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Uniform estimates for cscK metrics

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Abstract. — This note grew out of a series of lectures held in Cortona in 2019 and whose aim was to understand the recent breakthrough obtained by Chen and Cheng on the existence of constant scalar curvature Kähler metrics. We present a detailed version of the C^0 and C^2 a priori estimates within the realm of pluripotential theory.

Résumé. — Cette note est le fruit d'une série d'exposés donnés à Cortona en 2019 et dont le but visait à comprendre les avancées majeures due à Chen et Cheng sur l'existence de métriques kähleriennes à courbure scalaire constante. Nous donnons ici une preuve alternative et détaillée des estimées a priori dites C^0 et C^2 dans le cadre de la théorie du pluripotentiel.

1. Introduction

Finding "canonical" metrics on compact Kähler manifolds is one of the central questions in complex geometry (see for example [5, 21, 23]). Examples of these metrics are Kähler-Einstein metrics, constant scalar curvature and more generally extremal metrics. Given a Kähler metric ω on a compact Kähler manifold X of complex dimension n, one looks for a Kähler potential φ such that the curvature of the new metric $\omega_\varphi := \omega + \mathrm{dd}^c\varphi$ becomes more tractable in some sense. This general problem is known to admit a solution in some important particular cases (theorems of Aubin [1], Yau [23], Chen–Donaldson–Sun [10, 11, 12], Tian [22] to cite only a few) as well as obstructions (Futaki [17], Donaldson [16]).

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Recently there have been major breakthroughs related to a longstanding conjecture on the existence of constant scalar curvature Kähler metrics (cscK for short) and the properness of a functional, called K-energy (see [3, 7, 8, 9, 15]): it was shown in [3, 15] that the existence of a cscK metric implies the K-energy is proper (even coercive), while the breakthrough of Chen–Cheng in [7, 8, 9] show that this necessary condition is actually also a sufficient condition.

Given a Kähler form $\omega_\varphi \in \{\omega\}$ we define its *scalar curvature* as

$$S(\omega_\varphi) := n\frac{\mathrm{Ric}(\omega_\varphi) \wedge \omega_\varphi^{n-1}}{\omega_\varphi^n}.$$

We say that $\omega_\varphi = \omega + \mathrm{dd}^c\varphi$ is a *cscK metric* if ω_φ is a Kähler metric and $S(\omega_\varphi) = \overline{S}$, $\overline{S} \in \mathbb{R}$. Integrating both sides with respect to ω_φ^n, we find that $\overline{S}$ is a cohomological constant equal to $nc_1(X) \cdot \{\omega\}^{n-1}/\{\omega\}^n$.

The fact that the existence of a cscK metric implies the properness of the K-energy is due to [3] and [15], while the reverse implication was proved more recently by Chen and Cheng [7, 8, 9]. The first observation that has to be made is that the constant scalar curvature equation can be re-written as a system of two equations. Indeed, if we set $\omega_\varphi^n = e^F\omega^n$, then tracing the pointwise equality

$$\mathrm{Ric}(\omega_\varphi) = \mathrm{Ric}(\omega) - \mathrm{dd}^c \log \frac{\omega_\varphi^n}{\omega^n}$$

with respect to ω_φ leads to

$$\overline{S} = S(\omega_\varphi) = \mathrm{Tr}_{\omega_\varphi}(\mathrm{Ric}(\omega)) - \Delta_{\omega_\varphi} F.$$

It then follows that the cscK equation can be re-written as a system of coupled equations:

$$\omega_\varphi^n = e^F\omega^n, \qquad \Delta_{\omega_\varphi} F = -\overline{S} + \mathrm{Tr}_{\omega_\varphi}(\mathrm{Ric}(\omega)).$$

The (classical) idea is then to deform the above system using a continuity path in such a way that the initial system (at time $t = 0$) has an obvious solution while the system of equations at $t = 1$ is the one for which we want to prove existence of solutions. The goal is to show that the set S of parameters $t \in [0, 1]$ such that a smooth solution exists is open, closed and non-empty. This would imply in turn that $t = 1$ is in S, meaning that the desired solution exists.

The closedness part is historically the most difficult. In the framework of the continuity method (specific to this setting) it suffices to prove uniform estimates for cscK potentials. Indeed, such estimates generalize easily to potentials which are solutions of the intermediate equations we have to deal with in the continuity method.

The key result that Chen and Cheng [8] are able to obtain states as follows:

Theorem 1.1 (Chen–Cheng, [8]). — *Let* (X, ω) *be a compact Kähler manifold. Assume* ω_φ *is a cscK metric for some smooth function* φ *on* X *normalized such that* $\sup_X \varphi = 0$*. Then all the derivatives of* φ *can be estimated in terms of* $\mathrm{Ent}(\varphi)$*, i.e. for each* $k \geqslant 0$*, there exists a positive constant* $C_k = C(k, \mathrm{Ent}(\varphi))$ *such that*

$$\|\varphi\|_{C^k} \leqslant C_k.$$

Here $\mathrm{Ent}(\varphi)$ denotes the *entropy* of the measure ω_φ^n and it is defined as

$$\mathrm{Ent}(\varphi) := \int_X F e^F \omega^n = \int_X \log \frac{\omega_\varphi^n}{\omega^n}\, \omega_\varphi^n \geqslant 0.$$

In [8], the authors establish C^0 and C^2 a priori estimates by proving an intermediate C^1 a priori estimate on the Kähler potential φ.

Once the C^0 and C^2 estimates are in hand, higher order estimates follow from standard regularity results for complex Monge–Ampère equations: see [19, Chapter 14, Section 14.3] and the references therein for instance.

In this note we present a detailed version of the C^0 and C^2 a priori estimates that does not require an a priori C^1 estimate in between.

It is worth it to emphasize that, while the original proof of the C^0-estimate in [8, Theorem 5.1] uses the Alexandroff maximum principle (for the real Monge–Ampère operator), we present here an alternative proof that makes use in a crucial way of pluripotential theory and which is based on the recent paper [14].

This note grew out of a series of lectures held in Cortona in 2019 and whose aim was to understand the articles [7, 8, 9] by Chen and Cheng. By no means we intend here to claim new proofs of their results. Our goal is only to get a more self-contained proof of the aforementioned a priori results within the realm of pluripotential theory.

Outline of the paper

Section 2 introduces basic notations and preliminary results that will be used in the rest of this note. In particular, appropriate definitions and references to pluripotential theory that are needed in the sequel are given.

Section 3 gives a different proof of the a priori C^0 estimate which is based on the work [14]: see Theorem 3.1. This culminates in the statement of Corollary 3.2.

The proof of the a priori C^2 estimate from Theorem 1.1 requires an intermediate step which is taken care of in Section 4: Theorem 4 establishes a priori integral L^p estimates on the laplacian of φ for all $p \geqslant 1$. At that stage of the proof, notice that the L^p bounds one gets, might blow up as p tends to $+\infty$.

Finally, Section 5 ends the proof of the a priori C^2 estimate through a delicate De Giorgi–Nash–Moser iteration applied to the laplacian of φ: notice that it not only gives the desired C^2-estimate but also an a priori Lipschitz bound on the (log of the) volume ratio $\omega_\varphi^n/\omega^n$ required to complete the proof of the higher-order estimates. The last two sections essentially follow the work of Chen and Cheng in [8].

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2. Notations & Preliminaries

Let (X,ω) be a compact Kähler manifold of complex dimension $n \geqslant 2$ and ω be a reference Kähler form normalized so that

$$V := \mathrm{Vol}_\omega(X) = \int_X \omega^n = 1.$$

We denote by g the corresponding hermitian metric, i.e. $\omega = \sqrt{-1}\sum g_{i\bar{j}}\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$.

The assumption $n \geqslant 2$ will be crucial in the proof of Lemma 5.2. At the same time it is harmless since when $n = 1$, the Uniformization Theorem guarantees the existence of a constant curvature metric on a Riemann surface.

Definition 2.1. — *A function* $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ *is quasi-plurisubharmonic (*qpsh *for short) if it is locally given as the sum of a smooth and a psh function. Quasi-psh functions satisfying*

$$\omega_\varphi := \omega + dd^c\varphi \geqslant 0,$$

in the weak sense of currents are called ω*-psh functions.*

We let $\mathrm{PSH}(X,\omega)$ *denote the set of all* ω*-psh functions which are not identically* $-\infty$.

Quasi-psh functions are upper semi-continuous and Lebesgue-integrable. They are actually in L^p for all $p \geqslant 1$, and the induced topologies are all equivalent.

We say that φ is strictly ω-psh if and only if $\omega_\varphi := \omega + dd^c\varphi \geqslant \varepsilon\omega$, for some $\varepsilon > 0$.

Thanks to [18], for each $\varphi \in \mathrm{PSH}(X,\omega)$, one can make sense of its associated Monge–Ampère measure ω_φ^n, which by construction is a positive measure that does not charge mass on pluripolar sets. When φ is smooth and strictly ω-psh, then ω_φ is a genuine Kähler form and ω_φ^n is nothing but the wedge product of ω_φ with itself n-times.

For notational convenience we note $\Delta_\varphi := \Delta_{\omega_\varphi}$ and $\mathrm{Tr}_\varphi := \mathrm{Tr}_{\omega_\varphi}$. We simply denote $\Delta := \Delta_\omega$ and $\mathrm{Tr} := \mathrm{Tr}_\omega$. Analogously, we denote by ∇ the Levi-Civita connection associated to ω and by ∇^φ the one associated to ω_φ. We recall that given a smooth function u and a Kähler form η,

$$\Delta_\eta u := n\frac{dd^c u \wedge \eta^{n-1}}{\eta^n}, \quad |\nabla^\eta u|^2 = n\frac{du \wedge d^c u \wedge \eta^{n-1}}{\eta^n}.$$

Equivalently, in coordinates we have

$$\Delta_\eta u = \sum_{i,j} g^{i\bar{j}} u_{i\bar{j}}, \qquad |\nabla^\eta u|^2 = \sum_{i,j} g^{i\bar{j}} u_i u_{\bar{j}},$$

where g is the associated hermitian metric. For notational convenience, in what follow we drop the sum when working in local coordinates.

With the aforementioned conventions, the cscK equation then writes as

$$\omega_\varphi^n = e^F\omega^n, \qquad \Delta_\varphi F = -\overline{S} + \mathrm{Tr}_\varphi(\mathrm{Ric}(\omega)). \tag{2.1}$$

We now recall below some ingredients from pluripotential theory that are going to be crucial in what follows in order to establish uniform C^0-estimates. The first is a powerful integrability result which is known as a uniform version of Skoda's integrability theorem. We introduce

$$\nu_\omega := \sup_{u,x} \nu(u,x), \quad x \in X,\ u \in \mathrm{PSH}(X,\omega),$$

where $\nu(u,x)$ denotes the Lelong number of u at x. We note that from the proof of [19, Lemma 8.10] one can deduce that $\nu_\omega \geqslant 1$.

Theorem 2.2. — *Let $c < 2\nu_\omega^{-1}$. Then there exists a uniform constant $C > 0$ such that for all $u \in \mathrm{PSH}(X,\omega)$ with $\sup_X u = 0$ we have*

$$\int_X e^{-cu}\omega^n \leqslant C.$$

We refer to [19, Theorem 8.11] for a proof. The following result is due to Kołodziej [20]:

Theorem 2.3. — *Assume $\omega_u^n = f\omega^n$ with $f \in L^p$ for some $p > 1$. Then there exists $C > 0$ depending only on $\omega, n, \|f\|_{L^p}$ such that*

$$\mathrm{Osc}_X\, u \leqslant C.$$

Here $L^p := L^p(\omega^n)$. We specify the reference volume form in the notation of the L^p norms only if is different from the standard one.

At last, we recall [14, Theorem 3.3], that can be viewed as a generalization of Kołodziej's theorem:

Theorem 2.4. — *Fix $a \in [0,1), A > 0$, $\chi \in \mathrm{PSH}(X,\omega)$ and $0 \leqslant f \in L^p$ for some $p > 1$. Assume that $u \in \mathrm{PSH}(X,\omega)$, normalized by $\sup_X u = 0$, satisfies*

$$\omega_u^n \leqslant f\omega^n + a\omega_\chi^n.$$

Assume also that

$$\int_E f\omega^n \leqslant A[\mathrm{Cap}_\chi(E)]^2, \tag{2.2}$$

for every Borel subset $E \subset X$. If $P[u]$ is less singular than χ (i.e. $\chi \leqslant P[u] + C$, for some $C > 0$) then

$$\chi - \sup_X \chi - C\Big(\|f\|_{L^p}, p, (1-a)^{-1}, A\Big) \leqslant u.$$

It is worth it to mention that such a result is stated and proved in a much more general version in [14] to which we refer for a proof. Here $\mathrm{Cap}_\chi(E)$ is the χ-relative capacity of E and it is defined as

$$\mathrm{Cap}_\chi(E) := \sup\left\{ \int_E \omega_u^n \,\middle|\, u \in \mathrm{PSH}(X,\omega),\ \chi - 1 \leqslant u \leqslant \chi \right\}$$

and

$$P[u] = \left(\sup\{v \in \mathrm{PSH}(X,\omega),\ v \leqslant 0 \text{ and } v \leqslant u + C, \text{ for some } C > 0\}\right)^*,$$

where $*$ denotes the upper semi-continuous regularization. For later purposes we mention that $P[u] = 0$ if and only if u is such that $\int_X \omega_u^n = V$ ([13, Theorem 1.3]).

3. A priori C^0-estimate

Let φ and F be solutions to (2.1). Let ψ be the unique smooth solution of

$$\omega_\psi^n = b^{-1} e^F \sqrt{F^2+1}\,\omega^n = b^{-1}\sqrt{F^2+1}\,\omega_\varphi^n, \qquad \sup_X \psi = 0, \tag{3.1}$$

where $b = \int_X e^F \sqrt{F^2+1}\,\omega^n$ in order to have $\int_X \omega_\psi^n = \int_X \omega^n = 1$. The existence of a smooth solution ψ to (3.1) is guaranteed by Yau's theorem [23]. Observe that, since $F^2+1 \leqslant 2F^2$ on $\{F \geqslant 1\}$,

$$0 < b = \int_{\{F<1\}} e^F \sqrt{F^2+1}\,\omega^n + \int_{\{F\geqslant 1\}} e^F \sqrt{F^2+1}\,\omega^n \leqslant \sqrt{2}\left(e + \mathrm{Ent}(\varphi)\right).$$

Therefore, if $\mathrm{Ent}(\varphi)$ is uniformly bounded, so is b.

We now establish the following:

Theorem 3.1. — *Given $\varepsilon \in (0,1)$, there exists $C = C(\varepsilon, \omega, b)$ such that*

$$F + \varepsilon\psi - A\varphi \leqslant C,$$

where $A > 0$ is a uniform constant depending only on the lower bound of the Ricci curvature.

Proof. — Let $H := F + \varepsilon\psi - A\varphi$, A_0 be such that $\mathrm{Ric}(\omega) \geqslant -A_0\,\omega$ and $A = A_0 + 1$. An easy computation gives

$$\begin{aligned}
\Delta_\varphi H &= \Delta_\varphi F + \varepsilon\Delta_\varphi\psi - A\Delta_\varphi\varphi \\
&= -\overline{S} + \mathrm{Tr}_\varphi(\mathrm{Ric}(\omega)) + n\varepsilon\frac{\omega_\psi \wedge \omega_\varphi^{n-1}}{\omega_\varphi^n} - \varepsilon\,\mathrm{Tr}_\varphi\,\omega + A\,\mathrm{Tr}_\varphi\,\omega - nA \\
&\geqslant -(\overline{S} + nA) + (A - A_0 - \varepsilon)\,\mathrm{Tr}_\varphi\,\omega + n\varepsilon\frac{\omega_\psi \wedge \omega_\varphi^{n-1}}{\omega_\varphi^n} \\
&\geqslant -(\overline{S} + nA) + n\varepsilon\frac{\omega_\psi \wedge \omega_\varphi^{n-1}}{\omega_\varphi^n} \geqslant -(\overline{S} + nA) + n\varepsilon(F^2+1)^{1/2n}
\end{aligned}$$

where the last inequality follows from the mixed Monge–Ampère inequalities [4, Proposition 1.11] ensuring that $\omega_\psi \wedge \omega_\varphi^{n-1} \geqslant (\sqrt{F^2+1})^{1/n} e^F\,\omega^n$. By the maximum principle, applied to H, we can then infer that at a maximum point x_0 we have

$$n\varepsilon(F^2+1)^{1/2n}(x_0) \leqslant \overline{S} + nA.$$

Thus $F(x_0) \leqslant C_0$, $C_0 = C_0(\varepsilon, A_0, \omega)$.

We then claim that

$$\varepsilon\psi - A\varphi \leqslant C_1,$$

where $C_1 > 0$ depends on ε, A and b.

Let us now prove the claim. First of all we observe that, for any $a, \delta \in (0,1)$ we have either $\sqrt{F^2+1} \geqslant b/(a\delta^n)$ or $F \leqslant \sqrt{F^2+1} \leqslant b/(a\delta^n)$; thus

$$\begin{aligned}\omega_\varphi^n = e^F\omega^n \leqslant a\delta^n b^{-1} e^F\sqrt{F^2+1}\,\omega^n + e^{\frac{b}{a\delta^n}}\omega^n &= a\delta^n\omega_\psi^n + e^{\frac{b}{a\delta^n}}\omega^n \\ &\leqslant a\omega_{\delta\psi}^n + e^{\frac{b}{a\delta^n}}\omega^n.\end{aligned}$$

We are going to apply Theorem 2.4 with $u = \varphi$, $\chi = \delta\psi$ and $f = e^{b/(a\delta^n)}$. In fact, we have that $e^{b/(a\delta^n)} \in L^p$, for any $p \geqslant 1$ and, since $\int_X \omega_\varphi^n = V = 1$, $P[\varphi] = 0 \geqslant \delta\psi$ (in particular $P[\varphi]$ is less singular than $\delta\psi$). Moreover, the assumption in (2.2) is satisfied thanks to [14, Proposition 3.10] since $\int_X \omega_{\delta\psi}^n > 0$.

We can then infer that $\varphi \geqslant \delta\psi - C_3((1-a)^{-1}, e^{b/a\delta^n})$. Choosing δ small enough so that $\varepsilon - A\delta > 0$ we obtain the claim with $C_1 = A\,C_3$.

It then follows that for any $x \in X$

$$H(x) \leqslant H(x_0) \leqslant C_0 + C_1,$$

which concludes the proof. □

Corollary 3.2. — *The functions ψ, φ, F are uniformly bounded by a constant that only depends on ω and* $\mathrm{Ent}(\varphi)$.

Proof. — From Theorem 3.1 we know that $F \leqslant C - \varepsilon\psi + A\varphi \leqslant C - \varepsilon\psi$, since $\sup_X \varphi = 0$. Therefore

$$\int_X e^{2F}\omega^n \leqslant \widetilde{C}\int_X e^{-2\varepsilon\psi}\omega^n.$$

Choosing $\varepsilon < \nu_\omega^{-1}$, by Theorem 2.2 we get a uniform bound for $\|e^F\|_{L^2}$. It follows from Kołodziej uniform estimates (Theorem 2.3), applied to the equation $\omega_\varphi = e^F\omega^n$, that $\varphi \geqslant -C(\|e^F\|_{L^2}, \omega)$. In particular, since $\sup_X \varphi = 0$ we do get a uniform control on $\|\varphi\|_{L^\infty}$. Also,

$$\int_X e^{2F}(F^2+1)\,\omega^n \leqslant \int_X e^{4F}\,\omega^n \leqslant C'\int_X e^{-4\varepsilon\psi}\omega^n.$$

Once again, thanks to Theorem 2.2, choosing $\varepsilon \leqslant (2\nu_\omega)^{-1}$ we get a uniform bound for $\|e^F\sqrt{F^2+1}\|_{L^2}$. Theorem 2.3 then gives a uniform control for $\|\psi\|_{L^\infty}$.

We can then conclude from Theorem 3.1 together with the arguments above that

$$F \leqslant C - \varepsilon\psi + A\varphi \leqslant -\varepsilon\inf_X \psi \leqslant C_4$$

for some uniform positive constant C_4.

It remains to prove a uniform lower bound for F. For this purpose we apply the minimum principle to $F + t\varphi$, with $t = A_1 + 1$ where $A_1 > 0$ is such that $\mathrm{Ric}(\omega) \leqslant A_1\omega$. A standard computation gives

$$\begin{aligned}\Delta_\varphi(F + t\varphi) &= -\overline{S} + \mathrm{Tr}_\varphi(\mathrm{Ric}(\omega)) + tn - t\,\mathrm{Tr}_\varphi\,\omega \\ &\leqslant (tn - \overline{S}) - \mathrm{Tr}_\varphi\,\omega \leqslant C_5 - ne^{-F/n}\end{aligned}$$

where the last inequality is a simple consequence of the algebro-geometric inequality. Now, let x_0 be a minimum point of the function $F + t\varphi$, then $0 \leqslant C_5 - ne^{-F(x_0)/n}$, or equivalently $F(x_0) \geqslant -n\log(C_5/n)$. For any $x \in X$, $F(x) + t\varphi(x) \geqslant F(x_0) + t\varphi(x_0)$, hence $F \geqslant -n\log(C_5/n) - t\|\varphi\|_{L^\infty}$. □

4. Integral C^2-estimates

The theorem we are going to prove in this section states as follows:

Theorem 4.1. — *Let φ be a solution of* (2.1)*. Then, for any $p \geqslant 1$, there exists a constant $C > 0$, depending on p, $\|\varphi\|_{C^0}$, $\|F\|_{C^0}$, an upper bound on the Ricci form and a lower bound of the holomorphic bisectional curvature of ω so that*

$$\|\mathrm{Tr}\,\omega_\varphi\|_{L^p} \leqslant C. \tag{4.1}$$

Proof. — Consider

$$u := e^{-\gamma(F+\lambda\varphi)}\,\mathrm{Tr}\,\omega_\varphi \geqslant 0,$$

where $\gamma, \lambda > 1$ are uniform constants to be chosen in a suitable way in the following. Observe that, given a smooth function f, $\mathrm{dd}^c e^f = e^f \mathrm{dd}^c f + e^f \mathrm{d}f \wedge \mathrm{d}^c f$. Hence a simple computation gives

$$\Delta_\varphi u = \Delta_\varphi e^{\log u} \geqslant e^{\log u}\Delta_\varphi \log u = -\gamma u \Delta_\varphi(F + \lambda\varphi) + u\Delta_\varphi \log \mathrm{Tr}\,\omega_\varphi.$$

Also, by [6, Lemma 2.2]

$$\Delta_\varphi \log \mathrm{Tr}\,\omega_\varphi \geqslant \frac{\Delta F}{\mathrm{Tr}\,\omega_\varphi} - B\,\mathrm{Tr}_\varphi\,\omega,$$

where $B > 0$ is a lower bound for the holomorphic bisectional curvature of ω. Moreover, using (2.1) we see that

$$\Delta_\varphi(F + \lambda\varphi) = (\lambda n - \overline{S}) + \mathrm{Tr}_\varphi(\mathrm{Ric}(\omega)) - \lambda\,\mathrm{Tr}_\varphi\,\omega \leqslant (\lambda n - \overline{S}) + (A - \lambda)\,\mathrm{Tr}_\varphi\,\omega,$$

where $A > 0$ is such that $\mathrm{Ric}(\omega) \leqslant A\omega$. Thus, combining the above inequalities leads us to:

$$\Delta_\varphi u \geqslant e^{-\gamma(F+\lambda\varphi)}\big\{\gamma(\overline{S} - \lambda n)\,\mathrm{Tr}\,\omega_\varphi + \Delta F + (\lambda\gamma - A\gamma - B)\,\mathrm{Tr}\,\omega_\varphi\,\mathrm{Tr}_\varphi\,\omega\big\}.$$

Observe that using (2.1) and the fact that $n \geqslant 2$, we have

$$\mathrm{Tr}\,\omega_\varphi\,\mathrm{Tr}_\varphi\,\omega \geqslant e^{-\frac{F}{n-1}}(\mathrm{Tr}\,\omega_\varphi)^{1+\frac{1}{n-1}}.$$

We now choose $\lambda \geqslant 4\max(A,B)$ (in order to have $\lambda\gamma - A\gamma - B \geqslant \frac{\lambda\gamma}{2}$) so that

$$\Delta_\varphi u \geqslant \gamma(\overline{S} - \lambda n)\, u + \frac{\lambda\gamma}{2} e^{-\frac{F}{n-1}} (\operatorname{Tr}\omega_\varphi)^{\frac{1}{n-1}}\, u + e^{-\gamma(F+\lambda\varphi)}\Delta F. \tag{4.2}$$

Now, since $|\nabla^\varphi u|^2_\varphi \operatorname{Tr}\omega_\varphi \geqslant |\nabla u|^2$ holds pointwise, we write

$$\begin{aligned}\frac{1}{2p+1}\Delta_\varphi u^{2p+1} &= u^{2p}\Delta_\varphi u + 2pu^{2p-1}|\nabla^\varphi u|^2_\varphi \\ &\geqslant u^{2p}\Delta_\varphi u + 2pu^{2p-2}e^{-\gamma(F+\lambda\varphi)}|\nabla u|^2.\end{aligned}$$

Thus, by combining the above inequality with (4.2), we get

$$\begin{aligned} 0 &= \frac{1}{2p+1}\int_X \Delta_\varphi u^{2p+1}\omega_\varphi^n \\ &\geqslant 2p\int_X u^{2p-2}|\nabla u|^2 e^{-\gamma(F+\lambda\varphi)+F}\omega^n + \gamma(\overline{S}-\lambda n)\int_X u^{2p+1}e^F\omega^n \\ &\quad + \frac{\gamma\lambda}{2}\int_X u^{2p+1}e^{\left(\frac{n-2}{n-1}\right)F}(\operatorname{Tr}\omega_\varphi)^{\frac{1}{n-1}}\omega^n + \int_X u^{2p}e^{-\gamma(F+\lambda\varphi)+F}\Delta F\,\omega^n. \end{aligned} \tag{4.3}$$

Next, we focus on finding a suitable lower bound for the last term involving the laplacian of F.

A formal trick gives that

$$\begin{aligned} I &:= -\int_X u^{2p}\Delta F e^{(1-\gamma)F-\gamma\lambda\varphi}\,\omega^n \\ &= -\frac{1}{1-\gamma}\int_X u^{2p}\,\Delta\left((1-\gamma)F-\gamma\lambda\varphi\right)e^{(1-\gamma)F-\gamma\lambda\varphi}\,\omega^n \\ &\quad - \frac{\gamma\lambda}{1-\gamma}\int_X u^{2p}\Delta\varphi\, e^{(1-\gamma)F-\gamma\lambda\varphi}\,\omega^n \\ &:= I_1 + I_2. \end{aligned}$$

Set $G := (1-\gamma)F - \gamma\lambda\varphi$. From Stokes' theorem we get that

$$\begin{aligned} I_1 &= -\frac{1}{\gamma-1}\int_X u^{2p}|\nabla G|^2 e^G\,\omega^n - \frac{2pn}{\gamma-1}\int_X u^{2p-1}\,e^G\,\mathrm{d}u\wedge \mathrm{d}^cG\wedge\omega^{n-1} \\ &\leqslant -\frac{1}{2(\gamma-1)}\int_X u^{2p}|\nabla G|^2e^G\omega^n + \frac{2p^2}{\gamma-1}\int_X u^{2p-2}|\nabla u|^2e^G\,\omega^n \\ &\leqslant \frac{2p^2}{\gamma-1}\int_X u^{2p-2}|\nabla u|^2e^G\,\omega^n \end{aligned} \tag{4.4}$$

where in the first inequality we used the fact that

$$\left|2pu^{2p-1}n\frac{\mathrm{d}u\wedge\mathrm{d}^cG\wedge\omega^{n-1}}{\omega^n}\right| \leqslant \frac{(2p)^2}{2}u^{2p-2}|\nabla u|^2 + \frac{1}{2}u^{2p}|\nabla G|^2,$$

by Young's inequality. Also, since $\operatorname{Tr}\omega_\varphi = n + \Delta\varphi$,

$$I_2 = \frac{\gamma\lambda}{\gamma-1}\int_X u^{2p+1}e^F\,\omega^n - \frac{n\gamma\lambda}{\gamma-1}\int_X u^{2p}e^G\,\omega^n \leqslant \frac{\gamma\lambda}{\gamma-1}\int_X u^{2p+1}e^F\,\omega^n. \quad (4.5)$$

Combining (4.3), (4.4), (4.5) and choosing γ big enough (say $\gamma = ap$, with $a \gg 1$) we obtain

$$\begin{aligned} 0 &\geqslant 2\left(p - \frac{p^2}{\gamma-1}\right)\int_X u^{2p-2}|\nabla u|^2 e^G\,\omega^n \\ &\quad + \gamma\left(\overline{S} - \lambda n - \frac{\lambda}{\gamma-1}\right)\int_X u^{2p+1}e^F\,\omega^n \\ &\quad + \frac{\gamma\lambda}{2}\int_X u^{2p+1}e^{\left(\frac{n-2}{n-1}\right)F}(\operatorname{Tr}\omega_\varphi)^{\frac{1}{n-1}}\,\omega^n \\ &\geqslant -C_1\int_X (\operatorname{Tr}\omega_\varphi)^{2p+1}\,\omega^n + C_2\int_X (\operatorname{Tr}\omega_\varphi)^{2p+1+\frac{1}{n-1}}\,\omega^n, \end{aligned} \quad (4.6)$$

where the constant $C_1, C_2 > 0$ depends on $\|F\|_{C^0}$ and $\|\varphi\|_{C^0}$ only. Observe that in (4.6), the choice of γ ensures that $p - \frac{p^2}{\gamma-1} > 0$. Using Hölder inequality we can conclude that

$$\|\operatorname{Tr}\omega_\varphi\|^{2p+1+\frac{1}{n-1}}_{L^{2p+1+\frac{1}{n-1}}} \leqslant C\|\operatorname{Tr}\omega_\varphi\|^{2p+1}_{L^{2p+1}} \leqslant C'\|\operatorname{Tr}\omega_\varphi\|^{2p+1}_{L^{2p+1+\frac{1}{n-1}}}.$$

This gives the statement for $p > 3$, hence for $p \geqslant 1$ thanks to Hölder inequality. □

5. C^2-estimates

The main result of this section is the following C^2 a priori estimate as promised in the introduction.

Theorem 5.1. — *Let φ be a solution to* (2.1). *Then there exists a positive constant C depending on $\omega, \|F\|_{C^0}, \|\varphi\|_{C^0}$ and* $\operatorname{Ent}(\varphi)$ *such that,*

$$\max_X\,(|\nabla F| + \operatorname{Tr}\omega_\varphi) \leqslant C.$$

It is worth it to underline that, thanks to Theorem 3.1 and Corollary 3.2, the quantities $\|F\|_{C^0}$ and $\|\varphi\|_{C^0}$ are controlled by $\operatorname{Ent}(\varphi)$.

In order to prove Theorem 5.1, we need to establish several lemmata.

Lemma 5.2. — *Let $u := e^{\frac{F}{2}}|\nabla^\varphi F|^2 + K\operatorname{Tr}\omega_\varphi$ for $K > 0$. Then there exist positive constants K and c depending on ω,$\|F\|_{C^0}$ and $\|\varphi\|_{C^0}$ such that the function u satisfies the following differential inequality:*

$$\Delta_\varphi u \geqslant -c(\operatorname{Tr}\omega_\varphi)^{3n-3}u.$$

As a first remark, notice that the function u defined above is uniformly bounded from below. Indeed, by the arithmetic-geometric inequality,

$$u \geqslant K\operatorname{Tr}(\omega_\varphi) \geqslant nKe^{\frac{F}{n}} \geqslant nKe^{-\frac{\|F\|_{C^0}}{n}} > 0. \tag{5.1}$$

Proof of Lemma 5.2. — We start with the following claim:

Claim 5.3. — For some positive constants C_1 and C_2 depending on $\|F\|_{C^0}$ and the geometry of the Kähler form ω,

$$\Delta_\varphi\left(e^{\frac{F}{2}}|\nabla^\varphi F|^2\right) \geqslant C_1|\nabla^\varphi\overline{\nabla}^\varphi F|^2 - C_2\left((\operatorname{Tr}\omega_\varphi)^{3n-3}|\nabla^\varphi F|^2 + |\nabla^\varphi\overline{\nabla}^\varphi\nabla\varphi|^2 + 1\right),$$

where, in holomorphic normal coordinates with respect to the Kähler form ω,

$$|\nabla^\varphi\overline{\nabla}^\varphi\nabla\varphi|^2 := \frac{|\varphi_{\beta\bar{\alpha}i}|^2}{(1+\varphi_{\alpha\bar{\alpha}})(1+\varphi_{\beta\bar{\beta}})}, \quad |\nabla^\varphi\overline{\nabla}^\varphi F|^2 := \frac{|F_{i\bar{j}}|^2}{(1+\varphi_{i\bar{\imath}})(1+\varphi_{j\bar{j}})}.$$

Proof of Claim 5.3. — By Bochner formula applied to the function F:

$$\Delta_\varphi|\nabla^\varphi F|^2 = |\nabla^\varphi\overline{\nabla}^\varphi F|^2 + |\nabla^\varphi\nabla^\varphi F|^2 + \operatorname{Ric}(\omega_\varphi)(\nabla^\varphi F, \nabla^\varphi F) + 2\Re\left(\langle\nabla^\varphi F, \nabla^\varphi\Delta_\varphi F\rangle\right). \tag{5.2}$$

Now, for some real constant c,

$$\begin{aligned}\Delta_\varphi&\left(e^{cF}|\nabla^\varphi F|^2\right)\\ &= (\Delta_\varphi e^{cF})|\nabla^\varphi F|^2 + 2ce^{cF}\Re(\langle\nabla^\varphi F, \nabla^\varphi|\nabla^\varphi F|^2\rangle) + e^{cF}\Delta_\varphi|\nabla^\varphi F|^2\\ &= e^{cF}\{c\Delta_\varphi F|\nabla^\varphi F|^2 + c^2|\nabla^\varphi F|^4\\ &\qquad + 2c\Re(\langle\nabla^\varphi F, \nabla^\varphi|\nabla^\varphi F|^2\rangle) + \Delta_\varphi|\nabla^\varphi F|^2\}.\end{aligned} \tag{5.3}$$

One can check that in holomorphic normal coordinates with respect to ω_φ:

$$\begin{aligned}2\Re&\left(\langle\nabla^\varphi F, \nabla^\varphi|\nabla^\varphi F|^2\rangle\right)\\ &= F_{i\bar{j}}F_{\bar{\imath}}F_j + F_{\bar{\imath}j}F_iF_{\bar{j}} + F_{ij}F_{\bar{\imath}}F_{\bar{j}} + F_{\bar{\imath}\bar{j}}F_iF_j\\ &= 2\Re\left(\nabla^\varphi\overline{\nabla}^\varphi F(\nabla^\varphi F, \nabla^\varphi F)\right) + 2\Re\left(\nabla^\varphi\nabla^\varphi F(\nabla^\varphi F, \nabla^\varphi F)\right)\\ &= 2\left(\nabla^\varphi\overline{\nabla}^\varphi F(\nabla^\varphi F, \nabla^\varphi F)\right) + 2\Re\left(\nabla^\varphi\nabla^\varphi F(\nabla^\varphi F, \nabla^\varphi F)\right).\end{aligned} \tag{5.4}$$

where the last equality follows from the fact that the quantity $\left(\nabla^\varphi\overline{\nabla}^\varphi F(\nabla^\varphi F,\nabla^\varphi F)\right)$ is real. In particular, by choosing $c=\frac{1}{2}$, one can complete the square as follows:

$$c^2|\nabla^\varphi F|^4+2c\Re\left(\nabla^\varphi\nabla^\varphi F(\nabla^\varphi F,\nabla^\varphi F)\right)+|\nabla^\varphi\nabla^\varphi F|^2 = \left|\nabla^\varphi\nabla^\varphi F+\frac{1}{2}\nabla^\varphi F\otimes\nabla^\varphi F\right|^2 := T\geqslant 0.$$

Therefore, by combining the above identity with (5.2) and (5.3), we have

$$\begin{aligned} e^{-F/2}\Delta_\varphi\left(e^{cF}|\nabla^\varphi F|^2\right) &= T+\frac{\Delta_\varphi F}{2}|\nabla^\varphi F|^2+\Re\left(\nabla^\varphi\overline{\nabla}^\varphi F(\nabla^\varphi F,\nabla^\varphi F)\right)+|\nabla^\varphi\overline{\nabla}^\varphi F|^2 \\ &\quad+\mathrm{Ric}(\omega_\varphi)(\nabla^\varphi F,\nabla^\varphi F)+2\Re\left(\langle\nabla^\varphi F,\nabla^\varphi\Delta_\varphi F\rangle\right). \end{aligned}$$

Moreover, since $\mathrm{Ric}(\omega_\varphi)=\mathrm{Ric}(\omega)-\mathrm{dd}^cF$, the term $\mathrm{dd}^cF(\nabla^\varphi F,\nabla^\varphi F)$ introduces a cubic term in F by the Bochner formula (5.2). Choosing $c=\frac{1}{2}$ again lets us absorb this cubic term with the help of (5.2) based on (5.4).

Consequently,

$$\begin{aligned} e^{-\frac{F}{2}}\Delta_\varphi\left(e^{\frac{F}{2}}|\nabla^\varphi F|^2\right) &\geqslant \frac{\Delta_\varphi F}{2}|\nabla^\varphi F|^2+|\nabla^\varphi\overline{\nabla}^\varphi F|^2 \\ &\quad+\mathrm{Ric}(\omega)(\nabla^\varphi F,\nabla^\varphi F)+2\Re\left(\langle\nabla^\varphi F,\nabla^\varphi\Delta_\varphi F\rangle\right). \end{aligned}$$

Now, by (2.1),

$$|\Delta_\varphi F|\leqslant|\overline{S}|+|\operatorname{Tr}_\varphi\mathrm{Ric}(\omega)|\leqslant|\overline{S}|+C|\operatorname{Tr}_\varphi\omega|\leqslant|\overline{S}|+C\left(\operatorname{Tr}\omega_\varphi\right)^{n-1}, \quad (5.5)$$

where C is a positive constant depending on $\sup_M|\mathrm{Ric}(\omega)|$ and $\|F\|_{C^0}$ that may vary from line to line.

Similarly, since $\mathrm{Ric}(\omega)(\nabla^\varphi F,\nabla^\varphi F)=g_\varphi^{i\bar{\imath}}g_\varphi^{j\bar{\jmath}}\,\mathrm{Ric}_{i\bar{\jmath}}F_{\bar{\imath}}F_{\bar{\jmath}}$ in holomorphic normal coordinates with respect to ω, one has:

$$|\mathrm{Ric}(\omega)(\nabla^\varphi F,\nabla^\varphi F)|\leqslant C\operatorname{Tr}_\varphi\omega\cdot|\nabla^\varphi F|^2\leqslant C\left(\operatorname{Tr}\omega_\varphi\right)^{n-1}|\nabla^\varphi F|^2. \quad (5.6)$$

Using (2.1) again, in holomorphic normal coordinates with respect to ω, we have

$$\begin{aligned} \langle\nabla^\varphi F,\nabla^\varphi\Delta_\varphi F\rangle &= \langle\nabla^\varphi F,\nabla^\varphi\operatorname{Tr}_\varphi\mathrm{Ric}(\omega)\rangle \\ &= -\frac{\varphi_{\bar{k}li}\,\mathrm{Ric}_{k\bar{l}}\,F_{\bar{\imath}}}{(1+\varphi_{k\bar{k}})(1+\varphi_{l\bar{l}})(1+\varphi_{i\bar{\imath}})}+\frac{\mathrm{Ric}_{k\bar{k},i}\,F_{\bar{\imath}}}{(1+\varphi_{k\bar{k}})(1+\varphi_{i\bar{\imath}})}. \end{aligned}$$

By Young's inequality:

$$
\begin{aligned}
|\langle \nabla^{\varphi} F, \nabla^{\varphi} \operatorname{Tr}_{\varphi} \operatorname{Ric}(\omega)\rangle| & \\
&\leqslant C|\nabla^{\varphi}\overline{\nabla}^{\varphi}\nabla\varphi|^2 + C\left(\operatorname{Tr}_{\varphi}\omega\right)^3|\nabla^{\varphi}F|^2 + C\\
&\leqslant C|\nabla^{\varphi}\overline{\nabla}^{\varphi}\nabla\varphi|^2 + C\left(\operatorname{Tr}\omega_{\varphi}\right)^{3(n-1)}|\nabla^{\varphi}F|^2 + C, \quad (5.7)
\end{aligned}
$$

where C is a positive constant depending on $\sup_M|\operatorname{Ric}(\omega)|_{\omega}$ and $\sup_M|\nabla\operatorname{Ric}(\omega)|_{\omega}$. Using once again the fact that $\operatorname{Tr}\omega_{\varphi}$ is uniformly bounded from below as we noticed in (5.1), we can infer that there exists a constant $C>0$ such that $1\leqslant C(\operatorname{Tr}\omega_{\varphi})^{3n-3}$ and $(\operatorname{Tr}\omega_{\varphi})^{n-1}\leqslant C(\operatorname{Tr}\omega_{\varphi})^{3n-3}$. This ends the proof of Claim 5.3 by combining (5.5), (5.6) together with (5.7):

$$
\begin{aligned}
\Delta_{\varphi}\left(e^{\frac{F}{2}}|\nabla^{\varphi}F|^2\right) & \\
&\geqslant e^{\frac{F}{2}}|\nabla^{\varphi}\overline{\nabla}^{\varphi}F|^2 - Ce^{\frac{F}{2}}\left(\left(\operatorname{Tr}\omega_{\varphi}\right)^{3n-3}|\nabla^{\varphi}F|^2 + |\nabla^{\varphi}\overline{\nabla}^{\varphi}\nabla\varphi|^2 + 1\right)\\
&\geqslant C_1|\nabla^{\varphi}\overline{\nabla}^{\varphi}F|^2 - C_2\left(\left(\operatorname{Tr}\omega_{\varphi}\right)^{3n-3}|\nabla^{\varphi}F|^2 + |\nabla^{\varphi}\overline{\nabla}^{\varphi}\nabla\varphi|^2 + 1\right),
\end{aligned}
$$

where C_1 and C_2 are uniform positive constants depending on the geometry of ω and $\|F\|_{C^0}$. □

We recall Yau's C^2 estimates on $\operatorname{Tr}\omega_{\varphi}$ in the form we need:

$$
\Delta_{\varphi}\operatorname{Tr}\omega_{\varphi} \geqslant -C(\operatorname{Tr}\omega_{\varphi})^n + |\nabla^{\varphi}\overline{\nabla}^{\varphi}\nabla\varphi|^2 + \Delta F - C, \tag{5.8}
$$

where C is a positive constant depending on a lower bound of the bisectional curvature of the Kähler form ω: see [23, (2.10)] for a proof. By Claim (5.3) together with (5.8),

$$
\begin{aligned}
\Delta_{\varphi}u \geqslant C_1|\nabla^{\varphi}\overline{\nabla}^{\varphi}F|^2 - C_2\left(\left(\operatorname{Tr}\omega_{\varphi}\right)^{3n-3}|\nabla^{\varphi}F|^2_{\varphi} + |\nabla^{\varphi}\overline{\nabla}^{\varphi}\nabla\varphi|^2 + 1\right)&\\
+ K|\nabla^{\varphi}\overline{\nabla}^{\varphi}\nabla\varphi|^2 - KC(\operatorname{Tr}\omega_{\varphi})^n + K(\Delta F - C).&
\end{aligned}
$$

Choose K large enough so that one can drop the term $|\nabla^{\varphi}\overline{\nabla}^{\varphi}\nabla\varphi|^2$. Now, by (5.1), any power of $\operatorname{Tr}\omega_{\varphi}$ can be bounded from above uniformly by a higher power.

Therefore,

$$
\Delta_{\varphi}u \geqslant C_1|\nabla^{\varphi}\overline{\nabla}^{\varphi}F|^2 - C_2\left(\operatorname{Tr}\omega_{\varphi}\right)^{3n-3}u + K(\Delta F - C) - C_2, \tag{5.9}
$$

where C_2 may vary from line to line.

It remains to bound from below the last term on the righthand side:

$$
K|(\Delta F - C)| \leqslant K\left(|\Delta F| + C\right).
$$

In holomorphic normal coordinates with respect to ω:

$$K|F_{i\bar{\imath}}| = K\left|\frac{F_{i\bar{\imath}}}{(1+\varphi_{i\bar{\imath}})}\cdot(1+\varphi_{i\bar{\imath}})\right|,$$

so that by Young's inequality,

$$\begin{aligned}K|\Delta F| &\leqslant \varepsilon|\nabla^\varphi\overline{\nabla}^\varphi F|^2 + C\varepsilon^{-1}K^2(\operatorname{Tr}\omega_\varphi)^2\\ &\leqslant \varepsilon|\nabla^\varphi\overline{\nabla}^\varphi F|^2 + C\varepsilon^{-1}K^2(\operatorname{Tr}\omega_\varphi)^{3n-3}u,\end{aligned}$$

where ε will be chosen later and where we used the fact that u is bounded from below (5.1) in the last inequality. Here we have used $n \geqslant 2$. Let us choose ε less than or equal to the constant C_1 from (5.9) so that one can drop the term $|\nabla^\varphi\overline{\nabla}^\varphi F|^2$ to get the expected result:

$$\Delta_\varphi u \geqslant -C\,(\operatorname{Tr}\omega_\varphi)^{3n-3}\,u - C\varepsilon^{-1}K^2(\operatorname{Tr}\omega_\varphi)^{3n-3}u - C_2 \geqslant -C(\operatorname{Tr}\omega_\varphi)^{3n-3}u,$$

where C is a positive constant that may vary from line to line and which depends on the parameter K, ε, $\|F\|_{C^0}$ and the geometry of the Kähler form ω. □

The next lemma establishes a priori L^1 bound for the function u:

Lemma 5.4. — *The following estimate holds true:*

$$\|u\|_{L^1} \leqslant C(\omega, \|F\|_{C^0}, \operatorname{Ent}(\varphi)).$$

Proof of Lemma 5.4. — Since we assume that the volume is normalized with $V = 1$, we notice first that:

$$\|\operatorname{Tr}\omega_\varphi\|_{L^1} = \int_M \operatorname{Tr}\omega_\varphi\,\omega^n = n\int_X \omega_\varphi\wedge\omega^{n-1} = n\operatorname{Vol}_\omega(X) = n, \quad (5.10)$$

$$\|\operatorname{Tr}_\varphi\omega\|_{L^1(\omega_\varphi^n)} = \int_X \operatorname{Tr}_\varphi\omega\;\omega_\varphi^n = n\operatorname{Vol}_\omega(X) = n. \quad (5.11)$$

Now, by using (2.1), one has:

$$\Delta_\varphi F^2 = 2F\Delta_\varphi F + 2|\nabla^\varphi F|^2 = 2F(-\overline{S} + \operatorname{Tr}_\varphi(\operatorname{Ric}(\omega))) + 2|\nabla^\varphi F|^2.$$

After integrating the previous identity with respect to ω_φ^n and by using (2.1) once more,

$$\begin{aligned}\int_X |\nabla^\varphi F|^2\omega_\varphi^n &= \overline{S}\int_X F\omega_\varphi^n - \int_X \operatorname{Tr}_\varphi(\operatorname{Ric}(\omega))F\omega_\varphi^n\\ &\leqslant \overline{S}\int_X Fe^F\omega^n + C\|F\|_{C^0}\int_X \operatorname{Tr}_\varphi\omega\,\omega_\varphi^n\\ &= \overline{S}\operatorname{Ent}(\varphi) + nC\|F\|_{C^0},\end{aligned}$$

where C is a positive constant depending on an upper bound of the norm of the Ricci curvature $\operatorname{Ric}(\omega)$ and where we have used (5.11) in the last line.

This bound together with (5.10) proves the expected bound on the L^1 norm of the function u. □

We are now in a position to prove Theorem 5.1.

Proof of Theorem 5.1. — The proof consists in applying a De Giorgi–Nash–Moser iteration to the auxiliary function u defined in Lemma 5.2 as follows for $p \geqslant 1$:

$$\begin{aligned}\frac{1}{2p+1}\Delta_\varphi u^{2p+1} &= u^{2p}\Delta_\varphi u + 2p\, u^{2p-1}|\nabla^\varphi u|^2\\ &= u^{2p}\Delta_\varphi u + \frac{8p}{(2p+1)^2}\left|\nabla^\varphi\left(u^{p+\frac{1}{2}}\right)\right|^2.\end{aligned}$$

Integrating over X with respect to ω_φ^n and using Lemma 5.2 we get:

$$\int_X \left|\nabla^\varphi\left(u^{p+\frac{1}{2}}\right)\right|^2 \omega_\varphi^n \leqslant Cp\int_X (\operatorname{Tr}\omega_\varphi)^{3n-3}u^{2p+1}\omega_\varphi^n, \tag{5.12}$$

for $p \geqslant 1$. Let us apply Hölder's inequality in the perspective of invoking a suitable Sobolev inequality with respect to ω^n in the following. Let $\varepsilon \in (0,2)$ which will be chosen later. Then,

$$\begin{aligned}\int_X (\operatorname{Tr}\omega_\varphi)^{3n-3}u^{2p+1}\omega_\varphi^n &\leqslant \|(\operatorname{Tr}\omega_\varphi)^{3n-3}\|_{L^{\frac{2+\varepsilon}{\varepsilon}}}\cdot\|u^{2p+1}\|_{L^{\frac{2+\varepsilon}{2}}}\\ &= \|\operatorname{Tr}\omega_\varphi\|^{3n-3}_{L^{\frac{(2+\varepsilon)(3n-3)}{\varepsilon}}}\cdot\|u^{2p+1}\|_{L^{\frac{2+\varepsilon}{2}}},\end{aligned}$$

which implies by (5.12) and the fact that $\|F\|_{C^0}$ is under control (Corollary 3.2),

$$\||\nabla^\varphi u^{p+\frac{1}{2}}|\|^2_{L^2} \leqslant Cp\|\operatorname{Tr}\omega_\varphi\|^{3n-3}_{L^{\frac{(2+\varepsilon)(3n-3)}{\varepsilon}}}\cdot\|u^{p+\frac{1}{2}}\|^2_{L^{2+\varepsilon}}. \tag{5.13}$$

For $p \geqslant 1$, define the auxiliary function $v := u^{p+\frac{1}{2}}$ and recall that $|\nabla v|^2 \leqslant \operatorname{Tr}\omega_\varphi\cdot|\nabla^\varphi v|^2$ is true pointwise.

Using Hölder's inequality once more,

$$\begin{aligned}\int_X |\nabla v|^{2-\varepsilon}\omega^n &\leqslant \int_X (\operatorname{Tr}\omega_\varphi)^{\frac{2-\varepsilon}{2}}|\nabla^\varphi v|^{2-\varepsilon}\omega^n\\ &\leqslant \|(\operatorname{Tr}\omega_\varphi)^{\frac{2-\varepsilon}{2}}\|_{L^{\frac{2}{\varepsilon}}}\||\nabla^\varphi v|^{2-\varepsilon}\|_{L^{\frac{2}{2-\varepsilon}}}\\ &= \|\operatorname{Tr}\omega_\varphi\|^{\frac{2-\varepsilon}{2}}_{L^{\frac{2-\varepsilon}{\varepsilon}}}\||\nabla^\varphi v|\|^{2-\varepsilon}_{L^2},\end{aligned}$$

that gives $\|\nabla v\|^2_{L^{2-\varepsilon}} \leqslant \|\operatorname{Tr}\omega_\varphi\|_{L^{\frac{2-\varepsilon}{\varepsilon}}}\||\nabla^\varphi v|\|^2_{L^2}$. Therefore, by (5.13)

$$\|\nabla v\|^2_{L^{2-\varepsilon}} \leqslant Cp\|\operatorname{Tr}\omega_\varphi\|^{3n-3}_{L^{\frac{(2+\varepsilon)(3n-3)}{\varepsilon}}}\cdot\|\operatorname{Tr}\omega_\varphi\|_{L^{\frac{2-\varepsilon}{\varepsilon}}}\cdot\|v\|^2_{L^{2+\varepsilon}}.$$

Using the a priori integral bounds from Theorem 4.1, one arrives at:

$$\|\nabla v\|_{L^{2-\varepsilon}}^2 \leqslant Cp\|v\|_{L^{2+\varepsilon}}^2. \tag{5.14}$$

Let us apply the following Sobolev inequality with respect to ω^n to the function v [2, Theorem 2.21]:

$$\|v\|_{L^\theta} \leqslant C\left(\|\nabla v\|_{L^{2-\varepsilon}} + \|v\|_{L^{2-\varepsilon}}\right),$$

where $\theta := \frac{2n(2-\varepsilon)}{2n-(2-\varepsilon)}$. With the help of (5.14) and the previous Sobolev inequality, one gets:

$$\|v\|_{L^\theta}^2 \leqslant C\left(p\|v\|_{L^{2+\varepsilon}}^2 + \|v\|_{L^{2-\varepsilon}}^2\right),$$

which implies (by definition of v in terms of u),

$$\begin{aligned}\|u\|_{L^{\left(p+\frac12\right)\theta}}^{2p+1} &\leqslant Cp\|u\|_{L^{\left(p+\frac12\right)(2+\varepsilon)}}^{2p+1} + C\|u\|_{L^{\left(p+\frac12\right)(2-\varepsilon)}}^{2p+1}\\ &\leqslant Cp\|u\|_{L^{\left(p+\frac12\right)(2+\varepsilon)}}^{2p+1},\end{aligned}$$

since $p \geqslant 1$ and where we used Hölder's inequality on the second norm of the righthand side. Here, C denotes a positive constant that may vary from line to line which is independent of p.

Choose $\varepsilon \in (0,2)$ so small such that $\theta > 2+\varepsilon$ and define the quotient $\chi := \frac{\theta}{2+\varepsilon}$ which is larger than 1. Consider the diverging sequence $(\chi^i)_{i\geqslant 0}$ and define a sequence $(p_i)_{i\geqslant 0}$ by $p_i + \frac12 = \chi^i$. Let $i_0 \geqslant 0$ such that if $i \geqslant i_0$, $p_i \geqslant 1$. Then the previous estimate applied to this sequence $(p_i)_{i\geqslant i_0}$ reads:

$$\|u\|_{L^{(2+\varepsilon)\chi^{i+1}}} \leqslant (C\chi^i)^{\frac{1}{2\chi^i}}\|u\|_{L^{(2+\varepsilon)\chi^i}}, \quad i \geqslant i_0.$$

Consequently,

$$\limsup_{i\to+\infty}\|u\|_{L^{(2+\varepsilon)\chi^i}} \leqslant (C\chi)^{\sum_{i\geqslant i_0}\frac{1+i}{2\chi^i}}\|u\|_{L^{(2+\varepsilon)\chi^{i_0}}},$$

which implies in turn by Hölder's inequality,

$$\|u\|_{L^\infty} \leqslant C\|u\|_{L^{(2+\varepsilon)\chi^{i_0}}} \leqslant C\|u\|_{L^1}^{\frac{1}{p_\varepsilon}}\cdot\|u\|_{L^\infty}^{1-\frac{1}{p_\varepsilon}}, \quad p_\varepsilon := (2+\varepsilon)\chi^{i_0}.$$

This leads to the desired estimate on u by invoking Lemma 5.4: $\|u\|_{L^\infty} \leqslant C$. This ends the proof of the theorem. Indeed, on one hand, one has a priori C^2 bounds on the potential φ: this implies that norms of tensors defined with respect to either the metric g or the metric g_φ are uniformly equivalent. On the other hand, $\|\nabla^\varphi F\|_{C^0}$ being uniformly bounded, the previous remark leads to the expected uniform bound on the first derivatives of F (with respect to the background metric g). □

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