet along a side of type* $a, c$ *or* $d$ *in* $S_2$, *then* $B_{m+k}$ *and* $B_{n+k}$ *meet along the same side in* $S_1$, $k = 1, 3, 5, 7$.

Again we can label $U^1, \ldots, U^s$ such that for suitable $\tau$, $1 \leqslant \tau \leqslant s$, the images $\varphi \circ J(U^1), \ldots, \varphi \circ J(U^\tau)$ and $\varphi^{-1} \circ J(U^{\tau+1}), \ldots, \varphi^{-1} \circ J(U^s)$ become the components of $U_{p_1} \cap S_1^*$ for a circular neighbourhood $U_{p_1}$ in $S_1$. This follows from (iii), (iv). At the same time the images $\varphi^3 \circ J(U^1), \ldots, \varphi^3 \circ J(U^\tau)$ and $\varphi^{-3} \circ J(U^{\tau+1}), \ldots, \varphi^{-3} \circ J(U^s)$ then become the components of $U_{p_3} \cap S_1^*$ for a circular neighbourhood $U_{p_3}$ in $S_1$. This proves 4.3.2.

Now let $p$ be an arbitrary point on $S_2$ and let $U_p$ be a neighbourhood of $p$ such that 4.3.1 and 4.3.2 hold. If $f \in E_1^+$ then 4.3.1 implies

$$f \circ j = f \circ J = f^t \quad \text{on} \quad U_p \cap S_2^*.$$

In this case, $f \circ j$ is the desired $C^\infty$-extension of $f$ on $U_p$. If $f \in E_1^-$, then 4.3.2 implies

$$f \circ j_1 - f \circ j_3 = f \circ \varphi \circ J - f \circ \varphi^3 \circ J = (\hat{\varphi} f - \hat{\varphi}^3 f)^t \quad \text{on} \quad U_p \cap S_2^*$$

(because $f \circ \varphi^{-1} - f \circ \varphi^{-3} = -f \circ \varphi^3 + f \circ \varphi$). In this case $f \circ j_1 - f \circ j_3$ is the desired $C^\infty$-extension. This proves Lemma 4.3.

4.4. LEMMA. — *The linear mapping* $L : E_1 \to E_2$ *defined by* $L(f) = f^t$ *for* $f \in E_1^+$ *and* $L(f) = \frac{1}{\sqrt{2}}(\hat{\varphi}f - \hat{\varphi}^3 f)^t$ *for* $f \in E_1^-$ *is an isomorphism.*

*Proof.* — Observe first by Lemma 4.3 that indeed L maps $E_1$ to $E_2$.

Let $L' : E_2 \to E_1$ be the linear mapping defined by

$$L'(g) = g^t \quad \text{if} \quad g \in E_2^+, \quad L'(g) = \frac{1}{\sqrt{2}}(\hat{\varphi}g - \hat{\varphi}^3 g)^t \quad \text{if} \quad g \in E_2^-,$$

(Lemma 4.2, 4.3). For $f \in E_1^+$ we have $Lf \in E_2^+$, and $L' \circ Lf = f$. For $f \in E_1^-$ we have $Lf \in E_2^-$, and

$$L' \circ Lf = \frac{1}{\sqrt{2}}(\hat{\varphi}Lf - \hat{\varphi}^3 Lf)^t = 1/2(\hat{\varphi}^2 f - \hat{\varphi}^4 f - \hat{\varphi}^4 f + \hat{\varphi}^6 f) = f.$$

Hence $L' \circ L = \mathrm{id}$ and similarly one proves $L \circ L' = \mathrm{id}$, q.e.d.

With Lemma 4.4 we have $\dim E_1 = \dim E_2$ for the eigenspaces with eigenvalue $\lambda$, for any $\lambda \geqslant 0$. This accomplishes the proof of Proposition 4.1.

## 5. The length spectrum.

For a compact Riemannian manifold we define the *length spectrum* to be the function $\ell \mapsto n_M(\ell)$, which to each positive real number $\ell$ associates the cardinality $n_M(\ell)$ of the set of all closed geodesics of length $\ell$ on M.

If M is a compact Riemann surface of genus $g \geqslant 2$, in each free homotopy class of a homotopically non trivial closed curve there is a unique closed geodesic (this is due to the negative curvature). From this it follows that the number of closed geodesics of length $\leqslant \ell$ is finite for all $\ell$ so that the length spectrum here may also be defined by giving the list of all possible lengths, arranged in increasing order. The following is a classical result :

([6], [11]). *Two compact Riemann surfaces of genus* $g \geqslant 2$ *are isospectral with respect to the Laplacian if and only if they have the same length spectrum.* On a general Riemannian manifold the relation between the two spectra is less strict, but still the length spectrum carries a lot of

information. In [12] Sunada showed that his examples are isospectral with respect to the length spectrum as well. We shall prove the same result here. Instead of covering arguments as in [12] we use a transplantation technique which is similar to the one in the preceding section, hoping that it will give a closer insight to the phenomenon of isospectrality.

5.1. PROPOSITION. — *The surfaces* $S_1, S_2$ *obtained in section* 2 *have the same length spectrum.*

*Proof.* — For each closed geodesic on $S_1$, and more generally for any closed curve, we shall draw a copy on $S_2$ which has the same length and vice versa.

5.2. DEFINITION. — *Two curves* $t \mapsto \gamma(t) \in S_1$ *and* $t \mapsto \gamma'(t) \in S_2$, $t \in [0, 1]$, *are locally congruent, if for a suitable subdivision* $0 = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_n = 1$ *we have local isometries* $\varphi_i$ *satsifying* $\gamma'(t) = \varphi_i(\gamma(t))$, $t \in [t_{i-1}, t_i]$; $i = 1, \ldots, n-1$.

Here, each $\varphi_i$ is an isometry of a neighbourhood of $\gamma([t_{i-1}, t_i]$ onto a neighbourhood of $\gamma'([t_{i-1}, t_i])$. In the definition it is allowed that one of the curves is closed and the other is not.

For the proof of Proposition 5.1 we let

$$j_{kk'} : B_k \to B_{k'}$$

be the natural identification mapping of building block $B_k$ in $S_1$ onto building block $B_{k'}$ in $S_2$: $k, k' = 1, \ldots, 8$. Strictly speaking $j_{kk'}$ is only defined in the interior of $B_k$ since the boundary of $B_k$ in $S_1$ is not pasted the same way as the boundary of $B_{k'}$ in $S_2$. For each closed curve $t \to \gamma(t) \in S_1$; $t \in [0, 1]$, we draw a locally congruent copy $\gamma'$ in $S_2$ in the most obvious way:

Suppose for simplicity that an initial part $\gamma|[0,\delta]$ and a final part $\gamma|[1-\delta,1]$ are contained in the same block, for sufficiently small $\delta$.

Select some initial block $B_{k'}$ in $S_2$ —the correct choice of an initial block will be described in 5.3 below — and define $\gamma'(t) = j_{kk'}(\gamma(t))$ for $t \in [0,t_1]$ where $t_1 > 0$ is such that $\gamma([0,t_1]) \subset B_k$ and $\gamma$ crosses a side of $B_k$ transversally for $t = t_1$. We then let $t_2 > t_1$ be such that $\gamma([t_1, t_2])$ is contained in the next block $B_\ell$ and such that $\gamma$ crosses a side of $B_\ell$ transversally for $t = t_2$. (Here we may well have $B_\ell = B_k$). We let $B_{\ell'}$ be the block in $S_2$ such that $B_k, B_\ell$ in $S_1$ and $B_{k'}, B_{\ell'}$ in $S_2$ meet along the

same side. The definition of $\gamma'$ now continues as $\gamma'(t) = j_{\ell\ell'}(\gamma(t))$, $t \in [t_1, t_2]$. Then we select $t_3 > t_2$ such that $\gamma([t_2, t_3])$ is contained in the next block, crossing its boundary for $t = t_3$, and so on. Homotoping the curve slightly if necessary, we may assume that $\gamma$ has only a finite number of such crossings. (If, e.g. $\gamma$ is a geodesic, this homotopy is not necessary). The copy process then ends after finitely many steps.

As a very simple example of this process we consider the closed geodesic $\gamma = \gamma_2$ on the boundary of $B_2$ in $S_1$ (see section 3), parametrized such that it starts and ends in the middle of the side. We have $B_k = B_2$, $B_\ell = B_2$ and $\gamma$ crosses a side of type $a$ transversally. The copy process ends after two steps. If we take as initial block $B_{k'} = B_2$ in $S_2$, then the terminal block is $B_{\ell'} = B_6$, and $\gamma'$ is not closed. If we take $B_{k'} = B_3$ then $B_{\ell'} = B_3$ and $\gamma'$ is closed.

If $\gamma$ is closed in $S_1$ then the terminal block and the initial block coincide, and $\gamma'$ on $S_2$ is closed if and only if the initial and terminal block coincide also. We shall now see under what conditions this is the case.

To this end we let $\# a$ denote the number of times the curve $\gamma$ crosses a side $a$ of type $a$ transversally. Correspondingly, $\# b$ is the number of crossings of type $b$. We claim that the following receipe yields closed image curves :

5.3. INITIATION. —

(i) *If* $\# a$ *is even, take* $k' = k$.

(ii) *If* $\# a$ *is odd and* $\# b$ *is even, take* $k' = k + 1$.

(iii) *If* $\# a$ *and* $\# b$ *are odd, take* $k' = k + 2$.

*Proof.* — Let $\kappa(t) \in \{1, \ldots, 8\}$ denote the momentary block number of $\gamma(t)$ (i.e. if $\gamma([t_{i-1}, t_i])$ lies on $B_m$ then $\kappa(t) = m$ for $t \in [t_{i-1}, t_i)]$ and let $\kappa'(t)$ be the block number of $\gamma'(t)$ in $S_2$, $0 \leqslant t \leqslant 1$. We define

$$\delta(t) = \kappa'(t) - \kappa(t) \quad \text{(integers mod 8)}.$$

We have to prove that with the initiation 5.3 we have $\delta(1) = \delta(0)$.

Case (i) : We have $\delta(0) = 0$. As long as $\delta$ is even, a crossing of type $a$ replaces $\delta$ by $\delta + 4$ (always mod 8). As long as $\delta = 0$ or $\delta = 4$, any crossing of type $b$ leaves $\delta$ invariant. Hence we *always* have $\delta = 0$ or $\delta = 4$. Since $\# a$ is even this proves $\delta(1) = 0 = \delta(0)$.

Case (ii) : We have $\delta(0) = 1$. As long as $\delta$ is odd, crossings of type $a$ leave $\delta$ invariant. As long as $\delta = \pm 1$, any crossing of type $b$ changes $\delta$ from $\pm 1$ to $\mp 1$. Hence, we *always* have $\delta = \pm 1$. Since $\# b$ is even, this proves $\delta(1) = 1 = \delta(0)$.

Case (iii) : We have $\delta(0) = 2$. As long as $\delta$ is even, each crossing of type $a$ replaces $\delta$ by $\delta + 4$. As long as $\delta = \pm 2$ each crossing of type $b$ replaces $\delta$ by $\delta + 4$. Hence we *always* have $\delta = \pm 2$. Since $\# a + \# b$ is even, this proves $\delta(1) = 2 = \delta(0)$, q.e.d.

In the same way we can, for each closed curve $\eta$ on $S_2$ draw a closed copy $\eta'$ on $S_1$ taking in 5.3 (i), (ii), (iii) respectively $k' = k$, $k' = k - 1$, $k' = k - 2$. We then have

$$(\gamma')' = \gamma, \qquad (\eta')' = \eta.$$

Consequently, the above copy process establishes a one-to-one correspondence of locally congruent closed curves on $S_1$ and $S_2$. By restricting this correspondence to closed geodesics, we have a one-to-one correspondence of closed geodesics of any given length. Hence $S_1$ and $S_2$ have the same length spectrum and Proposition 5.1 is proved.

5.4. *Remark.* — On non orientable surfaces, a so called weighted length spectrum is more frequently used, where the length of a closed geodesic is multiplied by a positive factor which depends on the eigenvalue $(= \pm 1)$ of the holonomy of $\gamma$. It is not difficult to see that the above curves $\gamma$ and $\gamma'$ always have the same holonomy. Hence Proposition 5.1 holds also for the weighted spectrum. (This is of course only interesting if non orientable building blocks are used).

## 6. Examples of higher genus.

In this paragraph we construct isospectral Riemann surfaces for any genus $g \geqslant 7$ ($g = 6$ has resisted all efforts so far). The pasting scheme will be the one of section 2. We shall use various types of polygonal domains in the hyperbolic plane. To keep the description reasonably short, their definition will be given via a figure. These figures represent domains in the hyperbolic plane; however we made no effort to make them look as they do in the Poincaré model.

A) *Odd genus.* For $g = 5 + 2n$, $n = 1, 2, \ldots$, the domain B is a generalization of the one used in section 3. Its sides are shown in figure 4.

B is composed of rectangular pentagons of only two types: Four pentagons at the outer ends with sides $p, q, q', p', r$, meeting along $p$, and $2n$ pentagons in between with sides $u, v, w, w', v'$. The matching conditions are

$$c' = c'' = 2r, \qquad a' = a'' = a^{**} = a^{*} = p', \qquad v = 2q.$$

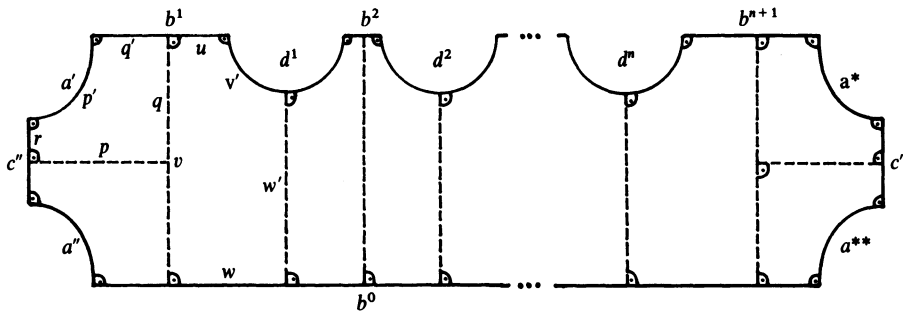


Fig. 4.

As mentioned in section 3, pentagons exist for any lengths of two given non adjacent sides. It is also easy to see that the lengths of two adjacent sides, like $u, v$ above, may be prescribed as long as $\sinh u . \sinh v > 1$. For simplicity we shall use the values

$$u = v = 1.$$

This determines $q$ to be $q = 1/2$, and $r$ is still a free parameter. From the trigonometric formulae 3.1, 3.2 we see that $r \to 0$ implies $p \to \infty$, $p' \to \infty$ and $q' \to q'_0$ for some $q'_0 > 0$ (the correct value is given by $\sinh q'_0 . \sinh 1/2 = 1$). Since all other pentagons remain invariant as $r \to 0$, we have the following analogue of Lemma 3.3:

6.1. LEMMA. — *If $r > 0$ is sufficiently small, then any curve $\delta$ which connects two non adjacent sides of* B (fig. 4) *has length $\ell(\delta) \geqslant c' = c''$. Equality holds only if $\delta = c'$ or $\delta = c''$.*

The pasting conditions for the sides of type $b$ and $d$ are

$$(6.2.1) \qquad \begin{cases} \pi(b_1^{\nu}) = b_4^{\sigma(\nu)}, & \pi(b_2^{\nu}) = b_3^{\sigma(\nu)}, \\ \pi(b_5^{\nu}) = b_8^{\sigma(\nu)}, & \pi(b_6^{\nu}) = b_7^{\sigma(\nu)}; \end{cases} \qquad \nu = 1, \ldots, n+2,$$

with the permutation

$$(6.2.2)\quad \sigma(\nu) = n + 2 - \nu; \quad \nu = 1, \ldots, n+1, \quad \sigma(n+2) = n+2,$$

and

$$(6.2.3)\quad \pi(d_i^\nu) = d_{i+4}^\nu; \quad i = 1, 2, 3, 4; \quad \nu = 1, \ldots, n.$$

Like in section 3, we check that pasted sides have the same length and that the number of meeting vertices is always 4 so that indeed $S_1$, $S_2$ are Riemann surfaces.

By Lemma 6.1, the shortest closed geodesics on $S_1$ are those composed of $c_2'$, $c_4'$, $c_6'$, $c_8'$ and on $S_2$ those composed of $c_1'$, $c_3'$, $c_5'$, $c_7'$ so that $S_1$ and $S_2$ are non isometric by the same reasoning like in section 3. We have 8 faces, $8n + 32$ edges and $4n + 16$ vertices. This yields Euler characteristic $-8 - 4n$ and genus $g = 5 + 12n$. Note that $r$ is a variable so that indeed $\dim V_g > 0$ for these values of $g$.

B) *Even genus.* For even genus the construction is more difficult. Rectangular domains are no longer suitable. For the more general domains described here, we use so called *trirectangles*, i.e. geodesic quadrangles in the hyperbolic plane with 3 right angles and one acute angle $\varphi$.

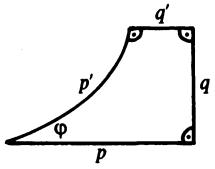


Fig. 5.

The trigonometric formulae of a trirectangle are similar to those of a pentagon. We have (e.g. [10]).

6.3. *Trirectangle.*

(i) $\cosh q = \cosh p' . \sin \varphi$

(ii) $\cos \varphi = \tanh p . \tanh p'$

(iii) $\cot \varphi = \sinh p . \tanh q$

(iv) $\sinh p' = \cosh p . \sinh q$

(v) $\cos \varphi = \sinh q . \sinh q'$.

It is well known that such domains exist for any values of $q > 0$ and

$\varphi \in (0,\pi/2)$. As $q$ and $\varphi$ range in these domains, trirectangles are obtained for all values of $q$ and $q'$ satisfying $\sinh q . \sinh q' < 1$ ((6.3(v)). We also obtain trirectangles for all values of $p'$ and $q$ satisfying $\sinh p'/\sinh q > 1$ (6.3(iv)) etc.

In this part B) we construct surfaces for $g = 12 + 2n$, $n = 0, 1, 2, \ldots$.

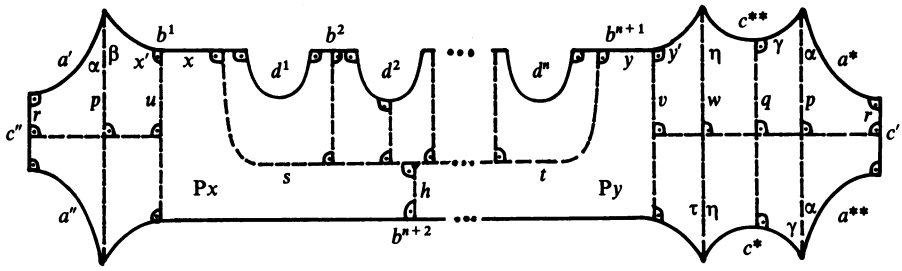


Fig. 6.

The building block is given in figure 6. We have 6 pairs of trirectangles: two equal pairs at the outer ends with acute angle $\alpha$, and four different pairs with acute angles $\beta$, $\tau$, $\eta$, $\gamma$. Here we consider $\alpha$ and $\tau$ as variable parameters. The remaining angles are determined by the condition

$$(6.4) \quad \alpha + \beta = \pi/16, \qquad \tau + \eta = \pi/16, \qquad \alpha + \gamma = \pi/8.$$

The middle section of the building block is composed of $2n$ pentagons, if $n \geqslant 1$. In the case $n = 0$ these pentagons are absent and we have the matching condition

$$(6.5.1) \qquad u = v \quad (\text{if } n=0).$$

In the case $n = 1$ we have two pentagons $P_x$, $P_y$. Side $s$ of $P_x$ together with side $t$ of $P_y$ yields the only side $d^1$ of type $d$ in this case. In the case $n \geqslant 2$ we have $2n - 2$ additional pentagons arranged as shown in figure 6. We have $n + 2$ sides of type $b$ which are subject to the condition

$$(6.5.2) \quad b^1 = b^{n+1}, \quad b^2 = b^n, \quad b^3 = b^{n-1}, \ldots, \quad (\text{if } n \geqslant 1).$$

The existence of such domains is less obvious and will be shown below. For $b^1, \ldots, \vert b^{n+2}$ and for $d^1, \ldots, \vert d^n$ the pasting scheme is again 6.2 like in part A). For $c^*, c^{**}$ the pasting scheme is $\pi(c_i^*) = c_{i+1}^{**}$, $i = 1, \ldots, 8$ (mod 8) (c.f. 2.5).

By drawing a figure like figure 1 we see that the 32 vertices of $B_1, \ldots, B_8$ which have angle $\pi/16$ (c.f. 6.4) meet at a single point of $S_1$ resp. $S_2$. Similarly the 16 vertices of $B_1, \ldots, B_8$ which have angle $\pi/8$ (c.f. 6.4) meet at a single point of $S_1$ resp. $S_2$. The remaining vertices have meeting number 4 and angle $\pi/2$. Hence the angle condition 2.2 is satisfied, and by 6.5.2 and 6.2, pasted sides have the same length so that $S_1$, $S_2$ are Riemann surfaces. We have 8 faces, $4n + 10$ vertices and $8n + 40$ edges. This yields characteristic $-4n - 22$ and genus $g = 2n + 12$.

To make sure that $S_1$ and $S_2$ are not isometric we proceed like in part A) of this section by taking $c' = c'' = 2r$ sufficiently small so that Lemma 6.1 holds in the present case too.

Thus it now remains to prove that a domain with all the required properties exists. By 6.3(i) we have the following formulae for the trirectangles :

$$\cosh p = \cosh r/\sin\alpha, \qquad \cosh q = \cosh p.\sin\gamma$$
$$\cosh w = \cosh q/\sin\eta, \qquad \cosh v = \cosh w.\sin\tau$$
$$\cosh u = \cosh p.\sin\beta$$

from which we deduce (c.f. 6.4)

$$\text{(6.6)} \qquad \begin{aligned} \cosh u &= \cosh r \cdot \frac{\sin(\pi/16 - \alpha)}{\sin\alpha} \\ \cosh v &= \cosh r \cdot \frac{\sin(\pi/8 - \alpha).\sin\tau}{\sin\alpha.\sin(\pi/16 - \tau)}. \end{aligned}$$

We start with the case $n \geqslant 1$. Here we choose the fixed values

$$\alpha = \pi/64, \qquad \tau = \pi/40 \quad (\text{if } n \geqslant 1)$$

and let $r = c'/2 = c''/2$ be arbitrarily small. As $r \to 0$, $p$ converges to a limiting value $p_0$ given by $\cosh p_0 = 1/\sin\alpha$. With the above choice of $\alpha$ and $\tau$, all trirectangles are well defined for small $r$, and, except for those with side $r$, converge to a well defined *non degenerate* limiting trirectangle as $r \to 0$. The above formulae together with 6.3(ii) yield the limiting values $u_0 = 1.75\ldots, x_0' = 2.66\ldots$; $v_0 = 2.20\ldots, y_0' = 3.26\ldots$. For the pentagons $P_x$, $P_y$ we have the matching condition

(*) $\sinh x.\sinh(2u) = \sinh y.\sinh(2v) = \cosh h > 1$

(by 3.2); and in order to satisfy $b^1 = b^{n+1}$ (6.5.2) we impose the additional (for $n \geqslant 2$ somewhat stronger than necessary) condition

$$(**) \qquad x + x' = y + y'.$$

Since $\sinh(y+y'-x') = \sinh y.\cosh(y'-x') + \cosh y.\sinh(y'-x')$ a solution $y$ of

$$(***) \quad \coth y = \frac{\sinh 2v}{\sinh 2u.\sinh(y'-x')} - \coth(y'-x')$$

together with $x = y + y' - x'$ will satisfy conditions (*) and (**). Plugging in the above limiting values $u_0$, $v_0$, $y'_0$, $x'_0$ shows that (***) indeed has a solution with the limiting value $y_0 = |3.26\ldots,|$ provided $r$ is sufficiently small.

For $n \geqslant 2$ we add the remaining $2n-2$ rectangular pentagons such that 6.5.2 is satisfied. We may e.g. take all these pentagons such that one side has length $(s+t)/(2n-2)$ and such that the two opposite sides are equal to one another. Clearly, as $r \to 0$ these pentagons converge to non degenerate limiting pentagons so that Lemma 6.1 holds for sufficiently small $r$. This concludes the existence proof in the case $n \geqslant 1$.

In the case $n = 0$, condition 6.5.2 is replaced by 6.5.1. By 6.6 the condition reads

$$\frac{\sin(\pi/16 - \alpha)}{\sin(\pi/8 - \alpha)} = \frac{\sin\tau}{\sin(\pi/16 - \tau)}.$$

This is easy to satisfy. We may e.g. choose $\alpha = \pi/40$ and solve for $\tau$. The rest is as before.

Since $r$ is variable, Theorem 1.2 is now proved for all $g$ except $g = 8$ and 10.

C) *Genus* 8 *and* 10. The arguments are slightly different here. Since these examples are for completeness, we only give a brief outline and leave the details to the untiring reader. The building block is sketched in figure 7.

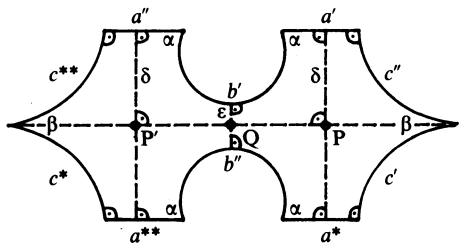


Fig. 7.

For the example with genus 8 the angles $\alpha$ and $\beta$ are

$$\alpha = \pi/2, \qquad \beta = \pi/4 \quad (\text{for } g=8).$$

For the example with genus 10 we shall take

$$\alpha = \pi/4, \qquad \beta = \pi/4 \quad (\text{for } g=10).$$

All other angles are right angles. The pasting scheme is from section 2, the pasting of sides of type $b$ being as follows: If $\pi(b_i) = b_k$ then

$$\pi(b_i') = b_k'', \qquad \pi(b_i'') = b_k', \quad (\text{for } g=8)$$

respectively

$$\pi(b_i') = b_k'', \qquad \pi(b_i'') = b_k'', \qquad (\text{for } g=10).$$

In either example, $\delta$ denotes the length of the common perpendicular of $a'$, $a^*$ resp. $a''$, $a^{**}$, and $\varepsilon$ is the length of the common perpendicular of $b'$, $b''$. We can choose $\varepsilon < \delta$ with $\varepsilon$ and $\delta$ arbitrarily small. The perpendiculars then extend to closed geodesics of length $4\delta$ resp. $2\varepsilon$. All other closed geodesics (on all surfaces) are longer. It follows that these geodesics can be detected by rules which only make use of the intrinsic geometry of the surfaces. The globally shortest connecting curves between a geodesic of length $4\delta$ and a geodesic of length $2\varepsilon$ are precisely the segments marked PQ resp. P'Q (fig. 7). Hence the midpoints Q of the building blocks can also be detected intrinsically. It follows that any isometry from $S_1$ to $S_2$ sends building blocks to building blocks. The difference in the pasting schemes makes this impossible, and thus $S_1$, $S_2$ are not isometric.

Theorem 1.2 is now proved.

6.7. Remark. — A detailed analysis of the number of free parameters in the above examples gives roughly the lower bound $\dim V_g \geqslant \frac{1}{8} \dim T_g$ for large $g$.

## 7. Isospectral surfaces in $\mathbf{R}^3$.

In this section we use the pasting technique to give examples of isospectral surfaces which are embedded in ordinary 3-space. We shall also give examples of flat bordered domains in $\mathbf{R}^3$ with smooth boundary.

These examples improve results by Urakawa [16] who found 3-dimensional domains with piecewise smooth boundary in $\mathbf{R}^4$ and also 4-dimensional domains in $\mathbf{R}^4$ in the ordinary sense. The bordered domains are very easily realized by paper models.

$\mathbf{R}^3$ denotes the standard Euclidean 3-space. A surface in $\mathbf{R}^3$ will always be an embedded surface with the induced Riemannian metric. The geometry of the building block is to a great extent arbitrary so that we restrict ourselves to describe the procedure which leads to such surfaces rather than to go into the details of a singular example.

A) *Closed surfaces.* The building block is a bordered surface in $\mathbf{R}^3$ with cylindrical ends. By this we mean that each boundary component $\gamma$ is a closed geodesic and that a suitable neighbourhood $U_\gamma$ of $\gamma$ is isometric to the circular cylinder

$$Z = \{(x_1, x_2, x_3) \in \mathbf{R}^3 | x_1^2 + x_2^2 = \rho^2, 0 \leqslant x_3 \leqslant \varepsilon\},$$

where $\varepsilon$ and $\rho$ are positive constants. Each boundary geodesic is considered as a side of a given type (*a*,*b* or *c*). If all ends have the same radius $\rho$, the pasting is locally the same as in the preceding sections and yields a smooth metric. The surfaces are isospectral. If the building block has no intrinsic isometry other than the identity, and if the boundary curves are so small that on $S_1$ resp $S_2$ they become the globally smallest closed geodesics, then any isometry $S_1 \to S_2$ would induce natural identifications of building blocks. However, this is made impossible by the difference in the two pasting schemes. Hence pasting in space yields isospectral non isometric surfaces in the same way as before.

However, there is the additionnal problem that the pasted surfaces be *embedded*. That is, all boundary components which are to be identified with each other actually need to have contact. As long as we try to arrange *congruent* blocks, this is not possible. We shall therefore use building blocks which are intrinsically isometric but not congruent, i.e. bordered surfaces which admit different isometric embeddings in $\mathbf{R}^3$.

The standard construction of isometric non congruent surfaces is by pasting together smaller congruent pieces. The method is based on the following principle: Let F, G be bordered surfaces in $\mathbf{R}^3$ which are pasted together along a common closed boundary curve $\gamma$. Assume that $\gamma$ is contained in a plane $\alpha$ and that neighbourhoods $U_\gamma \subset F$, $V_\gamma \subset G$ are also contained in $\alpha$ so that $U_\gamma$, $V_\gamma$ together form a plane annular region $(U_\gamma \cap V_\gamma = \gamma)$. Now consider the mirror image G′ of G

with respect to plane $\alpha$. If $G' \cap F = \gamma$ then $F \cup G$ and $F \cup G'$ are isometric to each other but, in general, not congruent.

This principle will now be exploited to construct building blocks. The idea is to build long and thin « arms » which one can bend in different directions without affecting the intrinsic geometry. Here « bending » means that finitely many different positions are possible. It is not known whether continuous isometric deformations exist also. One such pair of arms or rather « hinges » in different positions is drawn schematically in figure 8.

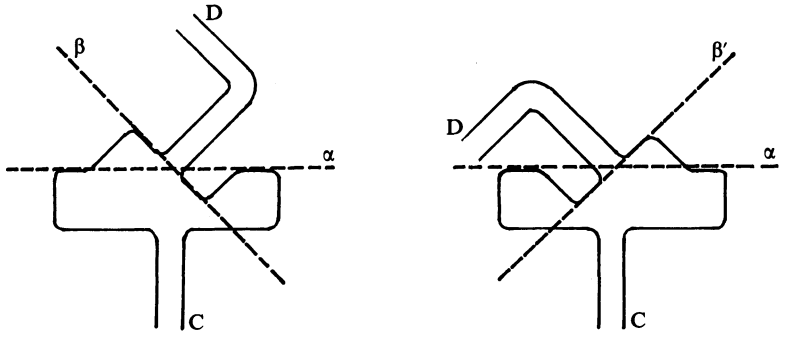


Fig. 8.

C and D denote cylindrical ends of the same radius. The hinges consist of three parts $F_1$, $F_2$, $F_3$ which are the same for either hinge. These parts satisfy the above matching conditions. One end of $F_1$ is C, the other end is contained in plane $\alpha$. Part $F_2$ has one end in $\alpha$ the other end in $\beta$. Part $F_3$ has one end in $\beta$, the other end is D. The angle $\varphi$ between the planes $\alpha$ and $\beta$ may be any angle in the interval $(0,\pi/2)$. (In fig. 8 we have $\varphi=\pi/4$). To go from the first hinge to the second, we first replace $F_3$ by its mirror image $F'_3$ with respect to plane $\beta$ (in this intermediate step the surface may have self intersection). Then we replace $F_2 \cup F'_3$ by its mirror image with respect to plane $\alpha$. End C stays where it is but D is now rotated by the angle $\omega = 2\varphi$ about the axis $\alpha \cap \beta$. Calling these surfaces *hinges with angle* $\omega$, we have:

7.1. LEMMA. — *A pair of hinges with angle* $\omega$ *exists for all* $\omega \in [0,\pi)$.

In order to get a relatively simple example we may take the standard sphere, cut out 24 circular holes with equidistant centers on the equator and deform the metric near the boundary such that all ends become

cylindrical. We can achieve that an isometry of order 8 with north and south pole as fixed points operates. At each hole we attach an arm which is obtained by pasting together a number of suitable hinges. We use two types of arms which we call type $a$ and $b$. Arms of the same type are isometric, arms of different types are not. The building block is then a 3 holed sphere sector (with vertices = north and south pole) on the above 24 holed sphere with two arms of type $a$ and one arm of type $b$. We attach the arms such that the building block has no intrinsic symmetry. If one takes a *large* number of hinges, it becomes relatively simple to determine angles $\omega$ and lengths of ends C, D such that, in fact, the pasting of section 2 can be carried out in $\mathbf{R}^3$ without any self intersections. This yields

7.2. THEOREM. — *There exist isospectral non isometric closed surfaces of genus* 12 *which are isometrically embedded in* $\mathbf{R}^3$.

B) *Paper models.* In order to obtain paper models of isospectral surfaces one may replace the class of smooth surfaces by the class of piecewise linear surfaces on which isospectral problems still make sense. However the construction is very tedious.

A much simpler way is to extend the pasting procedure to bordered surfaces. In fact, if the building block is such that $S_1$ and $S_2$ have non empty boundary, the proof of Proposition 4.1 is still valid for the Laplacian for both, Neumann or Dirichlet boundary conditions.

The advantage of bordered surfaces is that they admit continuous isometric deformations in space so that the construction of hinges is not necessary. One such example is shown via its building block in figure 9 : The block is a plane Euclidean circle sector with angle $\pi/4$ and with three rectangular sufficiently long strips attached. The corners have been smoothed such that $S_1$, $S_2$ have smooth boundary. Paper is stiff and assumes the shape

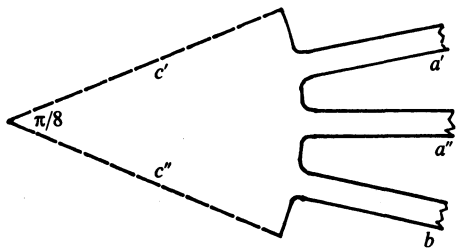


Fig. 9.

of developable surfaces. By practical experience it is easy to stick the building blocks together in space. We thus obtain :

7.3. THEOREM. — *There exist isometrically embedded flat two dimensional domains with smooth boundary in* $\mathbf{R}^3$ *which are not isometric and have the same spectrum of the Laplacian for Neumann as well as Dirichlet boundary conditions.*

It does not seem to be possible to obtain isospectral domains in $\mathbf{R}^2$ along these lines.

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