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Moser–Trudinger type inequalities for complex Monge–Ampère operators and Aubin's "hypothèse fondamentale"

Robert J. Berman [(1)] and Bo Berndtsson [(2)]

Dedicated to Ahmed Zeriahi on the occasion of his retirement

Abstract. — We prove a version of Aubin's "Hypothèse fondamentale" concerning the existence of Moser–Trudinger type inequalities on any integral compact Kähler manifold X. In the case of the anti-canonical class on a Fano manifold the constants in the inequalities are shown to only depend on the dimension of X (but there are counterexamples to the precise value proposed by Aubin). In the different setting of pseudoconvex domains in complex space we also obtain a quasi-sharp version of the inequalities and relate it to Brezis–Merle type inequalities for the complex Monge–Ampère operator, recently considered by Demailly and Åhag–Cegrell–Kołodziej–Phạm–Zeriahi. The inequalities are shown to be sharp for S^1-invariant functions on the unit ball.

1. Introduction

As shown by Trudinger in the seminal work [62] there is a limiting exponential version of the critical Sobolev inequalities which, in the case of the plane, may be formulated as the existence of positive constants c and C such that

$$\int_\Omega e^{c\left(\frac{u}{\|\nabla u\|_\Omega}\right)^2} \mathrm{d}V \leqslant C \tag{1.1}$$

for any, say smooth, function u vanishing on the boundary of a domain Ω in $\mathbb{R}^2$. Motivated by the Nirenberg problem for constructing conformal metrics on a real surface with prescribed positive curvature, Moser [53] obtained the sharp constant $c = 4\pi$ in Trudinger's inequality (1.1). The relation to

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the Nirenberg problem appears in the following consequence of the previous inequality:

$$\log \int_{\Omega} e^{-u} \mathrm{d}V \leqslant A \left\|\nabla u\right\|_{\Omega}^{2} + B. \tag{1.2}$$

Here e^{-u} plays the role of the conformal factor of a metric on Ω. As shown by Moser the inequalities also hold when the domain Ω is replaced by the two-sphere - which is the setting for the Nirenberg problem - and then the extremals u of the inequality correspond to metrics g_u with constant positive curvature (with $A = 1/16\pi$, the sharp constant). Conversely, the latter inequality (1.2), with the sharp constant, implies an inequality of the form (1.1), but only with quasi-sharp constants, i.e. the two inequalities are equivalent "modulo ϵ".

There has been a wealth of work on extending Moser–Trudinger inequalities in various directions in real analysis and conformal geometry, as well as CR-geometry (see for example [4, 5, 39] and references therein). However, the present paper is concerned with a different *complex* variant of these inequalities first proposed by Aubin [3], motivated by the existence problem for Kähler–Einstein metrics with positive Ricci curvature on complex (Fano) manifolds; see also [36, 37, 54]. More precisely, we will consider two different settings:

(1) compact complex (Kähler) manifolds
(2) pseudoconvex domains in $\mathbb{C}^n$.

A characteristic feature of the complex setting is that it is considerably more non-linear than the real one (when $n > 1$). Indeed, the corresponding inequalities (see below) only hold for a *convex* subspace $\mathcal{H}_0$ of functions u and moreover the Laplacian Δ appearing in the Dirichlet energy $\|\nabla u\|_{\Omega}^{2}$ $(= \int_{\Omega} -u\Delta u \mathrm{d}V)$ has to be replaced by fully non-linear complex Monge–Ampère operators. Moreover, in the compact setting (1) the space $\mathcal{H}_0$ is not even a cone and the corresponding Monge–Ampère operator is not n-homogeneous (in contrast to the setting (2)).

Before turning to the precise formulation of our main results, it may be worth emphasizing that the differences that appear in the two different settings referred to above are not merely technical. Indeed, as we will show, the optimal multiplicative constant for the Moser–Trudinger type inequalites in the case of pseudoconvex domains is *universal*, i.e. the same one as in the model case of the unit ball. Moreover, there is also a special role played by S^1-symmetry in this setting. However, the corresponding universality property fails in the setting of compact Fano manifolds, where the role of the unit ball is played by complex projective space $\mathbb{P}^n$. The point is that on a compact Fano manifold X the optimal multiplicative constant in the Moser–Trudinger type inequality depends on the *volume* of X (i.e. the top intersection number of the first Chern class of X); compare the discussion in Section 6. In fact, the differences between the two settings can be exploited as the authors realized during the revision of the first version of the present paper and it leads to a sharp volume bound for Fano manifolds admitting a Kähler–Einstein and a suitable S^1-symmetry, saying that the volume is maximal on $\mathbb{P}^n$ [10]. Briefly, the bound is obtained by studying Moser–Trudinger type inequalities on a large pseudoconvex domain in X, containing an attractive fixed point.

1.1. Statement of the main results

1.1.1. The setting of a compact Kähler manifold

Let (X, ω) be a compact Kähler manifold without boundary of complex dimension n and recall that a smooth function u on X is called a *Kähler potential* if

$$\omega_u := \omega + \frac{i}{2\pi}\partial\bar{\partial}u := \omega + \mathrm{dd}^c u > 0,$$

i.e. ω_u is a Kähler metric in the cohomology class $[\omega] \in H^2(X, \mathbb{R})$. We will denote by $\mathcal{H}_0(X, \omega)$ the convex space of all such u normalized so that $\sup_X u = 0$ and we will consider the following well-known functional on $\mathcal{H}_0(X, \omega)$:

$$\mathcal{E}_\omega(u) := \frac{1}{(n+1)!}\sum_{j=0}^{n}\int_X u(\omega_u)^j \wedge (\omega)^{n-j} \tag{1.3}$$

that we will refer to as (minus) the *Monge–Ampère energy.*

Theorem 1.1. — *Let (X, ω) be a Kähler manifold such that $[\omega]$ is an integral class and fix a volume form* $\mathrm{d}V$*. Then the following Moser–Trudinger type inequality holds for any function u in $\mathcal{H}_0(X, \omega)$ and positive number $k \geqslant k_0 > 0$ for a fixed k_0 :*

$$\log \int_X e^{-ku}\mathrm{d}V \leqslant Ak^{n+1}(-\mathcal{E}_\omega(u)) + B \tag{1.4}$$

for some positive constants A and B. More precisely, the constant A may be replaced by $(1 + C_1/k)$ and B by $(1 + C_2/k)$ for certain invariants C_1 and C_2 of ω (see (2.6)*).*

The first part of the theorem essentially establishes a conjecture of Aubin (called "Hypothèse fondamentale" in [3]) under the assumption that the class

$[\omega]$ be integral (see the discussion in Section 6). The inequalities (1.4) are equivalent to the existence of positive constants c and C such that

$$\int_X e^{c\left(\frac{-u}{(-\mathcal{E}_\omega(u))^{1/(n+1)}}\right)^{(n+1)/n}} \mathrm{d}V \leqslant C, \tag{1.5}$$

providing a variant of Trudinger's inequality (1.1) in the Kähler setting. It appears to be new even in the case of two-dimensional projective space. In particular we deduce the following "non-linear" Sobolev type inequalities of independent interest: for any u in $\mathcal{H}_0(X,\omega)$

$$\|u\|^{n+1}_{L^p(X)} \leqslant Cp^n(-\mathcal{E}_\omega(u)) \tag{1.6}$$

for all p in $[1,\infty[$, for some constant C only depending on ω. Note that one interesting feature of the constants in Theorem 1.1 is that the leading asymptotics of the constants A and B is universal, i.e. independent of (X,ω) and even the dimension n of X.

The starting point of the proof of the previous theorem is the basic fact that, in the integral case when $[\omega] \in H^2(X,\mathbb{Z})$, the space $k\mathcal{H}_0(X,\omega)$ may, when k is a positive integer, be identified (modolo constants) with the space $\mathcal{H}(kL)$ of all positively curved metrics on the kth tensor of an ample line bundle $L \to X$ with Chern class $c_1(L) = [\omega]$. Moreover, it is clearly enough to establish the theorem in the case when k is a positive integer. The proof then exploits convexity properties along geodesics of certain functionals on the space $\mathcal{H}(L)$ equipped with the Mabuchi metric (see Section 1.4 for an outline of the proof).

As pointed out above Aubin's main motivation for his conjecture came from the existence problem for positively curved Kähler–Einstein metrics on a Fano manifold where the Kähler class $[\omega]$ is the integral class $c_1(-K_X)$, i.e. the first Chern class of the anti-canonical line bundle $-K_X$ of X. In this setting, which we will refer to as the *Fano setting,* he also conjectured an explicit optimal value for A which only depends on the dimension n of the Fano manifold. However, as explained in Section 6 there are simple counter-examples to the explicit value proposed by Aubin. Still, combining our arguments with previous work on finiteness properties of Fano manifolds [24, 48, 61] we deduce the following partial confirmation of Aubin's latter conjecture:

Theorem 1.2. — *When X is an n-dimensional Fano manifold and $[\omega]$ is the anti-canonical class the constant A can be taken to only depend on n (if B is allowed to depend on k).*

Coming back to the general setting in Theorem 1.1 we note that an immediate consequence of Theorem 1.1 is the first part of the following corollary (see Section 5 for the volume estimate).

Corollary 1.3. — *Let (X,ω) be a compact Kähler manifold such that $[\omega]$ is an integral class. For any non-positive function u in* $\mathrm{PSH}(X,\omega)$ *with a uniform lower bound on its energy, i.e. $\mathcal{E}_\omega(u) \geqslant -A$, there is a constant B such that for any $k \in [1,\infty[$*

$$\int_X e^{-ku}\mathrm{d}V \leqslant Bk^{n+1}, \tag{1.7}$$

where the constant B only depends on A. Equivalently, there is a positive constant δ only depending on A such that

$$\mathrm{Vol}_\omega\{u < -s\} \leqslant Ce^{-\delta s^{(n+1)/n}}$$

for any positive number s. In particular, it follows from (1.7) *that any function u in* $\mathrm{PSH}(X,\omega)$ *with finite (pluricomplex) energy has vanishing Lelong numbers.*

The notion of finite energy is recalled in the beginning of Section 1.6. The vanishing of the Lelong numbers in the previous corollary was first obtained by Guedj–Zeriahi [43] and the estimate

$$\int_X e^{-ku}\mathrm{d}V \leqslant C_k,$$

which is not as quantiative as the inequality (1.7), was observed in [11] (where it was used in the variational construction of Kähler–Einstein metrics on Fano manifolds). The main point of the present approach is thus that it furnishes the sharp asymptotic growth in k and s (the sharpness can be verified in simple examples). Of course, in the "local setting", i.e. the $\mathbb{C}^n$-setting considered below, the growth in k and s comes for free, as it follows immediately from homogenity. The point of the previous corollary is thus that it reveals that, even if there is a lack of homogenity, the growth in k and s in the global Kähler setting mimics the growth in the local setting. Interestingly, this is in contrast to capacity estimates where there is a discrepency between the growth behaviour in the Kähler setting and the local setting (compare [11, 43]).

1.1.2. The setting of a pseudoconvex domain in $\mathbb{C}^n$

Let now Ω be a pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary (for example the unit ball) and set $\omega := 0$. In this setting we let $\mathcal{H}_0(\Omega)$ be the convex cone of all smooth plurisubharmonic functions, i.e. $dd^c u \geqslant 0$, vanishing on the boundary $\partial\Omega$. Then the $(n+1)$-homogeneous functional

$$n!\mathcal{E}_0(u) = \frac{1}{(n+1)} \int_\Omega u(dd^c u)^n \tag{1.8}$$

is the usual generalization to $\mathbb{C}^n$ of (minus) the squared Dirichlet norm in the unit disc. In the paper [3] Aubin claims that the conjectured inequality holds in the setting of the unit ball in $\mathbb{C}^n$, but it appears that he only proved this under radial symmetry ([4, Cor. 8.3]), and, in fact, with a non-optimal constant (as explained in Section 6). Assuming only circular symmetry, i.e. invariance under the diagonal S^1-action on $\mathbb{C}^n$, our method of proof of Theorem 1.1 also yields the following generalization of Moser's inequality on the disc:

Theorem 1.4. — *The following Moser–Trudinger inequality holds for any S^1-invariant function in $\mathcal{H}_0(\mathcal{B})$, where $\mathcal{B}$ is the unit ball in $\mathbb{C}^n$:*

$$\log \int_{\mathcal{B}} e^{-u} dV \leqslant \frac{1}{(n+1)^{(n+1)}} \int_\Omega (-u)(dd^c u)^n + C_n \tag{1.9}$$

for a constant C_n. Moreover the multiplicative constant in the inequality is sharp.

Note that the sharp multiplicative constant in (1.9) coincides with the well-known sharp multiplicative constant in the Fano setting when $X = \mathbb{P}^n$, $[\omega] = c_1(-K_X)$ and $k = 1$ (and our proof shows that this is no coincidence). We conjecture that the symmetry assumption in the previous theorem may be removed. In this direction we will prove the following quasi-sharp Moser–Trudinger inequality for a general pseudoconvex domain (or more generally a hyperconvex one):

Theorem 1.5. — *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. Then, for any $\delta > 0$ there is a constant C (depending on δ) such*

$$\log \int_\Omega e^{-u} dV \leqslant \frac{1+\delta}{(n+1)^{(n+1)}} \int_\Omega (-u)(dd^c u)^n - (n-1)\log\delta + C \tag{1.10}$$

for any function u in $\mathcal{H}_0(\Omega)$. Moreover, for any domain Ω the limiting multiplicative constant $\frac{1}{(n+1)^{(n+1)}}$ is sharp. In particular, for any $\delta > 0$ there is a constant C_δ such that

$$\int_\Omega e^{(1-\delta)n(-u)^{(n+1)/n}} dV \leqslant C_\delta$$

for any u in $\mathcal{H}_0(\Omega)$ such that $\int_\Omega (-u)(dd^c u)^n = 1$.

The proof of the latter theorem is completely different than the previous one. The starting point is the observation that if the sharp Moser–Trudinger inequality holds in dimension $n-1$ then so does the following sharp *Brezis–Merle–Demailly type inequality* [(1)] in dimension n :

$$\int_\Omega e^{-u}\mathrm{d}V \leqslant A\left(1-\frac{1}{n^n}\mathcal{M}(u)\right)^{-1} \tag{1.11}$$

for any u in $\mathcal{H}_0(\Omega)$ such that $\mathcal{M}(u)^{1/n} < n$, where

$$\mathcal{M}(u) := \int_\Omega (\mathrm{dd}^c u)^n. \tag{1.12}$$

We then show that, conversely, a quasi-sharp version of the Brezis–Merle–Demailly inequality in dimension n implies the quasi-sharp Moser–Trudinger inequality above in the same dimension n and Theorem 1.5 then follows directly from induction over n. More precisely, the induction argument gives the following quasi-sharp version of the conjectural Brezis–Merle–Demailly type inequality above.

Theorem 1.6. — *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary, where $n > 1$. Then there is a constant A such*

$$\int_\Omega e^{-u}\mathrm{d}V \leqslant A\left(1-\frac{1}{n^n}\mathcal{M}(u)\right)^{-(n-1)} \tag{1.13}$$

for any function in $\mathcal{H}_0(\Omega)$ such that $\mathcal{M}(u)^{1/n} < n$.

In particular, this proves the sharp inequality in the case when $n = 2$.

In Section 7 we consider the problem of finding extremals for Moser–Trudinger type functionals that are parametrized by the multiplicative constants in the corresponding inequalities. In particular, we obtain solutions to the Euler–Lagrange equations for these functionals which are Monge–Ampère equations with exponential non-linearities. We also establish a "concentration/compactness" principle for the case of pseudoconvex domains.

1.2. Relations to previous results

The Kähler setting

On the two-sphere the inequality in Theorem 1.1 was first shown by Moser with the sharp constant $A = 1/2$. Subsequently, the general Riemann surface case was settled by Fontana [39] with the same sharp constant. Strictly

(1) See [22] for the case when $n = 1$, where the inequalities were introduced in context of blow-up analysis of PDEs - inequalities of a similar form for the complex Monge–Ampère operator in higher dimensions were first obtained by Demailly [32] and accordingly we will refer to such inequalities as Brezis–Merle–Demailly type inequalities (see Section 1.2 for further references to previous work).

speaking these latter inequalities were shown to hold for *any* smooth function u, under the different (but equivalent) normalization condition $\int_X u\omega = 0$. Then $-\mathcal{E}_\omega(u)$ coincides with the usual two-homogeneous Dirichlet energy and the growth rate with respect to k can hence be reduced, by scaling, to the case $k = 1$. It should however be emphasized that in higher dimensions this reduction argument breaks down, since the space $\mathcal{H}_0(X, \omega)$ is not preserved under scaling with positive numbers k. The sharp form of the Sobolev inequalities on the two-sphere in (1.6) was obtained by Beckner [5].

In the case when X admits a Kähler–Einstein metric the Moser–Trudinger inequality, for the anti-canonical class and for $k = 1$, was first shown by Ding–Tian [37] with $A = 1/V(X)$ equal to the inverse of the volume of $-K_X$. This is the sharp constant in case X admits holomorphic vector fields (see Lemma 6.1). More precisely, they showed that any potential of a Kähler–Einstein metric on X optimizes the corresponding Moser–Trudinger inequality (when $\mathrm{d}V$ is taken to depend on ω in a standard way). In case X has no holomorphic vector field the constant $A = 1/V(X)$ may be improved slightly as shown in the coercivity estimate of Phong–Song–Sturm–Weinkove [54] (confirming a previous conjecture of Tian).

In the case of a general Fano manifold Ding [36] obtained, using the Green function estimate of Bando–Mabuchi, a Moser–Trudinger inequality for all u in $\mathcal{H}_0(X, \omega)$ with a uniform positive lower bound ϵ on the Ricci curvature of the corresponding Kähler metric ω_u (for $k = 1$). The case of Theorem 1.1 for the anti-canonical class (but possibly no Kähler–Einstein metric) and with $k = 1$ was shown in [7], building on [17]. The approach in [7, 15] will be further developed in the present paper.

The setting of domains

Demailly [32] originally showed that a weaker version of inequality (1.11) is equivalent to a local algebra inequality previously obtained in [38] in the context of the study of birational rigidity of Fano manifolds. This latter inequality says that

$$\mathrm{lc}(\mathcal{I}) \geqslant n/(\mathrm{e}(\mathcal{I}))^{1/n}, \tag{1.14}$$

where $\mathrm{lc}(\mathcal{I})$ is the log canonical threshold of an ideal $\mathcal{I}$ of germs of holomorphic functions and $\mathrm{e}(\mathcal{I})$ is its Samuel multiplicity. A quasi-sharp version of the Brezis–Merle–Demailly type inequality (1.11) was recently shown by Åhag–Cegrell–Kołodziej–Phạm–Zeriahi [2]. More precisely it was shown that the inequality holds when raising the bracket in (1.11) to the power n. However the relation to the Moser–Trudinger inequality does not seem to have been noted before and we use it, among other things, to improve the inequality in [2] with one power. The proof uses the "thermodynamical formalism" recently introduced in [8] (in the Kähler setting) and shows that the Moser–Trudinger inequality is equivalent to yet another inequality, coinciding with the classical *logarithmic Hardy–Sobolev inequality* when $n = 1$. As explained in [8] the corresponding inequality in the Kähler setting amounts to the boundedness from below of Mabuchi's K-energy functional.

1.3. Added in the revision

In the first version of the present paper, as a preprint on ArXiv, the vanishing of Lelong numbers in Corollary 1.3 was claimed to hold in the more general setting when ω is only assumed to be semi-positive and big. But as kindly pointed out to us by Sébastien Boucksom there was a gap in our argument in the more general setting. However, recently the vanishing in question has been established in [30] in an even more general setting. The proof in [30] also uses weak geodesics, but in a rather different way than in the present paper.

After the first appearence of the present paper there has also been several other interesting developments. In the compact Kähler setting the sharp volume bound for Fano manifolds admitting Kähler–Einstein metrics, discussed in Section 6.3, has been settled in [40] (improving on the result in [10], where the existence of suitable S^1-action was assumed). Moreover, very recently the inequality in Theorem 1.1 was extended to general Kähler classes using an elegant approach involving quasi-plurisubharmonic envelopes [35]. The latter proof does not, however, appear to give a universal asymptotic control on the constants A and B. As for the setting of domains, Moser–Trudinger inequalities in the case of pseudoconvex domains have also been obtained independently in [42] (for non-sharp constants).

1.4. Outline of the proof of Theorems 1.1, 1.2

As is well-known a Kähler form ω is integral precisely when it can be realized as the (normalized) curvature form of a metric h on an ample line bundle $L \to X$. Abusing notation slightly this means that

$$\omega = dd^c \phi_0$$

where $h = e^{-\phi_0}$ is the expression of the metric h wrt a local holomorphic frame. Hence, ω_u is the curvature form of the metric on L with weight $\phi := \phi_0 + u$. The proof of Theorem 1.1 follows the same outline as the proof of the Moser–Trudinger inequality in [7, 17] concerning the case when $L = -K_X$ and ϕ_0 is the weight of a Kähler–Einstein metric - with some important modifications. The proof in [7, 17] is based on consideration of the functional

$$\mathcal{G}(\phi) := \log \int_X e^{-\phi} + \frac{1}{V}\mathcal{E}(\phi, \phi_0),$$

where we have used that $e^{-\phi}$ defines a global volume form on X (since $L = K_X$) and where $\mathcal{E}(\phi, \phi_0) := \mathcal{E}_\omega(\phi - \phi_0)$. The Moser–Trudinger inequality says that $\mathcal{G}$ is negative on the space $\mathcal{H}(-K_X)$ of positively curved metrics on $-K_X$. But $\mathcal{G}$ is geodesically concave on the space $\mathcal{H}(-K_X)$ equipped with the Mabuchi metric (see the next section) and the Kähler–Einstein condition says that ϕ_0 is a critical point of $\mathcal{G}$. Moreover, by definition $\mathcal{G}$ vanishes at $\phi = \phi_0$ and that ends the proof.

At first glance, not much of this argument works in our situation of a general line bundle $L \to X$. The functional

$$\phi \mapsto \log \int_X e^{-(\phi - \phi_0)} \mathrm{d}V$$

has no obvious concavity properties and we have in general nothing that corresponds to the Kähler–Einstein condition. To handle the lack of concavity, we use a different functional, defined for each point x in X :

$$\phi \mapsto \log(K_{\phi_0}(x)/K_\phi(x)),$$

where K_ϕ is the restriction to the diagonal of the Bergman kernel for the space of global sections $H^0(X, L + K_X)$ of the adjoint line bundle $L + K_X$, which is known to be concave by the results in [15, 16]. It then turns out that we can replace the Kähler–Einstein condition by a standard estimate for the Bergman kernel in terms of the volume form; see [7] where a similar argument was used. The remaining problem is then to get from an estimate of the Bergman kernel to an estimate of the metric on L itself. On a compact manifold, this can be done using the basic formula

$$\int_X K_\phi(x) e^{-\phi} = N$$

where N is the dimension of $H^0(X, K_X + L)$. The growth rate in k in the inequality of the theorem is a consequence of a the Bergman kernel estimate, using that $k\phi$ is the weight of a metric on the k th tensor power of L, written as kL in our additive notation.

As for Theorem 1.2 it is proved by noting that the Bergman kernel estimate can be made to be uniform over all Fano manifolds of the same dimension by picking a reference metric ϕ_0 such that the corresponding Kähler metric $\mathrm{dd}^c\phi_0$ has a universal lower bound on its Ricci curvature.

1.5. Acknowledgements

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1.6. Notation and preliminaries

Here we will briefly recall the notions of (quasi-) psh functions and finite energy spaces in the setting of compact manifolds X and domains Ω. In practice, it will, by approximation, be enough to prove the inequalities we will be interested in for smooth (or bounded) functions.

The setting of a compact manifold X

Let X be a compact complex manifold and ω a smooth real closed $(1,1)$-form on X such that $\omega \geqslant 0$. We will mainly be concerned with the case when $\omega > 0$, i.e. when (X, ω) is a Kähler manifold. Denote by $\mathrm{PSH}(X, \omega)$ the space of all ω-psh functions u on X, i.e. $u \in L^1(X)$ and u is upper-semicontinuous (usc) and

$$\omega_u := \omega + \frac{i}{2\pi}\partial\bar{\partial}u := \omega + dd^c u \geqslant 0,$$

in the sense of currents (the normalizations are made so that $dd^c \log |z|^2$ is a probability measure when $n = 1$). We will write $\mathcal{H}(X, \omega)$ for the interior of $\mathrm{PSH}(X, \omega) \cap \mathcal{C}^\infty(X)$ (called the space of Kähler potentials when $\omega > 0$) and $\mathcal{H}_0(X, \omega)$ for its subspace defined by the normalization $\sup_X u = 0$. We will also use the (non-standard) notion $\mathcal{H}(X, \omega)_b := \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ for the bounded functions in $\mathrm{PSH}(X, \omega)$. By the local theory of Bedford–Taylor the Monge–Ampère operator

$$\mathrm{MA}(u) := \omega_u^n / n!$$

is well-defined on $\mathcal{H}(X, \omega)_b$ and continuous under sequences decreasing to elements in $\mathcal{H}(X, \omega)_b$ as are all powers ω_u^p. In particular, the functional $\mathcal{E}_\omega$ (formula (1.3)) is well-defined and continuous in the previous sense. Following [11, 21] $\mathcal{E}_\omega$ may be extended to all of $\mathrm{PSH}(X, \omega)$ by setting

$$\mathcal{E}_\omega(u) := \inf_{v \in \mathcal{H}(X,\omega)_b,\, v \geqslant u} \mathcal{E}_\omega(v) \in [-\infty, \infty[.$$

Now the space $\mathcal{E}^1(X, \omega)$ of all ω-psh functions of *finite energy* may be defined as the set of all u such that $\mathcal{E}_\omega(u) > -\infty$. As explained in [11, 21] it coincides with the space with the same name introduced in [44].

Metrics/weights on a line bundle vs. ω-psh functions

In the integral case, i.e. when $[\omega] = c_1(L)$ for an ample holomorphic line bundle $L \to X$, the space $\mathrm{PSH}(X, \omega)$ may be identified with the space of (singular) Hermitian metrics on L with positive curvature current. More precisely, let s be a trivializing local holomorphic section of L, i.e. s is non-vanishing in a given open set U in X. First we identify an Hermitian metric $h_0 = \|\cdot\|$ on L with its *weight* ϕ_0, which is locally defined by the relation

$$\|s\|^2 = e^{-\phi_0}.$$

The (normalized) curvature ω of the metric is the globally well-defined $(1,1)$-current defined by the following local expression:

$$\omega = \mathrm{dd}^c\phi_0.$$

The identification with $\mathrm{PSH}(X, \omega)$ referred to above is now obtained by fixing ϕ_0 and letting $\phi \mapsto u := \phi - \phi_0$ so that $\mathrm{dd}^c\phi = \omega_u$. We will denote by $\mathcal{H}(L)$ the space of all metrics (/weights) on L with positive curvature form, $\mathrm{dd}^c\phi > 0$.

The setting of a domain Ω in $\mathbb{C}^n$

Let Ω be a bounded domain $\mathbb{C}^n$ (in this setting $\omega = 0$) which is *hyperconvex*, i.e. it admits a negative continuous psh exhaustion function (for example a pseudoconvex domain with Lipschitz continuous boundary). The main reason that we will consider general hyperconvex domains (with possibly non-smooth boundary) is that this property is preserved under Cartesian products. When Ω has smooth boundary we let $\mathcal{H}_0(\Omega)$ be the subspace of all smooth psh functions on $\overline{\Omega}$ such that $u = 0$ on $\partial\Omega$. Following [2, 25] (see also [6] for a comparison with the Kähler setting) it will also be convenient to use two singular versions of $\mathcal{H}_0(\Omega)$, namely $\mathcal{F}(\Omega)$ and $\mathcal{E}_1(\Omega)$, where the Monge–Ampère mass $\mathcal{M}(u)$ (1.12) and energy $\mathcal{E}_0(:= \mathcal{E})$ (1.8) are well-defined and finite, respectively. More precisely, let first $\mathcal{H}_0(\Omega)_b$ be the space all u in $\mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ such that $\mathcal{M}(u) < \infty$ and such that $\lim_{\zeta \to z} u(z) = 0$ for any $z \in \partial\Omega$ (called the space of psh "test-functions" $\mathcal{E}_0(\Omega)$ in [25]). Now $\mathcal{F}(\Omega)$ is defined as the space of all u such that there exists $u_j \in \mathcal{H}_0(\Omega)_b$ decreasing to u with $\mathcal{M}(u_j) \leqslant C$. The Monge–Ampère operator extends to $\mathcal{E}_0(\Omega)$ and is continuous under decreasing limits. As for the space $\mathcal{E}_1(\Omega)$ it is defined in a similar manner, but by demanding that $-\mathcal{E}(u_j) \leqslant C$ (instead of $\mathcal{M}(u_j) \leqslant C$). There is also an alternative characterization of $\mathcal{F}(\Omega)$ as the set of all u in the "domain of definition of the Monge–Ampère operator" such that u has finite total Monge–Ampère mass and with smallest maximal plurisubharmonic majorant equal to zero. For the purpose of the present paper it will in practice be enough to know that if $u \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$ such that $\lim_{\zeta \to z} u(z) = 0$ for any $z \in \partial\Omega$, then $u \in \mathcal{F}(\Omega)$ if $\int_\Omega (\mathrm{dd}^c u)^n < \infty$ and similarly $u \in \mathcal{E}^1(\Omega)$ if $\int_\Omega (-u)(\mathrm{dd}^c u)^n < \infty$ (see [2, 25]).

It may also be convenient to recall (even if, strictly speaking, it will not be needed) the approximation result in [26] saying that *any* negative psh function u on a hyperconvex domain Ω can be written as decreasing limit of "smooth test functions", i.e. psh functions u_j in $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^\infty(\Omega)$, vanishing on the boundary and with finite Monge–Ampère mass. As a consequence one may as well replace the space $\mathcal{H}_0(\Omega)_b$ in the previous definitions with the space of "smooth test functions" in the previous sense.

2. Moser–Trudinger inequalities on Kähler manifolds

Let X be an n-dimensional compact Kähler manifold and let L be ample line bundle over X. We will use the noation introduced in Section 1.6 and thus denote by $\mathcal{H}(L)$ the space of all metrics ϕ on L with positive curvature form. We fix an element $\phi_0 \in \mathcal{H}(L)$ and set $\omega := \mathrm{dd}^c\phi_0$, which defines a Kähler form on X. We also set

$$V = \int_X (\mathrm{dd}^c\phi)^n/n! = \int_X \omega^n/n! > 0$$

Finally, note that if ϕ, ϕ_0 are in $\mathcal{H}(L)$, then $\phi - \phi_0$ is a globally well-defined function on X.

2.1. Energy, geodesics and Bergman kernels (preliminaries)

Given ϕ and ϕ_0 in $\mathcal{H}(L)$ we define (minus) the relative energy by

$$\mathcal{E}(\phi,\phi_0) = \frac{1}{(n+1)!}\int_X (\phi-\phi_0)\sum_0^n (\mathrm{dd}^c\phi_0)^k \wedge (\mathrm{dd}^c\phi)^{n-k}$$

If $t \to \phi_t$ is a smooth curve in the affine space $\mathcal{H}(L)$, in the sense that the local time-derivatives

$$\dot\phi_t := \frac{\partial \phi_t}{\partial t}$$

exist and define smooth functions on X, then [10, Prop. 4.1]

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\phi_t,\phi_0) = \int_X \dot\phi_t (\mathrm{dd}^c\phi_t)^n/n!.$$

This formula, together with the normalization $\mathcal{E}(\phi_0,\phi_0) = 0$ can also be used to define $\mathcal{E}$.

A basic property of $\mathcal{E}$ is that it is linear along *geodesics* in $\mathcal{H}(L)$ and convex along subgeodesics defined wrt Mabuchi's Riemannian metric on $\mathcal{H}(L)$. For technical reasons we will work with the following weaker notion of geodesics. Given two smooth metrics ϕ_0 and ϕ_1 the corresponding geodesic ϕ_t is defined as the following envelope:

$$\phi_t := \Phi(z,t) := \sup_{\psi \in \mathcal{K}} \{\Psi(z,t)\}$$

where we have extended t to the strip $\mathcal{T} = [0,1] + i\mathbb{R}$ in $\mathbb{C}$ and $\mathcal{K}$ is the set of all continuous semi-positively curved metrics Ψ on the pull-back of L to $X \times \mathcal{T}$ such that $\psi_0 \leqslant \phi_0$ and $\psi_1 \leqslant \phi_1$. We will sometimes refer to a curve $\psi_t := \Psi(\,\cdot\,,t)$ above as a *subgeodesic*. When L is ample it was shown in [13] that Φ is a continuous solution to the Dirichlet problem for the Monge–Ampère operator on $M := X \times \mathcal{T}$, i.e.

$$(\mathrm{dd}^c\Phi)^{n+1} = 0$$

in the interior of M (in the usual sense of pluripotential theory) and on the boundary ∂M the metric Φ coincides with the $i\mathbb{R}$ invariant boundary data determined by ϕ_0 and ϕ_1. However, we will only need some very modest regularity properties of Φ, namely that Φ is locally bounded and that $\Phi(t,\cdot\,) = \phi_t$ converges uniformly to the given boundary data as t approaches $\partial\mathcal{T}$. As shown by a simple barrier argument this is always the case as long as L is semi-positive (see [17]). Indeed,

$$\chi_t := \max\{\phi_0 - A\Re t, \phi_1 - A(1-\Re t)\} \tag{2.1}$$

gives a candidate for the sup defining ϕ_t converging uniformly towards the right boundary values. Hence so does ϕ_t. Also note that, by imposing S^1-symmetry in the complex variable t we might as well replace $\mathcal{T}$ with an annulus $\mathcal{A}$.

Lemma 2.1. — *Let ϕ_t be a (weak) geodesic as above. Then $t \mapsto \mathcal{E}(\phi_t,\phi_0)$ is affine and continuous up to the boundary of* $[0,1]$. *Moreover, if* $\dot{\phi}_0$ denotes the right derivative of ϕ_t at $t=0$ (which exists by convexity), then

$$\frac{\mathrm{d}}{\mathrm{d}t}_{t=0^+}\mathcal{E}(\phi_t) \leqslant \int_X \dot{\phi}_0(\mathrm{dd}^c\phi_0)^n/n!,$$

As pointed out above this is well-known in the case when ϕ_t is smooth and follows immediately from the formula

$$\mathrm{d}_t\mathrm{d}_t^c\mathcal{E}(\phi_t,\phi_0) = \int_X (\mathrm{dd}^c\Phi)^{n+1}/n!, \tag{2.2}$$

where $\int_X \alpha$ denotes the push-forward (fiber-integral) of the top form α on $X \times \mathcal{T}$ under the projection from $X \times \mathcal{T}$ to $\mathcal{T}$.

The general case is shown in [14, Prop. 3.1]; see also Lemma 2.4 for the corresponding properties in the setting of domains.

Any element ϕ in $\mathcal{H}(L)$ defines an L^2 metric on $H^0(X, K_X + L)$,

$$\|u\|_\phi^2 = i^{n^2} \int u \wedge \bar{u} e^{-\phi},$$

where we have identified u with a holomorphic $(n, 0)$-form with values in the line bundle L. The Bergman kernel for this L^2-metric is denoted $K_\phi(x)$ is defined by

$$K_\phi(x) = i^{n^2} \sum u_j(x) \wedge \bar{u}_j(x)$$

where u_j is an orthonormal basis for $H^0(X, K_X + L)$. Alternatively,

$$K_\phi(x) = \sup_{H^0(X, K_X+L)} \{|u(x)|^2; \|u\|_\phi \leqslant 1\}. \tag{2.3}$$

Here the expression $|u(x)|^2$ depends on the choice of a trivialization of L near x, but $\log K_\phi$ is invariantly defined as the weight of a metric on $K_X + L$. As a consequence, the quotient of two Bergman kernels

$$K_\phi(x)/K_{\phi_0}(x)$$

is a global function on X, smooth if the sections in $H^0(X, K_X + L)$ have no common zeros.

We will use a result from [15, 19] saying that the weight

$$t \to \log K_{\phi_t}(x)$$

is, for any x fixed, convex along (sub)geodesics ϕ_t. Equivalently, this means that, for any fixed ϕ_0, the logarithm of $K_{\phi_t}(x)/K_{\phi_0}(x)$ is convex in t.

We will also have use for the following simple formula for the derivative of the Bergman kernel along a curve (see for example [18] or the appendix in [7]):

Lemma 2.2. — *Let ϕ_t be a smooth curve in $\mathcal{H}(L)$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}t} K_{\phi_t}(x) = \int_X \dot{\phi}_t |K_{\phi_t}(x, y)|^2 e^{-\phi_t}$$

where the off-diagonal Bergman kernel is

$$K_{\phi_t}(x, y) := \sum c_n u_j(x) \wedge \bar{u}_j(y)$$

for any orthonormal basis of $H^0(X, K_X + L)$, defining a section over $(K_X + L) \boxtimes \overline{(K_X + L)}$.

More generally, the lemma above also holds (with the same proof) if one-sided, say *right,* derivatives are used (as long as the right derivatives of ϕ_t are assumed to exist).

2.2. Moser–Trudinger type inequalities

The next proposition is the crux of the proof of the Moser–Trudinger inequalities.

Proposition 2.3. — *Let ϕ and ϕ_0 be two metrics in $\mathcal{H}(L)$, satisfying the normalizing condition*

$$\phi - \phi_0 \leqslant 0.$$

Assume that the Bergman kernel for ϕ_0 satisfies

$$K_{\phi_0} e^{-\phi_0} \leqslant C_1 (\mathrm{dd}^c \phi_0)^n / n! \tag{2.4}$$

Then

$$\inf_X (K_\phi / K_{\phi_0}) \geqslant e^{C_1 \mathcal{E}(\phi, \phi_0)}$$

Proof. — Join ϕ_0 and ϕ with a geodesic ϕ_t such that $\phi_1 = \phi$. By the previous lemma

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \log K_{\phi_t}(x) = \int_X -\dot{\phi}_0 \frac{|K_{\phi_0}(x,y)|^2}{K_{\phi_0}(x)} e^{-\phi_0}.$$

Since ϕ_t is a geodesic, ϕ_t is convex in t, so

$$\dot{\phi}_0 \leqslant \phi - \phi_0 \leqslant 0.$$

Hence, since by Cauchy's inequality

$$|K_{\phi_0}(x,y)|^2 \leqslant K_{\phi_0}(x) K_{\phi_0}(y),$$

$$-\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \log K_{\phi_t}(x) \leqslant \int_X -\dot{\phi}_0 K_{\phi_0}(y) e^{-\phi_0},$$

which in turn is dominated by

$$C_1 \int_X -\dot{\phi}_0 (\mathrm{dd}^c \phi_0)^n / n! \leqslant -C_1 \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{E}(\phi_t, \phi_0)$$

by the definition of C_1 (formula (2.4)) and Lemma 2.1 which also gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} \mathcal{E}(\phi_t, \phi_0) = \mathcal{E}(\phi, \phi_0),$$

since $\mathcal{E}(\phi_t, \phi_0)$ is affine in t. Now we use that $f(t) := -\log K_{\phi_t}$ is concave. Therefore

$$f(1) - f(0) \leqslant f'(0)$$

which means that

$$\log K_{\phi_0} - \log K_\phi \leqslant f'(0) \leqslant -C_1 \mathcal{E}(\phi, \phi_0)$$

which completes the proof. □

Now it only remains to convert this estimate of the Bergman kernel to an estimate of the integral of $e^{-\phi}$. Here we use

$$\int_X K_\phi e^{-\phi} = N := \dim H^0(X, L + K_X) \tag{2.5}$$

for any locally bounded ϕ. Let C_1 and C_2 be constants satisfying

$$C_2 \mathrm{d}V \leqslant K_{\phi_0} e^{-\phi_0} \leqslant C_1 (\mathrm{dd}^c \phi_0)^n / n! \tag{2.6}$$

where $\mathrm{d}V$ is a fixed volume form on X (the same constant C_1 appeared in the previous proposition). Note that $L+K_X$ is basepoint free precisely when C_2 can be taken to be strictly positive.

By the previous proposition and (2.6) we have for any x in X

$$K_\phi \geqslant K_{\phi_0} e^{C_1 \mathcal{E}(\phi,\phi_0)} \geqslant C_2 e^{\phi_0} \mathrm{d}V e^{C_1 \mathcal{E}(\phi,\phi_0)}, \tag{2.7}$$

so it follows that

$$\int_X e^{-(\phi-\phi_0)} \mathrm{d}V \leqslant C_2^{-1} N e^{-C_1 \mathcal{E}(\phi,\phi_0)}.$$

We collect this in the next theorem which, as explained below, implies Theorem 1.1 in the introduction.

Theorem 2.4. — *Assume that the Bergman kernel for ϕ_0 satisfies* (2.6)*. Then for any other $\phi \in \mathcal{H}(L)$, satisfying*

$$\phi - \phi_0 \leqslant 0.$$

we have that

$$\log \int_X e^{-(\phi-\phi_0)} \mathrm{d}V \leqslant \log(N/C_2) - C_1 \mathcal{E}(\phi, \phi_0).$$

We say (cf [7, 16]) that the metric ϕ_0 is *balanced in the adjoint sense* if there is a constant C such that

$$K_{\phi_0} e^{-\phi_0} = C (\mathrm{dd}^c \phi_0)^n / n!.$$

When $\mathrm{d}V := (\mathrm{dd}^c \phi_0)^n / n!$ this amounts to saying that the constants C_1 and C_2 in (2.6) can both be chosen to be equal to C and integrating over X we see that in this case $C = N/V$. We thus immediately get the next corollary.

Corollary 2.5. — *With assumptions as in Theorem 2.3, assume in addition that ϕ_0 is balanced in the adjoint sense. Then*

$$\log \int_X e^{-(\phi-\phi_0)} (\mathrm{dd}^c \phi_0)^n / n! \leqslant -\frac{N}{V} \mathcal{E}(\phi, \phi_0) + \log V$$

As an example of this, let us look at the case $L = -K_X$. Then $H^0(X, K_X + L) = \mathbb{C}$, i.e. $N = 1$, and

$$K_{\phi_0}(x) = 1/\int_X e^{-\phi_0}.$$

Hence the condition that ϕ_0 be balanced in the adjoint sense means that

$$(\mathrm{dd}^c\phi_0)^n/(Vn!) = \left(\int_X e^{-\phi_0}\right)^{-1} e^{-\phi_0}$$

which means that ϕ_0 is the potential of a Kähler–Einstein metric. Then the corollary becomes

$$\log\int_X e^{-\phi} \leqslant \log\int_X e^{-\phi_0} - \mathcal{E}(\phi,\phi_0)$$

since $N = 1$. This is the Moser–Trudinger inequality first proved in [37] (using a different method). Note that the assumption that $\phi \leqslant \phi_0$ is unnecessary here since both sides scale the same way if we subtract a constant from ϕ.

2.2.1. Proof of Theorem 1.1

Next we consider asymptotic versions of Theorem 2.3, when we replace L be kL, with k a large integer. Then it follows from well-known Bergman kernel asymptotics due to Bouche and Tian (see [63] and references therein for various refinements) that for any fixed smooth and strictly positively curved ϕ_0

$$K_{k\phi_0}e^{-k\phi_0} = (\mathrm{dd}^c k\phi_0)^n/n!(1 + O(k^{-1})) \tag{2.8}$$

Hence in (2.6) we can take C_1 equal to

$$1 + O(k^{-1})$$

and $C_2 = C_2(k)$ equal to

$$k^n(1 + O(k^{-1}).$$

Integrating (2.8) we also get the well known formula

$$N_k = Vk^n + o(k^{n-1})$$

for the dimension of the space of global sections of $K_X + kL$. Hence $N_k/C_2(k)$ can be estimated by a constant independent of k. Altogether this finishes the proof of Theorem 1.1 when k is a sufficently large integer. In fact, this implies the case of a general $k \in [1,\infty[$ by a simple comparison argument.

2.3. Uniformity over all Fanos (proof of Theorem 1.2)

We start with the following essentially well-known lemma which is proved using Moser iteration (see [51, Thm. 7] which is stated for eigenfunctions, but the proof in general is the same):

Lemma 2.6. — *Let (X,g) be a Riemannian manifold of real dimension $2n>2$ and let a_g and b_g be constants such that the following Sobolev inequality holds for any function F on X such that F and its gradient are in L^2 :*

$$\left(\int_X |F|^{2\sigma}\mathrm{d}V_g\right)^{1/\sigma} \leqslant \left(a_g\int_X |\nabla_g F|^2\mathrm{d}V_g) + b_g\int_X |F|^2\mathrm{d}V_g\right), \quad \sigma = n/(n-1)$$

For any positive function H such that $\Delta_g H \geqslant -\lambda H$ there is a constant C_g only depending on a_g and b_g such that

$$\|H\|_{L^\infty(X)} \leqslant C_g\lambda^n \|H\|_{L^1(X,g)} \tag{2.9}$$

Let us now assume that $L\to X$ is a an ample line bundle with a fixed smooth positively curved weight ϕ_0 such that the Kähler form $\omega_0 := \mathrm{dd}^c\phi_0$ has a lower bound δ on its Ricci curvature:

$$\mathrm{Ric}\,\omega_0 \geqslant \delta\omega_0 \tag{2.10}$$

Then we claim that there is a constant C_δ only depending on δ such that the Bergman kernel $K_{k\phi_0}(x)$ of the space $H^0(kL+K_X)$ has the following point-wise upper bound:

$$K_{k\phi_0} \leqslant C_\delta k^n(\mathrm{dd}^c\phi_0)^n/n! \tag{2.11}$$

To see this let g be the Riemannian metric on X corresponding to ω_0. By [46] the corresponding constants a_g and b_g only depend on the lower bound δ of the Ricci curvature of g and the lower bound on the volume V of g :

$$a_g := \frac{2n-1}{n(n-1)\delta V^{1/n}}, \quad b_g = \frac{1}{V^{1/n}}$$

Let now f_k be an element in $H^0(kL+K_X)$ and write

$$H := |f_k|^2 e^{-k\phi_0}/((\mathrm{dd}^c\phi_0)^n/n!)$$

Then it follows immediately from the definition of Ricci curvature and the fact that $\log|f_k|^2$ is locally psh that

$$\mathrm{dd}^c\log H \geqslant -k\omega_0 - \delta\omega_0$$

and hence $\mathrm{dd}^c H \geqslant -(k+\delta)H\omega_0$. Applying the previous Lemma to H with $\lambda := n(k+\delta)$ now gives

$$|f_k|^2 e^{-k\phi_0} \leqslant C_\delta k^n \frac{(\mathrm{dd}^c\phi_0)^n}{n!}\int_X |f_k|^2 e^{-k\phi_0}.$$

By the extremal definition of $K_{k\phi_0}$ this finally proves the inequality (2.11).

Let us now assume that X is a Fano manifold and take $L := -K_X$ so that $V := c_1(-K_X)^n/n!$. As shown by Tian–Yau [61] one may always choose $\omega := \omega_0 \in c_1(-K_X)$ so that $1/\delta$ in (2.10) only depends on an upper bound on V (since changing ϕ_0 only changes the additive constant B_k we are allowed to choose ϕ_0 and $\mathrm{d}V$). As later shown in [24, 48] the volume V of an n-dimensional Fano has a universal bound $V \leqslant c_n$ and hence ϕ_0 may be chosen so that the Bergman kernel estimate (2.11) holds with a constant C_δ only depending on the dimension n. The proof of Theorem 1.2 is now concluded by invoking Theorem 2.4.

Remark 2.7. — One may also ask whether there is universal *lower* bound on $\inf_X(K_{k\phi}e^{-k\phi}/(\mathrm{dd}^c\phi)^n)$ in terms of a positive lower bound δ of the Ricci curvature of $\mathrm{dd}^c\phi$ and the dimension n of the Fano manifold? If one instead considers the Bergman kernel $\widetilde{K}_{k\phi}$ defined wrt the L^2-norm $\int_X |f|^2 e^{-k\phi}(\mathrm{dd}^c\phi)^n$ on $H^0(X, kK)$ then a lower bound for $\widetilde{K}_{k\phi}e^{-k\phi}$ was obtained by Tian [59] when $n = 2$, for all $\phi = \phi_t$ appearing in Aubin's continuity path, and the case of a general dimension n was recently settled in [56].

3. Moser–Trudinger inequality in the ball under S^1-invariance

In this section we will look at estimates for integrals of $e^{-\phi}$, where ϕ is plurisubharmonic in appropriate pseudoconvex domains Ω in $\mathbb{C}^n$, eventually specializing to the case of the ball $\mathcal{B}$. As in the previous section we let $K_\phi(x)$ be the Bergman kernel at the diagonal for the plurisubharmonic weight function ϕ. It follows from the results in [15] that $\log K_{\phi_t}(x)$ is convex in t if $t \to \phi_t$ is a geodesic in the space of plurisubharmonic functions in Ω (see below).

We say that a function f is S^1-invariant if $f(e^{i\theta}z) = f(z)$. (Here $e^{i\theta}$ acts diagonally so that $e^{i\theta}(z_1, \ldots, z_n) := (e^{i\theta}z_1, \ldots, e^{i\theta}z_n)$).

3.1. Bergman kernels and plurisubharmonic variations

Proposition 3.1. — *Assume ϕ is plurisubharmonic in an S^1-invariant domain Ω that contains the origin in its interior and that ϕ is also S^1-invariant. Then*

$$K_\phi(0, \zeta) = 1/\int e^{-\phi}$$

for all ζ in Ω.

Proof. — By definition, $K_\phi(0,\zeta)$ is antiholomorphic in ζ and by uniqueness of Bergman kernels it must also be S^1-invariant. Hence it is a constant, and since

$$\int K_\phi(0,\cdot)e^{-\phi} = 1$$

the proposition follows. □

The next proposition then follows immediately from the plurisubharmonic variation of Bergman kernels (cf [15]).

Proposition 3.2. — *Let ϕ_t be a subgeodesic of S^1-invariant plurisubharmonic functions in an S^1-invariant pseudoconvex domain Ω that contains the origin in its interior. Then*

$$t \mapsto \log\left(\int e^{-\phi_t}\right)$$

is concave.

3.2. Energy and geodesics

In this section we will adapt the results about geodesics and energy in the compact Kähler setting to the setting of domains. In principle all the previous properties go through in this latter setting. The main technical difference is that one has to be a bit careful when performing integration by parts, due to the presence of the boundary. For this reason it will be convenient to work in the singular setting of the finite energy class $\mathcal{E}(\Omega)$ (compare Section 1.6).

In a domain Ω we have a variant of the energy $\mathcal{E}$, which in case ϕ_0 and ϕ are smooth is defined by

$$\mathcal{E}(\phi,\phi_0) = \frac{1}{(n+1)!}\int_\Omega (\phi-\phi_0)\sum_0^n (\mathrm{dd}^c\phi_0)^k \wedge (\mathrm{dd}^c\phi)^{n-k}$$

and when $\phi = \phi_0 = 0$ on $\partial\Omega$ integration by parts show that $\mathcal{E}(\phi,\phi_0) = \mathcal{E}(\phi) - \mathcal{E}(\phi_0)$ (compare the lemma below), where

$$\mathcal{E}(\phi) := \mathcal{E}_0(\phi) := \mathcal{E}(\phi, 0)$$

so that

$$\mathcal{E}(\phi) = \frac{1}{(n+1)!}\int_\Omega \phi(\mathrm{dd}^c\phi)^n.$$

Moreover, integration by parts also give

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\phi_t,\phi_0) = \int_{\mathcal{B}} \dot{\phi}_t(\mathrm{dd}^c\phi_t)^n/n!$$

We will need the following generalization:

Lemma 3.3. — *Let ϕ and ψ be in $\mathcal{E}^1(\Omega)$). Then*

$$\frac{\mathrm{d}}{\mathrm{d}t}_{t=0^+}\mathcal{E}(\phi+t(\psi-\phi))=\int_\Omega(\psi-\phi)(\mathrm{dd}^c\phi)^n/n!$$

Moreover, the following cocycle relation holds $\mathcal{E}(\phi)-\mathcal{E}(\psi)=\mathcal{E}(\phi,\psi)$.

Proof. — Assume first that ϕ and ψ are in $\mathcal{H}_0(\Omega)_b$. In this class one may integrate by parts just as in the smooth case (using the assumption on finite Monge–Ampère mass; see [25] and references therein) and hence expanding $\mathcal{E}(\phi+t(\psi-\phi))$ and integrating by parts gives

$$\mathcal{E}(\phi+t(\psi-\phi))=t\int_\Omega(\psi-\phi)(\mathrm{dd}^c\phi)^n+O(t^2)I$$

where I is a sum of terms of the form $\int(\psi-\phi)(\mathrm{dd}^c\phi)^{n-j}(\mathrm{dd}^c\psi)^j$ which are finite since ϕ and ψ are in $\mathcal{H}(\Omega)_b$. This finishes the proof in the case of the class $\mathcal{H}_0(\Omega)_b$. Finally, given ϕ and ψ in $\mathcal{E}^1(\Omega)$) we take sequences ϕ_j and ψ_k in $\mathcal{H}(\Omega)_b$, decreasing to ϕ and ψ respectively. By the previous case we have

$$\mathcal{E}(\phi_j+t(\psi_k-\phi_j))=\int_0^t\int_\Omega(\psi_k-\phi_j)(\mathrm{dd}^c(\phi_j+s(\psi_k-\phi_j)))^n\mathrm{d}s=:\int_0^t g_{k,j}(s)\mathrm{d}s$$

By well-known continuity properties [25] and the finite energy assumptions letting first j and then k tend to infinity shows that the previous formula holds with ϕ_j and ψ_k replaced with ϕ and ψ, respectively. Moreover, for the same reason the corresponding density $g(s)$ is continuous wrt s and that ends the proof of the derivative formula in the general case. Finally, the previous formula implies the cocycle relation by integrating along the line $t\mapsto\phi+t(\psi-\phi)$ (note that by a well-known Cauchy–Schwartz type estimate all terms in $\mathcal{E}(\phi,\psi)$ are finite). □

Next we turn to the definition of geodesic segments in the setting of domains. Given, say ϕ_0 and ϕ_1 on Ω which are psh and smooth up to the boundary, where they vanish, the corresponding geodesic ϕ_t is defined by replacing the space $\mathcal{H}(L)_b$ with the space of all bounded psh functions tending to zero at the boundary. More precisely, a geodesic is defined as the following regularized envelope, where $M:=\Omega\times\mathcal{A}$ (with $\mathcal{A}$ denoting an annulus):

$$\phi_t:=\Phi(z,t):=\sup_{\Psi\in\mathcal{K}}\{\Psi(z,t)\}^*$$

where $\mathcal{K}$ is the set of all psh functions $\Psi\in\mathrm{PSH}\cap L^\infty(M)$ such that $\Psi^*\leqslant f$ on ∂M, where f is the function on ∂M defined as follows: decomposing $\partial M:=B_1\cup B_2:=\partial\Omega\times\mathcal{A}\cup\Omega\times\partial\mathcal{A}$ we let $f=0$ on B_1 and $f=\phi_i$ for $i=1,2$ on the two different components of B_2. In particular, if ϕ_0 and ϕ_1 are continuous on $\overline{\Omega}$ then so is the boundary data f. Just as in the setting of compact Kähler manifolds we may as well, by symmetry, replace the bounded domain $\mathcal{A}$ with a strip so that, for t real, ϕ_t gets identified with a function on $\Omega \times [0,1]$. In this setting there is a similar construction of a barrier χ_t as in the compact case, namely

$$\chi_t := \max\{\phi_0 - A\Re t, \phi_1 - A(1-\Re t), A\rho\} \tag{3.1}$$

where ρ is a psh defining function of Ω (e..g. $\rho = |z|^2 - 1$ in the ball case). The barrier induces, as before, an extension F of f such that $F \in \mathcal{C}^0(\overline{M}) \cap \mathrm{PSH}(M)$ and hence Φ is bounded on M and converges uniformly towards the right boundary values. In fact, given the extension F above it follows from Theorem 1.1 in [20] (since M is hyperconvex) that $\Phi \in \mathcal{C}^0(\overline{M}) \cap \mathrm{PSH}(M)$ with

$$(\mathrm{dd}^c\Phi)^{n+1} = 0, \quad \text{in } M$$

(but strictly speaking we will not need the continuity, only the boundedness and the uniform boundary behavior as $t \to 0$ and $t \to 1$). In particular we obtain a continuous curve ϕ_t in the space $\mathrm{PSH} \cap L^\infty(\Omega)$.

Lemma 3.4. — *Let ϕ_t be a geodesic segment as above.*

- *For any fixed t we have that $\phi_t \in \mathcal{E}^1(\Omega)$ and if $\dot{\phi}_0$* denotes the right derivative of ϕ_t at $t = 0$ (which exists by convexity), then

$$\frac{\mathrm{d}}{\mathrm{d}t}_{t=0^+}\mathcal{E}(\phi_t) \leqslant \int_{\mathcal{B}} \dot{\phi}_0 (\mathrm{dd}^c\phi_0)^n/n!,$$

- *$t \mapsto \mathcal{E}(\phi_t)$ is affine and continuous on* $[0,1]$.

Proof. — As explained above $\chi_t \leqslant \phi_t \leqslant 0$ where χ_t is a maximum of functions in $\mathcal{E}^1(\Omega)$ and hence χ_t is also in the space $\mathcal{E}^1(\Omega)$ [25]. By Lemma 3.3 the functional $\mathcal{E}$ is increasing on $\mathcal{E}^1(\Omega)$ (since its differential is a positive measure) and hence $-\infty < \mathcal{E}(\chi_t) \leqslant \mathcal{E}(\phi_t^j) \leqslant 0$ for any sequence ϕ_t^j in $\mathcal{H}_0(\Omega)_b$ decreasing to ϕ, which proves the first claim. Next, we recall that $\mathcal{E}$ is concave on $\mathcal{E}^1(\Omega)$ (wrt the usual affine structure) which for example follows from the formula for $d_t d_t^c \mathcal{E}(\phi_t)$ discussed below. In particular,

$$\frac{1}{t}(\mathcal{E}(\phi_t) - \mathcal{E}(\phi_0)) \leqslant \frac{1}{t}\int_\Omega (\phi_t - \phi)(\mathrm{dd}^c\phi_0)^n/n!,$$

so that letting $t \to 0^+$ proves the first point. As for the last point integration by parts show that the formula (2.2) for $\mathrm{d}_t\mathrm{d}_t^c\mathcal{E}(\phi_t)$ is still valid in the smooth case. However, as we will need the formula in a singular setting we instead refer to the result proved in [2] which implies that if $\Phi \in \mathcal{F}(\Omega \times \mathcal{A})$ whose slices ϕ_t are in $\mathcal{E}^1(\Omega)$ then the analogue of formula (2.2) holds (i.e. for $X = \Omega$) in the sense of currents. Finally, since $\phi_t \to \phi_0$ uniformly as $t \to 0$ we have that $\mathcal{E}(\phi_t) \to \mathcal{E}(\phi_0)$ as $t \to 0$ [25] and similarly for $t \to 1$ and that ends the proof. □

3.3. The case of the ball and the proof of Theorem 1.4

We now take Ω to be the unit ball and make a special choice of reference function ϕ_0 as

$$\phi_0 = (n+1)[\log(1+|z|^2) - \log 2].$$

This is a potential of the Fubini–Study metric on $\mathbb{P}^n$ and satisfies the Kähler–Einstein equation

$$(\mathrm{dd}^c\phi_0)^n/n! = a_n e^{-\phi_0} \tag{3.2}$$

We first prove an estimate for the Bergman kernel at the origin. By Proposition 3.1 this amounts to an estimate of the integral of $e^{-\phi}$ in the S^1-invariant case, but we prefer to argue first in the general case, since we feel the estimate for Bergman kernels has independent interest.

Proposition 3.5. — *Let ϕ be a smooth plurisubharmonic function in the ball $\mathcal{B}$ that vanishes on the boundary. Then*

$$-\log K_\phi(0) \leqslant \log \int_{\mathcal{B}} e^{-\phi_0} - b_n \mathcal{E}(\phi, \phi_0)$$

where $b_n = (a_n \int e^{-\phi_0})^{-1}$.

Proof. — As in the compact setting the proof uses geodesics. We connect ϕ and ϕ_0 by a geodesic ϕ_t such that $\phi_1 = \phi$. Then

$$g(t) := \log K_{\phi_t} - b_n \mathcal{E}(\phi_t, \phi_0)$$

is a convex function of t. We claim that $g'(0) \geqslant 0$. For this we use the same formula as before for the derivative of the Bergman kernel

$$\frac{\mathrm{d}}{\mathrm{d}t} K_{\phi_t}(x) = \int_{\mathcal{B}} \dot{\phi}_t |K_{\phi_t}(x,y)|^2 e^{-\phi_t}.$$

Take $x = 0$ and $t = 0$. Then, since ϕ_0 is S^1-symmetric

$$K_{\phi_0}(0,y) = 1 \Big/ \int_{\mathcal{B}} e^{-\phi_0}$$

by Proposition 3.1. Therefore

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \log K_{\phi_t}(0) = \int_{\mathcal{B}} \dot{\phi}_0 e^{-\phi_0} \Big/ \int_{\mathcal{B}} e^{-\phi_0}.$$

Combining this with the Kähler–Einstein condition (3.2) we get, also using Lemma 3.4, that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \log K_{\phi_t}(0) = b_n \int_X \dot{\phi}_0 (\mathrm{dd}^c\phi_0)^n/n! \geqslant b_n \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathcal{E}(\phi_t, \phi).$$

so $g'(0) \geqslant 0$ as claimed. Since g is moreover convex we get $g(1) \geqslant g(0)$ or explicitly

$$\log K_\phi(0) - b_n \mathcal{E}(\phi, \phi_0) \geqslant \log K_{\phi_0}(0).$$

Invoking Proposition 3.1 again the proposition follows. □

From here we can not continue as in the compact case since we have no counterpart of (2.5). It seems plausible to conjecture that for any compact K in the ball

$$\int_K K_\phi(z,z)e^{-\phi(z)} \leqslant C(K,\phi)$$

where the constant depends only on K and, say,

$$\int_{\mathcal{B}} (\mathrm{dd}^c(\phi + |z|^2))^n.$$

If this were true we could follow a route similar to what we did in the case of a compact manifold and obtain sharp estimates for

$$\int_K e^{-\phi}$$

for functions that are not necessarily S^1-invariant. The most one could hope for in this direction would be

$$\int_{\mathcal{B}} (1-|z|^2)^{n+1} K_\phi(z,z)e^{-\phi(z)} \leqslant C(\phi)$$

with the same dependence on ϕ. We do not know if either of these estimates hold.

Instead we now introduce the additional assumption that ϕ be S^1-invariant. We then get, by Proposition 3.1, that

$$\log \int_{\mathcal{B}} e^{-\phi} \leqslant \log \int_{\mathcal{B}} e^{-\phi_0} - b_n \mathcal{E}(\phi,\phi_0)$$

if ϕ is any smooth plurisubharmonic function in the ball, vanishing on the boundary and S^1-invariant.

As it stands the constant here is not optimal. An easy way to improve it is to replace our "reference" ϕ_0 by

$$\phi_0^\epsilon := (n+1)[\log(\epsilon^2 + |z|^2) - \log(\epsilon^2 + 1)]. \tag{3.3}$$

This amounts to replacing the unit ball by a larger ball of radius $1/\epsilon$ which brings us closer and closer to all of $\mathbb{P}^n$, where the same argument is known to give an optimal constant. Then

$$(\mathrm{dd}^c\phi_\epsilon)^n/n! = a_n(\epsilon)e^{-\phi_\epsilon},$$

and as before we let

$$b_n(\epsilon) = 1/\left(a_n(\epsilon)\int e^{-\phi_\epsilon}\right).$$

By the Kähler–Einstein equation for ϕ_ϵ

$$b_n(\epsilon) = n!/\int (\mathrm{dd}^c\phi_\epsilon)^n.$$

The integral here is easily computed using Stokes' theorem

$$\begin{aligned}\int_{|z|<1} (\mathrm{dd}^c\phi_\epsilon)^n &= \int_{|z|=1} \mathrm{d}^c\phi_\epsilon \wedge (\mathrm{dd}^c\phi_\epsilon)^{n-1} \\ &= (n+1)^n(1+\epsilon^2)^{-n}\int_{|z|=1} \mathrm{d}^c|z|^2 \wedge (\mathrm{dd}^c|z|^2)^{n-1} \\ &= (n+1)^n(1+\epsilon^2)^{-n}\int_{|z|<1} (\mathrm{dd}^c|z|^2)^n \\ &= (n+1)^n(1+\epsilon^2)^{-n} n!\pi^{-n}|\mathbb{B}_n| = (n+1)^n(1+\epsilon^2)^{-n}.\end{aligned}$$

Hence $b_n(\epsilon)$ is asymptotic to $n!/(n+1)^n$ as ϵ goes to zero (coinciding with the inverse of the volume of $\mathbb{P}^n$, as it must). We have

$$\mathcal{E}(\phi_\epsilon) = (n+1)^{-1}\int_{\mathcal{B}} \phi_\epsilon (\mathrm{dd}^c\phi_\epsilon)^n/n!$$

which by the Kähler–Einstein equation equals

$$-a_n(\epsilon)\int_{\mathcal{B}} \log(\epsilon^2+|z|^2)e^{-\phi_\epsilon}$$

plus a quantity tending to zero with ϵ. Thus

$$b_n\mathcal{E}(\phi_\epsilon) = -\int_{\mathcal{B}} \log(\epsilon^2+|z|^2)e^{-\phi_\epsilon}/\int_{\mathcal{B}} e^{-\phi_\epsilon}.$$

This is the integral of $-\log(\epsilon^2+|z|^2)$ against a sequence of measures that tend to a Dirac unit mass at the origin, and it is easily seen to be asymptotic to a constant plus $-\log\epsilon^2$. On the other hand

$$-\log\int_{\mathcal{B}} e^{-\phi_\epsilon}$$

is also asymptotic to $-\log\epsilon^2$ plus a constant. All in all this proves Theorem 1.4 stated in the introduction.

Notice that there seems to be no extremal function for the inequality. For any nonzero ϵ, ϕ_0^ϵ is an extremal by construction, but these functions tend to $(n+1)\log|z|^2$, which has infinite energy.

We do not know if Theorem 1.4 holds without our assumption of S^1-symmetry except for $n=1$, see [53] where a symmetrization argument can be used. Our methods also have bearings on symmetrization properties in the present higher dimensional setting of domains in $\mathbb{C}^n$ [9]. In Section 4 we shall use a different argument to prove the inequality "modulo ϵ" without assuming S^1-invariance.

4. Moser–Trudinger and Brezis–Merle–Demailly type inequalities on domains in $\mathbb{C}^n$

Let Ω be a bounded hyperconvex domain in $\mathbb{C}^n$ (we will use the notation introduced in Section 1.6 for the corresponding function spaces and functionals). We may then set the reference form ω_0 to be the zero-form: $\omega_0 = 0$ and use the notation

$$\mathcal{E}(u) := \mathcal{E}_{\omega_0}(u) = \frac{1}{(n+1)!}\int_\Omega u(\mathrm{dd}^c u)^n$$

It will also be convenient to write

$$\mathcal{M}(u) := \int_\Omega (\mathrm{dd}^c u)^n$$

We will say that the *sharp Moser–Trudinger (M-T) inequality* holds for the domain Ω if there is a constant C such that

$$\log \int_\Omega e^{-u}\mathrm{d}V \leqslant -\frac{n!}{(n+1)^n}\mathcal{E}(u) + C \tag{M-T}$$

for any $u \in \mathcal{E}_1(\Omega)$. Similarly, the *quasi-sharp M-T inequality* is said to hold on Ω if for any $\delta > 0$ the previous inequality holds when the factor $n+1$ in front of $\mathcal{E}(u)$ is replaced by $n+1-\delta$ and the constant C by $C - \log(\delta^{(n-1)})$.

The *sharp Brezis–Merle–Demailly (B-M-D) inequality* is said to hold for the domain Ω if there is a constant A such that

$$\int_\Omega e^{-u}\mathrm{d}V \leqslant A\left(1 - \frac{1}{n^n}\mathcal{M}(u)\right)^{-1} \tag{B-M-D}$$

for any $u \in \mathcal{F}(\Omega)$ such that $\mathcal{M}(u) := \int_\Omega (\mathrm{dd}^c u)^n < n^n$.

It will also be convenient to use the following equivalent formulations of the quasi-sharp Moser–Trudinger and Brezis–Merle–Demailly inequalities:

$$\int_\Omega e^{-(n+1-\delta)u}\mathrm{d}V \leqslant C\delta^{-(n-1)}e^{-(n+1-\delta)n!\mathcal{E}(u)} \tag{M-T$'$}$$

for some positive constant C and (when $n > 1$) there is a positive constant A such that

$$\int_\Omega e^{-(n-\delta)u}\mathrm{d}V \leqslant A\delta^{-(n-1)} \tag{B-M-D$'$}$$

for all $u \in \mathcal{F}(\Omega)$ such that $\mathcal{M}(u) = 1$

4.1. M-T in $\mathbb{C}^n$ implies B-M-D in $\mathbb{C}^{n+1}$

Proposition 4.1. — *The (quasi-) sharp Moser–Trudinger inequality on $\Omega \subset \mathbb{C}^n$ implies the (quasi-) sharp Brezis–Merle–Demailly inequality on $\Omega \times D \subset \mathbb{C}^{n+1}$. More generally, the (quasi-) sharp Moser–Trudinger inequality on the ball in $\mathbb{C}^n$ implies the (quasi-) sharp Brezis–Merle–Demailly inequality on any hyperconvex domain in $\mathbb{C}^{n+1}$.*

Proof. — Let us start with the sharp case. Given $u \in \mathcal{F}(\Omega_z \times D_t)$ we let $v(t) := \mathcal{E}(u(\,\cdot\,, t)$ and to fix ideas we first assume that u is smooth on the closure of $\Omega \times D$. Applying the sharp M-T inequality to $u(\,\cdot\,, t)$ for t fixed and integrating over $t \in D$ gives

$$\int_D \left(\int_\Omega e^{-u(t,z)} \mathrm{d}V(z) \right) \mathrm{d}V(t) \leqslant \int_D \exp\left(-\frac{n!}{(n+1)^n} v(t) \right) \mathrm{d}V(t),$$

By (2.2) the function $v(t)$ is a subharmonic function on D with

$$\int_D \mathrm{d}_t \mathrm{d}_t^c v = \int_{\Omega \times D} (\mathrm{dd}^c u)^{n+1}/(n+1)!.$$

Hence applying the sharp B-M-D inequality on the disc D for $n = 1$ (which is follows from Green's formula and Jensen's inequality [22] or alternatively from Polya's inequality [2]) and using that $\frac{n!}{(n+1)^n} \frac{1}{(n+1)!} = \frac{1}{(n+1)^{n+1}}$ finishes the proof under the smoothness assumption above. The general case is proved in a similar way, but using the singular variant of (2.2) proved in [2] (Theorem 3.1); compare the proof of Lemma 3.4. To prove the last statement we recall the *subextension theorem* [28] saying that given Ω and $\widetilde{\Omega}$ two hyperconvex domains such that $\Omega \subset \widetilde{\Omega}$ and a function $u \in \mathcal{F}(\Omega)$ there is a function $\widetilde{u} \in \mathcal{F}(\widetilde{\Omega})$ such that $\widetilde{u} \leqslant u$ on Ω and $\int_{\widetilde{\Omega}} \mathrm{MA}(\widetilde{u}) \leqslant \int_\Omega \mathrm{MA}(u)$ (up to taking approximations $\widetilde{u}$ is obtained by solving the Dirichlet problem $\mathrm{MA}(\widetilde{u}) = 1_\Omega \mathrm{MA}(u)$ on $\widetilde{\Omega}$). Applying subextension to $\Omega \subset r(\mathcal{B} \times D)$ for r sufficiently large thus shows that the sharp B-M-D inequality holds on any hyperconvex domain Ω. Finally, if we instead assume that the quasi-sharp M-T holds in dimension $n-1$ and take u such that $\mathcal{M}(u) = 1$ then repeating the same argument gives, with $v = (n+1-\delta)\mathcal{E}(u(\,\cdot\,, t)$ that

$$\int_D \left(\int_\Omega e^{-(n+1-\delta)u(t,z)} \mathrm{d}V(z) \right) \mathrm{d}V(t) \leqslant C' \delta^{-n} \left(1 - \frac{(n+1-\delta)^n}{(n+1)^n} \right)^{-1}$$

and expanding $1 - t^n = (1-t)(1 + \cdots + t^n)$ then concludes the proof. $\square$

4.2. Quasi B-M-D in $\mathbb{C}^n$ implies quasi M-T in $\mathbb{C}^n$ and the free energy functional

In this section it will be convenient to use a different normalization of $\mathcal{E}$ obtained by multiplication by $n!$, i.e. we let

$$\mathcal{E}(u) := \frac{1}{n+1}\left\langle u, (\mathrm{dd}^c u)^n \right\rangle, \quad \langle u, \mu\rangle := \int_\Omega u\mu$$

With this new normalization $\mathrm{d}\mathcal{E}_{|u} = (\mathrm{dd}^c u)^n$ and the sharp M-T inequality may be formulated as $\int_\Omega e^{-(n+1)u}\mathrm{d}V \leqslant Ce^{-(n+1)\mathcal{E}(u)}$.

Proposition 4.2. — *If the quasi-sharp Brezis–Merle–Demailly inequality holds on $\Omega \subset \mathbb{C}^n$ than so does the quasi-sharp Moser–Trudinger inequality.*

The proof uses the "thermodynamical formalism" recently introduced in a the setting of compact Kähler manifolds in [8]. The key point is to show that, by Legendre duality, the (sharp) Moser–Trudinger inequality is equivalent to yet another inequality, namely one which coincides with the classical *logarithmic Hardy–Sobolev (LHS) inequality* when $n = 1$. To make this precise we first define, for any given positive number γ,

$$\mathcal{G}_\gamma(u) := \mathcal{E}(u) - \mathcal{L}_\gamma(u), \quad \mathcal{L}_\gamma(u) = -\frac{1}{\gamma}\log\int_\Omega e^{-\gamma u}\mathrm{d}V,$$

where $u \in \mathcal{E}^1(\Omega)$ so that $\mathcal{G}_\gamma$ is bounded from above for $\gamma = n+1$ precisely when the sharp Moser–Trudinger inequality holds. As for the LHS type inequality referred to above it is said to hold when the following *free energy functional* F_γ is bounded from above:

$$F_\gamma(\mu) := E(\mu) - \frac{1}{\gamma}D(\mu)$$

where μ is a probability measure on $\overline{\Omega}$ with $E(\mu) < \infty$, where $E(\mu)$ is the (pluricomplex) energy of μ and $D(\mu)$ is its relative entropy, whose definitions we next recall. Following [25] a measure μ on Ω is said to have finite *(pluricomplex) energy* $E(\mu)$ if it admits a finite energy potential u_μ, i.e. $u_\mu \in \mathcal{E}^1(\Omega)$ and

$$(\mathrm{dd}^c u_\mu)^n = \mu \tag{4.1}$$

One may then define its energy by

$$E(\mu) := -\frac{n}{n+1}\left\langle u_\mu, \mu\right\rangle$$

which is finite and non-negative (the reason for our normalization appears in formula (4.3) below). If u_μ does not exist one sets $E(\mu) = \infty$. We also recall the classical notion of relative entropy: given a measure μ its *relative entropy* (wrt $\mathrm{d}V$) is defined as

$$D(\mu) := \int_\Omega \log(\mu/\mathrm{d}V)\mu$$

if μ is a measure which is absolutely continuous wrt $\mathrm{d}V$ (with density $\mu/\mathrm{d}V$) and otherwise $D(\mu) := \infty$. To see the relation to the Moser–Trudinger inequality we recall that E and $\frac{1}{\gamma}D$ can be realized as Legendre type transforms of the concave functionals $\mathcal{E}$ and $\mathcal{L}_\gamma$, respectively. Indeed, it is a classical fact (see [8] and references therein) that

$$\frac{1}{\gamma}D(\mu) = {\mathcal{L}_\gamma}^*(\mu) := \sup_{u\in C^0(\overline{\Omega})} \left(-\frac{1}{\gamma}\log\int_X e^{-\gamma u}\mu_0 - \langle u,\mu\rangle\right) \tag{4.2}$$

Moreover, it follows from the concavity of $\mathcal{E}$ and the solvability of equation (4.1) that [(2)]

$$E(\mu) = \sup_{u\in\mathcal{E}^1(\Omega)} \left(\mathcal{E}(u) - \langle u,\mu\rangle\right). \tag{4.3}$$

The idea is now to first show that

$$F_\gamma \leqslant C_\gamma \implies \mathcal{G}_\gamma := \mathcal{E} - \mathcal{L}_\gamma \leqslant C_\gamma \tag{4.4}$$

and then prove that $F_\gamma \leqslant C_\gamma$ for $\gamma < n+1$, giving the desired M-T inequality. If E were a proper Legendre transform of $\mathcal{E}$ (i.e. if the sup in (4.3) could be taken over $C^0(\overline{\Omega})$ then (4.4) would follow immediately from the fact that the Legendre transform is involutive together with the trivial implication

$$f \leqslant g + C \implies f^* \leqslant g^* + C$$

In the Kähler setting it was explained in [8] how to use a certain projection operator P to realize E the Legendre transform of $\mathcal{E}\circ P$, but here we give a direct argument, relying on the solvability of (4.1). First observe that, by approximation, it will be enough to prove the Moser–Trudinger inequality in question for $u \in \mathcal{H}_0(\Omega)$. Now, by the concavity of $\mathcal{E}$ on $\mathcal{E}^1(\Omega)$ we have, for any fixed measure μ,

$$\begin{aligned}\mathcal{E}(u) \leqslant \mathcal{E}(u_\mu) + \langle u - u_\mu, \mu\rangle &= E(\mu) + \langle u,\mu\rangle \\ &= F_\gamma(\mu) + \left(\frac{1}{\gamma}D(\mu) + \langle u,\mu\rangle\right)\end{aligned} \tag{4.5}$$

Next, we rewrite (4.2) as

$$\inf_\mu \left(\frac{1}{\gamma}D(\mu) + \langle u,\mu\rangle\right) = \mathcal{L}_\gamma(u)$$

---

(2) In fact, using a variational approach the potential u_μ above may be obtained directly by maximizing the functional in the rhs of (4.3). This was recently shown in the Kähler setting in [11] and in the setting of domains in [1].

where the infimum is taken over all measures on Ω. Since we have assumed that u is in $\mathcal{H}_0(\Omega)$ and, in particular, continuous, the measure $\mu = e^{-\gamma u}/\int e^{-\gamma u}\mathrm{d}V$ realizes the inf in (4.5). Hence,

$$\mathcal{E}(u) \leqslant F_\gamma\left(e^{-\gamma u}/\int e^{-\gamma u}\mathrm{d}V\right) + \mathcal{L}_\gamma(u),$$

which gives

$$\mathcal{G}_\gamma(u) \leqslant F_\gamma\left(e^{-\gamma u}/\int e^{-\gamma u}\mathrm{d}V\right),$$

proving (4.4).

Remark 4.3. — A similar argument also shows that $\mathcal{G}_\gamma \leqslant C_\gamma \implies F_\gamma \leqslant C_\gamma$ and hence *the M-T inequality holds iff the LHS inequality holds.* Indeed, writing $F_\gamma(\mu) = (\mathcal{E}(u_\mu) - \langle u_\mu, \mu\rangle) - \frac{1}{\gamma}D(\mu)$ one just uses that $\frac{1}{\gamma}D(\mu) \geqslant -\langle u_\mu, \mu\rangle + \mathcal{L}_\gamma(u_\mu)$, since u_μ is a candidate for the sup in (4.2) (strictly speaking a simple approximation argument has to be used, since $u_\mu$is not continuous in general; compare [8]). Moreover, the argument also shows that a probability measure μ is a maximizer of F_γ iff its potential u_μ is a maximizer of $\mathcal{G}_\gamma$.

Finally, to estimate F_γ we next define the following general invariant of a pair (Ω, μ_0) where μ_0 is a measure on Ω :

$$\alpha := \sup\left\{t : \begin{array}{l} \exists\, C_t : \int_\Omega e^{-tu}\mathrm{d}\mu_0 \leqslant C_t \\ \qquad \forall\, u \in \mathcal{H}_0(\Omega)_b \text{ such that } \int_\Omega (\mathrm{dd}^c u)^n = 1 \end{array}\right\} \tag{4.6}$$

Lemma 4.4. — *If $\gamma < \alpha\frac{(n+1)}{n}$, then $F_\gamma(\mu)$ is bounded from above, i.e. $F_\gamma(\mu) \leqslant C_\gamma$. More precisely, for any $t < \alpha$*

$$F_\gamma(\mu) \leqslant \left(\frac{t}{\gamma} - \frac{n}{n+1}\right)\langle u_\mu, \mu\rangle + \frac{t}{\gamma}C_t$$

where C_t is the minimum of $\mathcal{L}_t(u)$ over all $u \in \mathcal{H}_0(\Omega)_b \cap \{\int_\Omega (\mathrm{dd}^c u)^n = 1\}$.

Proof. — Given γ we fix $t < \alpha := \alpha(\Omega, \mu_0)$. By the definition of α we have $\mathcal{L}_t(u) \geqslant -C_t$ if $u \in \mathcal{H}_0(\Omega) \cap \{\int_\Omega (\mathrm{dd}^c u)^n = 1\}$ and hence

$$\frac{1}{t}D(\mu) = \mathcal{L}_t^*(\mu) \geqslant \mathcal{L}_t(u_\mu) - \langle u_\mu, \mu\rangle \geqslant -\langle u, \mu_\mu\rangle - C_t$$

As a consequence

$$F_\gamma(\mu) \leqslant \left(-\frac{n}{n+1} + \frac{t}{\gamma}\right)\langle u_\mu, \mu\rangle + tC_t$$

Given γ such that $\gamma < \alpha\frac{(n+1)}{n}$, we may now choose t sufficiently close to α so that the multiplicative constant above is strictly positive, thus concluding the proof. □

Assume now that the quasi-sharp BM-inequality holds in Ω. The point is that this implies that $\alpha(\Omega, \mathrm{d}V) = n$ and the previous Lemma then shows that $F_{n+1-\delta}$ is bounded from above. We can actually be more precise wrt the dependence on δ. Indeed, according to the formulation (B-M-D′) we have that $C_{n-\epsilon} \leqslant C + \log(1/\epsilon^{n-1})$ where C_t is defined as in the previous lemma (with $\mu_0 = \mathrm{d}V$). Applying the previous lemma with $\gamma = n + 1 - \delta$ and $t = n - \delta/2$ hence gives

$$F_{n+1-\delta}(\mu) \leqslant C_{n-\delta/2} \leqslant C' + \log(1/\delta^{n-1})$$

The proof of Proposition 4.2 is now concluded by using (4.4).

Remark 4.5. — When $\mu_0 = \mathrm{d}V$ is any volume form on $\overline{\Omega}$ $\alpha := \alpha(\Omega, \mathrm{d}V)$ defines an invariant of a domain Ω which can be seen as a variant of Tian's α-invariant for a Kähler manifold (X, ω) (or rather the class $[\omega]$). The difference is that in the latter case the Monge–Ampère mass is determined by $[\omega]$ and hence independent of u. Moreover, $-\gamma F_\gamma(\mathrm{MA}(u))$ is a variant of Mabuchi's K-energy functional, which plays a key role in Kähler geometry (compare the discussion in [8])

4.3. Proof of Theorem 1.5

The sharp Moser–Trudinger inequality holds when $n = 1$ in the disc D [53]. Hence combining Proposition 4.1 and Proposition 4.1 simultaneously prove the inequalities in Theorem 1.5 and Theorem 1.6.

As for the sharpness of the multiplicative constants in inequalities we make the following remark which concludes the proof of Theorem 1.5.

Remark 4.6. — Let $\Omega := \mathcal{B}$ be the unit ball in $\mathbb{C}^n$ and set $u := \log|z|^2$ so that $(\mathrm{dd}^c u)^n = \delta_0$. Letting $u_t := tu$ for $t < 1$ gives $\int_B e^{-u_t} = \frac{1}{1-t/n} \sim \frac{1}{(1-\frac{t^n}{n^n})}$ as $t \to 1^-$. Moreover, since $\mathrm{MA}(u_t) = t^n$ this shows that the sharp Brezis–Merle–Demailly inequality cannot hold on $\mathcal{B}$ with a better coefficient than $\frac{1}{n^n}$, nor with a smaller power in the rhs. An application of the subextenstion theorem (as in the proof of Proposition 4.1) gives the same conclusion for any hyperconvex domain Ω (alternatively we can apply the same argument as above with u replaced by the pluricomplex Green function g_z with a pole at any fixed point z in Ω). Finally, by Proposition 4.1 this also shows that the coefficient $n!/(n+1)^n$ in the sharp M-T inequality cannot be improved for any hyperconvex domain Ω.

5. Relations between the various inequalities

Let $(X, \mathrm{d}V)$ be a measure space and $u \leqslant 0$ a measurable function on X in $L^1_{loc}(X, \mathrm{d}V)$. Set

$$E(t) := \int e^{-tu}\mathrm{d}V$$

and

$$V(s) := \mathrm{Vol}\{u < -s\} := \int_{\{u<-s\}} \mathrm{d}V$$

Then $E(t)/t$ and $V(s)$ are (up to signs) related by Laplace transforms. Indeed, by the push-forward formula and integration by parts

$$E(t) := t\int_0^\infty e^{ts}V(s)\mathrm{d}s + V(0), \quad V(0) = \int_X \mathrm{d}V$$

According to a well-known principle the Laplace transform is asymptotically described by the Legendre transform:

$$E(t) \lesssim e^{f(t)} \text{ “} \Longleftrightarrow \text{” } V(s) \lesssim e^{-f^*(s)}$$

(as t and s tend to infinity), where f is assumed convex and $f^*(s)$ is its Legendre transform:

$$f^*(s) := \sup_t (st - f(t))$$

There are various ways of formulating this principle precisely but for our purposes the following basic lemma will be sufficient:

Lemma 5.1. — *If $E(t) \leqslant Ce^{f(t)}$, then $V(s) \leqslant Ce^{-f^*(s)}$. Conversely, if $V(s) \leqslant Ce^{-g(s)}$ then for any $\delta > 0$ there is a constant C_δ such that $E(t) \leqslant C_\delta e^{g^*(t+\delta)}$.*

Proof. — Fix $t \in \mathbb{R}$. On the subset $\{u < -s\}$ of X we have $1 < e^{-st}e^{-tu}$ and hence $V(s) \leqslant e^{-st}\int_X e^{-tu} \leqslant Ce^{-st+f(t)}$. Taking the infimum over all t then proves the first inequality. The second inequality follows immediately from the definitions if we rewrite $ts - g(s) = ((t+\delta)s - g(s)) - \delta s$ and let $C_\delta = C\int_0^\infty e^{-\delta s}\mathrm{d}s = C/\delta$. □

We will apply the previous lemma to the case when $f(t)$ is homogeneous and use the following basic relations (assuming $p > 1$)

$$f(t) = \frac{1}{a}t^p/p \iff f^*(s) := a^{(q-1)}s^q/q \tag{5.1}$$

where $1/p + 1/q = 1$ (the case $a = 1$ is immediate and implies the general case by scaling). More precisely, in our case we will have $p = (n+1)/n$ and hence $q = n+1$ and vice versa.

Corollary 5.2 (of Theorem 1.1). — *Let (X,ω) be a compact Kähler manifold and $u \in \mathcal{H}_0(X,\omega)$. Then there are constants A and B such that*

$$\mathrm{Vol}_\omega\{u < -s\} \leqslant Ce^{-B\frac{1}{(-\mathcal{E}_\omega(u))^{1/n}}s^{(n+1)/n}}$$

More precisely, we may replace the exponent above by

$$-\frac{n}{(-\mathcal{E}_\omega(u))^{1/n}(n+1)^{(1+1/n)}}s^{(n+1)/n}(1+o(1))$$

as $s \to \infty$.

From the first volume estimate in the previous corollary we see that the L^p-norms of u may be estimated as

$$\int_X (-u)^p \mathrm{d}V = \int_0^\infty V(s)\mathrm{d}(s^p) \leqslant C\Gamma\Big(\frac{n}{n+1}p\Big)\Big(\frac{1}{B}\Big)^{pn/(n+1)}(-\mathcal{E}_\omega(u))^{p/(n+1)}$$

(after setting $x = s^{(n+1)/n}$ and using $\Gamma(x)x = \Gamma(x+1)$, for $\Gamma(x) := \int_0^\infty s^{x-1}e^{-s}\mathrm{d}s$). Using that $\Gamma(x) \leqslant C_n x^x$ (e.g. by Stirling's approximation) hence gives the Sobolev type inequality (1.6) from the introduction.

The inequality (1.5) can now be deduced from the previous Sobolev type inequality (compare [62]). Indeed, assuming first that $-\mathcal{E}_\omega(u) = 1$ gives

$$\int e^{B(1-\delta)(-u)^{n+1/n}}\mathrm{d}V = \sum_{p=1}^{\infty}\frac{B^j}{j!}\int_X (-u)^{j(n+1)/n}\mathrm{d}V$$
$$\leqslant \sum_{p\in\mathbb{N}(n+1)/n}\frac{1}{p}(1-\delta)^{pn/(n+1)}$$

which is finite for any $\delta > 0$ and the general case then follows by scaling. Note in particular, that when $E(t) \leqslant e^{At^{n+1}}$ with $A = (n+1)^{-(n+1)}$ then $V(s) \leqslant e^{-Bs^{(n+1)/n}}$ with $B = n$ which proves the last statement in Theorem 1.5.

6. Remarks on the optimal constants

In this section we will compare our results with Aubin's conjectures [3, 4] (and his partial results). To this end we first have to compare our notations, which differ slightly. There are two reasons for the differences which come from (1), the choice of energy functional and (2), the normalizations of the energy functional. We therefore start with a discussion of the various energy functionals involved.

Given the functional $\mathcal{E}_\omega$ which we recall may be defined as a primitive of the Monge–Ampère operator one defines

$$J_\omega(u) := -\mathcal{E}_\omega(u) + \int u\omega^n/n!$$

and

$$I_\omega(u) := \frac{1}{n!}\int (-u)\left({\omega_u}^n - \omega^n\right)$$

The functionals J_ω and I_ω are both invariant under addition of constants and semi-positive [3] and when $\omega = 0$ (as in the $\mathbb{C}^n$-setting) they coincide. In general, they are equivalent up to multiplicative factors [3]:

$$J_\omega \leqslant I_\omega \leqslant (n+1)J_\omega \tag{6.1}$$

However, Aubin's normalizations are slightly different and obtained by replacing the factor $1/n!$ above by $(2\pi)^n/(n-1)!$. We will denote by $I_\omega^{(A)}$ and $J_\omega^{(A)}$ the corresponding functionals with Aubin's normalizations, so that

$$J_\omega = d_n J_\omega^{(A)}, \quad I_\omega = d_n I_\omega^{(A)}, \quad \text{where } d_n := \frac{1}{n}\frac{1}{(2\pi)^n} \tag{6.2}$$

In this notation Aubin's general "Hypothèse fondamentale" as formulated in [3] asserts that there exist positive constants ξ and C such that the following Moser–Trudinger type inequality holds for a given smooth volume form $\mathrm{d}V$ and Kähler metric ω :

$$\log \int e^{-ku}\mathrm{d}V \leqslant \xi k^{n+1} I_\omega^{(A)}(u)) + C \tag{6.3}$$

for all $k \geqslant 1$ and all $u \in \mathcal{H}(X,\omega)$ normalized such that $\int_X u\omega^n = 0$. Of course, if the inequality holds for some fixed volume form $\mathrm{d}V$ (for example ω^n) then it also holds for any other volume form $\mathrm{d}V$, but with a different additive constant C. Anyway, here we will focus on the (limiting) optimal multiplicative constant ξ.

To see that Theorem 1.1 essentially confirms Aubin's conjecture (in the case when $[\omega]$ is an integral class) we recall that there is a constant C' such that

$$\sup u \leqslant \frac{1}{V}\int u\omega^n + C', \tag{6.4}$$

if $u \in \mathcal{H}(X,\omega)$. Now, Theorem 1.1 applied to $u - \sup u$ gives

$$\log \int e^{-k(u-\sup u)}\mathrm{d}V \leqslant Ak^{n+1}\left(-\mathcal{E}_\omega(u)\right) + A(\sup u)k^{n+1} + B).$$

If

$$\int u\omega^n = 0,$$

then $0 \leqslant \sup u \leqslant C'$ and $-\mathcal{E}_\omega(u) = J_\omega(u) = d_n J_\omega^{(A)}(u)$. Therefore, under this assumption,

$$\log \int e^{-ku} \mathrm{d}V \leqslant A k^{n+1} J_\omega(u) + (AC' k^{n+1} + B)$$
$$= A k^{n+1} d_n J_\omega^{(A)}(u) + (AC' k^{n+1} + B).$$

Thus, since $J_\omega^{(A)} \leqslant I_\omega^{(A)}$, (6.3) holds with $\xi = Ad_n$ and $C = C_k := AC' k^{n+1} + B$. This means that the constant C_k depends on k while Aubin's hypothesis, strictly speaking, says that it should be independent of k. Anyway, in applications to existence problems for PDEs the precise value of C_k is immaterial, it is only the independence of ξ of k that is important ([3, §V]).

6.1. Counter-example to Aubin's explicit conjecture in the Fano case

In his paper Aubin also conjectured that in the Fano setting (with $[\omega] = c_1(-K_X)$) the limiting optimal (multiplicative) constant $\xi^{(A)}(X)$ for the Moser–Trudinger type inequality (6.3) (with respect to the functional $I_\omega^{(A)}$), i.e. the infimum over all constants ξ satisfying (6.3) for some C_ξ, is explicitly given by

$$\xi^{(A)}(X) = \xi_n := \pi^{-n}(n-1)! n^n (n+1)^{-(2n+1)}$$
$$= \pi^{-n}(n-1)! \left(1 + \frac{1}{n}\right)^{-n} (n+1)^{-(n+1)}. \quad (6.5)$$

A counter-example to this conjecture has been found by Y. Sano (thanks to Y. Odaka for informing us about this). For completeness and in order to illustrate the relation to volume bounds we will here provide a simple way to produce counter-examples.

First, we denote by $\eta^{(A)}$ the limiting constant in (6.3), if we replace the functional $I_\omega^{(A)}$ by $J_\omega^{(A)}$, and by η the limiting constant when $I_\omega^{(A)}$ is replaced by J_ω. Then $\eta^{(A)} \leqslant (n+1)\xi^{(A)}$, so Aubin's explicit conjecture in the Fano setting implies that

$$\eta^{(A)}(X) \leqslant (n+1)\xi_n, \quad (6.6)$$

and $\eta^{(A)}(X) = d_n \eta(X)$. We will now give a counter example to (6.6) for $k = 1$. The main idea is (see the lemma below) that $\eta(X) \geqslant 1/V(X)$, unless the *Ding functional* for X is coercive, and the Ding functional cannot be coercive e. g. if X has non trivial holomorphic vector fields. Thus, all we need to contradict (6.6) is to find a Fano manifold of small volume and non trivial holomorphic vector fields.

Lemma 6.1. — *Let X be a Fano manifold and $\omega \in c_1(-K_X)$. If one replaces the functional I_ω in (6.3) by the functional J_ω then the corresponding limiting optimal (multiplicative) constant $\eta(X)$ satisfies (when specializing to the case $k=1$)*

$$\eta(X) \geqslant 1/V(X)$$

if X admits non-trivial holomorphic vector fields, i.e. if $H^0(TX) \neq \{0\}$. (By (6.4), $\eta(X)$ coincides with the limiting optimal constant obtained for the functional $-\mathcal{E}_\omega$ together with the normalization condition $\sup_X u = 0$).

Proof. — This follows from [36, Prop. 6], but here we provide a different proof using geodesics (our $\eta(X)$ differs from the one in [36, Prop. 6] by the factor $1/V(X)$). First observe that since the optimal constants in question are independent of the choice of volume form we may as well take $\mathrm{d}V$ to be the metric on $-K_X$ induced by the reference metric ϕ_0 on $-K_X$ with curvature form ω, i.e. the local density of $\mathrm{d}V$ is represented by $e^{-\phi_0}$. We then define the *Ding functional* by

$$\mathcal{D}(u) := -\frac{1}{V}\mathcal{E}_\omega(u) - \log \int_X e^{-u}\mathrm{d}V,$$

which is invariant under the $\mathbb{R}$-action $u \mapsto u + c$ (and hence insensitive to the different normalization conditions for u). By definition, $\mathcal{D}(u)$ is bounded from below iff the Moser–Trudinger inequality for the functional $(-\mathcal{E}_\omega)$ holds with a multiplicative constant $1/V(X)$. Now assume that the corresponding limiting optimal constant $\eta(X)$ is *strictly smaller* than $1/V(X)$, i.e. $\eta(X) = 1/V(X) - \delta$ for some $\delta > 0$. This equivalently means that $\mathcal{D}$ is *coercive* in the following sense: there exist positive constants δ and C such that, for all $u \in \mathcal{H}_\omega(X)$, normalized so that $\sup_X u = 0$,

$$\mathcal{D}(u) \geqslant \delta\,(-\mathcal{E}_\omega(u)) - C$$

Equivalently (by (6.4)), for all $u \in \mathcal{H}_\omega(X)$,

$$\mathcal{D}(u) \geqslant \delta J_\omega(u) - C' \tag{6.7}$$

for a (possibly different) constant C'. Next, we show that this leads to a contradiction if X admits a non-trivial holomorphic vector field W. Indeed, fix u_0 in $\mathcal{H}_\omega(X)$ and denote by u_t^W the curve in $\mathcal{H}_\omega(X)$ obtained by acting on u_0 with the flow of the given holomorphic vector field (the action may be obtained by identifying $\mathcal{H}_\omega(X)$ with the space of positiely curved metrics on $-K_X$). Now on one hand,

$$\lim_{t\to\infty} J_\omega(u_t^W) = \infty$$

and on the other

$$\mathcal{D}(u_t^W) = a_W t + \mathcal{D}(u_0^W)$$

where $a_{-W} = -a_W \in \mathbb{R}$. This is well-known and follows form the fact that u_t^W defines a geodesic in $\mathcal{H}_\omega(X)$, using that $\mathcal{E}_\omega$ is affine along geodesics (while $\int_X e^{-u_t^W} \mathrm{d}V$ is independent of t, by invariance under automorphisms). Accordingly, after perhaps replacing W with $-W$ we may assume that $\mathcal{D}(u_t^W)$ is bounded from above as $t \to \infty$. But this contradicts the coercivity inequality (6.7). □

It follows from the lemma that if X has non trivial holomorphic vector fields and (6.6) holds, then

$$\frac{d_n}{V(X)} \leqslant d_n \eta(X) = \eta^{(A)}(X) \leqslant (n+1)\xi_n.$$

To get a contradiction it will hence be enough to exhibit an n-dimensional Fano manifold X with $H^0(TX) \neq \{0\}$ such that

$$V(X) < \frac{1}{(n+1)} \frac{d_n}{\xi_n},$$

where d_n is the constant appearing in formula (6.2). Using that $n!V(X)$ equals the top-intersection number $(-K_X)^n$ this means that

$$(-K_X)^n < ((n+1)^2/2n)^n := C_n \tag{6.8}$$

To find such Fano manifold X we simply start with a Fano manifold Y with very small volume and form the product with $\mathbb{P}^1$, to ensure the existence of holomorphic vector fields (using the basic formula $(-K_{Y\times\mathbb{P}^1})^{n+1} = 2(n+1)(-K_Y)^n)$). For example, since $C_3 > 18$ can take $Y = S_d$ for $d = 6,7,8$, where S_d denotes the blow-up of $\mathbb{P}^2$ in d generic points (using $(-K_{S_d})^2 = 9-d$). Similarly, since $C_4 > 95$ we can for $n=4$ take Y as a general smooth hypersurface of degree 6 in the weighted projective space $\mathbb{P}(1,1,1,1,3)$, which has $(-K_Y)^3 = 2$, according, to the "big table" in [29], where Y appears first in the list.

6.2. Comparison with Aubin's constant for the ball

Let us now turn to the setting of the unit ball, where $\omega = 0$ and consider the corresponding functional $I_0^{(A)}$ (called $\mathcal{J}$ with the same normalizations in [4]), i.e.

$$I_0^{(A)}(u) := \frac{1}{(n-1)!} \int (-u) \left(i\partial\bar{\partial}u\right)^n$$

In the case of a radial psh function u in the ball Aubin showed [4] that

$$\log \int e^{-u} \mathrm{d}V \leqslant a_n I_0^{(A)}(u) + C, \quad a_n = 2n^n(n+1)^{-(2n+1)}\sigma_{2n-1}^{-1}, \tag{6.9}$$

where σ_p denotes the volume of the unit p-sphere, giving $a_n = \xi_n$ (formula (6.5)). Moreover, in [3, §3] Aubin claims that he has proved that the optimal constant in the setting of the ball is indeed given by formula (6.5). But by Theorem 1.5 the (limiting) optimal constant c_n in the equality (6.9) is equal to

$$c_n = \frac{1}{(2\pi)^n}\frac{(n-1)!}{(n+1)!}\frac{n!}{(n+1)^n} = \frac{(n-1)!}{(2\pi)^n}\frac{1}{(n+1)^{n+1}}$$

which satisfies $a_n \geqslant c_n$ with strict inequality when $n > 1$. Indeed,

$$c_n = \left(\frac{1}{2}(1+1/n)\right)^n a_n,$$

Accordingly, Aubin's constant a_n is not optimal for $n > 1$ (moreover Aubin, in fact, only proved his inequality in the radial case, but, of course, his constant cannot be optimal in the subclass of radial functions either).

6.3. Discussion

It is natural to ask why Aubin expected that the particular value in formula (6.5) gives the optimal constant in the Fano case? We can only speculate on this. But it seems that Aubin was expecting that the optimal constant in the Fano case coincides with the optimal constant in the setting of the ball. Unfortunately, as pointed out in the previous section the constant found in the setting of the ball by Aubin is, however, not the optimal one (in contrast to Aubin's claim). Anyway, as we next explain (using an argument which has the virtue of avoiding comparing normalizations) it is *not* the case that the optimal constant in the Fano case coincides with the (correct!) optimal constant in the setting of the ball. Indeed, if the optimal constant coincided with the one in the setting of the ball, then, by Lemma 6.1, this would force

$$V(X) \geqslant V(\mathbb{P}^n)$$

for any Fano X such that $H^0(TX) \neq \{0\}$. But this bound is violated by many Fano manifolds and as explained in the previos section we can even arrange that $V(X)$ is sufficently small to compensate for Aubin's mistake concerning the optimal constant in the case of the ball. In fact, according to various conjectures in complex geometry one expects that the volume $V(X)$ is *maximized* on $\mathbb{P}^n$ for large classes of Fano manifolds, for example (1) all Fano n-folds with Picard number equal to one (see [45] for a proof when $n \leqslant 4$) and (2) among all Kähler–Einstein Fano manifolds.

Remark 6.2. — Building on the results in the present paper the authors established a generalized form of the latter conjecture (2) in [10] in the toric case and the general case of (2) was then settled in [40], using a different algebro-geometric argument.

7. Existence of extremals and applications to Monge–Ampère equations

7.1. The Kähler setting

Let (X,ω) be an integral Kähler manifold and fix a smooth volume form $\mathrm{d}V$ on X with unit total volume. For a given sequence $a_k \in \mathbb{R}$ we consider the following Moser–Trudinger type functional on $\mathcal{H}(X,\omega)$:

$$\mathcal{G}_{a_k}(u) := \frac{1}{k}\log\int e^{-ku}\mathrm{d}V + \frac{1}{V}\int u\frac{\omega^n}{n!} - \frac{k^n}{a_k}J_\omega(u)$$

which is $\mathbb{R}$-invariant (and hence descends to a functional on space of all Kähler metrics in $[\omega]$). We let $a_k(X)$ be the supremum over all a_k such that the functional above is bounded from above. By Theorem 1.1 (and the discussion in the beginning of Section 6) $a_k(X) \geqslant 1/A$ or more precisely $\liminf_k a_k(X) \geqslant 1$.

In this section we will be concerned with the question of existence of maximizers for $\mathcal{G}_{a_k}$ and solutions to the corresponding Euler–Lagrange equation

$$0 = (\mathrm{d}\mathcal{G}_{a_k})_{|u} = -\frac{e^{-ku}\mathrm{d}V}{\int e^{-ku}\mathrm{d}V} + \frac{1}{V}\frac{\omega^n}{n!} + \frac{k^n}{a_k}\left(\frac{\omega_u}{n!} - \frac{\omega^n}{n!}\right) \tag{7.1}$$

Breaking the $\mathbb{R}$-invariance (i.e. the invariance under $u \mapsto u + c$) by the introducing the normalization $\int_X e^{-ku}\mathrm{d}V = V$, the previous equation can hence be written as the following PDE:

$$\frac{\omega_u^n}{n!} = \frac{a_k}{Vk^n}e^{-ku}\mathrm{d}V + \left(1 - \frac{a_k}{Vk^n}\right)\frac{\omega^n}{n!} \tag{7.2}$$

for $u \in \mathcal{H}(X,\omega)$.

Theorem 7.1. — *Assume that $a_k < a_k(X)$ and $a_k < Vk^n$ (for example, this is the case if $a_k < \max\{A^{-1}, Vk^n\}$, where A is the constant appearing in Theorem 1.1). Then there is a solution to (7.2) in $\mathcal{H}(X,\omega)$. Moreover, the solution can be taken to maximize the functional $\mathcal{G}_{a_k}$. In particular, if $a_k < 1$ then there is such a solution for all k sufficiently large.*

Given the Moser–Trudinger inequalities in Theorem 1.1 the proof of the previous theorem follows from the variational approach to complex Monge–Ampère equation introduced in [11]. We outline the argument and refer to [11] for further details.

Existence of a maximizer u_* in $\mathcal{E}^1(X,\omega)$

We proceed in two steps. The first step amounts to the following *coercivity estimate:* there exists $\delta, C > 0$ such that

$$\mathcal{G}_{a_k} \leqslant \delta\mathcal{E}_\omega + C \tag{7.3}$$

on the space $\mathcal{E}^1_0(X,\omega) := \mathcal{E}^1(X,\omega) \cap \{\sup_X = 0\}$ (which we equip with the L^1-topology). This follows directly from the assumption that $a_k < a_k(X)$ and the inequality (6.4). The second step is to establish the following *semi-continuity property:* for any constant C the functional $\mathcal{G}_{a_k}$ is upper semi-continuous (usc) on $\{-\mathcal{E}_\omega \leqslant C\}$ in $\mathcal{E}^1_0(X,\omega)$ (wrt the L^1-topology). To this end first recall that $\mathcal{E}_\omega$ is usc on $\mathrm{PSH}(X,\omega)$ (in particular it follows from weak compactness that $\{-\mathcal{E}_\omega \leqslant C\}$ is compact) [11, 21]. All that remains is then to prove that $u \mapsto \int e^{-ku}\mathrm{d}V$ is usc on $\{-\mathcal{E}_\omega \leqslant C\}$. To this end it is enough to establish a uniform bound

$$\int e^{-(k+\delta)u} \leqslant C_\delta \tag{7.4}$$

for some $\delta > 0$ (compare the proof of Lemma 6.4 in [11] or Lemma 3.6 in [8]). But since we have assumed that $-\mathcal{E}_\omega(u) \leqslant C\}$ this is an immediate consequence of the Moser–Trudinger inequality in Theorem 1.1 (which shows that any $\delta > 0$ will do). The existence of a maximizer u_* is now rather immediate: take u_j in $\mathcal{E}^1_0(X,\omega)$ such that

$$\mathcal{G}_{a_k}(u_j) \to \sup_{\mathcal{E}^1(X,\omega)} \mathcal{G}_{a_k},$$

(note that, by the scale invariance of $\mathcal{G}_{a_k}$ we may indeed assume that $\sup_X u_j = 0$). By the coercivity estimate the sup is finite and moreover $(u_j) \subset \{-\mathcal{E}_\omega \leqslant C\}$ for some $C > 0$. But then it follows from the upper semi-continuity that the sup is attained on any accumulation point u_* of (u_j) (which exists by compactness). This concludes the proof of the existence of a maximizer.

The maximizer u_* is a weak solution of equation (7.2)

We will use the projection argument in [11] to see that u_* is a weak solution in $\mathcal{E}^1_0(X,\omega)$ to the variational equation (7.1), in the sense of pluripotential theory [11, 21] (shifting u_* by a constant hence gives a solution to the equation (7.2)). To this end we first decompose

$$\mathcal{G}_{a_k}(u) = \frac{k^n}{a_k}\mathcal{E}_\omega + \mathcal{I}_{a_k}, \qquad \mathcal{I}_{a_k}(u) = \frac{1}{k}\log\int e^{-ku}\mathrm{d}V + \left(1 - \frac{k^n}{Va_k}\right)\int u\omega^n/n!$$

Fixing $v \in \mathcal{C}^\infty(X)$ let $f(t) := k^n\mathcal{E}_\omega(P_\omega(u_* + tv)) + \mathcal{I}_{a_k}(u_* + tv)$, where

$$P_\omega(u)(x) := \sup\{v(x) : v \leqslant u, \ v \in \mathrm{PSH}(X,\omega)\}^* \in \mathrm{PSH}(X,\omega)$$

where the star denotes the upper semi-continuous regularization. By the assumption $a_k \leqslant k^n V$ the functional $\mathcal{I}_{a_k}(u)$ is decreasing in u and hence the sup of $f(t)$ on $\mathbb{R}$ is attained for $t = 0$:

$$\sup_{t\in\mathbb{R}} f(t) = f(0) \tag{7.5}$$

Indeed, for any $u \in \mathcal{E}^1(X,\omega) + C^\infty(X)$

$$k^n\mathcal{E}_\omega(P_\omega(u)) + \mathcal{I}_{a_k}(u) \leqslant k^n\mathcal{E}_\omega(P_\omega(u)) + \mathcal{I}_{a_k}(P_\omega(u)),$$

using that $P_\omega u \leqslant u$ a.e. $\mathrm{d}V$ and that $\mathcal{I}$ is decreasing. Since $P_\omega u$ is in $\mathcal{E}^1(X,\omega)$ the right hand side above is bounded from above by the sup of $\mathcal{G}_{a_k}(u)$ i.e. by $f(0)$, which proves (7.5). Next we recall that the functional $\mathcal{E}_\omega \circ P_\omega$ is Gateaux differentiable with differential $\mathrm{MA}(P_\omega u)$ at u [11]. Hence, the condition $\mathrm{d}f/\mathrm{d}t = 0$ for $t = 0$ gives that the variational equation (7.1) holds when integrated against any $v \in \mathcal{C}^\infty(X)$.

Regularity

Now, by the previous estimate (7.4), $\omega_{u_*}^n$ has a density in L^p for some $p > 1$ (or even all $p > 1$) and hence it follows from Kolodziejs L^∞-estimate [49] that u_* is in $L^\infty(X)$ (and is even continuous). Finally the higher order regularity $u \in \mathcal{C}^\infty(X)$ then follows from [57], using that the rhs in equation (7.2) is of the form $F(u)$ for $F(t)$ smooth and strictly positive (using the assumption $a_k < k^n V$).

7.2. Remarks on the Fano setting

Let now X be Fano with $[\omega] = c_1(-K_X)$. In the case when $k = 1$ and $a_k := V$ the functional $\mathcal{G}_{a_k}$ above becomes

$$\mathcal{G}_{a_k} := \mathcal{G}_V(u) := \log\int e^{-u}\mathrm{d}V + \frac{1}{V}\mathcal{E}_\omega(u)$$

with Euler–Lagrange equation

$$\omega_u^n/n! = Ve^{-u}\mathrm{d}V$$

In particular, if $\mathrm{d}V$ is taken as $e^{-h_\omega}\omega^n/n!$ where h_ω is the Ricci potential of ω then the previous equation may be written as the Kähler–Einstein equation

$$(\mathrm{dd}^c\phi)^n/n! = Ve^{-\phi}\mathrm{d}z \wedge \mathrm{d}\bar{z}$$

for the local weight ϕ of the metric ω, saying that $\operatorname{Ric}\omega = \omega$. In this setting it is well-known that the corresponding coercivity estimate (7.3) is *equivalent* to the existence of a Kähler–Einstein metric, which in turn is equivalent to X being "analytically K-stable" in the sense of Tian (which means that Mabuchi's K-energy functional is proper); see [60, Thm. 7.13] and [54].

Now, the coercivity estimate holds for $\mathcal{G}_V$ precisely when a Moser–Trudinger inequality holds for some $a_k := a$ (i.e. $\mathcal{G}_a \leqslant C$) satisfying

$$V < a \tag{7.6}$$

In other words, if a could be chosen uniformly over all Fano manifolds X of dimension n then the previous inequality would give an existence criterion for Kähler–Einstein metrics on a Fano manifold X, in terms of the volume of X. This follows for example from the variational approach above, but a proof using the continuity method already appears in Aubin's paper [3] (see also [36] where the functional $\mathcal{G}_V$ seems to first have appeared explicitly). As explained in Section 6 Aubin also proposed an explicit value for a, which however cannot be correct.

Unfortunately, it can be shown that the uniform constant provided by Theorem 1.2 (at least in its present form) is not useful for this kind of application. On the other hand the existence of Moser–Trudinger type inequalities established in Theorems 1.1 and 1.2 are very useful in other regards, for example for establishing semi-continuity properties and uniform estimates as in the proof of Theorem 7.1. In particular, it plays an important role in [12] in the construction of Kähler–Einstein metrics on "analytically K-stable" log-Fano varieties.

7.2.1. Relations to the α-invariant

Before turning to the setting of domains in $\mathbb{C}^n$ we briefly recall Tian's [58] existence criterion for Kähler–Einstein metrics which has proved to be very useful:

$$\alpha(X) > n/(n+1), \tag{7.7}$$

where

$$\alpha(X) := \sup\left\{t : \exists\, C_t : \int_X e^{-t(u-\sup_X)}\mathrm{d}V \leqslant C_t,\ \forall\, u \in \mathrm{PSH}(X,\omega)\right\}$$

As is well-known it is enough to consider u with analytic singularities in the sup above (and hence $\alpha(X)$ coincides with the algebraically defined log canonical threshold $\mathrm{lc}(X)$). Now, if it would be enough to take the sup above over all u with *isolated* singularities, then it would follow from the inequality (1.14) (see also below) that

$$\alpha(X) > n/(n!V)^{1/n}$$

and hence Tian's criterion (7.7) would be satisfied if $n!V < (n+1)^n$. However, this latter condition is satisfied for *any* Fano manifold when $n = 2$ (i.e. Del Pezzo surfaces) and in particular for those which do not admit a Kähler–Einstein metric (like $\mathbb{P}^2$ blown-up in one point). Still, as we will see next a similar approach turns out to be very fruitful in the setting of domains. At least on a heuristic level this could perhaps be expected as all analytic singularities are indeed isolated in this setting.

7.3. The setting of domains in $\mathbb{C}^n$ and Mean Field Equations

Let now Ω be a hyperconvex domain in $\mathbb{C}^n$ with $\mathrm{d}V$ the Euclidean volume form and recall (see Section 4.2) that

$$\mathcal{G}_\gamma(u) := \frac{1}{\gamma}\log\int_\Omega e^{-\gamma u}\mathrm{d}V + \frac{1}{n+1}\int(-u)(\mathrm{dd}^c u)^n$$

so that the corresponding Euler–Lagrange equation reads

$$(\mathrm{dd}^c u)^n = \frac{e^{-\gamma u}\mathrm{d}V}{\int_\Omega e^{-\gamma u}\mathrm{d}V} \tag{7.8}$$

with the boundary condition $u = 0$. Equivalently setting $v = \gamma u$ gives the Euler–Lagrange equation corresponding to the non-scaled Moser–Trudinger inequality $(M-T)$ in the beginning of Section 4 (it is obtained by setting $\gamma = 1$ and inserting a multiplicative constant $a = \gamma^n$ in the rhs). Ideally, we would like to look for smooth solutions (in $\mathcal{H}_0(\Omega)$) to the previous equation, but as the corresponding higher order regularity theory does not seem to be sufficiently developed we will merely be able to produce continuous solutions (vanishing on the boundary). Note that in this setting there is no invariance under additive scalings of u (due to the boundary conditions $u = 0$).

In the case when $n = 1$ the previous equation is often referred to as the *mean field equation* as it appears in a statistical model of mean field type, with γ playing the role of (minus) the temperature [23, 47]. In the one-dimensional case it is well-known [23, 47] that $\gamma = 2$ appears as a *critical value* (the value is 8π when dd^c is replaced by the usual non-normalized Laplacian in the plane). It should be emphasized that from the statistical mechanical point of view only the solutions u such that $(\mathrm{dd}^c u)^n$ maximizes the corresponding *free energy functional* F_γ are relevant (which is equivalent to u maximizing $\mathcal{G}_\gamma$; see Remark 4.3).

THEOREM 7.2. — *Let Ω be a hyperconvex domain and assume that $\gamma < n+1$. Then there exists $u_\gamma \in \mathcal{C}^0(\overline{\Omega})$ solving equation (7.8) in Ω with $u_\gamma = 0$ on $\partial\Omega$ and which maximizes the corresponding functional $\mathcal{G}_\gamma$.*

Proof. — Assume that $\gamma < n+1$. By Theorem 1.5 the coercivity estimate corresponding to (7.3) still holds for $\mathcal{G}_\gamma$ and it is well-known that $\mathcal{E}$ is usc and its sub-level sets $\{\mathcal{E} \geqslant -C\}$ are compact (wrt the L^1_{loc}-topology); see [1] and references therein. Hence, all the previous arguments still apply in the present setting of domains to give the existence of a maximizer u_γ for $\mathcal{G}_\gamma$ on the space $\mathcal{E}^1(\Omega)$. To see that u_γ satisfies the equation (7.8) one applies a projection argument as in the Kähler setting above (see [1] where the projection argument from [11] was adapted to the setting of hyperconvex domains). Finally, by the M-T inequality $\mathrm{MA}(u)$ has an L^p-density for $p > 1$ and hence when Ω is strictly pseudoconvex the continuity statement follows from [49]. As for the general hyperconvex case it follows from [20]. □

Remark 7.3. — After the first preprint version of the present paper had appeared it was shown in [42] that u_γ above can be taken to be smooth in the case when Ω has a smooth and strictly pseudoconvex boundary.

Next we will establish a "concentration/compactness principle" for the behavior of the solutions above when γ approaches the critical value $n+1$. We firs recall, following standard terminilogy in the PDE-litterature [22], that a point $z_0 \in \overline{\Omega}$ is said to be a *blow up-point* for a sequence u_j of non-negative functions in Ω if there exists a sequence of points $z_j \in \Omega$ converging to z_0 such that

$$\lim_{j\to\infty} u(z_j) = -\infty$$

THEOREM 7.4. — *Let γ_j be a sequence increasing to $n+1$ and $u_j := u_{\gamma_j}$ a sequence of solutions of equation (7.8) as in the previous theorem such that u_j has no blow-up point at the boundary and u_j converges to $u \in \mathcal{F}(\Omega))$ in the L^1_{loc}-topology. Then, precisely one of the following two alternatives holds:*

(1) *The limit u is a solution of equation (7.8) for $\gamma = n+1$, maximizing the functional $\mathcal{G}_{n+1}$ (and u_j converges uniformly to u).*
(2) *The limit u is the pluricomplex Green function with a logarithmic pole at some point $z_0 \in \Omega$, i.e.*

$$(\mathrm{dd}^c u)^n = \delta_{z_0}, \quad u(z) = \log|z - z_0|^2 + O(1)$$

Proof. — Let us first show that if there exists $\delta > 0$ such that

$$\int_\Omega e^{-(n+\delta)u_j}\mathrm{d}V \leqslant C_\delta, \tag{7.9}$$

then alternative 1 in the theorem holds. We will use the notation from Section 4.2. First note that since $\gamma \mapsto \mathcal{L}_\gamma$ is decreasing we have that $\mathcal{G}_\gamma \leqslant \mathcal{G}_{\gamma_*}$ if $\gamma < \gamma_*$ and in particular $\sup \mathcal{G}_\gamma \leqslant \sup \mathcal{G}_{\gamma_*}$. Hence, for $\gamma_1 < \gamma_i < n+1$ we get

$$-C := \mathcal{G}_{\gamma_1}(u_1) \leqslant \mathcal{G}_{\gamma_i}(u_i) \leqslant \mathcal{G}_{n+1}(u_i) \tag{7.10}$$

Now if the bound (7.9) holds then Lemma 4.4 shows that there exists δ such that the free energy functional $F_{n+1+\delta}$ is uniformly bounded from above along (u_j) and hence so are the functionals $\mathcal{G}_{n+1+\delta}$ (as explained in connection to Lemma 4.4). Combined with the lower bound (7.10) this means that

$$\mathcal{E}(u_j) \geqslant -C.$$

Hence, the Moser–Trudinger inequality applied to a fixed $\gamma_1 < n+1$ (i.e. the bound $\mathcal{G}_{\gamma_1} \leqslant C$) shows that $\int e^{-pu_j} \leqslant C_p$ for any $p > 0$. But then it follows from general principles (for the same reasons as in the Kähler case) that $\int e^{-pu_j} \to \int e^{-pu}$, i.e. $\|e^{-u_j}\|_{L^p(\Omega)} \to \|e^{-u}\|_{L^p(\Omega)}$ and even more precisely that

$$e^{-u_j} \to e^{-u}, \quad \text{in } L^p(\Omega). \tag{7.11}$$

In particular $\mathcal{L}_{n+1}(u_j) \to \mathcal{L}_{n+1}(u)$, as $j \to \infty$. Moreover, a similar argument shows that u is a maximizer of $\mathcal{G}_{n+1}$ and hence the projection argument gives, as above, that u solves the equation (7.8). Moreover, the convergence (7.11) for $p = 2$ gives that the $L^2(\Omega)$-norm of $\mathrm{MA}(u_j)/\mathrm{d}V - \mathrm{MA}(u)/\mathrm{d}V$ tends to zero and hence the stability result in [27] shows that $u_j \to u$ in $L^\infty(\Omega)$.

We next consider the case when the bound (7.9) does not hold, i.e. for any $\delta > 0$ the sequence $\int_\Omega e^{-(n+\delta)u_j}\mathrm{d}V$ is unbounded. Since we have assumed that there is no blow-up point in $\partial\Omega$, there is a constant M and a compact subset K of Ω such that $u \geqslant -M$ on $\Omega - K$ and hence $\int_K e^{-(n+\delta)u_\gamma}\mathrm{d}V$ is also unbounded. By the semi-continuity of complex singularity exponents [33] it follows that there is a neighborhood U of K (compactly contained in Ω) such that

$$\int_U e^{-(n+\delta)u} = \infty \tag{7.12}$$

for any $\delta > 0$. Since, by assumption, $u_j \to u$ in L^1_{loc}, where $\int_\Omega (\mathrm{dd}^c u_j)^n = 1$ it also follows that

$$\int_\Omega (\mathrm{dd}^c u)^n \leqslant 1 \tag{7.13}$$

(see for example the appendix in [32]). We claim that (7.12) implies that there exists a point $z_0 \in \overline{U}$ such that

$$\int_{\{z_0\}} (\mathrm{dd}^c u) \geqslant 1. \tag{7.14}$$

To see this first recall that if u is psh in a neighborhood of a point z_0 then its *complex singularity exponent* $c_{z_0}(u)$ at z_0 is defined as

$$c_{z_0}(u) := \sup\left\{t : \int_{U_0} e^{-tu}\mathrm{d}V < \infty\right\},$$

for U_0 some neighborhood of z_0. As shown in [2] (Thm 5.5) the Brezis–Merle–Demailly type inequality proved there may be localized to give that

$$c_{z_0}(u) \geqslant n/\left(\int_{\{z_0\}}(\mathrm{dd}^c u)^n\right)^{1/n}$$

for any point $z_0 \in \Omega$ and function $u \in \mathcal{F}(\Omega)$ (more generally the boundary conditions on u are not needed). Now, assume to get a contradiction that (7.14) does not hold. Then it follows from the previous inequality and the compactness of $\overline{U}$ that $\int_U e^{-(n+\delta)u} < \infty$, which contradicts (7.12). Now combining the lower bound (7.14) with the upper bound (7.13) we deduce that $(\mathrm{dd}^c u)^n = \delta_{z_0}$. Moreover, since $\int_\Omega(\mathrm{dd}^c u)^n \leqslant 1$ we already know, by the quasi-sharp B-M-D inequality that, for any $\delta > 0$ $e^{-(n-\delta)\phi}$ is in $L^1(\Omega)$ and hence $c_{z_0}(u) \geqslant n$. All in all this means that $u \in \mathcal{F}(\Omega)$ satisfies

$$(\mathrm{dd}^c u)^n = \delta_{z_0}, \quad c_{z_0}(u) = n \tag{7.15}$$

But, according to the results in [55], this can only happen if u is the pluricomplex Green function with a logarithmic pole in z_0. Indeed, according to [55, Cor. 1.3], if $u \in \mathcal{F}(\Omega)$ satisfies

$$\int_\Omega (\mathrm{dd}^c u)^n = \left(\frac{n}{c_{z_0}(u)}\right)^n,$$

then $u(z) = a\log|z - z_0|^2 + O(1)$ as $z \to z_0$ for some positive number a. In the present setting $n/c_{z_0}(u) = 1$ and hence it follows that $a = 1$, which concludes the proof. □

Remark 7.5. — In the first preprint version of the present paper on ArXiv a weaker form of the previous theorem appeared saying that, in the second alternative, u satisfies the equations (7.15) and it was conjectured that this implies that u is the pluricomplex Green function. We are grateful to Alexander Rashkovskii for pointing out to us that the validity of this conjecture follows from his recent results [55] (which use the results in [34]).

In the case when $n = 1$ it is well-known that there cannot be any blow-points on $\partial\Omega$ (see [52, Prop. 4]) and we expect this to be true in general. When $n = 1$ it is well-known that the question which of the two alternatives in the previous theorem holds depends on the nature of the domain Ω (see [23]). For example, for the disc the second alternative holds (with $z_0 = 0$), while the first alternative holds for an annulus (remarkably, there are convex domains for which the first alternative holds, for example any sufficently "thin" rectangle).

In the case of a general dimension n the sharpness part of Theorem 1.5 gives that, in the super critical case $\gamma > n+1$, the functional $\mathcal{G}_a$ is not bounded from above and in particular it has no maximizers (i.e. the last part of Theorem 7.2 cannot hold in this range). As for the critical case $\gamma = n+1$ one would expect that there is no solution of the equation (7.8) when Ω is the ball. [(3)] For radial solutions this is straight-forward to check. Indeed, an explicit calculation then reveals that, for any $\gamma < n+1$, a radial solution u_γ is uniquely determined and hence given by $u_\gamma = \phi_0^\epsilon$ (formula (3.3)) for some ϵ, where $\gamma \to n+1$ corresponds to $\epsilon \to 0$. Moreover, when $\gamma = n+1$ there is no radial solution and $u_\gamma \to (n+1)\log|z|^2$ as $\gamma \to n+1$ where u has infinite energy, i.e. it is not an element in $\mathcal{E}^1(\Omega)$. In fact, in the case $n = 1$ *any* solution is radial, as follows from the method of moving planes [41] (which also applies to the corresponding equation associated to the *real* Monge–Ampère operator [31]). Accordingly, it seems natural to make the following

Conjecture 7.6. — *In the case of the ball in $\mathbb{C}^n$ any solution to equation* (7.8) *is radial and hence given by u_γ above.*

If true the previous conjecture implies the validity of the sharp Moser–Trudinger inequality (without assuming S^1-invariance), i.e. that $\mathcal{G}_\gamma$ is bounded in the critical case $\gamma = n+1$. Indeed, given $u \in \mathcal{H}_0(\mathcal{B})$ we have

$$\mathcal{G}_\gamma(u) = \lim_{\epsilon\to 0} \mathcal{G}_{\gamma(\epsilon)}(u) \leqslant \lim_{\epsilon\to 0} \sup \mathcal{G}_{\gamma(\epsilon)}$$

But by the previous theorem the sup of $\mathcal{G}_{\gamma(\epsilon)}$ is attained for some function $u_{\gamma(\epsilon)}$ satisfying the equation (7.8), which if the conjecture above is correct has to be radial and thus coincides with ϕ_0^ϵ above. Finally, as shown towards the end in Section 3 $\mathcal{G}_a(\phi_0^\epsilon) \to C_n$ and hence $\mathcal{G}_a(u) \leqslant C_n$. Note also that by Theorem 1.4 it would be enough to know that any solution is S^1-invariant in order to deduce the sharp Moser-inequality using the previous argument.

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(3) Added in the revision: this has now been shown to be true by Chi Li [50], under some regularity assumptions on u, using a Pohozaev type identity.

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