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SIDON SETS AND RIESZ PRODUCTS

by Jean BOURGAIN

1. Notations.

In what follows, G will be a compact abelian group and $\Gamma = \hat{G}$ the dual group. According to the context, we will use the additive or multiplicative notation for the group operation in Γ . For $1 \leq p \leq \infty$, $L^p(G)$ denotes the usual Lebesgue space. For $\mu \in M(G)$, let $\|\mu\|_{PM} = \sup_{\gamma \in \Gamma} |\hat{\mu}(\gamma)|$.

A subset Λ of Γ is called a Sidon set provided there is a constant C such that the inequality

$$\sum_{\gamma \in \Lambda} |\alpha_\gamma| \leq C \left\| \sum_{\gamma \in \Lambda} \alpha_\gamma \gamma \right\|_\infty \quad (1)$$

holds for all finite scalar sequences $(\alpha_\gamma)_{\gamma \in \Lambda}$. The smallest constant $S(\Lambda)$ fulfilling (1) is called the Sidon constant of Λ . The reader is referred to [3] for elementary Sidon set theory.

$|A|$ stands for the cardinal of the set A .

Assume A a subset of Γ and $d \geq 0$. We will consider the set of characters

$$P_d[A] = \left\{ \sum_{\gamma \in A} z_\gamma \gamma \mid z_\gamma \in \mathbb{Z} (\gamma \in A) \text{ and } \sum_{\gamma \in A} |z_\gamma| \leq d \right\}.$$

Then $|P_d[A]| \leq \left(\frac{C|A|}{d} \right)^d$ if $d \leq |A|$

and $|P_d[A]| \leq \left(\frac{C d}{|A|} \right)^{|A|}$ if $d > |A|$

where C is a numerical constant (cf. [7], p. 46).

Mots-clefs : Ensemble de Sidon, Ensemble quasi-indépendant, Produits de Riesz.

We say that $A \subset \Gamma$ is quasi-independent, if the relation $\sum_A z_\gamma \gamma = 0$, $z_\gamma = -1, 0, 1$ ($\gamma \in A$) implies $z_\gamma = 0$ ($\gamma \in A$).

If A is quasi-independent, the measure

$$\mu = \prod_{\gamma \in A} (1 + \operatorname{Re} a_\gamma \gamma)$$

where $a_\gamma \in \mathbf{C}$, $|a_\gamma| \leq 1$, is positive and $\|\mu\|_{M(G)} = 1$.

We call it a Riesz product.

Say that $A \subset \Gamma$ tends to infinity provided to each finite subset Γ_0 of Γ corresponds a finite subset A_0 of A such that

$$\gamma, \delta \in A \setminus A_0, \gamma \neq \delta \implies \gamma - \delta \notin \Gamma_0.$$

A Sidon set Λ is of first type provided there is a constant $C < \infty$ and, for each nonempty open subset I of G , there is a finite subset Λ_0 of Λ so that

$$\sum_{\gamma \in \Lambda \setminus \Lambda_0} |\alpha_\gamma| \leq C \left\| \sum_{\gamma \in \Lambda \setminus \Lambda_0} \alpha_\gamma \gamma \right\|_{C(I)} \quad (2)$$

for finite scalar sequences $(\alpha_\gamma)_{\gamma \in \Lambda \setminus \Lambda_0}$, where

$$\|f\|_{C(I)} = \sup_{x \in I} |f(x)|.$$

2. Interpolation by averaging Riesz products.

In this section, we will prove the following result:

THEOREM. — *For a subset Λ of Γ , the following conditions are equivalent:*

- (1) Λ is a Sidon set
- (2) $\|\sum_\Lambda \alpha_\gamma \gamma\|_p \leq C p^{1/2} (\sum |\alpha_\gamma|^2)^{1/2}$ for all finite scalar sequences $(\alpha_\gamma)_{\gamma \in \Lambda}$ and $p \geq 1$.
- (3) There is $\delta > 0$ such that each finite subset A of Λ contains a quasi-independent subset B with $|B| \geq \delta |A|$.
- (4) There is $\delta > 0$ such that if $(\alpha_\gamma)_{\gamma \in \Lambda}$ is a finite sequence of scalars, there exists a quasi-independent subset A of Λ such that

$$\sum_{\gamma \in \Lambda} |\alpha_\gamma| \geq \delta \sum_{\gamma \in \Lambda} |\alpha_\gamma|.$$

Implication (1) \implies (2) is a consequence of Khintchine's inequalities and is due to W. Rudin [8]. The standard argument that quasi-independent sets are Sidon sets yields (4) \implies (1). We will not give it here since it will appear in the next section in the context of an application. Finally, the results (2) \implies (1) and (1) \implies (3) are due to G. Pisier (see [4], [5] and [6]. The characterization (4) is new. It has the following consequence (by a duality argument):

COROLLARY 1. — *If Λ is a Sidon set, there is $\delta > 0$ such that whenever $(a_\gamma)_{\gamma \in \Lambda}$ is a finite scalar sequence and $|a_\gamma| \leq \delta$, then we have*

$$\hat{\mu}(\gamma) = \int_G \bar{\gamma}(x) \mu(dx) = a_\gamma \quad \text{for } \gamma \in \Lambda$$

where μ is in the σ -convex hull of a sequence of Riesz products.

Recall that the σ -convex hull of a bounded subset P of a complex Banach space X is the set of all elements $\sum_{i=1}^{\infty} \lambda_i x_i$ where $x_i \in P$, $\sum_{i=1}^{\infty} |\lambda_i| \leq 1$.

The remainder of the paragraph is devoted to the proof of (2) \implies (3) \implies (4).

Let us point out that in the case of bounded groups, i.e. which elements are of bounded order, they can be simplified using algebraic arguments.

LEMMA 1. — *Condition (2) implies (3) with $\delta \sim C^{-2}$.*

Proof. — We first exhibit a subset A_1 of A , $|A_1| \geq C^{-2} |A|$, such that if $\sum_{\gamma \in A_1} \epsilon_\gamma \gamma = 0$ and $\epsilon_\gamma = -1, 0, 1$, then $\sum |\epsilon_\gamma| < \frac{1}{2} |A_1|$. If $\sum_{\gamma \in A_2} \epsilon_\gamma \gamma = 0$, $\epsilon_\gamma = \pm 1$ and $A_2 \subset A_1$ is chosen with $|A_2|$ maximum, the set $B = A_1 \setminus A_2$ will be quasi-independent and $|B| > \delta |A|$.

The set A_2 is obtained using a probabilistic argument. Fix $\tau = C_1^{-1} C^{-2}$ and $\ell = \frac{1}{4} \tau |A|$ (C_2 is a fixed constant, chosen to fulfil a next estimation). Let $(\xi_\gamma)_{\gamma \in A}$ be independent $(0, 1)$ -valued random variables in ω and define

$$F_\omega(x) = \sum_{m=\ell}^{|A|} \sum_{\substack{S \subset A \\ |S|=m}} \prod_{\gamma \in S} \xi_\gamma(\omega) (\gamma(x) + \bar{\gamma}(x)).$$

Notice that the property $\int_G F_\omega(x) dx = 0$ is equivalent to the fact $\int_G \prod_{\gamma \in S} (\gamma + \bar{\gamma}) = 0$ whenever S is a subset of the random set $\{\gamma \in A \mid \xi_\gamma(\omega) = 1\}$ with $|S| \geq \ell$.

Thus the random set does not present (± 1) -relations of length at least ℓ .

Using condition (2) and the choice of τ, ℓ , we may evaluate

$$\begin{aligned} \iint_G F_\omega(x) dx d\omega &\leq \sum_{m=\ell}^{|A|} \tau^m \frac{1}{m!} \int_G |\Sigma_A(\gamma + \bar{\gamma})|^m \\ &\leq \sum_{m \geq \ell} \tau^m (6C)^m \left(\frac{|A|}{m}\right)^{m/2} < 2^{-\ell/2}. \end{aligned}$$

Hence

$$\frac{\tau |A|}{2} + 2^{\ell/2} \iint_G F_\omega(x) dx d\omega < \int \sum_{\gamma \in A} \xi_\gamma(\omega)$$

implying the existence of ω s.t.

$$|A_1| > \frac{\tau |A|}{2}$$

where $A_1 = \{\gamma \in A \mid \xi_\gamma(\omega) = 1\}$

and

$$\int_G F_\omega(x) < 2^{-\ell/2} |A| < 1, \text{ so } \int_G F_\omega(x) = 0.$$

By definition of F_ω and the choice of ℓ , it follows that A_1 has the desired properties.

The key step is the following construction:

LEMMA 2. — Assume $\Lambda_1, \dots, \Lambda_J$ disjoint quasi-independent subsets of Γ and

$$\frac{|\Lambda_{j+1}|}{|\Lambda_j|} > R \quad \text{for } j = 1, \dots, J - 1$$

where the ratio $R > 10$ is some fixed numerical constant (appearing through later computations).

Then there are subsets Λ'_j of $\Lambda_j (1 \leq j \leq J)$ s.t.

$$(1) \quad |\Lambda'_j| > \frac{1}{10} |\Lambda_j| \quad \text{and} \quad (2) \quad \bigcup_{j=1}^J \Lambda'_j \text{ is quasi-independent.}$$

Proof. — Fixing $j = 1, \dots, J$, we will exhibit a subset Λ'_j of Λ_j satisfying the following condition (*)

$$\eta_1 \cdots \eta_{j-1} \eta_{j+1} \cdots \eta_J \neq 0$$

if

$$0 \neq \eta_j = \sum_{\gamma \in \Lambda'_j} \epsilon_\gamma \gamma (\epsilon_\gamma = -1, 0, 1)$$

and for each $k \neq j$

$$\eta_k \in P_{d_k}(\Lambda_k) \quad \text{where} \quad d_k = \frac{|\Lambda_j|}{|\Lambda_k|} \sum_{\gamma \in \Lambda'_j} |\epsilon_\gamma|.$$

Those sets Λ'_j satisfy (2). Indeed if

$$\eta_1 \cdots \eta_J = 0 \quad \text{and} \quad \eta_j = \sum_{\gamma \in \Lambda'_j} \epsilon_\gamma \gamma (\epsilon_\gamma = -1, 0, 1)$$

then, defining $d_j = \sum_{\Lambda'_j} |\epsilon_\gamma|$, either $d_j = 0$ or $d_k |\Lambda_k| > d_j |\Lambda_j|$ for some $k \neq j$. If the d_j are not all 0, we may consider j' s.t. $d_{j'} |\Lambda_{j'}|$ is maximum, leading to a contradiction.

The construction of Λ'_j for fixed j is done in the spirit of Lemma 1. It suffices to construct first $\bar{\Lambda}_j \subset \Lambda_j$, $|\bar{\Lambda}_j| > \frac{1}{5} |\Lambda_j|$, fulfilling (*) under the additional restriction

$$\sum_{\gamma \in \bar{\Lambda}_j} |\epsilon_\gamma| > \frac{1}{2} |\bar{\Lambda}_j|. \tag{**}$$

This set $\bar{\Lambda}_j$ is again found randomly. Consider independent $(0, 1)$ -valued random variable $\{\xi_\gamma \mid \gamma \in \Lambda_j\}$ of mean $\frac{1}{4}$ and define the random function on G

$$F_\omega = \sum_{m=|\Lambda_j|/10}^{|\Lambda_j|} \sum_{\substack{S \subset \Lambda_j \\ |S|=m}} \prod_{\gamma \in S} \xi_\gamma(\omega) (\gamma + \bar{\gamma}) \prod_{k \neq j} \sum \{\eta \in P_{d_k(m)}(\Lambda_k)\}$$

where $d_k(m) = \frac{|\Lambda_j|}{|\Lambda_k|} m$. Write

$$\int \int_G F_\omega(x) dx d\omega \leq \sum_{m=|\Lambda_j|/10}^{|\Lambda_j|} 2^{-m} \int_G \prod_{\gamma \in \Lambda_j} (1 + \operatorname{Re} \gamma) \prod_{k \neq j} \sum \{\eta \in P_{d_k(m)}(\Lambda_k)\},$$

and using the estimation on $|P_d(A)|$ mentioned in the introduction, it follows the majoration by

$$2^{-\frac{|\Lambda_j|}{10}} \prod_{k \neq j} |P_{d_k}(\Lambda_k)| \quad (d_k = d_k(|\Lambda_j|) = \frac{|\Lambda_j|^2}{|\Lambda_k|})$$

$$\leq 2^{-\frac{|\Lambda_j|}{10}} \exp \left\{ 2 \sum_{k < j} |\Lambda_k| \log C \frac{|\Lambda_j|}{|\Lambda_k|} + 2 \sum_{k > j} \frac{|\Lambda_j|^2}{|\Lambda_k|} \log C \frac{|\Lambda_k|}{|\Lambda_j|} \right\}.$$

Since $\log x < 2\sqrt{x}$ for $x \geq 1$, we may further estimate by

$$2^{-\frac{|\Lambda_j|}{10}} \exp \left\{ C_1 \sum_{k < j} \left(\frac{|\Lambda_k|}{|\Lambda_j|} \right)^{1/2} + C_1 \sum_{k > j} \left(\frac{|\Lambda_j|}{|\Lambda_k|} \right)^{1/2} \right\} |\Lambda_j| < 2^{-\frac{|\Lambda_j|}{11}}$$

for an appropriate choice of the ratio R .

So again, since we may assume $|\Lambda_j| > 20$

$$\frac{1}{5} |\Lambda_j| + 2^{\frac{|\Lambda_j|}{11}} \int \int_G F_\omega(x) dx d\omega < \int_{\Lambda_j} \sum \xi_\gamma(\omega) d\omega$$

and there exists therefore some ω s.t. if $\bar{\Lambda}_j = \{\gamma \in \Lambda_j | \xi_\gamma(\omega) = 1\}$ we have

$$|\bar{\Lambda}_j| > \frac{1}{5} |\Lambda_j| \quad \text{and} \quad \int_G F_\omega(x) dx = 0.$$

But the latter property means that (*) holds under the restriction (**).

This proves lemma 2.

We derive now the implication (3) \implies (4).

LEMMA 3. — *If (3) of the theorem holds, then (4) is valid with $\delta(4) \sim \delta(3)$.*

Proof. — From Lemma 2, the argument is routine. Let R be the constant appearing in Lemma 2 and fix a sequence $(\alpha_\gamma)_{\gamma \in \Lambda}$ s.t. $\sum |\alpha_\gamma| = 1$.

Define for $k = 0, 1, 2, \dots$

$$\Lambda_k = \{\gamma \in \Lambda \mid R_1^{-k} \geq |\alpha_\gamma| > R_1^{-k-1}\}$$

where R_1 is a numerical constant with $R_1 > 4R$.

By hypothesis, there exists for each k a quasi-independent subset Λ_k^1 of Λ_k s.t.

$$|\Lambda_k^1| > \delta |\Lambda_k|. \quad (1)$$

Defining

$$\Omega_e = \bigcup_{k \text{ even}} \Lambda_k^1 \quad \text{and} \quad \Omega_o = \bigcup_{k \text{ odd}} \Lambda_k^1$$

we have

$$\sum_{\gamma \in \Omega_e} |\alpha_\gamma| + \sum_{\gamma \in \Omega_o} |\alpha_\gamma| \geq \frac{\delta}{R_1}$$

and may for instance assume

$$\sum_{\gamma \in \Omega_e} |\alpha_\gamma| \geq \frac{\delta}{2R_1}. \quad (2)$$

Define inductively the sequence $(k_j)_{j=1,2,\dots}$ by

$$k_1 = 0 \quad \text{and} \quad k_{j+1} = \min \{k > k_j \mid |\Lambda_{2k}^1| > R |\Lambda_{2k_j}^1|\}.$$

If we take $\Lambda_j^2 = \Lambda_{2k_j}^1$, it follows by construction that $\frac{|\Lambda_{j+1}^2|}{|\Lambda_j^2|} > R$.

Moreover

$$\begin{aligned}
 \sum_j \sum_{k_j < k < k_{j+1}} \sum_{\gamma \in \Lambda_{2k}^1} |\alpha_\gamma| &\leq \sum_j \sum_{k > k_j} R_1^{-2k} R |\Lambda_{2k_j}^1| \\
 &\leq \frac{2R}{R_1} \sum_j R_1^{-2k_j-1} |\Lambda_{2k_j}^1| \\
 &\leq \frac{2R}{R_1} \sum_{\gamma \in \Omega_e} |\alpha_\gamma|
 \end{aligned}$$

and since $R_1 > 4R$, it follows thus by (2)

$$\sum_j \sum_{\gamma \in \Lambda_j^2} |\alpha_\gamma| > \frac{1}{4R_1} \delta. \tag{3}$$

Application of Lemma 2 to the sequence $(\Lambda_j^2)_{j=1,2,\dots}$ leads to further subsets $\Lambda_j^3 \subset \Lambda_j^2$ satisfying

$$|\Lambda_j^3| \geq \frac{1}{10} |\Lambda_j^2| \quad \text{and} \quad A = \cup \Lambda_j^3 \text{ is quasi-independent.}$$

It remains to write

$$\begin{aligned}
 \sum_{\gamma \in A} |\alpha_\gamma| &\geq \sum_j R_1^{-2k_j-1} |\Lambda_j^3| \geq \frac{1}{10R_1} \sum_j R_1^{-2k_j} |\Lambda_j^2| \\
 &\geq \frac{1}{10R_1} \sum_j \sum_{\gamma \in \Lambda_j^2} |\alpha_\gamma|
 \end{aligned}$$

and use (3).

Remark. – Say that a subset A of the dual group Γ is d -independent ($d = 1, 2, \dots$) provided the relation

$$\sum_{\gamma \in A} \epsilon_\gamma \gamma = 0 \quad (\epsilon_\gamma = -d, -d+1, \dots, d)$$

implies $\epsilon_\gamma = 0$ ($\gamma \in A$).

With this terminology, 1-independent corresponds to quasi-independent.

Assume G a torsion-free compact, abelian group. Fixing an integer d , statements (3) and (4) of the theorem can be reformulated for d -independent sets. The proof is a straightforward modification.

3. Sidon sets of first type.

As an application of previous section, we show

COROLLARY 2. — *A sidon set tending to infinity is a Sidon set of first type.*

Notice that conversely each set of first type tends to infinity (see [2]). Also, each Sidon set is the finite union of sets tending to infinity (see [3], p. 141 and [1] for the general case).

Proof of Cor. 2. — Fix a Sidon set Λ tending to infinity and a nonempty open subset I of G . Choose $\delta > 0$ s.t. (4) of the previous theorem holds.

Let $p \in L^1(G)$ be a polynomial s.t. $p \geq 0$, $\hat{p} \geq 0$, $\int_G p = 1$ and $|p| < \epsilon$ on $G \setminus I$ (where $\epsilon > 0$ will be defined later). Denote Γ_0 the spectrum of p . By hypothesis, we may assume

$$\gamma - \delta \notin \Gamma_0 \quad \text{for } \gamma \neq \delta \quad \text{in } \Lambda. \quad (1)$$

We claim the existence of a finite subset Λ_0 of Λ s.t. if $(\alpha_\gamma)_{\gamma \in \Lambda \setminus \Lambda_0}$ is a finite scalar sequence, there exists a quasi-independent subset A of $\Lambda \setminus \Lambda_0$ s.t.

$$\sum_{\gamma \in A} |\alpha_\gamma| > \frac{\delta}{2} \sum |\alpha_\gamma| \quad (2)$$

and

$$\int p \prod_{\gamma \in A} (1 + \operatorname{Re} \gamma) < 2. \quad (3)$$

The existence of Λ_0 is shown by contradiction. Indeed, one should otherwise obtain finite disjointly supported systems

$$(\alpha_\gamma)_{\gamma \in \Lambda_1}, \dots, (\alpha_\gamma)_{\gamma \in \Lambda_r}, \dots \quad (\Lambda_r \subset \Lambda)$$

with

$$\sum_{\gamma \in \Lambda_r} |\alpha_\gamma| = 1$$

and for which a quasi-independent set fulfilling (2), (3) does not exist.

Fix R large and apply (4) of the Theorem to the system

$$\left\{ \alpha_\gamma \mid \gamma \in \bigcup_{r=1}^R \Lambda_r \right\}.$$

This yields a quasi-independent set $B \subset \Lambda$ so that

$$\sum_{r=1}^R \sum_{\gamma \in \Lambda_r \cap B} |\alpha_\gamma| > \delta R. \tag{4}$$

Also, since $\hat{p} \geq 0$

$$\begin{aligned} \sum_{r=1}^R \int p \left\{ \prod_{\gamma \in B \cap \Lambda_r} (1 + \operatorname{Re} \gamma) - 1 \right\} \\ \leq \int p \prod_{\gamma \in B} (1 + \operatorname{Re} \gamma) \leq \|p\|_\infty \leq |\Gamma_0|. \end{aligned} \tag{5}$$

As a consequence of (4), (5), there must be some $r = 1, \dots, R$ for which $\sum_{\gamma \in \Lambda_r \cap B} |\alpha_\gamma| > \frac{\delta}{2}$ as well as

$$\int p \prod_{\gamma \in B \cap \Lambda_r} (1 + \operatorname{Re} \gamma) < 1 + \int p = 2,$$

provided R is chosen large enough. Since $A = B \cap \Lambda_r$ is quasi-independent, a contradiction follows. This ensures the existence of Λ_0 . We assume $\Gamma_0 \subset \Lambda_0$.

Let now $(\alpha_\gamma)_{\gamma \in \Lambda \setminus \Lambda_0}$ a finite scalar sequence and A a quasi-independent set fulfilling (2), (3). Clearly, whenever $|a_\gamma| \leq 1$ ($\gamma \in A$), by construction of p ,

$$\left| \int \prod_{\gamma \in A} (1 + \operatorname{Re} a_\gamma \gamma) (\sum \alpha_\gamma \gamma)^p \right| \leq 2 \|\sum \alpha_\gamma \gamma\|_{C(I)} + \epsilon \sum |\alpha_\gamma|.$$

We now analyze the left side, defining $a_\gamma = \kappa b_\gamma$ ($|b_\gamma| = 1$), κ to be specified later. Write

$$\prod_{\gamma \in A} (1 + \operatorname{Re} a_\gamma \gamma) = 1 + \kappa \sum_{\gamma \in A} \operatorname{Re} b_\gamma \gamma + \sum_{\ell > 2} \kappa^\ell Q_\ell$$

where $Q_\ell = \sum_{\substack{S \subset A \\ |S| = \ell}} \prod_{\gamma \in S} \operatorname{Re} b_\gamma \gamma$ and, since $\int (\sum \alpha_\gamma \gamma)^p = 0$,

minorate consequently the left member as

$$\kappa \left| \int \left(\sum_{\gamma \in A} \operatorname{Re} b_\gamma \gamma \right) (\Sigma \alpha_\gamma \gamma) p \right| - \sum_{\ell > 2} \kappa^\ell \left| \int Q_\ell p (\Sigma \alpha_\gamma \gamma) \right|. \quad (*)$$

Since $\hat{p} \geq 0$, we have for fixed ℓ (from (3))

$$\left| \int Q_\ell p (\Sigma \alpha_\gamma \gamma) \right| \leq \|Q_\ell p\|_{PM} \cdot \Sigma |\alpha_\gamma|$$

and

$$\begin{aligned} \|Q_\ell p\|_{PM} &\leq \left\| \left(\sum_{\substack{S \subset A \\ |S| = \ell}} \prod_{\gamma \in S} \operatorname{Re} \gamma \right) p \right\|_{PM} \\ &\leq \left\| \prod_{\gamma \in A} (1 + \operatorname{Re} \gamma) \cdot p \right\|_1 < 2. \end{aligned}$$

Thus (*) can be minorated as

$$\kappa \left| \int \left(\sum_{\gamma \in A} \operatorname{Re} b_\gamma \gamma \right) (\Sigma \alpha_\gamma \gamma) p \right| - 3 \kappa^2 \Sigma |\alpha_\gamma|.$$

Since $\operatorname{Re} b_\gamma \gamma$ can be replaced by $\operatorname{Im} b_\gamma \gamma$, we see that

$$2 \|\Sigma \alpha_\gamma \gamma\|_{C(I)} \geq \frac{\kappa}{2} \left| \int (\Sigma_A b_\gamma \bar{\gamma}) (\Sigma_\Lambda \alpha_\gamma \gamma) p \right| - (\epsilon + 3\kappa^2) \Sigma |\alpha_\gamma|.$$

Now, for $\gamma \in A \subset \Lambda$ and $\delta \in \Lambda$, either $\gamma = \delta$ or $\int \bar{\gamma} \delta p = 0$.

This as a consequence of (1). Thus, taking $b_\gamma = \frac{\overline{\alpha_\gamma}}{|\alpha_\gamma|}$,

$$\int (\Sigma_A b_\gamma \bar{\gamma}) (\Sigma_\Lambda \alpha_\gamma \gamma) p = \Sigma_A |\alpha_\gamma| > \frac{\delta}{2} \Sigma |\alpha_\gamma|.$$

Choosing ϵ, κ appropriately, the proof is completed.

Remark. — Let G be a compactly generated, locally compact abelian group and B the dual group. A subset Λ of Γ is called a topological Sidon set provided there exists a compact subset K of G satisfying $\sum_{\gamma \in \Lambda} |\alpha_\gamma| \leq C \sup_{x \in K} \left| \sum_{\gamma \in \Lambda} \alpha_\gamma \gamma(x) \right|$ where C is a fixed constant.

Similarly to the case of compact groups, we define Sidon sets of first type. Then Cor. 2 remains valid. It is indeed easy using the stability property of topological Sidon sets for small perturbations (see [2] for details) to reduce the problem to the periodic case.

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