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Integrability on Direct Limits of Banach Manifolds ^(*)

PATRICK CABAU ⁽¹⁾ AND FERNAND PELLETIER ⁽²⁾

ABSTRACT. — In this paper, we study several objects in the framework of direct limits of anchored Banach bundles over particular convenient manifolds (direct limits of Banach manifolds). In particular, we give a criterion of integrability for distributions on such convenient manifolds which are locally direct limits of particular sequences of Banach anchor ranges.

RÉSUMÉ. — Dans cet article, on s'intéresse à l'étude de divers objets rencontrés dans le cadre de limites directes de fibrés de Banach, munis d'une ancre, au dessus de certaines variétés apparaissant comme limites directes de variétés de Banach. En particulier, on donne un critère d'intégrabilité pour des distributions sur de telles variétés qui sont localement des limites directes de suites particulières d'images d'ancres banachiques.

1. Introduction and results

In classical differential geometry, a *distribution* on a smooth manifold M is an assignment

$$\mathcal{D} : x \mapsto \mathcal{D}_x \subset T_x M$$

on M , where \mathcal{D}_x is a subspace of $T_x M$. This distribution is *integrable* if, for any $x \in M$, there exists an immersed submanifold $f : L \rightarrow M$ such that $x \in f(L)$ and for any $z \in L$, we have $Tf(T_z L) = \mathcal{D}_{f(z)}$. On the other hand, \mathcal{D} is called *involutive* if, for any vector fields X and Y on M tangent to \mathcal{D} , their Lie bracket $[X, Y]$ is also tangent to \mathcal{D} .

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On a finite dimensional manifold, when \mathcal{D} is a subbundle of TM , the classical Frobenius Theorem gives an equivalence between integrability and involutivity. In the other case, the distribution is *singular* and, even under assumptions of smoothness on \mathcal{D} , in general, the involutivity is not a sufficient condition for integrability (one needs some more additional local conditions). These problems were clarified and resolved essentially in [30] and [31].

In the context of Banach manifolds, the Frobenius Theorem is again true for distributions which are complemented subbundles in the tangent bundle. For singular Banach distributions closed and complemented (i.e. \mathcal{D}_x is a complemented Banach subspace of $T_x M$) we also have the integrability property under some natural geometrical conditions (see [5] for instance). In a more general way, for weak Banach distributions \mathcal{D} , the integrability property is again true under some geometrical criterion (see [26] for more details).

The notion of *Lie algebroid* $\mathcal{A} = (E, \pi, M, \rho, [\cdot, \cdot]_E)$, where $\pi : E \rightarrow M$ is a fiber bundle and where the anchor ρ is a morphism of Lie algebras, was first introduced by Pradines in [29]. Such objects can be seen as generalizations of both Lie algebras and tangent vector bundles. This context is an adapted framework for different problems one can meet in Mechanics (e.g. non holonomic lagrangian systems, [6]) or in symplectic Geometry in view of the symplectization of Poisson manifolds and applications to quantization ([20], [33]).

The Stefan–Sussmann’s Theorem implies the integrability of the distribution $\rho(E)$ for a finite dimensional Lie algebroid $(E, \pi, M, \rho, [\cdot, \cdot]_E)$. Moreover one also gets the existence of symplectic leaves for Lie–Poisson Banach manifolds under comparable assumptions. For a Banach Lie algebroid $(E, \pi, M, \rho, [\cdot, \cdot]_E)$ the same result is also true under some additional assumptions (see [26]).

However, the Banach context is not necessarily the most appropriate: for instance, in the framework of Lie–Poisson structure on the dual of the Lie algebra of an infinite-dimensional Lie group, the adapted model is not anymore the Banach one. A lot of infinite-dimensional Lie groups G , linked with symmetries depending on infinitely many parameters one can meet in Mathematical Physics, can often be expressed as the union of an ascending sequence $G_1 \subset G_2 \subset \dots \subset G_i \subset \dots$ of finite or infinite-dimensional Lie groups. Various examples of such objects can be found in papers of Glöckner (see [11], [12] and [14]). The convenient setting as defined by [10] and [21] seems well adapted to this framework (see for instance [11]).

The context of this paper concerns the study of direct limits of anchored Banach bundles over direct limits of Banach manifolds endowed with convenient structures and the results of [11], [12] and [14]. More precisely, *essentially* we consider sequences $(E_n, \pi_n, M_n, \rho_n)_{n \in \mathbb{N}^*}$ of anchored Banach bundles where $(E_n, \pi_n, M_n)_{n \in \mathbb{N}^*}$ is a strong ascending sequence of Banach bundles (cf. Definition 4.17) where the anchors ρ_n and the bonding maps $\lambda_n^m : E_n \rightarrow E_m$ and $\epsilon_n^m : M_n \rightarrow M_m$ fulfill conditions of compatibility given in Definition 6.9, (2).

Given such a sequence $(E_n, \pi_n, M_n, \rho_n)_{n \in \mathbb{N}^*}$, we get an anchored convenient bundle $(E = \varinjlim E_n, \pi = \varinjlim \pi_n, M = \varinjlim M_n, \rho = \varinjlim \rho_n)$ (cf. Theorem 6.10). Note that, according to Glöckner's results, in order to get an interesting (convenient) structure on the direct limit of Banach manifolds, an essential hypothesis is *the existence of direct limit charts* (cf. Definition 4.3). In particular, this assumption is true if each member M_n of the ascending sequence $M_1 \subset \dots \subset M_n \subset \dots$ of Banach manifolds can be endowed with a Koszul connection ∇^n (cf. Proposition 4.5).

Another problem in the context of direct limit of an ascending sequence $\{X_n\}_{n \in \mathbb{N}^*}$ of topological spaces is the following one: even if each X_n is a Hausdorff topological space the direct limit $X = \varinjlim X_n$, provided with the direct limit topology, can be *not Hausdorff*. This leads us to introduce the notion of *non necessary Hausdorff convenient manifold structure* (cf. Definition 2.13).

When each E_n can be endowed with an almost Lie bracket $[\cdot, \cdot]_n$ (resp. a Koszul connection ∇^n) such that the restriction of $[\cdot, \cdot]_{n+1}$ (resp. ∇^{n+1}) to E_n is $[\cdot, \cdot]_n$ (resp. ∇^n) we obtain an almost Lie bracket $[\cdot, \cdot] = \varinjlim [\cdot, \cdot]_n$ (resp. a Koszul connection $\nabla = \varinjlim \nabla^n$) on E . Moreover, if for each $n \in \mathbb{N}$ we have $[\rho_n(X), \rho_n(Y)] = \rho_n[X, Y]_n$ (resp. $[\cdot, \cdot]_n$ satisfies the Jacobi identity) the same property is true for the direct limit $[\cdot, \cdot]$.

Now, according to Theorem 5 of [26], we obtain the main result of this paper:

THEOREM (Criterion of integrability (cf. Theorem 7.5)). — *Let Δ be a distribution on a convenient manifold M with the following properties:*

- (1) *for any $x \in M$, there exists an open neighborhood U of x , a strong ascending sequence of anchored Banach bundles $(E_n, \pi_n, U_n, \rho_n)_{n \in \mathbb{N}^*}$ endowed with a Koszul connection ∇^n , such that $U = \varinjlim U_n$, $\varinjlim \rho_n(E_n) = \Delta|_U$ and such that E_n is a complemented subbundle of E_{n+1} ;*

(2) *there exists an almost Lie bracket $[\cdot, \cdot]_n$ on $(E_n, \pi_n, U_n, \rho_n)$ such that:*

- *$(E_n, \pi_n, U_n, \rho_n, [\cdot, \cdot]_n)$ is a Banach Lie algebroid;*
- *over each point $y_n \in U_n$ the kernel of ρ_n is complemented in the fiber $\pi_n^{-1}(y_n)$.*

Then the distribution Δ is integrable and each maximal integral manifold satisfies the direct limit chart property at any point and so is endowed with a non necessary Hausdorff convenient manifold structure.

Note that in the framework of finite-dimensional or Hilbert manifolds, this criterion of integrability requires much weaker assumptions (cf. Corollary 7.7).

In order to make this article as self-contained as possible we first recall various notions: the convenient differential calculus setting as defined by Frölicher, Kriegl and Michor (Section 2), direct limits of topological vector spaces (Section 3) or manifolds (Section 4) and linear connections on Banach bundles (Section 5). In Section 6, Theorem 6.10, we prove that certain limits of Almost Lie Banach algebroids can be endowed with a structure of Almost Lie convenient algebroid. In the last part, we prove the previous theorem which is a criterion of integrability for distributions and we give an application to actions of direct limits of Banach Lie groups.

2. Convenient differential calculus

Differential calculus in infinite dimensions has already a long history which goes back to the beginnings of variational calculus developed by Bernoulli and Euler. During the last decades, a lot of theories of differentiation have been proposed in order to differentiate in spaces more general than Banach ones; the traditional calculus for Banach spaces is not satisfactory for the categorical point of view since the space $C^\infty(E, F)$ of smooth maps between Banach spaces is no longer a Banach space.

The setting of convenient differential calculus discovered by A. Frölicher and A. Kriegl (see [10]) is chosen. The reference for this section is the tome [21] which includes some further results.

In order to define the smoothness on locally convex topological vector spaces (l.c.t.v.s.) E , the basic idea is to test it along smooth curves (cf. Definition 2.6), since this notion in this realm is a concept without problems.

On the one hand, a curve $c : \mathbb{R} \rightarrow E$ is *differentiable* if, for all t , the derivative $c'(t)$ exists where $c'(t) = \lim_{h \rightarrow 0} \frac{1}{h}(c(t+h) - c(t))$. It is *smooth* if all iterative derivatives exist.

On the other hand, if J is an open subset of \mathbb{R} , we say that c is *Lipschitz* on J if the set $\{\frac{c(t_2)-c(t_1)}{t_2-t_1}; t_1, t_2 \in J, t_1 \neq t_2\}$ is bounded in E . The curve c is *locally Lipschitz* if every point in \mathbb{R} has a neighborhood on which c is Lipschitz. For $k \in \mathbb{N}$, the curve c is of class Lip^k if c is derivable up to order k , and if the k^{th} -derivative $c : \mathbb{R} \rightarrow E$ is locally Lipschitz.

We then have the following link between both these notions ([21, Section 1.2]):

PROPOSITION 2.1. — *Let E be a l.c.t.v.s, and let $c : \mathbb{R} \rightarrow E$ be a curve. Then c is C^∞ if and only if c is Lip^k for all $k \in \mathbb{N}$.*

The space $C^\infty(\mathbb{R}, E)$ of such curves does not depend on the locally convex topology on E but only on its associated *bornology* (system of bounded sets). Note that the topology can vary considerably without changing the bornology; the *bornologification* E_{born} of E is the finest locally convex structure having the same bounded sets.

One can note that the link between continuity and smoothness in infinite dimension is not as tight as in finite dimension: there are smooth maps which are not continuous!

The c^∞ -topology on a l.c.v.s. is the final topology with respect to all smooth curves $\mathbb{R} \rightarrow E$; it is denoted by $c^\infty E$. Its open sets will be called c^∞ -open.

For every absolutely convex closed bounded set B , the linear span E_B of B in E is equipped with the Minkowski functional $p_B(v) = \inf\{\lambda > 0 : v \in \lambda.B\}$ which is a norm on E_B .

We then have the following characterization of c^∞ -open sets ([21, Theorem 2.13]):

PROPOSITION 2.2. — *$U \subset E$ is c^∞ -open if and only if $U \cap E_B$ is open in E_B for all absolutely convex bounded subsets $B \subset E$.*

Remark 2.3. — The c^∞ -topology is in general finer than the original topology and E is not a topological vector space when equipped with the c^∞ -topology.

For Fréchet spaces and so Banach spaces, this topology coincides with the given locally convex topology.

DEFINITION 2.4. — *A locally convex vector space E is called bornological if any bounded linear mapping⁽¹⁾ $f : E \rightarrow F$ (where F is any Banach space) is continuous.*

LEMMA 2.5. — *Let E be a bornological vector space. The c^∞ -topology and the locally convex topology coincide (i.e. $c^\infty E = E$) if the closure of subsets in E is formed by all limits of sequences in the subset.*

DEFINITION 2.6. — *Let E and F be l.c.t.v.s. A mapping $f : E \rightarrow F$ is called conveniently smooth if it maps smooth curves into smooth curves, i.e. if $f \circ c \in C^\infty(\mathbb{R}, F)$ for all $c \in C^\infty(\mathbb{R}, E)$.*

Note that in finite dimensional spaces E and F this corresponds to the usual notion of smooth mappings as proved by Boman (see [3]).

In finite-dimensional analysis, we use the Cauchy condition, as a necessary condition for the convergence of a sequence, to define completeness of the space. In the infinite-dimensional framework, we use the notion of Mackey–Cauchy sequence (cf. [21, Section 2]).

DEFINITION 2.7. — *A sequence (x_n) in E is called Mackey–Cauchy if there exists a bounded absolutely convex subset B of E such that (x_n) is a Cauchy sequence in the normed space E_B .*

DEFINITION 2.8. — *A locally convex vector space is said to be c^∞ -complete or convenient if any Mackey–Cauchy sequence converges (c^∞ -completeness).*

We then have the following characterizations:

PROPOSITION 2.9. — *A locally convex vector space is convenient if one of the following equivalent conditions is satisfied:*

- (1) *For every absolutely convex closed bounded set B the linear span E_B of B in E , equipped with the norm p_B is complete.*
- (2) *A curve $c : \mathbb{R} \rightarrow E$ is smooth if and only if $\lambda \circ c$ is smooth for all $\lambda \in E'$ where E' is the dual consisting of all continuous linear functionals on E .*
- (3) *Any Lipschitz curve in E is locally Riemann integrable.*

Example 2.10. — The vector space \mathbb{R}^∞ , also denoted by $\mathbb{R}^{(\mathbb{N})}$ or Φ , of all finite sequences is a countable convenient vector space ([21, 47.1]) which is not metrizable. A basis of \mathbb{R}^∞ is $(e_i)_{i \in \mathbb{N}^*}$ where $e_i = (0, \dots, 0, 1_{i^{\text{th}} \text{ term}}, 0, \dots)$.

⁽¹⁾ A linear map between locally convex vector spaces is *bounded* if it maps every bounded set to a bounded one.

THEOREM 2.11. — *Let U be a c^∞ -open set of a convenient vector space E and let F and G be convenient vector spaces.*

- (1) *The space $C^\infty(U, F)$ may be endowed with a structure of convenient vector space. The subspace $L(E, F)$ of all bounded linear mappings from E to F is closed in $C^\infty(E, F)$.*
- (2) *The category is cartesian closed, i.e. we have the natural diffeomorphism:*

$$C^\infty(E \times F, G) \simeq C^\infty(E, C^\infty(F, G)).$$

- (3) *The differential operator*

$$d : C^\infty(E, F) \longrightarrow C^\infty(E, L(E, F))$$

$$df(x)v = \lim_{t \rightarrow 0} \frac{f(x + tv) - f(x)}{t}$$

exists and is linear and smooth.

- (4) *The chain rule holds:*

$$d(f \circ g)(x)v = df(g(x))dg(x)v.$$

PROPOSITION 2.12. — *The following constructions preserve c^∞ -completeness: limits, direct sums, strict direct limits of sequences of closed embeddings.*

In general, an inductive limit of c^∞ -complete spaces needs not be c^∞ -complete (cf. [21, 2.15] for example).

According to [21, Section 27.1], a C^∞ -atlas modelled on a set M modelled on a convenient space E is a family $\{(U_\alpha, u_\alpha)\}_{\alpha \in A}$ of subsets U_α of M and maps u_α from U_α to E such that:

- u_α is a bijection of U_α onto a c^∞ -open subset of E for all $\alpha \in A$;
- $M = \bigcup_{\alpha \in A} U_\alpha$;
- for any α and β such that $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$, $u_{\alpha\beta} = u_\alpha \circ u_\beta^{-1} : u_\beta(U_{\alpha\beta}) \rightarrow u_\alpha(U_{\alpha\beta})$ is a conveniently smooth map.

Classically, we have a notion of equivalent C^∞ -atlases on M . An equivalent class of C^∞ -atlases on M is a maximal C^∞ -atlas. Such an atlas defines a topology on M which is not in general Hausdorff.

DEFINITION 2.13. — *A maximal C^∞ -atlas on M is called a non necessary Hausdorff convenient manifold structure on M (n.n.H. convenient manifold M for short); it is called a Hausdorff convenient manifold structure on M when the topology defined by this atlas is a Hausdorff topological space.*

Following the classical framework, when E is a Banach space (resp. a Fréchet space) we say that M is a *Banach manifold* (resp. *Fréchet manifold*) if M is provided with a C^∞ -atlas (modelled on E) which generates a Hausdorff topological space.

The notion of vector bundle modelled on a convenient space over a n.n.H. convenient manifold is defined in a classic way (cf. ([21, 29])). Note that since a convenient space is Hausdorff, a vector bundle modelled on a convenient space has a natural structure of n.n.H. convenient manifold which is Hausdorff if and only if the base is a Hausdorff convenient manifold.

3. Direct limits of topological vector spaces

In this section the reader is referred to [4], [11] and [12].

Let (I, \leq) be a directed set. A *direct system* in a category \mathbb{A} is a pair $\mathcal{S} = (X_i, \varepsilon_i^j)_{i \in I, j \in I, i \leq j}$ where X_i is an object of the category and $\varepsilon_i^j : X_i \rightarrow X_j$ is a morphism (*bonding map*) where:

- (1) $\varepsilon_i^i = \text{Id}_{X_i}$;
- (2) $\forall (i, j, k) \in I^3, i \leq j \leq k \Rightarrow \varepsilon_j^k \circ \varepsilon_i^j = \varepsilon_i^k$.

A *cone* over \mathcal{S} is a pair $(X, \varepsilon_i)_{i \in I}$ where $X \in \text{ob } \mathbb{A}$ and $\varepsilon_i : X_i \rightarrow X$ is such that $\varepsilon_j \circ \varepsilon_i^j = \varepsilon_i$ whenever $i \leq j$.

A cone $(X, \varepsilon_i)_{i \in I}$ is a *direct limit* of \mathcal{S} if for every cone $(Y, \theta_i)_{i \in I}$ over \mathcal{S} there exists a unique morphism $\psi : X \rightarrow Y$ such that $\psi \circ \varepsilon_i = \theta_i$. We then write $X = \varinjlim \mathcal{S}$ or $X = \varinjlim X_i$.

When $I = \mathbb{N}$ with the usual order relation, countable direct systems are called *direct sequences*.

3.1. Direct limit of sets

Let $\mathcal{S} = (X_i, \varepsilon_i^j)_{i \in I, j \in I, i \leq j}$ be a direct system of sets (we then have $\mathbb{A} = \text{SET}$).

Let $\mathcal{U} = \coprod_{i \in I} X_i = \{(x, i) : x \in X_i\}$ be the disjoint union of the sets X_i with the canonical inclusion

$$\begin{aligned} \nu_i : X_i &\longrightarrow \mathcal{U} \\ x &\longmapsto (x, i). \end{aligned}$$

We define an equivalence relation on \mathcal{U} as follows: $\mathbf{u}_i(x) \sim \mathbf{u}_j(y)$ if there exists $k \in I$: $i \leq k$ and $j \leq k$ s.t. $\varepsilon_i^k(x) = \varepsilon_j^k(y)$. We then have the quotient set $X = \mathcal{U}/\sim$ and the map $\varepsilon_i = \pi \circ \mathbf{u}_i$ where $\pi : \mathcal{U} \rightarrow \mathcal{U}/\sim$ is the canonical quotient map. Then (X, ε_i) is the direct limit of \mathcal{S} in the category $\mathbb{S}\mathbb{E}\mathbb{T}$. If each ε_i^j is injective then so is ε_i , whence \mathcal{S} is equivalent to the direct system of the subsets $\varepsilon_i(X_i) \subset X$, together with the inclusion maps.

3.2. Direct limit of topological spaces

If $\mathcal{S} = (X_i, \varepsilon_i^j)_{i \in I, j \in I, i \leq j}$ is a direct system of topological spaces and continuous maps, then the direct limit $(X, \varepsilon_i)_{i \in I}$ of the sets becomes the direct limit in the category $\mathbb{T}\mathbb{O}\mathbb{P}$ of topological spaces if X is endowed with the direct limit topology (*DL-topology* for short), i.e. the finest topology which makes the maps ε_i continuous. So $O \subset X$ is open if and only if $\varepsilon_i^{-1}(O)$ is open in X_i for each $i \in I$.

When $\mathcal{S} = (X_n, \varepsilon_n^m)_{n \in \mathbb{N}^*, m \in \mathbb{N}^*, n \leq m}$ is a direct sequence of topological spaces such that each ε_n^m is injective, without loss of generality, we may assume that we have

$$X_1 \subset X_2 \subset \dots \subset X_n \subset X_{n+1} \subset \dots$$

and ε_n^{n+1} becomes the natural inclusion. Therefore \mathcal{S} will be called an *ascending sequence of topological spaces* and simply denoted $(X_n)_{n \in \mathbb{N}^*}$.

Moreover, if each ε_n^m is a topological embedding, then we will say that \mathcal{S} is a *strict ascending sequence of topological spaces* (*expanding sequence* in the terminology of [17]). In this situation, each ε_n is a topological embedding on X_n in $X = \varinjlim X_n$.

Let us give some properties of ascending sequences of topological spaces ([12, Lemma 1.7]):

PROPOSITION 3.1. — *Let $(X_n)_{n \in \mathbb{N}^*}$ be an ascending sequence of topological spaces. Equip $X = \bigcup_{n \in \mathbb{N}^*} X_n$ with the final topology with respect to the inclusion maps $\varepsilon_n : X_n \rightarrow X$ (i.e. the DL-topology). Then we have:*

- (1) *If each X_n is T_1 , then X is T_1 .*
- (2) *If $O_n \subset X_n$ is open and $O_1 \subset O_2 \subset \dots$, then $O = \bigcup_{n \in \mathbb{N}^*} O_n$ is open in X and the DL-topology on $O = \varinjlim O_n$ coincides with the topology induced by X .*
- (3) *If each X_n is locally compact, then X is Hausdorff.*
- (4) *If each X_n is T_1 and $K \subset X$ is compact, then $K \subset X_n$ for some n .*

Unfortunately, in general, a direct limit of Hausdorff topological spaces is not Hausdorff (see [18] for an example of such a situation). Sufficient conditions on $(X_n)_{n \in \mathbb{N}^*}$ under which the direct limit $X = \bigcup_{n \in \mathbb{N}^*} X_n$ is Hausdorff can be found in [16]. However, we have:

PROPOSITION 3.2. — *Let $(X_n)_{n \in \mathbb{N}^*}$ be a strict ascending sequence of topological spaces; equip $X = \bigcup_{n \in \mathbb{N}^*} X_n$ with the DL-topology. Then we have:*

- (1) *Assume that for each n , X_n is closed in X_{n+1} .*
 - (a) *If each X_n is normal then X is normal.*
 - (b) *If each X_n is Hausdorff and paracompact then X is normal.*
In particular, in each previous situation, X is Hausdorff.
- (2) *Assume that for each n , X_n is open in X_{n+1} and Hausdorff, then X is Hausdorff.*

Proof.

(1a). — See [17, Proposition 4.3(i)].

(1b). — It is well known that any Hausdorff and paracompact topological space is normal and then such a topological space is Hausdorff.

(2). — See [17, Proposition 4.2]. □

Remark 3.3. — The direct limit of an ascending sequence $(X_n)_{n \in \mathbb{N}^*}$ is equal to the direct limit of $(X_n)_{n \in \mathbb{N}^*, n \geq n_0}$

Let $(X_n, i_n^m)_{n \leq m, m \in \mathbb{N}^*, n \in \mathbb{N}^*}$ and $(Y_n, j_n^m)_{n \leq m, m \in \mathbb{N}^*, n \in \mathbb{N}^*}$ be two ascending sequences of topological spaces. Then assume that we are given a sequence of maps $f_n : X_n \rightarrow Y_n$ which is *consistent*, i.e. that we have for any $n \leq m$, $f_m \circ i_n^m = j_n^m \circ f_n$. Then these sequences induce a map $f : X = \varinjlim X_n \rightarrow Y = \varinjlim Y_n$ s.t. $f \circ i_n = i_n \circ f_n$ where $i_n : X_n \rightarrow X = \varinjlim X_n$ and $j_n : Y_n \rightarrow Y = \varinjlim Y_n$ are the associate inclusions respectively.

If every map f_n is continuous, then the induced map f is continuous with respect to the DL-topologies on X and Y (continuity criterion, [19]).

3.3. Direct limit of Banach spaces

Let $(E_n)_{n \in \mathbb{N}^*}$ be an ascending sequence of Banach spaces. It is easy to see that we can choose a norm $\|\cdot\|_n$ on E_n , for $n \in \mathbb{N}^*$, such that:

$$\|\cdot\|_{n+1} \leq \|\cdot\|_n \text{ on } E_n \text{ for each } n \in \mathbb{N}^*.$$

In this paper, we always make such a choice.

Given such an ascending sequence of Banach spaces, then $E = \bigcup_{n \in \mathbb{N}^*} E_n$ is called the *direct limit* of this sequence. The finest locally convex vector topology making each inclusion map $E_n \rightarrow E$ continuous, is called the *locally convex direct limit topology* and denoted *LCDL-topology* for short.

A convex set $O \subset E$ is open in this topology if and only if $O \cap E_n$ is open in E_n for each $n \in \mathbb{N}^*$.

If E_n is a Banach subspace of E_{n+1} for each $n \in \mathbb{N}^*$, we have a *strict ascending sequence of Banach spaces*.

DEFINITION 3.4. — *A locally convex limit of ascending sequence of Banach spaces is called an (LB)-space. If the sequence is strict we speak of LB-space or strict (LB)-space.*

Of course, an (LB)-space does not have a structure of convenient space in general. However, since every Banach space is convenient, then each LB-space has a structure of convenient space (see [21, Theorem 2.15]). In particular the direct limit of an ascending sequence of finite dimensional Banach spaces has a structure of convenient space.

Note that if $E = \varinjlim E_n$ and $F = \varinjlim F_n$ are LB-spaces, then we can identify $E \times F = \varinjlim (E_n \times F_n)$ with $\varinjlim E_n \times \varinjlim F_n$ as locally convex topological spaces (cf. [19, Theorem 4.3]) and so $E \times F$ is a convenient space (cf. [21]).

We now consider a general situation which gives rise to a convenient structure on a direct limit of an ascending sequence of Banach spaces.

PROPOSITION 3.5. — *Let $(E_n)_{n \in \mathbb{N}^*}$ be an ascending sequence of Banach spaces. Assume that there exists an infinite subset $I \subset \mathbb{N}^*$ such that $E_I = \bigcup_{i \in I} E_i = \varinjlim E_i$ is an LB-space. Then $E = \bigcup_{n \in \mathbb{N}^*} E_n = E_I$ and E is an LB-space.*

Proof. — We set $J = \mathbb{N}^* \setminus I$. An index of I (resp. J) will be denoted i_l , $l \in \mathbb{N}^*$ (resp. j_k , $k \in \mathbb{N}^*$). In the category of $\mathbb{S}\mathbb{E}\mathbb{T}$, we have $E_I = \bigcup_{l \in \mathbb{N}^*} E_{i_l} = \varinjlim E_{i_l}$ and $E_J = \bigcup_{k \in \mathbb{N}^*} E_{j_k} = \varinjlim E_{j_k}$. From our assumption, E_I is a convenient space and for any $i_l \in I$ there exists $j_k \in J$ such that $E_{i_l} \subset E_{j_k}$ and conversely. Therefore we have the equality $E_I = E_J = E$. Thus in the category $\mathbb{S}\mathbb{E}\mathbb{T}$ we have

$$E = \varinjlim E_{i_l} = \varinjlim E_{j_k}.$$

Consider an open set $O = \bigcup_{l \in \mathbb{N}^*} O_{i_l}$ of E_I . Given any $j_k \in J$, the space E_{j_k} is contained in some $E_{i_{l_0}}$ for $i_{l_0} < j_k$. Therefore $O \cap E_{j_k} = O \cap O_{i_{l_0}}$. As the inclusion $\iota_{i_{l_0}}^{i_{l_0}} : E_{j_k} \rightarrow E_{i_{l_0}}$ is continuous (as composition of a finite number

of continuous inclusions), $O \cap E_{j_k}$ is an open set of E_{j_k} . It follows that the *DL*-topology of $E = \varinjlim E_n$ and E_I coincide. In particular E is Hausdorff.

Of course, the algebraic structure of vector space on each set E, E_i coincide and then E and E_I have the same convex sets. As a set, O is an open set of the *LCDL*-topology on E if and only if O is convex and $O \cap E_n$ is open in E_n for all $n \in \mathbb{N}^*$. Again, as each E_{j_k} is contained in some $E_{i_{l_0}}$ for some $i_{l_0} > j_k$, it follows that O is an open set of the *LCDL*-topology on E if and only if O is an open set for *LCDL*-topology on E_I . It follows that the *LCDL*-topology on E and E_I also coincide. Finally, the locally convex vector spaces E_I and E have the same convex bounded sets. It follows that E is a convenient space with the same structure as E_I and also the same c^∞ -topology. \square

LEMMA 3.6. — *If $E = \varinjlim E_n$ is an LB-space, then for the LCDL-topology we have:*

- (1) *E is Hausdorff and bounded regular (i.e. every bounded subset of E is contained in some E_n).*
- (2) *Let $f : E \rightarrow F$ be a linear map where F is a Banach space. The following properties are equivalent:*
 - (a) *f is bounded;*
 - (b) *each restriction f_n of f to E_n is continuous;*
 - (c) *f is continuous.*

Proof.

(1). — As E is endowed with a convenient structure, it must be Hausdorff. On the other hand, since E is an LB-space, it must be bounded regular (see for example [27]).

(2a) \Rightarrow (2b). — If f is bounded, then its restriction f_n of f to E_n is bounded, so $f_n : E_n \rightarrow F$ is continuous.

(2b) \Rightarrow (2c). — Assume that each restriction f_n of f to E_n is continuous. To prove that f is continuous, it is sufficient to show that for any ball $B(0, r)$ in F , then $f^{-1}(B(0, r))$ is an open convex set of E . But we have

$$f^{-1}(B(0, r)) = \bigcup_{n \in \mathbb{N}^*} f_n^{-1}(B(0, r)).$$

As each f_n is continuous, $U_n = f_n^{-1}(B(0, r))$ is an open set of E_n . Moreover, $U_n \subset U_{n+1}$. On the other hand, as $B(0, r)$ is convex, so is U_n . There $U = f^{-1}(B(0, r)) = \bigcup_{n \in \mathbb{N}^*} U_n$ is also convex. So U is an open set of E (relative to the *LCDL*-topology).

(2c) \Rightarrow (2a). — Assume now that f is continuous and consider a bounded set B of E . From part (1), B is contained in some E_n . But as the inclusion

of E_n in E is continuous, f_n is continuous and so $f(B) = f_n(B)$ is bounded in F . \square

PROPOSITION 3.7. — *On an LB-space $E = \varinjlim E_n$, the DL-topology coincides with the c^∞ -topology.*

Proof. — Let B be an absolutely convex bounded set of E . From Lemma 3.6, B is contained in some E_n and then E_B is a vector subspace of E_n equipped with the Minkowski norm p_B . The closed unit ball B_n in E_n is also an absolutely convex bounded set and there exists $\alpha \geq 1$ such that $B \subset \alpha.B_n$. Remark that if p_n is the given norm on E_n , then p_n is the Minkowski functional associated to B_n . If $v \in E_B$, then $\frac{v}{p_B(v)}$ belongs to $\alpha.B_n$; so we get

$$p_n \leq \alpha p_B.$$

Therefore the inclusion of E_B in E_n is continuous. Let U be an open set of the DL-topology on E . Then for any $n \in \mathbb{N}^*$, $U \cap E_n$ is open in E_n . Thus, given any absolutely convex bounded set B of E , if E_B is contained in E_n , then $U \cap E_B = U \cap E_n \cap E_B$ is open in E_B . It follows from Proposition 2.2 that U is a c^∞ -open.

Conversely, if U is a c^∞ -open, as $E_n = E_{B_n}$ and the norm p_{B_n} is the given norm on E_n , again from Proposition 2.2, it follows that U is an open set of the DL-topology on E . \square

Remark 3.8. — In Proposition 3.7, the fact that an LB-space is regular is essential. More generally, if an (LB)-space is convenient and bounded regular, such a result is also true. However, for the sake of simplicity, we limit ourselves to the LB-space context.

Recall that from the classical differential calculus in locally convex topological spaces, for $r \in \mathbb{N} \cup \{\infty\}$, the map f is of class C^r (C^r -map for short) if it is continuous and, for all $k \in \mathbb{N}$ such that $k \leq r$, the iterated directional derivatives $d_k f(x, y_1, \dots, y_k) := D_{y_1} \dots D_{y_k} f(x)$ exist for all $x \in U$ and $y_1, \dots, y_k \in E$ and the associated map $d_k f : U \times E^k \rightarrow F$ is continuous. When $r = \infty$ we say that f is smooth.

We have the following link between C^∞ -smoothness on each member E_n and conveniently smoothness on E for an LB-space (cf. [12, Lemma 1.9]):

LEMMA 3.9. — *Let $E = \varinjlim E_n$ be an LB-space, $U \subset E = \varinjlim E_n$ an open set of E (for the DL-topology) and $U_n = U \cap E_n$ the associated open set in E_n . Given a map $f : U \rightarrow F$ where F is a convenient space, f is conveniently smooth if and only if $f_n = f|_{U \cap E_n}$ is C^∞ for each $n \in \mathbb{N}^*$.*

Proof. — This proof is an adaption of the proof of Lemma 1.9 of [12].

First, note that on an open set of a Banach space, we have equivalence between conveniently smoothness and C^∞ -differentiability (cf. [3]).

Assume that f is conveniently smooth on U . Given any smooth curve $\gamma : \mathbb{R} \rightarrow U \cap E_n$, then γ is a smooth curve in U , so $f \circ \gamma$ is smooth. As $U \cap E_n$ is an open set of a Banach space, it follows that $f_n = f|_{U \cap E_n}$ is C^∞ .

Conversely⁽²⁾ assume that $f_n = f|_{U \cap E_n}$ is C^∞ and let $\gamma : \mathbb{R} \rightarrow U$ be a smooth curve. Fix $a < b$ and $k \in \mathbb{N}$. From the bounded regularity of E , it follows that there exists $N \in \mathbb{N}$, $N \geq n$, such that all the sets

$$\left\{ \frac{(\gamma^{(j)}(t) - \gamma^{(j)}(s))}{s - t} : (s, t) \in]a, b[\times]a, b[, s \neq t \right\} \quad \text{and } \gamma^{(j)}(]a, b[)$$

where $\gamma^{(j)} : \mathbb{R} \rightarrow E$ is the j^{th} derivative of γ , are contained and bounded in E_N for all $j = 0, \dots, k$. Since $\gamma|_{]a, b[}$ is a Lipschitz curve in the Banach space E_N , so there exists a primitive η_j on $]a, b[$. Thus η_j and $\gamma|_{]a, b[}^{(j-1)}$ have the same derivative $\gamma|_{]a, b[}^{(j)}$ and so these curves differ by a constant for all $j = 1, \dots, k$. We conclude that $\gamma^{(j-1)} :]a, b[\rightarrow E_N$ is a C^1 -curve whose derivative is $\gamma|_{]a, b[}^{(j)}$. As a consequence, $\gamma|_{]a, b[}$ is a Lip^k -curve in E_N . We can choose $b - a$ small enough such that $\gamma(]a, b[) \subset U \cap E_n \subset U \cap E_N$. Since the restriction of f to $U \cap E_n$ is smooth, it follows that, for any $k \in \mathbb{N}$, $f \circ \gamma :]a, b[\rightarrow \mathbb{R}$ is Lip^k for any $a < b$ where $b - a$ is small enough. From Proposition 2.1, we get that $f \circ \gamma$ is smooth in E and so f is conveniently smooth (cf. Definition 2.6). \square

PROPOSITION 3.10. — *Let $E = \varinjlim E_n$ and $F = \varinjlim F_n$ be LB-spaces and $U_1 \subset \dots \subset U_n \subset \dots$ an ascending sequence (U_n) of open sets of (E_n) and set $U = \bigcup_{n \in \mathbb{N}^*} U_n$. Assume that we have a sequence of $f_n : U_n = U \cap E_n \rightarrow F_n$ which are C^∞ .*

- (1) *Then $f = \varinjlim f_n$ is a conveniently smooth map from U to F .*
- (2) *Let $f_n : E_n \rightarrow F_n$ be a sequence of continuous linear maps; then f is a linear map from E to F which is conveniently smooth and continuous for the DL-topologies.*

Proof.

(1). — According to Proposition 3.1, U is an open set of E and $U = \varinjlim U_n$. So $f = \varinjlim f_n : U \rightarrow F$ is a well defined continuous map (for the DL-topology). From Lemma 3.9, f is then a conveniently smooth map from U to F .

⁽²⁾ We are grateful to the anonymous referee for this part of the proof.

(2). — Under the assumption of (2), the associated map $f = \varinjlim f_n$ is linear and continuous. Now, as f_n is continuous linear between Banach spaces, f_n must be C^∞ and by part (1), f must be conveniently smooth. \square

4. Direct limit of manifolds

4.1. Direct limit of ascending sequence of Banach manifolds

Let M be a n.n.H. convenient manifold modelled on a convenient space E and TM its kinematic tangent bundle (cf. [21, 28.1]). We adapt to our context the notion of weak submanifold used in [26].

DEFINITION 4.1. — *A weak submanifold of M is a pair (N, φ) where N is a n.n.H. convenient connected manifold (modelled on a convenient space F) and $\varphi : N \rightarrow M$ is a conveniently smooth map such that:*

- *there exists a continuous injective linear map $i : F \rightarrow E$ (for the structure of l.c.v.s. of E)*
- *φ is an injective conveniently smooth map and the tangent map $T_x\varphi : T_xN \rightarrow T_{\varphi(x)}M$ is an injective continuous linear map with closed range for all $x \in N$.*

Note that for a weak submanifold $\varphi : N \rightarrow M$, on the subset $\varphi(N)$ of M , we have two topologies:

- the induced topology from M ;
- the topology for which φ is a homeomorphism from N to $\varphi(N)$.

With this last topology, via φ , we get on $\varphi(N)$ a n.n.H. convenient manifold structure modelled on F . Moreover, the inclusion from $\varphi(N)$ into M is continuous as a map from the manifold $\varphi(N)$ to M . In particular, if U is an open set of M , then $\varphi(N) \cap U$ is an open set for the topology of the manifold on $\varphi(N)$. Therefore, if M is Hausdorff so is $\varphi(N)$.

LEMMA 4.2. — *Let $\mathcal{M} = (M_n)_{n \in \mathbb{N}^*}$ be an ascending sequence of Banach C^∞ -manifolds, where M_n is modelled on the Banach space E_n and where the inclusion $\varepsilon_n^{n+1} : M_n \rightarrow M_{n+1}$ is a C^∞ injective map such that $\varepsilon_n^{n+1}(M_n)$ is a weak submanifold of M_{n+1} .*

- (i) *There exist injective continuous linear maps $\iota_n^{n+1} : E_n \rightarrow E_{n+1}$ such that $(E_n)_{n \in \mathbb{N}}$ is an ascending sequence of Banach spaces.*
- (ii) *Assume that for $x \in M = \varinjlim M_n$, there exists a family of charts (U_n, ϕ_n) of M_n , for each $n \in \mathbb{N}^*$, such that:*

- $(U_n)_{n \in \mathbb{N}^*}$ is an ascending sequence of chart domains;
- $\phi_{n+1} \circ \varepsilon_n^{n+1} = \iota_n^{n+1} \circ \phi_n$.

Then $U = \varinjlim U_n$ is an open set of M endowed with the DL-topology and $\phi = \varinjlim \phi_n$ is a well defined map from U to $E = \varinjlim E_n$. Moreover, ϕ is a continuous homeomorphism from U onto the open set $\phi(U)$ of E .

Note that, from Remark 3.3, the direct limit of $\mathcal{M} = (M_n)_{n \in \mathbb{N}^*}$ is the same as the direct limit of $(M_n)_{n \in \mathbb{N}^*, n \geq n_0}$. The result of part (ii) of this Lemma is still true if there exists an integer n_0 such that the assumptions of part (ii) are satisfied for all $n \geq n_0$.

Proof.

(i). — As $(M_n, \varepsilon_n^{n+1})$ is a weak submanifold of M_n , there exists an injective continuous linear map $i_n^{n+1} : E_n \rightarrow E_{n+1}$ for each n . Therefore $(E_n)_{n \in \mathbb{N}^*}$ is an ascending sequence of Banach spaces.

(ii). — Under the assumption of part (ii), we set $V_n = \varphi_n(U_n)$. First, from Proposition 3.1, as V_n is an open of E , we have $U = \bigcup_{n \in \mathbb{N}^*} U_n = \varinjlim U_n$ and $V = \bigcup_{n \in \mathbb{N}^*} V_n = \varinjlim V_n$. Moreover, U (resp V) is an open neighborhood of x (resp. y). According to the continuity criterion, $f = \varinjlim f_n$ is a continuous map from U to V which is injective and surjective. As each f_n is a homeomorphism, we can apply the same arguments to the family f_n^{-1} , which ends the proof. □

DEFINITION 4.3. — We say that an ascending sequence $\mathcal{M} = (M_n)_{n \in \mathbb{N}^*}$ of Banach C^∞ -manifolds has the direct limit chart property at $x \in M = \varinjlim M_n$ if $(M_n)_{n \in \mathbb{N}^*}$ satisfies the assumptions of Lemma 4.2(ii).

Once more, note that the direct limit of $\mathcal{M} = (M_n)_{n \in \mathbb{N}^*}$ is the same as the direct limit of $(M_n)_{n \in \mathbb{N}^*, n \geq n_0}$ (cf. Remark 3.3).

Example 4.4. — The existence of a direct limit chart is a natural requirement which is satisfied in many examples. We give some of them below.

- (1) According to Theorem 3.1 of [12], if $(M_n)_{n \in \mathbb{N}^*}$ is an ascending sequence of C^∞ finite dimensional manifolds, then such a sequence has the direct limit chart property at any $x \in M$.
- (2) If M is a compact analytic manifold, it is well known that the set $\text{Diff}(M)$ of analytic diffeomorphisms of M can be described as a direct limit of an ascending sequence of Banach manifolds $(M_n)_{n \in \mathbb{N}^*}$ which has the direct limit chart property for any point

of $\text{Diff}(M)$. Note that, in this case, $(M_n)_{n \in \mathbb{N}^*}$ is modelled on a sequence $(E_n)_{n \in \mathbb{N}^*}$ of Banach spaces whose direct limit E is a Silva space⁽³⁾.

- (3) In [8], the reader can find examples of Lie groups which can be described as direct limits of ascending sequences of Banach manifolds $(M_n)_{n \in \mathbb{N}^*}$ modelled on sequences of Banach spaces l^p whose direct limits E are not Silva spaces.
- (4) In the introduction of [14], one can also find many examples of Lie groups which have the direct limit chart property at each point.
- (5) Let $(M_n)_{n \in \mathbb{N}^*}$ and $(N_n)_{n \in \mathbb{N}^*}$ be two ascending sequences of Banach manifolds which have the direct limit chart property at $x \in M = \varinjlim M_n$ and at $y \in N = \varinjlim N_n$ respectively. Then $(M_n \times N_n)_{n \in \mathbb{N}^*}$ has the direct limit chart property at $(x, y) \in M \times N = \varinjlim (M_n \times N_n)$. Therefore, given any Banach manifold M and any ascending sequence $(N_n)_{n \in \mathbb{N}^*}$ of C^∞ finite dimensional manifolds, $(M \times N_n)_{n \in \mathbb{N}^*}$ has the direct limit chart property at any point of $M \times N = \varinjlim (M \times N_n)$.

We now give a general context under which an ascending sequence $\mathcal{M} = (M_n)_{n \in \mathbb{N}^*}$ of Banach C^∞ -manifolds has the direct limit chart property at each point of $M = \varinjlim M_n$.

PROPOSITION 4.5. — *Let $\mathcal{M} = (M_n)_{n \in \mathbb{N}^*}$ be an ascending sequence of Banach C^∞ -manifolds modelled on the ascending sequence $(E_n)_{n \in \mathbb{N}^*}$. We assume that:*

- (i) E_n is a complemented subspace in E_{n+1} for each $n \in \mathbb{N}^*$;
- (ii) there exists a linear connection⁽⁴⁾ on TM_n for each $n \in \mathbb{N}^*$.

Then $E = \varinjlim E_n$ is an LB-space and \mathcal{M} has the direct limit chart property at each point of $M = \varinjlim M_n$.

Example 4.6. — By application of Proposition 4.5, $\mathcal{M} = (M_n)_{n \in \mathbb{N}^*}$ has the direct limit chart property at each point in the following cases:

- (1) Each space M_n is a paracompact finite dimensional manifold (cf. [12]).
- (2) Each manifold M_n is a smooth paracompact⁽⁵⁾ Hilbert submanifold of M_{n+1} .

(3) A locally convex limit of ascending sequence of Banach spaces where each inclusion is compact is called a Silva space or a (DFS)-space (for dual of Fréchet Schwartz space).

(4) For the definition of a connection on a Banach manifold see for instance [9]. For more details see also Subsection 5.2.

(5) i.e. M_n is paracompact and every locally finite open covering of M_n admits a conveniently smooth partition of unity subordinated to it.

- (3) Each manifold M_n is a smooth paracompact Banach submanifold of M_{n+1} .

The proof of Proposition 4.5 requires the following Lemma:

LEMMA 4.7. — *Let N_1 be a Banach complemented immersed submanifold, modelled on F_1 , of a Banach manifold N , modelled on F with $F_1 \subset F$. Assume that there exists a linear connection on TN . Given any chart (U_1, ϕ_1) of x in N_1 such that U_1 is a contractible set, there exists a chart (U, Φ) of $x \in N_1$ such that U is contractible, $U \cap N_1 = U_1$ and $\Phi|_{U_1} = \phi_1$.*

Proof. — Let N_1 be an immersed complemented submanifold of a Banach manifold N . If N_1 (resp. N) is modelled on F_1 (resp. F), there exists a Banach subspace F_2 of F such that $F = F_1 \oplus F_2$.

Assume that there exists a linear connection on TN . Therefore we have an exponential map $\text{Exp} : \mathcal{O} \subset TN \rightarrow N$ where \mathcal{O} is an open neighborhood of the zero section in TN . Note that $\text{Exp}|_{T_x N}$ is a local diffeomorphism.

Choose a chart (U_1, ϕ_1) of N_1 around x such that U_1 is a contractible set. Now, as N_1 is immersed in N , for each $z \in U_1$, there exists a chart (U_z, Φ_z) of N such that

$$\Phi_z(U_z \cap N_1) = \phi_1(U_z \cap U_1) \times \{0_{F_2}\} \quad \text{and} \quad \Phi_z|_{U_z \cap U_1} = \phi_1|_{U_z \cap U_1}$$

Then $U = \bigcup_{z \in U_1} U_z$ is an open neighborhood containing U_1 and we have

$$U \cap N_1 = \bigcup_{z \in U_1} (U_z \cap N_1) = U_1.$$

Therefore U_1 is a closed submanifold of U .

As U_1 is a contractible set, the restriction of TN to U_1 is trivial (cf. [1, Theorem 3.4.35]). Therefore we have a diffeomorphism $\Theta : TN|_{U_1} \rightarrow U_1 \times F$. In the trivial bundle $U_1 \times F$, we can consider the subbundle $U_1 \times F_2$ and we have $TN|_{U_1} = TU_1 \oplus \Theta^{-1}(U_1 \times F_2)$. As $\phi_1(U)$ is an open set of F_1 and F_1 is paracompact, so is U_1 . Therefore by same arguments used in the proof of Theorem 5.1 of [22, Chapter IV], we can build a diffeomorphism Ψ from an open neighborhood \mathcal{U} of the zero section of $\Theta^{-1}(U_1 \times F_2)$ on an open neighborhood U of U_1 in N . Note that \mathcal{U} is a fibration on the zero section of $\Theta^{-1}(U_1 \times F_2)$. Moreover, from the property of Exp , we can choose \mathcal{U} such that each fiber is a contractible set. We denote by Φ the composition defined by $\Phi^{-1} = \Theta \circ \Psi^{-1} \circ ((\phi_1)^{-1} \times \text{Id}_{F_2})$. As $\text{Exp}|_{T_u N}(0)(u) = u$ we finally have $\Phi^{-1}(v, 0) = (\phi_1)^{-1}(v)$. \square

Proof of Proposition 4.5. — We have already seen that $E = \varinjlim E_n$ is an LB-space (cf. Proposition 2.12).

Now consider any point $x \in M = \varinjlim M_n$. Then x belongs to some E_n . Let l_0 be the first integer l such that x belongs to M_l . Assume that for each integer $l_0 \leq l \leq k$ we have the following property: there exists a family of charts (U_n, ϕ_n) of M_n , for each $l_0 \leq n \leq l$, such that:

- $(U_n)_{l_0 \leq n \leq l}$ is an ascending sequence of chart domains around x ;
- $\phi_{n+1}|_{U_n} = \phi_n$ for all $l_0 \leq n < l$.

From Lemma 4.7, this assumption is true for $l = l_0 + 1$. The proof is obtained by induction using Lemma 4.7. □

Now we can prove the following result which generalizes [12, Theorem 3.1].

THEOREM 4.8. — *Let $(M_n)_{n \in \mathbb{N}^*}$ be an ascending sequence of Banach C^∞ -manifolds, modelled on the Banach spaces E_n . Assume that $(M_n)_{n \in \mathbb{N}^*}$ has the direct limit chart property at each point $x \in M = \varinjlim M_n$ and $E = \varinjlim E_n$ is an LB-space. Then there is a unique n.n.H. convenient manifold structure on $M = \varinjlim M_n$ modelled on the convenient space E such that the topology associated to this structure is the DL-topology on M . In particular, for each $n \in \mathbb{N}^*$, the canonical injection $\varepsilon_n : M_n \rightarrow M$ is an injective conveniently smooth map and (M_n, ε_n) is a weak submanifold of M . Moreover, if each M_n is locally compact or is open in M_{n+1} or is a paracompact Banach manifold closed in M_{n+1} , then $M = \varinjlim M_n$ is provided with a Hausdorff convenient manifold structure.*

A direct application of this theorem gives rise to the following result:

COROLLARY 4.9. — *Let $(M_n)_{n \in \mathbb{N}^*}$ be an ascending sequence of Banach paracompact C^∞ -manifolds where M_n is closed in M_{n+1} . If the sequence $(M_n)_{n \in \mathbb{N}^*}$ satisfies the assumptions of Proposition 4.5, then $M = \varinjlim M_n$, provided with the DL-topology, has a unique structure of Hausdorff convenient manifold modelled on an LB-space.*

Proof of Theorem 4.8. — As in Lemma 4.2(ii), we consider the set \mathcal{A} of all sequences of charts $\{(U_n^\alpha, \phi_n^\alpha)_{n \in \mathbb{N}^*}\}_{\alpha \in A}$ of $(M_n)_{n \in \mathbb{N}^*}$ such that (U_n) is an ascending sequence of chart domains. We set $V_n^\alpha = \phi_n^\alpha(U_n^\alpha)$, $U^\alpha = \varinjlim U_n^\alpha$ and $\phi^\alpha = \varinjlim \phi_n^\alpha$. From Lemma 4.2(ii), ϕ^α is a homeomorphism from U^α to the open set $V^\alpha = \phi^\alpha(U^\alpha) = \varinjlim V_n^\alpha$ of E . Then U^α and V^α are open sets of the DL-topology on M and E respectively (cf. Proposition 3.1(2)). From our assumption, \mathcal{A} is then a topological atlas of M modelled on the convenient space E . Note that, from Proposition 3.7, each V^α is also a c^∞ -open set.

Let us prove that the change of charts are conveniently smooth diffeomorphisms. Consider two charts (U^α, ϕ^α) and (U^β, ϕ^β) around $x \in M$. We

consider

$$\tau_n^{\beta\alpha} = \phi_n^\beta \circ (\phi_n^\alpha)^{-1} : \phi_n^\alpha(U_n^\alpha \cap U_n^\beta) \longrightarrow \phi_n^\beta(U_n^\alpha \cap U_n^\beta).$$

For each $n \in \mathbb{N}^*$, the pairs $(U_n^\alpha, \phi_n^\alpha)$ and $(U_n^\beta, \phi_n^\beta)$ are charts of M_n and the intersection of their domains is not empty. It follows that the map $\tau_n^{\beta\alpha}$ is a C^∞ local diffeomorphism of E_n . But the construction of U^α and U^β implies that $U^\alpha \cap U^\beta$ is the direct limit of $(U_n^\alpha \cap U_n^\beta)_{n \in \mathbb{N}^*}$ in M . It follows that $\phi^\alpha(U^\alpha \cap U^\beta)$ is the direct limit of $\phi_n^\alpha(U_n^\alpha \cap U_n^\beta)_{n \in \mathbb{N}^*}$. In the same way, we have $\phi^\beta(U^\alpha \cap U^\beta) = \varinjlim \phi_n^\beta(U_n^\alpha \cap U_n^\beta)$. Therefore we get a direct limit map $\tau^{\beta\alpha} = \varinjlim \tau_n^{\beta\alpha}$ from the open set $\phi^\alpha(U^\alpha \cap U^\beta)$ onto the open set $\phi^\beta(U^\alpha \cap U^\beta)$ of E . Again the sets $\phi^\alpha(U^\alpha \cap U^\beta)$ and $\phi^\beta(U^\alpha \cap U^\beta)$ are c^∞ -open sets of E . As each $\tau_n^{\beta\alpha}$ is a C^∞ diffeomorphism of E_n , Lemma 3.9 implies that $\tau^{\beta\alpha}$ is a conveniently smooth diffeomorphism from $\phi^\alpha(U^\alpha \cap U^\beta)$ onto $\phi^\beta(U^\alpha \cap U^\beta)$. Therefore we obtain that \mathcal{A} is convenient atlas on M . Note that the topology of M defined by such an atlas is exactly the *DL*-topology on M . Therefore, if M_n locally compact, then M is Hausdorff from Proposition 3.1. In the same way, from Proposition 3.2, if M_n is open in M_{n+1} or is paracompact and closed in M_{n+1} for each n , then M is Hausdorff. Thus in each of the previous particular cases, M is provided with a Hausdorff convenient manifold structure.

Now we prove the uniqueness of this convenient structure. Assume that Y is a convenient manifold structure modelled on the convenient vector space E and $h_n : M_n \rightarrow Y$ a C^∞ map for each $n \in \mathbb{N}^*$ s.t. $(Y, (h_n)_{n \in \mathbb{N}^*})$ is a cone over \mathcal{S} . Then there is a uniquely determined continuous map $h : M \rightarrow Y$ s.t. $h|_{M_n} = h_n$. Let $x \in M$; we can find a chart (U, φ) around x in the atlas \mathcal{A} where $f = \varinjlim f_n U$ for charts $\varphi_n : U_n \rightarrow E_n$. Let $\psi : W \subset Y \rightarrow V$ be a chart for Y (W is an open set of Y). Then $O = (h \circ \varphi^{-1})^{-1}(W)$ is an open set of $U \subset E$ and $O_n = O \cap E_n$ is open in E_n for each n . Consider $g = \psi \circ h \circ \varphi^{-1}|_O^V : O \rightarrow V$. Then $g|_{O_n} = \psi \circ h_n \circ \varphi_n^{-1}|_{O_n}^V : O_n \rightarrow V$ is C^∞ for each $n \in \mathbb{N}^*$. Hence g is c^∞ (cf. Proposition 3.10), so is h on the open neighborhood U of x and hence on all of M because x is arbitrary. Thus $(M, (\varepsilon_n)_{n \in \mathbb{N}^*}) = \varinjlim S$ in the category of c^∞ -manifolds. The uniqueness of a convenient structure of manifold on M follows from the universal property of direct limits. \square

4.2. Direct limit of Lie groups

The reader is referred to [14].

Interesting infinite-dimensional Lie groups often appears as direct limits $G = \bigcup_{n \in \mathbb{N}^*} G_n$ of ascending Lie groups $G_1 \subset G_2 \subset \dots$ where the bonding

maps (inclusion maps $\varepsilon_n^{n+1} : G_n \rightarrow G_{n+1}$) are smooth homomorphisms (e.g. the group $\text{Diff}_c(M)$ of compactly supported diffeomorphisms of a σ -compact smooth manifold M or the *test function* groups $C_c^\infty(M, H)$ of compactly supported smooth maps with values in a finite-dimensional Lie group H).

When the Lie groups G_n are finite dimensional it is well known that the direct limit $\varinjlim G_n$ can be endowed with a structure of Lie group (see [12]).

Here we give conditions on the direct sequences $\mathcal{G} = (G_n, \varepsilon_n^m)_{n, m \in \mathbb{N}^*, n \leq m}$ of Lie groups in order to obtain a structure of Lie group on their direct limit.

We first recall the essential notion of *candidate for a direct limit chart*:

DEFINITION 4.10. — *Let $G = \bigcup_{n \in \mathbb{N}^*} G_n$ be the union of an ascending sequence of C^∞ -Lie groups G_n where the inclusion maps $\varepsilon_n^m : G_n \rightarrow G_m$ are C^∞ -homomorphisms and G_n is a subgroup of G .*

We say that G has a candidate for a direct limit chart if there exist charts $\phi_n : G_n \supset U_n \rightarrow V_n \subset \mathfrak{g}_n$ of G_n around the identity for $n \in \mathbb{N}^$ (where \mathfrak{g}_n stands for the Lie algebra of G_n) such that $U_n \subset U_m$ and $\phi_m|_{U_n} = \mathbb{L}(i_n^m) \circ \phi_n$ if $n \leq m$ and $V = \bigcup_{n \in \mathbb{N}^*} V_n$ is open in the locally convex direct limit $\varinjlim \mathfrak{g}_n$ which we assume to be Hausdorff.*

Glöckner obtains the following result (cf. [14, Proposition 1.4.3]):

PROPOSITION 4.11. — *Let $G = \bigcup_{n \in \mathbb{N}^*} G_n$ be a group which is the union of an ascending sequence of C^∞ -Lie groups. Assume that G has a candidate $\phi : U \rightarrow V \subset \varinjlim \mathfrak{g}_n$ for a direct limit chart and assume that one of the following conditions is satisfied:*

- (i) G_n is a Banach Lie group for each $n \in \mathbb{N}^*$ and the inclusion map $\mathfrak{g}_n \rightarrow \mathfrak{g}_m$ is a compact linear operator for all $n < m$;
- (ii) \mathfrak{g}_n is a k_ω -space for each $n \in \mathbb{N}^*$.

Then on $G = \varinjlim G_n$ there exists a unique C^∞ -Lie group structure making $\phi|_W$ a direct chart limit for G around 1, where W is an open neighbourhood of 1 contained in U .

Glöckner gives also results in the convenient setting where $V = \bigcup_{n \in \mathbb{N}^*} V_n$ is a c^∞ -open set in $\varinjlim \mathfrak{g}_n$ endowed with a suitable locally convex topology (cf. [13, Remark 14.8]):

PROPOSITION 4.12. — *Let $G = \bigcup_{n \in \mathbb{N}^*} G_n$ be a group which is the union of an ascending sequence of (Hausdorff) convenient Lie groups. Equip the vector space $\varinjlim \mathfrak{g}_n$ with the locally convex vector topology associated with the direct limit bornology which is assumed to be Hausdorff. We require that*

G admits a candidate for a direct limit chart in the convenient sense and that each bounded subset in \mathfrak{g} is a bounded subset of some \mathfrak{g}_n . Then G may be endowed with a structure n.n.H. convenient Lie group

This criterion permits to obtain:

THEOREM 4.13. — *Let $G = \bigcup_{n \in \mathbb{N}^*} G_n$ be a group which is the union of an ascending sequence of Banach Lie groups. Assume that the direct limit $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ of the ascending sequence $(\mathfrak{g}_n)_{n \in \mathbb{N}^*}$ of associated Lie algebras is an LB-space. If G admits a candidate for a direct limit chart, then G can be endowed with a structure of n.n.H. convenient Lie group modelled on the LB-space \mathfrak{g} .*

Therefore, Proposition 4.11 (ii) can be seen as a corollary of this theorem. According to Proposition 3.5, if G admits a candidate for a direct limit chart then assume that there exists a countable subset $I \subset \mathbb{N}^*$ such that the direct limit $\mathfrak{g}_I = \varinjlim \{\mathfrak{g}_i, i \in I\}$ is an LB-space, then G is endowed with a structure of convenient Lie group modelled on the LB-space \mathfrak{g} . Now since \mathfrak{g}_n is a k_ω -space for each $n \in \mathbb{N}^*$, by direct chart limit property and using the fact that direct limits of ascending sequences of locally k_ω -spaces are locally k_ω -spaces and so are Hausdorff (see [14]), the topology on G must be Hausdorff.

On the other hand, the reader can find the following criterion in [8]:

THEOREM 4.14. — *Let $\mathcal{G} = (G_n)_{n \in \mathbb{N}^*}$ be an ascending sequence of Banach Lie groups such that all inclusion maps $j_n : G_n \rightarrow G_{n+1}$ are analytic group morphisms and assume that we have the following properties:*

- (1) *For each $n \in \mathbb{N}^*$, there exists a norm $\|\cdot\|_n$ on the Lie algebra \mathfrak{g}_n defining its Banach space structure, such that its Lie bracket satisfies the inequality $\|[x, y]\|_n \leq \|x\|_n \|y\|_n$ for all x and y in \mathfrak{g}_n and such that the bounded linear operator $\mathbb{L}(j_n) : \mathfrak{g}_n \rightarrow \mathfrak{g}_{n+1}$ has a norm operator bounded by 1;*
- (2) *The locally convex structure of vector space $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ is Hausdorff;*
- (3) *The map $\exp_G = \bigcup_{n \in \mathbb{N}^*} \exp_{G_n} : \bigcup_{n \in \mathbb{N}^*} \mathfrak{g}_n \rightarrow \bigcup_{n \in \mathbb{N}^*} G_n$ is injective on some neighborhood of 0.*

Then $G = \varinjlim G_n = \bigcup_{n \in \mathbb{N}^} G_n$ has an analytic structure of Lie group modelled on \mathfrak{g} and \exp_G is an analytic diffeomorphism from some neighborhood of 0 to a neighborhood of $1 \in G$.*

We end this subsection with an application of this result⁽⁶⁾:

⁽⁶⁾ This result is certainly well known by specialists but it is an easy corollary of Theorem 4.14 and so we give a proof here.

THEOREM 4.15. — *Let $(E_n)_{n \in \mathbb{N}^*}$ be an ascending sequence of Banach spaces such that E_n is a complemented Banach subspace of E_{n+1} . Then $E = \bigcup_{n \in \mathbb{N}^*} E_n$ is an LB-space and $L(E) = \bigcup_{n \in \mathbb{N}^*} L(E_n)$ is also an LB-space, where $L(E_n)$ is the Banach space of continuous linear operators of E_n . Moreover, $GL(E) = \bigcup_{n \in \mathbb{N}^*} GL(E_n)$ has a structure of convenient Lie group modelled on $L(E)$, where $GL(E_n)$ is the Banach Lie group of linear continuous automorphisms of E_n .*

For the proof of this theorem we need the following lemma:

LEMMA 4.16. — *Let E and F be two Banach spaces such that E is a complemented Banach subspace of F . Given a norm $\|\cdot\|_E$ on E , there exists a norm $\|\cdot\|_F$ on F and an embedding $\lambda : L(E) \rightarrow L(F)$ which is an isometry with respect to the corresponding operator norms on $L(E)$ and $L(F)$ respectively.*

Moreover, we have $[\lambda(T), \lambda(T')] = \lambda([T, T'])$ where, as classically, the bracket is given by $[T, T'] = T \circ T' - T' \circ T$.

Proof. — Let E' be a subspace of F such that $F = E \oplus E'$. We endow E' with a norm $\|\cdot\|'$ and let $\|\cdot\|_F$ be the norm on F defined by $\|x\|_F = \|x_1\|_E + \|x_2\|'$ if $x = x_1 + x_2$ with $x_1 \in E$ and $x_2 \in E'$. Denote by λ the natural inclusion of E in F . By construction, λ is an isometry. We define $\Lambda : L(E) \rightarrow L(F)$ where $\Lambda(T)$ is the operator on F whose restriction to E is T and whose restriction to E' is the null operator. Clearly Λ is injective and the operator norm of Λ is 1. Indeed if Π is the projection of F on E with kernel E' , we have

$$\frac{\|\Lambda(T)(x)\|_F}{\|x\|_F} \leq \frac{\|T \circ \Pi(x)\|_F}{\|\Pi(x)\|_E} \leq \|T\|_{L(E)}.$$

We deduce $\|\Lambda(T)\|_{L(F)} \leq \|T\|_{L(E)}$. On the other hand

$$\begin{aligned} \|T\|_{L(E)} &= \sup \left\{ \frac{\|T(x)\|_E}{\|x\|_E}, x \in E \right\} \\ &\leq \sup \left\{ \frac{\|T(x)\|_E}{\|x\|_F}, x \in F \right\} = \sup \left\{ \frac{\|\Lambda(T)(x)\|_F}{\|x\|_F}, x \in F \right\}. \end{aligned}$$

Finally, it is easy to verify that we have $\Lambda(T \circ T') = \Lambda(T) \circ \Lambda(T')$, which ends the proof. \square

Proof of Theorem 4.15. — According to Lemma 4.16, by induction, we can build a sequence of norms $\|\cdot\|_n$ on each E_n and an isometry $\Lambda_n : L(E_n) \rightarrow L(E_{n+1})$. For simplicity, we identify $L(E_n)$ with $\Lambda_n(L(E_n))$ in $L(E_{n+1})$. Then $L(E_n)$ is a Banach subspace of $L(E_{n+1})$ with the induced topology. It follows that $\mathfrak{G} = \bigcup_{n \in \mathbb{N}} L(E_n) = \varinjlim L(E_n)$ is a convenient space.

On the other hand, for the operator norm in each $L(E_n)$ we have

$$\|[T, T']\|_{L(E_n)} \leq 2\|T\|_{L(E_n)}\|T'\|_{L(E_n)}$$

On each $L(E_n)$, we consider the norm $\nu_n = 2\|\cdot\|_{L(E_n)}$. Then ν_n defines the topology of $L(E_n)$. The inclusion Λ_n is still an isometry and we have

$$\nu_n([T, T']) \leq \nu_n(T)\nu_n(T').$$

Given $T \in \mathfrak{G}$, then T belongs to some $L(E_n)$; we then have

$$\exp_{\mathfrak{G}}(T) = \sum_{k \in \mathbb{N}} \frac{T^k}{k!}$$

On one hand, classically, the exponential map $\exp_n : L(E_n) \rightarrow GL(E_n)$ is an analytic diffeomorphism over the ball $B_n(0, \ln(2))$ (relative to the norm ν_n on $L(E_n)$) in $GL(E_n)$. On the other hand, we have the relations: $\exp_{n+1}|_{L(E_n)} = \exp_n$ and $B_{n+1}(0, \frac{1}{2} \ln(2)) \cap E_n = B_n(0, \frac{1}{2} \ln(2))$. It follows that $\exp_{\mathfrak{G}}$ is injective on $\bigcup_{n \in \mathbb{N}} B_n(0, \frac{1}{2} \ln(2))$. Therefore all the assumptions of Theorem 4.14 are satisfied and we get the announced result. \square

4.3. Direct limit of Banach vector bundles

DEFINITION 4.17. — *A sequence $(E_n, \pi_n, M_n)_{n \in \mathbb{N}^*}$ of Banach vector bundles is called a strong ascending sequence of Banach vector bundles if the following assumptions are satisfied:*

- (A) $\mathcal{M} = (M_n)_{n \in \mathbb{N}^*}$ is an ascending sequence of Banach C^∞ -manifolds, where M_n is modelled on the Banach space \mathbb{M}_n such that \mathbb{M}_n is a complemented Banach subspace of \mathbb{M}_{n+1} and the inclusion $\varepsilon_n^{n+1} : M_n \rightarrow M_{n+1}$ is a C^∞ injective map such that $(M_n, \varepsilon_n^{n+1})$ is a weak submanifold of M_{n+1} ;
- (B) The sequence $(E_n)_{n \in \mathbb{N}^*}$ is an ascending sequence such that the sequence of typical fibers $(\mathbb{E}_n)_{n \in \mathbb{N}^*}$ of $(E_n)_{n \in \mathbb{N}^*}$ is an ascending sequence of Banach spaces such that \mathbb{E}_n is a complemented Banach subspace of \mathbb{E}_{n+1} ;
- (C) For each $n \in \mathbb{N}^*$, $\pi_{n+1} \circ \lambda_n^{n+1} = \varepsilon_n^{n+1} \circ \pi_n$ where $\lambda_n^{n+1} : E_n \rightarrow E_{n+1}$ is the natural inclusion ;
- (D) Any $x \in M = \varinjlim M_n$ has the direct limit chart property for $(U = \varinjlim U_n, \phi = \varinjlim \phi_n)$;

(E) For each $n \in \mathbb{N}^*$, there exists a trivialization $\Psi_n : (\pi_n)^{-1}(U_n) \rightarrow U_n \times \mathbb{E}_n$ such that the following diagram is commutative:

$$\begin{array}{ccc} (\pi_n)^{-1}(U_n) & \xrightarrow{\lambda_n^{n+1}} & (\pi_{n+1})^{-1}(U_{n+1}) \\ \Psi_n \downarrow & & \downarrow \Psi_{n+1} \\ U_n \times \mathbb{E}_n & \xrightarrow{(\varepsilon_n^{n+1} \times \iota_n^{n+1})} & U_{n+1} \times \mathbb{E}_{n+1}. \end{array}$$

For example, the sequence $(TM_n, \pi_n, M_n)_{n \in \mathbb{N}^*}$ is a strong ascending sequence of Banach vector bundles whenever $(M_n)_{n \in \mathbb{N}^*}$ is an ascending sequence which has the direct limit chart property at each point of $x \in M = \varinjlim M_n$ whose model \mathbb{M}_n is complemented in \mathbb{M}_{n+1} .

PROPOSITION 4.18. — *Let $(E_n, \pi_n, M_n)_{n \in \mathbb{N}^*}$ be a strong ascending sequence of Banach vector bundles. We have:*

- (1) $\varinjlim E_n$ has a structure of *n.n.H* convenient manifold modelled on the *LB*-space $\varinjlim \mathbb{M}_n \times \varinjlim \mathbb{E}_n$ which has a Hausdorff convenient structure if and only if M is Hausdorff.
- (2) $(\varinjlim E_n, \varinjlim \pi_n, \varinjlim M_n)$ can be endowed with a structure of convenient vector bundle whose typical fiber is $\varinjlim \mathbb{E}_n$ and whose structural group is a metrizable complete topological group, projective limit of a sequence of Banach-Lie groups.

Proof. —

(1). — Consider (x, v) in some E_n ; in particular x belongs to M_n . According to the assumptions (C) and (D), there exists a chart (U, ϕ) of $M = \varinjlim M_i$ around x where $\phi : U = \bigcup_{i \geq n} U_i \rightarrow V = \bigcup_{i \geq n} \phi_i(U_i)$, (U_i, ϕ_i) being a chart around x_i and $V_i = \phi_i(U_i) \subset \mathbb{M}_i$. A local trivialization $\Psi_i : \pi_i^{-1}(U_i) \rightarrow U_i \times \mathbb{E}_i$ gives rise, via the chart $\phi_i : U_i \rightarrow V_i$, to a chart $\psi_i : \pi_i^{-1}(U_i) \rightarrow V_i \times \mathbb{E}_i \subset \mathbb{M}_i \times \mathbb{E}_i$.

From the assumption (E), we get the commutativity of the diagram

$$\begin{array}{ccc} (\pi_i)^{-1}(U_i) & \xrightarrow{\lambda_i^j} & (\pi_j)^{-1}(U_j) \\ \Psi_i \downarrow & & \downarrow \Psi_j \\ U_i \times \mathbb{E}_i & \xrightarrow{(\varepsilon_i^j \times \iota_i^j)} & U_j \times \mathbb{E}_j \end{array}$$

The previous arguments imply that the sequence of Banach manifolds $\{E_n\}_{n \in \mathbb{N}^*}$ has the direct limit chart property at any point $(x, v) \in \varinjlim E_n$.

Therefore, from Theorem 4.8, there exists a unique structure of n.n.H. convenient manifold on $E = \varinjlim E_n$ whose topology coincides with the DL -topology on E . In particular this structure is Hausdorff if and only if M is so.

(2). — The main difficulty is to define the structural group⁽⁷⁾, say $G(\mathbb{E})$ where $\mathbb{E} = \varinjlim \mathbb{E}_n$.

Let $\mathbb{E}_1 \subset \mathbb{E}_2 \subset \dots$ be the direct sequence of complemented Banach spaces associated to the direct sequence $E_1 \subset E_2 \subset \dots$; so there exist Banach subspaces $\mathbb{E}'_1, \mathbb{E}'_2, \dots$ such that:

$$\begin{cases} \mathbb{E}_1 = \mathbb{E}'_1, \\ \forall i \in \mathbb{N}^*, \mathbb{E}_{i+1} \simeq \mathbb{E}_i \times \mathbb{E}'_{i+1}. \end{cases}$$

For $i, j \in \mathbb{N}^*$, $i \leq j$, we have the injection

$$\begin{aligned} \iota_i^j : \mathbb{E}_i \simeq \mathbb{E}'_1 \times \dots \times \mathbb{E}'_i &\longrightarrow \mathbb{E}_j \simeq \mathbb{E}'_1 \times \dots \times \mathbb{E}'_j \\ (x'_1, \dots, x'_i) &\longmapsto (x'_1, \dots, x'_i, 0, \dots, 0) \end{aligned}$$

Any $A_{n+1} \in GL(\mathbb{E}_{n+1})$ is represented by $\begin{pmatrix} A_n & B_{n+1} \\ A'_n & B'_{n+1} \end{pmatrix}$ where $A_n \in \mathcal{L}(\mathbb{E}_n, \mathbb{E}_n)$, $A'_n \in \mathcal{L}(\mathbb{E}_n, \mathbb{E}'_{n+1})$, $B_{n+1} \in \mathcal{L}(\mathbb{E}'_{n+1}, \mathbb{E}_n)$ and $B'_{n+1} \in \mathcal{L}(\mathbb{E}'_{n+1}, \mathbb{E}'_{n+1})$.

The group

$$GL_0(\mathbb{E}_{n+1}|\mathbb{E}_n) = \{A \in GL(\mathbb{E}_{n+1}) : A(\mathbb{E}_n) = \mathbb{E}_n\}$$

can be identified with the Banach–Lie sub-group of operators of type $\begin{pmatrix} A_n & B_{n+1} \\ 0 & B'_{n+1} \end{pmatrix}$ (cf. [5]).

The set

$$G_n = \{A_n \in GL(\mathbb{E}_n) : \forall k \in \{1, \dots, n-1\}, A_n(\mathbb{E}_k) = \mathbb{E}_k\}$$

can be endowed with a structure of Banach–Lie subgroup.

An element A_n of G_n can be seen as

$$A_n = \left(\begin{array}{cc|cc|c|c} A_1 & B_2 & B_3 & B_4 & & \\ 0 & B'_2 & B'_3 & B'_4 & & \\ \hline 0 & & B'_3 & & B_i & \\ \hline & 0 & & B'_4 & & B_n \\ \hline & & & & \ddots & \\ \hline & & & 0 & B'_i & \\ \hline & & & & & \ddots \\ \hline & & & & & 0 & B'_n \end{array} \right)$$

⁽⁷⁾ As the referee pointed out, the structural group is much larger than the direct limit of the linear groups $GL(\mathbb{E}_n)$.

For $1 \leq i \leq j \leq k$, we consider the following diagram

$$\begin{array}{ccc}
 \mathbb{E}_k & \xrightarrow{A_k} & \mathbb{E}_k \\
 \uparrow \iota_j^k & & \downarrow P_j^k \\
 \mathbb{E}_j & \xrightarrow{A_j} & \mathbb{E}_j \\
 \uparrow \iota_i^j & & \downarrow P_i^j \\
 \mathbb{E}_i & \xrightarrow{A_i} & \mathbb{E}_i
 \end{array}$$

where $P_i^j : \mathbb{E}_j \rightarrow \mathbb{E}_i$ is the projection along the direction $\mathbb{E}'_{i+1} \oplus \dots \oplus \mathbb{E}'_j$.

The map

$$\begin{aligned}
 \theta_i^j : G_j &\longrightarrow G_i \\
 A_j &\longmapsto P_i^j \circ A_j \circ \iota_i^j
 \end{aligned}$$

is perfectly defined and we have:

$$(\theta_i^j \circ \theta_j^k)(A_k) = \theta_i^j[\theta_j^k(A_k)] = \theta_i^j(P_j^k \circ A_j \circ \iota_j^k) = P_i^j \circ P_j^k \circ A_j \circ \iota_j^k \circ \iota_i^j.$$

Because $P_i^j \circ P_j^k = P_i^k$ (projective system) and $\iota_j^k \circ \iota_i^j = \iota_i^k$ (inductive system), we have

$$(\theta_i^j \circ \theta_j^k)(A_k) = P_i^k \circ A_j \circ \iota_i^k = \theta_i^k(A_k)$$

So $(G_i, \theta_i^j)_{i \leq j}$ is a projective system of Banach-Lie groups and the projective limit $G(\mathbb{E}) = \varprojlim G_n$ can be endowed with a structure of metrizable complete topological group.

From assumptions (C) and (D) it follows that we have a well defined conveniently smooth projection $\pi = \varinjlim \pi_i : \varinjlim E_i \rightarrow \varinjlim M_i$ given by $\pi(x, v) = x$ and, with the previous notations, we also have $\varinjlim (\pi_i)^{-1}(U_i) = \pi^{-1}(\varinjlim U_i)$.

The map $\Psi_i : \pi_i^{-1}(U_i) \rightarrow U_i \times \mathbb{E}_i$ can be written $\Psi_i(y_i, u_i) = (y_i, \tilde{\Psi}_i(y_i)(u_i))$ where $u_i \mapsto \tilde{\Psi}_i(y_i)(u_i)$ is an isomorphism of Banach spaces from $\pi_i^{-1}(y_i)$ to \mathbb{E}_i .

Consider an atlas $\mathcal{A} = \{(U^\alpha = \varinjlim U_i^\alpha, \phi^\alpha = \varinjlim \phi_i^\alpha)\}_{\alpha \in A}$ on M . From the proof of the first part, the set $\hat{\mathcal{A}} = \{(\pi^{-1}(U^\alpha) = \varinjlim \pi_i^{-1}(U_i^\alpha), \psi^\alpha = \varinjlim \psi_i^\alpha)\}_{\alpha \in A}$ is an atlas for the manifold E .

Now, if $U_i^\alpha \cap U_i^\beta \neq \emptyset$,

$$\Psi_i^\alpha \circ (\Psi_i^\beta)^{-1} : \phi_i^\beta(U_i^\alpha \cap U_i^\beta) \times \mathbb{E}'_1 \times \dots \times \mathbb{E}'_i \longrightarrow \phi_i^\alpha(U_i^\alpha \cap U_i^\beta) \times \mathbb{E}'_1 \times \dots \times \mathbb{E}'_i$$

can be written

$$(\bar{y}_i^\beta, \bar{u}_1^\beta, \dots, \bar{u}_i^\beta) \mapsto (\phi_i^\alpha \circ (\phi_i^\beta)^{-1}(\bar{y}_i^\beta), [\tilde{\Psi}_i^\alpha(y_i)] \circ [\tilde{\Psi}_i^\beta(y_i)]^{-1}(\bar{u}_1^\beta, \dots, \bar{u}_i^\beta))$$

where $\bar{y}_i^\beta = \phi_i^\beta(y_i)$. With these notations, $\bar{y}_i^\beta \mapsto \Theta_i^{\alpha\beta}(\bar{y}_i^\beta) = [\tilde{\Psi}_i^\alpha(y_i)] \circ [\tilde{\Psi}_i^\beta(y_i)]^{-1}$ is a conveniently smooth map.

From assumption (C) and assumption (E) written over the open sets U_i^α and U_j^α (resp. U_i^β and U_j^β), we have

$$(\tilde{\Psi}_j^\alpha(y_j))^{-1} \circ \iota_i^j = \lambda_i^j \circ (\tilde{\Psi}_i^\alpha(y_i))^{-1}.$$

Finally we get

$$\Theta_j^{\alpha\beta}(\bar{y}_j) \circ \iota_i^j = \iota_i^j \circ \Theta_i^{\alpha\beta}(\bar{y}_i)$$

So if $\bar{y} = \varinjlim \bar{y}_i$, from the above relation, one can define the transition function $\Theta^{\alpha\beta}(\bar{y})$ as an element of the metrizable complete topological group $G(\mathbb{E})$. \square

5. Linear connections on direct limit of anchored Banach bundles

5.1. Bundle structures on the tangent bundle to a vector bundle

Let M be a smooth Banach manifold modelled on a Banach space \mathbb{M} and let $\pi : E \rightarrow M$ be a smooth Banach vector bundle on M whose typical fiber is a Banach space \mathbb{E} . Let $p_E : TE \rightarrow E$ and $p_M : TM \rightarrow M$ be the canonical projections of each tangent bundle.

There exists an atlas $\{U^\alpha, \phi^\alpha\}_{\alpha \in A}$ of M for which $E|_{U^\alpha}$ is trivial; therefore we obtain a chart $(U_E^\alpha, \phi_E^\alpha)$ on E , where $U_E^\alpha = \pi^{-1}(U^\alpha)$ and s.t. ϕ_E^α is a diffeomorphism from U_E^α on $\phi^\alpha(U^\alpha) \times \mathbb{E}$. We also have a chart $(U_{TM}^\alpha, \phi_{TM}^\alpha)$ on TM where $U_{TM}^\alpha = p_M^{-1}(U^\alpha)$ and $\phi_{TM}^\alpha = (\phi^\alpha, Tp_M)$.

Hence the family $\{T(E|_{U^\alpha}), T\phi_E^\alpha\}_{\alpha \in A}$ where

$$T\phi_E^\alpha : T(E|_{U^\alpha}) \longrightarrow T(\phi^\alpha(U^\alpha) \times \mathbb{E}) = \phi^\alpha(U^\alpha) \times \mathbb{E} \times \mathbb{M} \times \mathbb{E}$$

is the atlas describing the canonical vector bundle structure of (TE, p_E, E) .

Let (x, u) be an element of $E_x = \pi^{-1}(x)$ where $x \in U^{\alpha\beta} = U^\alpha \cap U^\beta \neq \emptyset$ and let (x, u, y, v) be an element of $T_{(x,u)}E$. For $(x^\alpha, u^\alpha, y^\alpha, v^\alpha) = T\phi_E^\alpha(x, u, y, v)$, we have the transition functions:

$$\begin{aligned} & (T((\phi^\alpha \times \text{Id}_{\mathbb{E}}) \circ \phi_E^\alpha) \circ (T((\phi^\beta \times \text{Id}_{\mathbb{E}}) \circ \phi_E^\beta))^{-1})(x^\beta, u^\beta, y^\beta, v^\beta) \\ &= (\phi^{\alpha\beta}(x^\beta), \phi_E^{\alpha\beta}((\phi^\beta)^{-1}(y^\beta))u^\beta, d\phi^{\alpha\beta}(x^\beta)y^\beta, (d(\phi_E^{\alpha\beta} \circ (\phi^\beta)^{-1})(x^\beta)y^\beta)u^\beta \\ & \quad + \phi_E^{\alpha\beta}((\phi^\beta)^{-1}(x^\beta)v^\beta)) \end{aligned}$$

where $\phi^{\alpha\beta} = \phi^\alpha \circ (\phi^\beta)^{-1}$ and $(x^\beta, \phi_E^{\alpha\beta}(x^\beta)u^\beta) = (\phi_E^\alpha \circ (\phi_E^\beta)^{-1})(x^\beta, u^\beta)$ for $x^\beta \in \phi^\alpha(U^{\alpha\beta})$.

So, for fixed (x^β, u^β) , the transition functions are linear in $(y^\beta, v^\beta) \in \mathbb{M} \times \mathbb{E}$. This describes the vector bundle structure of the tangent bundle (TE, p_E, E) .

On the other hand, for fixed (x^β, y^β) the transition functions of TE are also linear in $(u^\beta, v^\beta) \in \mathbb{E} \times \mathbb{E}$ and we get a vector bundle structure on $(TE, T\pi, TM)$ which appears as the derivative of the original one on E .

5.2. Connections on a Banach bundle

The kernel of $T\pi : TE \rightarrow TM$ is denoted by VE and is called the *vertical bundle* over E . It appears as a vector bundle over M . It is well known that VE can also be seen as the pull-back of the bundle $\pi : E \rightarrow M$ over π as described by the following diagram:

$$\begin{array}{ccc} E \times_M E \simeq \pi^* E & \xrightarrow{\hat{\pi}} & E \\ \downarrow & & \downarrow \pi \\ E & \xrightarrow{\pi} & M \end{array}$$

We have a canonical isomorphism $E \times_M E \rightarrow VE$ called the *vertical lift* vl_E defined by

$$vl_E(x, u, v) = \dot{\gamma}(0)$$

where $\gamma(t) = u + tv$. This map is fiber linear over M .

Let $J : VE \rightarrow TE$ be the canonical inclusion. According to [32] we have:

DEFINITION 5.1. — *A (non linear) connection on E is a bundle morphism $V : TE \rightarrow VE$ such that $V \circ J = \text{Id}_{VE}$.*

The datum of a connection V on E is equivalent to the existence of a decomposition $TE = HE \oplus VE$ of the Banach bundle E with $HE = \ker V$.

We then have the following diagram:

$$\begin{array}{ccc} VE & \xleftarrow{V} & TE \\ \uparrow vl_E & & \downarrow D \\ \pi^* E & \xrightarrow{\hat{\pi}} & E \end{array}$$

The bundle morphism $D = \hat{\pi} \circ vl_E^{-1} \circ V : TE \rightarrow E$ is called the *connection map* or *connector* which is a smooth morphism of fibrations. Note that, in

each fiber $T_{(x,u)}E$, the kernel of D is exactly the subspace $H_{(x,u)}E$ of HE in $T_{(x,u)}E$. Therefore, the datum of D is equivalent to the datum of V .

We then have, modulo the identification $VE \simeq \pi^*E$ via vl_E :

$$D : \quad TE \longrightarrow E$$

$$(x, u, y, v) \longmapsto (x, v + \omega(x, u)y)$$

where $\omega(x, u) \in L(T_xM, T_{(x,u)}E)$.

If moreover, D is linear on each fiber, then the connection is called a *linear connection*.

Modulo the identification of $U \subset M$ and $\phi(U) \subset \mathbb{M}$ we have the following identifications:

- $E|_U \equiv U \times \mathbb{E}$
- $TM|_U \equiv U \times \mathbb{M}$
- $TE|_{\pi^{-1}(U)} \equiv (U \times \mathbb{E}) \times (\mathbb{M} \times \mathbb{E})$
- $VE|_{\pi^{-1}(U)} \equiv (U \times \mathbb{E}) \times \mathbb{E}$

According to these identifications, we obtain the following characterizations of V and D :

$$V(x, u, y, v) = (x, u, 0, v + \omega(x, u)y)$$

$$D(x, u, y, v) = (x, v + \omega(x, u)y)$$

where ω is a smooth map from $U \times \mathbb{E}$ to the space $L(\mathbb{M}, \mathbb{E})$ of bounded linear operators from \mathbb{M} to \mathbb{E} .

This connection is *linear* if and only if ω is linear in the second variable. In this case, the relation $\Gamma(x)(u, y) = \omega(x, u)y$ gives rise to a smooth map Γ from U to the space of bilinear maps $L^2(\mathbb{E}, \mathbb{M}; \mathbb{E})$ called *local Christoffel components* of the connection.

Conversely, a connection can be given by a collection $(U^\alpha, \omega^\alpha)$ of local maps $\omega^\alpha : U^\alpha \times \mathbb{E} \rightarrow L(\mathbb{M}, \mathbb{E})$ on a covering (U^α) of M with adequate classical conditions of compatibility between $(U^\alpha, \omega^\alpha)$ and (U^β, ω^β) where $U_\alpha \cap U_\beta \neq \emptyset$.

Remark 5.2. — It is classical that if M is smooth paracompact, then there always exists a connection on M and also on each Banach bundle over M . However, these assumptions impose the same assumptions on the Banach space \mathbb{M} .

On the other hand, it is well known that there exist linear connections on a Banach manifold without such assumptions. For instance, if $TM \equiv M \times \mathbb{M}$ there always exists a (trivial) connection on M . But there are further situations for which a linear connection exists on a Banach manifold. For example,

there exist linear connections on loop spaces (see for instance [7]) or on the manifold $\mathcal{M}(\mu)$ of strictly positive probability densities of a probability space (Ω, Σ, μ) (cf. [24]).

DEFINITION 5.3. — *A Koszul connection on E is a \mathbb{R} -bilinear map $\nabla : \mathfrak{X}(M) \times \underline{E} \rightarrow \underline{E}$ which fulfills the following properties:*

- (i) $\nabla_X(f\sigma) = df(X)\sigma + f\nabla_X\sigma$
- (ii) $\nabla_{fX}\sigma = f\nabla_X\sigma$

for any function f on M , $X \in \chi(M)$ and $\sigma \in \underline{E}$.

Given a linear connection D on a Banach bundle $\pi : E \rightarrow M$, we obtain a covariant derivative $\nabla : \mathfrak{X}(M) \times \underline{E} \rightarrow \underline{E}$ which is a Koszul connection. Since any (linear) connection induces naturally a (linear) connection on the restriction $E|_U$ of E to any open set U of M , we also obtain a covariant derivative $\nabla^U : \mathfrak{X}(U) \times \underline{E}|_U \rightarrow \underline{E}|_U$ with the correspondent previous properties for any function f on U , $X \in \mathfrak{X}(U)$ and $\sigma \in \underline{E}|_U$.

Unfortunately, in general, a Koszul connection may be not localizable in the following sense: since any local section of E (resp. any local vector field on M) can not be always extended to a global section of E (resp. to a global vector field on M), the previous operator ∇ can not always induce a (local) operator ∇^U as previously. Therefore, in this work, a *Koszul connection will always be taken in the sense of the covariant derivative associated to a linear connection D on E .* In particular, for any $x \in M$, the value $\nabla_X\sigma(x)$ only depends on of the value of X at x and the 1-jet of σ at x .

In a local trivialization $E|_U \equiv U \times \mathbb{E}$, a local section σ of E , defined on U , can be identified with a map $\sigma : U \rightarrow \mathbb{E}$. Then ∇ has the local expression:

$$\nabla_X\sigma = d\sigma(X) + \Gamma(\sigma, X)$$

where Γ , smooth map from U to $L^2(\mathbb{E}, \mathbb{M}; \mathbb{E})$, is the local Christoffel components of the connection D which will be also called the local *Christoffel components* of ∇ .

Remark 5.4. — If M is smooth regular, then, as classically in finite dimension, any covariant derivative $\nabla : \mathfrak{X}(M) \times \underline{E} \rightarrow \underline{E}$ which fulfills the previous properties (i) and (ii) is localizable. Therefore, in this case, there is a one-to-one correspondence between such covariant derivative and linear connection on E as in the finite dimensional framework.

Finally if $E|_U \equiv U \times \mathbb{E}$ and $E|_{U'} \equiv U' \times \mathbb{E}$ are local trivializations such that $U \cap U' \neq \emptyset$, then we have a smooth map $g : U \cap U' \rightarrow GL(\mathbb{E})$ such

that $\sigma|_{U'} = g\sigma|_U$ for any section defined on $U \cup U'$. Therefore the Christoffel component Γ and Γ' of ∇ on $U \cap U'$ are linked by the relation

$$\Gamma'(X, \sigma) = g^{-1}dg(X, \sigma) + g^{-1}\Gamma(X, g\sigma).$$

5.3. Direct limit of Banach connections

DEFINITION 5.5. — *Let $(E_n, \pi_n, M_n)_{n \in \mathbb{N}^*}$ be a strong ascending sequence of Banach vector bundles where $\varepsilon_n^{n+1} : M_n \rightarrow M_{n+1}$ and $\lambda_n^{n+1} : E_n \rightarrow E_{n+1}$ are the compatible bonding maps. A sequence of connections $D_n : TE_n \rightarrow E_n$ is called a strong ascending sequence of Banach connections if*

$$\lambda_n^{n+1} \circ D_n = D_{n+1} \circ T\lambda_n^{n+1}.$$

THEOREM 5.6. — *Let $(D_n)_{n \in \mathbb{N}^*}$ be a strong ascending sequence of Banach connections on an ascending sequence $(E_n, \pi_n, M_n)_{n \in \mathbb{N}^*}$ of Banach bundles and assume that $(M_n)_{n \in \mathbb{N}^*}$ has the direct limit chart property at each point of $x \in M = \varinjlim M_n$.*

Then the direct limit $D = \varinjlim D_n$ is a connection on the convenient vector bundle $(\varinjlim E_n, \varinjlim \pi_n, \varinjlim M_n)$.

Proof. — Let x be in $\varinjlim M_n$. We suppose that $x \in M_{n_0}$. According to Definition 4.17, let (U^α, ϕ^α) be a chart of $M = \varinjlim M_n$ around x which satisfies the assumption (D) and (E), where $\phi^\alpha : U^\alpha = \bigcup_{i \geq n_0} U_i^\alpha \rightarrow O^\alpha = \bigcup_{i \geq n_0} O_i^\alpha$ with $(U_i^\alpha, \phi_i^\alpha)$ is a chart around x_i and $O_i^\alpha = \phi_i^\alpha(U_i^\alpha) \subset \mathbb{M}_i$. Moreover, $E_i|_{U_i}$ is trivial.

Denote by D_i^α the expression of the connection D_i in local charts. We then have $D_i^\alpha(x_i^\alpha, u_i^\alpha, y_i^\alpha, v_i^\alpha) = (x_i^\alpha, v_i^\alpha + \omega_i^\alpha(x_i^\alpha, u_i^\alpha)y_i^\alpha)$ where ω_i^α is a smooth map from $O_i^\alpha \times \mathbb{E}_i$ to the space $L(\mathbb{M}_i, \mathbb{E}_i)$ of bounded linear operators from \mathbb{M}_i to \mathbb{E}_i .

Using the relations $\lambda_i^{i+1} \circ D_i = D_{i+1} \circ T\lambda_i^{i+1}$ we have the following diagram:

$$\begin{array}{ccccccc} O_i^\alpha \times \mathbb{E}_i \times \mathbb{M}_i \times \mathbb{E}_i & \xleftarrow{T\phi_{E_i}^\alpha} & T(E_i|_{U_i^\alpha}) & \xrightarrow{T\lambda_i^{i+1}} & T(E_{i+1}|_{U_{i+1}^\alpha}) & \xrightarrow{T\phi_{E_{i+1}}^\alpha} & O_{i+1}^\alpha \times \mathbb{E}_{i+1} \times \mathbb{M}_{i+1} \times \mathbb{E}_{i+1} \\ \downarrow D_i^\alpha & & \downarrow D_i & & \downarrow D_{i+1} & & \downarrow D_{i+1}^\alpha \\ O_i^\alpha \times \mathbb{E}_i & \xleftarrow{\phi_{E_i}^\alpha} & E_i|_{U_i^\alpha} & \xrightarrow{\lambda_i^{i+1}} & E_{i+1}|_{U_{i+1}^\alpha} & \xrightarrow{\phi_{E_{i+1}}^\alpha} & O_{i+1}^\alpha \times \mathbb{E}_{i+1} \end{array}$$

Using the expression in local coordinates $\widehat{\varepsilon_{i+1}^\alpha} : O_i^\alpha \rightarrow O_{i+1}^\alpha$ and the map $\widehat{\lambda_i^{i+1}} : \mathbb{E}_i \rightarrow \mathbb{E}_{i+1}$ we then obtain that $(D_i^\alpha)_{i \geq n}$ can be realized as a

direct limit because we have:

$$\begin{aligned}
 (\widehat{\varepsilon}_i^{i+1}{}^\alpha \times \widehat{\lambda}_i^{i+1}) \circ D_i^\alpha &= (\widehat{\varepsilon}_i^{i+1} \times \widehat{\lambda}_i^{i+1}) \circ (\phi_{E_i}^\alpha \circ D_i \circ (T\phi_{E_i}^\alpha)^{-1}) \\
 &= \phi_{E_{i+1}}^\alpha \circ \underline{\lambda}_i^{i+1} \circ D_i \circ (T\phi_{E_i}^\alpha)^{-1} \\
 &= \phi_{E_{i+1}}^\alpha \circ \underline{D_{i+1} \circ T\lambda_i^{i+1}} \circ (T\phi_{E_i}^\alpha)^{-1} \\
 &= \phi_{E_{i+1}}^\alpha \circ (\phi_{E_{i+1}}^\alpha)^{-1} \circ D_{i+1}^\alpha \circ T\phi_{E_{i+1}}^\alpha \circ T\lambda_i^{i+1} \circ (T\phi_{E_i}^\alpha)^{-1} \\
 &= D_{i+1}^\alpha \circ (\widehat{\varepsilon}_i^{i+1}{}^\alpha \times \widehat{\lambda}_i^{i+1} \times \widehat{\varepsilon}_i^{i+1} \times \widehat{\lambda}_i^{i+1})
 \end{aligned}$$

We obtain an analogous result for the smooth Banach local forms $\omega_i^\alpha : U_i^\alpha \times \mathbb{E}_i \rightarrow L(\mathbb{M}_i, \mathbb{E}_i)$. \square

Using the intrinsic link between a connection D and a Koszul connection ∇ we get the following result:

COROLLARY 5.7. — *Let $(D_n)_{n \in \mathbb{N}^*}$ be a strong ascending sequence of Banach connections on a direct sequence $(E_n, \pi_n, M_n)_{n \in \mathbb{N}^*}$ of Banach bundles and consider the associated Koszul connections $(\nabla_n)_{n \in \mathbb{N}^*}$. The direct limit $\nabla = \varinjlim \nabla_n$ is a Koszul connection on the convenient vector bundle $(\varinjlim E_n, \varinjlim \pi_n, \varinjlim M_n)$.*

Example 5.8. — Denote by $L_{loc}^p(\mathbb{R}^m)$ the space of locally L^p functions on \mathbb{R}^m ($1 \leq p < +\infty$). A function belongs to $L_{loc}^p(\mathbb{R}^m)$ if and only if its restriction to any compact set K of \mathbb{R}^m belongs to $L^p(K)$. Since \mathbb{R}^m is an ascending sequence of compact sets K_n^m (where $K_n^m \subset K_{n+1}^m$), we have

$$L_{loc}^p(\mathbb{R}^m) = \varinjlim L^p(K_n^m).$$

Moreover, the closure $\widehat{K_n^m}$ of the open set $K_{n+1}^m \setminus K_n^m$ is also compact and we have $L^p(K_{n+1}^m) = L^p(K_n^m) \oplus L^p(\widehat{K_n^m})$. Therefore the sequence of Banach spaces $(L^p(K_n^m))_{n \in \mathbb{N}^*}$ is an ascending sequence of complemented Banach spaces. Since the tangent bundle to each $L^p(K_n^m)$ is trivial, there exists a (trivial) Koszul connection on this Banach bundle. Therefore we get a Koszul connection on $L_{loc}^p(\mathbb{R}^m)$.

Example 5.9. — Let $(H_n)_{n \in \mathbb{N}^*}$ be a sequence of Banach Lie groups and consider the Banach Lie group of cartesian products $G_n = \prod_{k=1}^n H_k$. The weak direct product $\prod_{k \in \mathbb{N}^*}^* H_k$ is the set of all sequences $(h_n)_{n \in \mathbb{N}^*}$ such that $h_n = 1$ for all but finitely many n . The weak direct product is a topological group for the box topology (see [14, 4]). In fact, this weak direct product has a structure of Lie group modelled on the locally convex topological space $\bigoplus_{k \in \mathbb{N}^*} \mathfrak{H}_k$ where \mathfrak{H}_k is the Lie algebra of H_k . The tangent space TG_n is the vector bundle $G_n \times \bigoplus_{k=1}^n \mathfrak{H}_k$. Moreover, TG_n is a complemented subbundle of TG_{n+1} and is naturally endowed with the (trivial) Koszul connection.

5.4. Sprays on an anchored Banach bundle

We begin this subsection with a brief presentation of the theory of semi-sprays on a Banach anchored bundle according to [2].

Let $\pi : E \rightarrow M$ be a Banach vector bundle on a Banach manifold modelled on a Banach space \mathbb{M} whose fiber is modelled on a Banach space \mathbb{E} .

DEFINITION 5.10. — *A morphism of vector bundles $\rho : E \rightarrow TM$ is called an anchor. (E, π, M, ρ) is then called a Banach anchored bundle.*

DEFINITION 5.11. — *A semi-spray on an anchored bundle is a vector field S on E such that $T\pi \circ S = \rho$.*

This means that, in a local trivialization $E|_U \equiv U \times \mathbb{E}$, we have $T\pi(S(x, u)) = \rho(x)u$ for all $(x, u) \in E|_U$.

A smooth curve $c : I \subset \mathbb{R} \rightarrow E$ is called *admissible* if the tangent vector $\gamma'(t)$ of $\gamma = \pi \circ c$ is precisely $\rho(c(t))$.

From [2], we have the following characterization of a semi-spray:

THEOREM 5.12. — *A vector field S on E is a semi-spray if and only if each integral curve of S is an admissible curve.*

In a local trivialization $E|_U \equiv U \times \mathbb{E}$, a semi-spray can be written as

$$S(x, u) = (x, u, \rho(x)u, -2G(x, u)).$$

The *Euler field* C is the global vector field on E which is tangent to the fiber of π (i.e. *vertical*) and such that the flow of C is an infinitesimal homothety on each fiber. A semi-spray S is called a *spray* if S is invariant by the flow of C . This condition is equivalent to the nullity of the Lie bracket $[C, S]$. In this case, in a local trivialization, the function G in Definition 5.11 is linear in the second variable.

Conversely, a spray can be given by a collection (U^α, G^α) of local maps $G^\alpha : U^\alpha \times \mathbb{E} \rightarrow L(\mathbb{E}, \mathbb{E})$ on a covering U^α of M with adequate classical conditions of compatibility between (U^α, G^α) and (U^β, G^β) when $U^\alpha \cap U^\beta \neq \emptyset$ (cf. [2]).

Given a Koszul connection ∇ on E and an admissible curve $c : I \rightarrow E$ as in the infinite dimensional case, we associate an operator of differentiation ∇^c of the set of sections of E along $\gamma = \pi \circ c$ given by $\nabla^c \sigma = \nabla_{\dot{\gamma}} \sigma$. In particular c is a section along γ .

DEFINITION 5.13. — *An admissible curve c is called a geodesic of ∇ if $\nabla_{\dot{\gamma}} c = \nabla^c c \equiv 0$.*

In a local trivialization $E|_U \equiv V \times \mathbf{E}$, an admissible curve $c : I \rightarrow E|_U$ is a geodesic of ∇ if and only $\pi \circ c$ is a solution of the following differential equation:

$$\begin{cases} \dot{x} = \rho(x)u \\ \dot{u} = \Gamma(x)(u, \dot{x}) \end{cases}$$

where Γ is the local Christoffel component of ∇ on $E|_U$. Therefore, if we set $G(x, u) = -\frac{1}{2}\Gamma(x)(u, \rho(x)u)$, we get a vector field S_U on $E|_U$ which satisfies the relation given in Definition 5.11 and so is a spray on $E|_U$. Now, according to the compatibility conditions between the local Christoffel components, we obtain a unique global spray associated to ∇ . Conversely, as in the case of $E = TM$ (cf. [32]), given a spray S on E , we can associate a unique connection ∇ whose associated spray is S .

Taking into account the classical theorem of existence of a local flow of a vector field on a Banach manifold, we obtain:

THEOREM 5.14. — *Let (E, M, ρ) be a Banach anchored bundle. There exists a spray on E if and only there exists a Koszul connection ∇ on E . Moreover, there exists a canonical correspondence one-to-one between sprays and Koszul connections on E so that an admissible curve is a geodesic of the Koszul connection ∇ if and only if this curve is an integral curve of the unique S associated to ∇ .*

When $E = TM$, there exists an exponential map $\text{Exp} : \mathcal{U} \subset TM \rightarrow M$, defined on an open neighborhood \mathcal{U} of the zero section, such that $p_{M|\mathcal{U}} : \mathcal{U} \rightarrow M$ is a fibration whose each fiber \mathcal{U}_x is a star-shaped open neighborhood of 0 in T_xM . Moreover, the differential of the restriction Exp_x of Exp to \mathcal{U}_x is equal to Id_{T_xM} at 0. In particular, Exp_x is a diffeomorphism of a star-shaped open neighborhood of $0 \in T_xM$ onto an open neighborhood of $x \in M$.

6. Direct limits of sequences of almost Banach Lie algebroids

6.1. Almost Banach Lie algebroids

Let (E, π, M, ρ) be a Banach anchored bundle.

If \underline{E} denotes the $C^\infty(M)$ -module of smooth sections of E , the morphism ρ gives rise to a $C^\infty(M)$ -module morphism $\underline{\rho} : \underline{E} \rightarrow \underline{TM} = \mathfrak{X}(M)$ defined for every $x \in M$ and every section s of E by: $(\underline{\rho}(s))(x) = \rho(s(x))$ and still denoted by ρ .

DEFINITION 6.1. — An almost Lie bracket on an anchored bundle (E, π, M, ρ) is a bilinear map $[\cdot, \cdot]_E : \underline{E} \times \underline{E} \rightarrow \underline{E}$ which satisfies the following properties:

- (1) $[\cdot, \cdot]_E$ is antisymmetric;
- (2) (Leibniz property)
 $\forall s_1, s_2 \in \underline{E}, \forall f \in C^\infty(M), [s_1, fs_2]_E = f \cdot [s_1, s_2] + df(\rho(s_1)) \cdot s_2.$

DEFINITION 6.2. — A Lie bracket is an almost Lie bracket whose Jacobiator vanishes:

$$\forall s_1, s_2, s_3 \in \underline{E}, [s_1, [[s_2, s_3]] + [s_2, [[s_3, s_1]] + [s_3, [[s_1, s_2]]] = 0$$

DEFINITION 6.3. — An almost Banach Lie algebroid is an anchored bundle (E, π, M, ρ) provided with an almost Lie bracket $[\cdot, \cdot]_E$. When $[\cdot, \cdot]_E$ is in fact a Lie bracket the associated structure $(E, M, \rho, [\cdot, \cdot]_E)$ is called a Banach Lie algebroid.

If $(E, \pi, M, \rho[\cdot, \cdot]_E)$ is a Banach Lie algebroid, $\rho : \underline{E} \rightarrow \mathfrak{X}(M)$ is a Lie algebra morphism; in particular, we have $[\rho s_1, \rho s_2] = \rho([s_1, s_2]_E)$.

Notice that the converse is not true in general (take $\rho \equiv 0$ for instance).

DEFINITION 6.4. — When we have $[\rho s_1, \rho s_2] = \rho([s_1, s_2]_E)$ for all sections $s_1, s_2 \in \underline{E}$, we will say that ρ is a Lie morphism. In this case $(E, \pi, M, \rho, [\cdot, \cdot]_E)$ is called an algebroid.

In general the almost Lie bracket of an algebroid $(E, \pi, M, \rho, [\cdot, \cdot]_E)$ does not satisfy the Jacobi identity.

Remark 6.5. — Since the terminology of almost Poisson bracket seems generally adopted in the most recent papers on nonholonomic mechanics, in this work we have adopted the definition of an almost Lie algebroid given in [23]. Therefore taking into account the relation between almost Linear Poisson bracket and almost Lie bracket, this terminology seems to us well adapted. Therefore and according to [28], we use the denomination “algebroid” for an almost algebroid such that the anchor is a morphism of Lie algebras. Note that in [28] or in [15] an almost Lie algebroid corresponds to the previous definition of an algebroid and our denomination “almost algebroid” corresponds to “quasi-Lie algebroid” in [28] or in [15].

Example 6.6. — Consider a smooth right action $\psi : M \times G \rightarrow M$ of a connected Lie group G on a Banach manifold M . Denote by \mathcal{G} the Lie algebra of G . We then have a natural morphism ξ of Lie algebras from \mathcal{G} to $\mathfrak{X}(M)$ defined by:

$$\xi_X(x) = T_{(x,e)}\psi(0, X).$$

For any X and Y in \mathcal{G} , we have: $\xi_{\{X,Y\}} = [\xi_X, \xi_Y]$ where $\{\cdot, \cdot\}$ denotes the bracket on the Lie algebra \mathcal{G} (see for instance [21, 36.12]).

On the trivial bundle $M \times \mathcal{G}$, each section can be identified with a map $\sigma : M \rightarrow \mathcal{G}$. We then define a Lie bracket $\{\{\cdot, \cdot\}\}$ on the set of such sections by:

$$\{\{\sigma_1, \sigma_2\}\}(x) = \{\sigma_1(x), \sigma_2(x)\} + d\sigma_1(\xi_{\sigma_2(x)}) - d\sigma_2(\xi_{\sigma_1(x)}).$$

An anchor $\Psi : M \times \mathcal{G} \rightarrow TM$ is defined by $\Psi(x, X) = \xi_X(x)$. Then $(M \times \mathcal{G}, \text{pr}_1, M, \Psi, \{\{\cdot, \cdot\}\})$ is a Banach Lie algebroid.

Moreover, if we denote by G_x the closed subgroup of isotropy of a point $x \in M$ and by $\mathcal{G}_x \subset \mathcal{G}$ its Lie subalgebra, we have $\ker \Psi_x = \mathcal{G}_x$. If \mathcal{G}_x is complemented in \mathcal{G} for any $x \in M$ and ρ has closed range, then the weak distribution $\mathcal{D} = \Psi(M \times \mathcal{G})$ is integrable and the leaf through x is its orbit $\psi(x, G)$ (cf. [26, Example 4.3, 3]).

Note that in finite dimension it is classical that a Lie bracket $[\cdot, \cdot]_E$ on an anchored bundle (E, π, M, ρ) respects the sheaf of sections of $\pi : E \rightarrow M$ or, for short, is *localizable* (see for instance [25]), if the following properties are satisfied:

- (i) for any open set U of M , there exists a unique bracket $[\cdot, \cdot]_U$ on the space of sections $\underline{E}|_U$ such that, for any s_1 and s_2 in $\underline{E}|_U$, we have:

$$[s_1|_U, s_2|_U]_U = ([s_1, s_2]_E)|_U$$

- (ii) (compatibility with restriction) if $V \subset U$ are open sets, then, $[\cdot, \cdot]_U$ induces a unique Lie bracket $[\cdot, \cdot]_{UV}$ on $\underline{E}|_V$ which coincides with $[\cdot, \cdot]_V$ (induced by $[\cdot, \cdot]_E$).

By the same arguments as in finite dimension, when M is smooth regular any Lie bracket $[\cdot, \cdot]_E$ on an anchored bundle (E, π, M, ρ) is localizable (cf. [26]).

But, in general, for analog reasons as for Koszul connection, we can not prove that any Lie bracket is localizable. Unfortunately in the Banach framework, we have *no example of Lie algebroid* for which is not localizable. Therefore at least for finding conditions under which a Banach Lie algebroid is integrable this condition is necessary. This condition of localization implies also that a bracket depends on the one jets of sections. Therefore, in the sequel, we will assume that *all almost Lie bracket $[\cdot, \cdot]_E$ are localizable*.

Remark 6.7. — If there exists a Koszul connection ∇ on E , then we get an almost Lie bracket $[\cdot, \cdot]_\nabla$ defined by

$$[s_1, s_2]_\nabla = \nabla_{\rho s_1} s_2 - \nabla_{\rho s_2} s_1.$$

Note that since a ∇ is localizable, so is $[\cdot, \cdot]_{\nabla}$

When $(E, \pi, M, \rho, [\cdot, \cdot]_E)$ is an almost Banach Lie algebroid we can define the following operators:

- (i) *Lie derivative* L_s^ρ according to a section s of E :
for a smooth function $f \in \Omega^0(M, E) = \mathcal{F}$,

$$L_s^\rho(f) = L_{\rho \circ s}(f) = i_{\rho \circ s}(df);$$

for a q -form $\omega \in \Omega^q(M, E)$ (where $q > 0$)

$$(L_s^\rho \omega)(s_1, \dots, s_q) = L_s^\rho(\omega(s_1, \dots, s_q)) - \sum_{i=1}^q \omega(s_1, \dots, s_{i-1}, [s, s_i]_E, s_{i+1}, \dots, s_q). \quad (6.1)$$

- (ii) $\Omega(M, E)$ -*value derivative* according to a section s of E : for a smooth function $f \in \Omega^0(M, E) = \mathcal{F}$

$$d_\rho f = t_\rho \circ df;$$

for a q -form $\omega \in \Omega^q(M, E)$ (where $q > 0$)

$$(d_\rho \omega)(s_0, \dots, s_q) = \sum_{i=0}^q (-1)^i L_{s_i}^\rho(\omega(s_0, \dots, \widehat{s}_i, \dots, s_q)) + \sum_{0 \leq i < j \leq q} (-1)^{i+j} (\omega([s_i, s_j]_E, s_0, \dots, \widehat{s}_i, \dots, \widehat{s}_j, \dots, s_q)).$$

In general, we have $d_\rho \circ d_\rho \neq 0$. However, $(E, M, \rho, [\cdot, \cdot]_E)$ is a Banach Lie algebroid if and only if $d_\rho \circ d_\rho = 0$.

DEFINITION 6.8. — *Let $\psi : E \rightarrow E'$ be a linear bundle morphism over $f : M \rightarrow M'$.*

- (1) *A section s' of $E' \rightarrow M'$ and a section s of $E \rightarrow M$ are ψ -related if $s' \circ f = \psi \circ s$.*
- (2) *ψ is a morphism of almost Banach Lie algebroids from $(E, \pi, M, \rho, [\cdot, \cdot]_E)$ to $(E', \pi', M', \rho', [\cdot, \cdot]_{E'})$ if:*
 - (a) $\rho' \circ \psi = Tf \circ \rho$;
 - (b) *for any pair of ψ -related sections s'_i and s_i ($i = 1, 2$), we have: $\psi([s_1, s_2]) = [s'_1, s'_2]' \circ f$, i.e. the Lie bracket $[s'_1, s'_2]'$ and $[s_1, s_2]$ are ψ -related.*

In a dual way, a morphism $\psi : E \rightarrow E'$ which satisfies property (a) is an almost Banach Lie algebroid morphism if the mapping $\psi^* : \Omega^q(M, E') \rightarrow \Omega^q(M, E)$ defined by:

$$(\psi^* \alpha')_x(s_1, \dots, s_q) = \alpha'_{f(x)}(\psi \circ s_1, \dots, \psi \circ s_q)$$

commutes with the differentials:

$$d_\rho \circ \psi^* = \psi^* \circ d_{\rho'}.$$

Notice that an almost Banach Lie algebroid $(E, \pi, M, \rho, [\cdot, \cdot]_E)$ is a Banach algebroid if and only if the anchor ρ is a morphism of Banach Lie algebroids from $(E, \pi, M, \rho, [\cdot, \cdot]_E)$ to the canonical Banach Lie algebroid $(TM, p_M, M, \text{Id}_{TM}, [\cdot, \cdot])$.

6.2. Direct limit of almost Banach Lie algebroids

As in the Banach framework, if $\pi : E \rightarrow M$ is a convenient bundle over a n.n.H. convenient manifold M , then we can define *the convenient algebroid or Lie algebroid structure*⁽⁸⁾ $(E, \pi, M, \rho, [\cdot, \cdot]_E)$ in an obvious way. Now coming back to the context of sequence of Banach anchored bundles, we have:

DEFINITION 6.9.

- (1) A sequence $(E_n, \pi_n, M_n, \rho_n)_{n \in \mathbb{N}^*}$ is called a strong ascending sequence of anchored Banach bundles if
 - (a) $(E_n, \pi_n, M_n)_{n \in \mathbb{N}^*}$ is a direct sequence of Banach bundles;
 - (b) For all $n \leq m$, we have

$$\rho_m \circ \lambda_n^m = T\epsilon_n^m \circ \rho_n.$$

where $\lambda_n^m : E_n \rightarrow E_m$ and $\epsilon_n^m : M_n \rightarrow M_m$ are the bonding morphisms.

- (2) A sequence $(E_n, \pi_n, M_n, \rho_n, [\cdot, \cdot]_n)_{n \in \mathbb{N}^*}$ is called a strong ascending sequence of almost Banach Lie algebroids if $(E_n, \pi_n, M_n, \rho_n)_{n \in \mathbb{N}^*}$ is a strong ascending sequence of anchored Banach bundles with the additional property: $\lambda_n^m : E_n \rightarrow E_m$ is an almost Banach algebroid morphism between the almost Banach Lie algebroids $(E_n, \pi_n, M_n, \rho_n, [\cdot, \cdot]_n)$ and $(E_m, \pi_m, M_m, \rho_m, [\cdot, \cdot]_m)$.

⁽⁸⁾ In this case E has a structure of n.n.H. convenient manifold.

THEOREM 6.10.

- (1) *If $(E_n, \pi_n, M_n, \rho_n)_{n \in \mathbb{N}^*}$ is a strong ascending sequence of anchored bundles, then $(\varinjlim E_n, \varinjlim \pi_n, \varinjlim M_n, \varinjlim \rho_n)$ is a convenient anchored bundle. Moreover, $(\varinjlim E_n, \varinjlim \pi_n, \varinjlim M_n, \varinjlim \rho_n, \varinjlim [\cdot, \cdot]_n)$ is a convenient algebroid (resp. a convenient Lie algebroid) if each $(E_n, \pi_n, M_n, \rho_n, [\cdot, \cdot]_n)$ is a Banach algebroid (resp. a Banach Lie algebroid) for $n \in \mathbb{N}^*$.*
- (2) *If $(E_n, \pi_n, M_n, \rho_n, [\cdot, \cdot]_n)_{n \in \mathbb{N}^*}$ is a strong ascending sequence of almost Banach Lie algebroids, then $(\varinjlim E_n, \varinjlim \pi_n, \varinjlim M_n, \varinjlim \rho_n, \varinjlim [\cdot, \cdot]_n)$ is an almost convenient Lie algebroid. Moreover, $(\varinjlim E_n, \varinjlim \pi_n, \varinjlim M_n, \varinjlim \rho_n, \varinjlim [\cdot, \cdot]_n)$ is a convenient algebroid (resp. a convenient Lie algebroid) if each $(E_n, \pi_n, M_n, \rho_n, [\cdot, \cdot]_n)$ is a Banach algebroid (resp. a Banach Lie algebroid) for $n \in \mathbb{N}^*$.*

Proof.

Step 1. — According to Proposition 4.18, $(\varinjlim E_n, \varinjlim \pi_n, \varinjlim M_n)$ can be endowed with a structure of convenient vector bundle whose base is modelled on the LB-space $\varinjlim M_n$ and whose structural group is the metrizable complete topological group $G(\mathbb{E})$.

Step 2. — Let $(s_n^1)_{n \in \mathbb{N}^*}$ and $(s_n^2)_{n \in \mathbb{N}^*}$ be sequences of sections of the linear bundles $\pi_n : E_n \rightarrow M_n$, i.e. fulfilling the conditions :

$$\begin{cases} \lambda_n^m \circ s_n^1 = s_m^1 \circ \varepsilon_n^m \\ \lambda_n^m \circ s_n^2 = s_m^2 \circ \varepsilon_n^m. \end{cases} \quad (6.2)$$

In order to define a structure of almost convenient Lie structure on the direct limit we have to prove the compatibility of the brackets

$$\lambda_n^m \circ [s_n^1, s_n^2]_{E_n} = [s_m^1, s_m^2]_{E_m} \circ \varepsilon_n^m \quad (6.3)$$

and the compatibility of the Leibniz properties:

$$\lambda_n^m \circ [s_n^1, g_n \times s_n^2]_{E_n} = [s_m^1, g_m \times s_m^2]_{E_m} \circ \varepsilon_n^m \quad (6.4)$$

Step 2a. — In order to prove (6.3) we use the morphisms $\lambda_n^m : E_n \rightarrow E_m$ of Lie algebroids over $\varepsilon_n^m : M_n \rightarrow M_m$:

$$d_{\rho_n} \circ (\lambda_n^m)^* = (\lambda_n^m)^* \circ d_{\rho_m} \quad (6.5)$$

applied to $\alpha_m \in \Omega^1(M_m, E_m)$.

We then have $(d_{\rho_n} \circ (\lambda_n^m)^*(\alpha_m))(s_n^1, s_n^2) = ((\lambda_n^m)^* \circ d_{\rho_m}(\alpha_m))(s_n^1, s_n^2)$,

For the LHS, we have:

$$\begin{aligned}
 & (d_{\rho_n} \circ (\lambda_n^m)^*(\alpha_m))(s_n^1, s_n^2) \\
 &= L_{\rho_n \circ s_n^1}(((\lambda_n^m)^*(\alpha_m))(s_n^2)) - L_{\rho_n \circ s_n^2}(((\lambda_n^m)^*(\alpha_m))(s_n^1)) \\
 & \quad - ((\lambda_n^m)^*(\alpha_m))[s_n^1, s_n^2]_{E_n} \\
 &= X_m^1(\alpha_m(\lambda_n^m \circ s_n^2)) - X_m^2(\alpha_m(\lambda_n^m \circ s_n^1)) - \alpha_m(\lambda_n^m \circ [s_n^1, s_n^2]_{E_n})
 \end{aligned}$$

where $X_m^a = \rho_m \circ s_m^a$ with $a = 1, 2$ fulfill the relation $X_m^a(f_m) = X_n^a(f_n)$ for $f_m = \alpha_m \circ s_m$.

For the RHS, we get:

$$\begin{aligned}
 & ((\lambda_n^m)^*(d_{\rho_m}(\alpha_m)))(s_n^1, s_n^2) \\
 &= d_{\rho_m}(\alpha_m)(\lambda_n^m \circ s_n^1, \lambda_n^m \circ s_n^2). \\
 &= L_{\rho_m \circ \lambda_n^m \circ s_n^1}(\alpha_m(\lambda_n^m \circ s_n^2)) - L_{\rho_m \circ \lambda_n^m \circ s_n^2}(\alpha_m(\lambda_n^m \circ s_n^1)) \\
 & \quad - \alpha_m[\lambda_n^m \circ s_n^1, \lambda_n^m \circ s_n^2]_{E_m} \\
 &= L_{\rho_m \circ s_n^1}(\alpha_m(\lambda_n^m \circ s_n^2)) - L_{\rho_m \circ s_n^2}(\alpha_m(\lambda_n^m \circ s_n^1)) - \alpha_m[\lambda_n^m \circ s_n^1, \lambda_n^m \circ s_n^2]_{E_m} \\
 &= X_m^1(\alpha_m(\lambda_n^m \circ s_n^2)) - X_m^2(\alpha_m(\lambda_n^m \circ s_n^1)) - \alpha_m[\lambda_n^m \circ s_n^1, \lambda_n^m \circ s_n^2]_{E_m}.
 \end{aligned}$$

Finally, we have for all $\alpha_m \in \Omega^1(M_m, E_m)$, $\alpha_m(\lambda_n^m([s_n^1, s_n^2]_{E_n})) = \alpha_m[\lambda_n^m \circ s_n^1, \lambda_n^m \circ s_n^2]_{E_m}$ and we obtain: $\lambda_n^m \circ [s_n^1, s_n^2]_{E_n} = [\lambda_n^m \circ s_n^1, \lambda_n^m \circ s_n^2]_{E_m}$. Using $\lambda_n^m \circ s_n^a = s_m^a \circ \varepsilon_n^m$, we have: $\lambda_n^m \circ [s_n^1, s_n^2]_{E_n} = [s_m^1, s_m^2]_{E_m} \circ \varepsilon_n^m$.

Step 2b. — To prove (6.4) we are going to establish that

$$\begin{aligned}
 & \lambda_n^m \circ (g_n \times [s_n^1, s_n^2]_{E_n} + (\rho_n(s_n^1))(g_n) \times s_n^2) \\
 &= (g_m \times [s_m^1, s_m^2]_{E_m} + (\rho_m(s_m^1))(g_m) \times s_m^2) \circ \varepsilon_n^m
 \end{aligned}$$

We can write:

$$\begin{aligned}
 & \lambda_n^m \circ (g_n \times [s_n^1, s_n^2]_{E_n} + (\rho_n(s_n^1))(g_n) \times s_n^2) \\
 &= \lambda_n^m \circ (g_n \times [s_n^1, s_n^2]_{E_n}) + \lambda_n^m \circ ((\rho_n(s_n^1))(g_n) \times s_n^2) \\
 &= g_n \times (\lambda_n^m \circ [s_n^1, s_n^2]_{E_n}) + \lambda_n^m(X_n^1(g_n)) \times \lambda_n^m \circ s_n^2 \quad (\lambda_n^m \text{ is a morphism}) \\
 &= g_n \times ([s_m^1, s_m^2]_{E_m} \circ \varepsilon_n^m) + X_m^1(g_m) \circ \varepsilon_n^m \times s_m^2 \circ \varepsilon_n^m \quad \text{cf. (6.3)} \\
 &= (g_m \circ \varepsilon_n^m) \times ([s_m^1, s_m^2]_{E_m} \circ \varepsilon_n^m) + (X_m^1(g_m) \times s_m^2) \circ \varepsilon_n^m \\
 &= (g_m \times [s_m^1, s_m^2]_{E_m}) \circ \varepsilon_n^m + (\rho_m(s_m^1)(g_m) \times s_m^2) \circ \varepsilon_n^m \\
 &= (g_m \times [s_m^1, s_m^2]_{E_m} + (\rho_m(s_m^1))(g_m) \times s_m^2) \circ \varepsilon_n^m.
 \end{aligned}$$

Step 3. — Now, from the previous construction of $\varinjlim[\cdot, \cdot]_n$, it is clear that if ρ_n is a morphism of almost algebroids from $(E_n, \pi_n, M_n, \rho_n, [\cdot, \cdot]_n)$ to

the canonical Banach Lie algebroid $(TM_n, \pi_n, M_n, \text{Id}_{TM_n}, [\cdot, \cdot])$, then $\varinjlim \rho_n$ satisfies

$$\varinjlim \rho_n(\varinjlim[\cdot, \cdot]_n) = [\varinjlim \rho_n(\cdot), \varinjlim \rho_n(\cdot)].$$

Moreover, it is also easy to show that if each bracket $[\cdot, \cdot]_n$ satisfies the Jacobi identity, then $\varinjlim[\cdot, \cdot]_n$ satisfies also a Jacobi identity. These last proofs are left to the reader. \square

COROLLARY 6.11. — *Let $(D_n)_{n \in \mathbb{N}^*}$ be a strong ascending sequence of Banach connections on a strong ascending sequence $(E_n, \pi_n, M_n)_{n \in \mathbb{N}^*}$ of Banach bundles. Then there exists an almost convenient Lie algebroid structure on the bundle $(\varinjlim E_n, \varinjlim \pi_n, \varinjlim M_n)$.*

Proof. — On each anchored bundle $(E_n, \pi_n, M_n, \rho_n)$, we denote by ∇_n the E_n -Koszul connection associated to D_n . Therefore $[s_n^1, s_n^2]_n = \nabla_{\rho_n(s_n^1)}^n s_n^2 - \nabla_{\rho_n(s_n^2)}^n s_n^1$ defines an almost Lie bracket on $(E_n, \pi_n, M_n, \rho_n)$. Since $(D_n)_{n \in \mathbb{N}^*}$ is a direct sequence of Banach connections, it follows that the sequence $([\cdot, \cdot]_n)_{n \in \mathbb{N}^*}$ of almost brackets satisfies a property of Definition 6.9. Therefore, from Theorem 6.10, $\varinjlim[\cdot, \cdot]_n$ is an almost Lie bracket on the convenient anchored bundle $(\varinjlim E_n, \varinjlim \pi_n, \varinjlim M_n, \varinjlim \rho_n)$. \square

7. Integrability of distributions which are direct limit of local Koszul Banach bundles

7.1. Integrability of the range of an anchor

We first recall the classical definitions of distribution, integrability and involutivity.

DEFINITION 7.1. — *Let M be a Banach manifold.*

- (1) *A distribution Δ on M is an assignment $\Delta : x \mapsto \Delta_x \subset T_x M$ on M where Δ_x is a subspace of $T_x M$.*
- (2) *A vector field X on M , defined on an open set $\text{Dom}(X)$, is called tangent to a distribution Δ if $X(x)$ belongs to Δ_x for all $x \in \text{Dom}(X)$.*
- (3) *A distribution Δ on M is called integrable if, for all $x_0 \in M$, there exists a weak submanifold (N, ϕ) of M such that $\phi(y_0) = x_0$ for some $y_0 \in N$ and $T\phi(T_y N) = \Delta_{\phi(y)}$ for all $y \in N$. In this case (N, ϕ) is called an integral manifold of Δ through x .*
- (4) *A distribution Δ is called involutive if for any vector fields X and Y on M tangent to Δ the Lie bracket $[X, Y]$ defined on $\text{Dom}(X) \cap \text{Dom}(Y)$ is tangent to Δ .*

Classically, in Banach context, when Δ is a complemented subbundle of TM , according to the Frobenius Theorem, involutivity implies integrability.

In finite dimension, the famous results of H. Sussman and P. Stefan give necessary and sufficient conditions for the integrability of smooth distributions.

A generalization of these results in the context of Banach manifolds can be found in [5] and [26].

We are now in a position to prove the following theorem which will be useful for the proof of the main theorem on the integrability of a distribution on a direct limit of Banach manifolds endowed with Koszul connections.

THEOREM 7.2. — *Let $(E, \pi, M, \rho, [\cdot, \cdot]_E)$ be a Banach algebroid (cf. Subsection 6.1). Assume that for each $x \in M$, the kernel of ρ_x is complemented in each fiber E_x and $\mathcal{D}_x = \rho(E_x)$ is closed in T_xM . Then \mathcal{D} is an integrable weak distribution of M .*

Assume that there exists a linear connection on E . Then there exists a non linear connection on the tangent bundle of each leaf of the distribution \mathcal{D} .

Proof. — The first part of this theorem is an easy adaptation of the proof of Theorem 5 in [26].

We consider a leaf L of \mathcal{D} . If $\iota : L \rightarrow M$ is the natural inclusion, it is a smooth immersion of L in M . Let x be any point of L and denote K_x the kernel of $\rho_x : \pi^{-1}(x) = E_x \rightarrow T_xM$. According to the assumption on E , we have a decomposition $E_x = K_x \oplus F_x$. From the proof of Theorem 2 of [26], L is a Banach manifold modelled on $F := F_x$. Consider the pull back E_L of E over L via $\iota : L \rightarrow M$. We have a bundle morphism $\hat{\iota}$ from E_L in E over ι which is an isomorphism on each fiber. Therefore, the kernel of $\hat{\rho} = \rho \circ \iota$ is a Banach subbundle K_L of E_L and we have a subbundle F_L of E_L such that $E_L = K_L \oplus F_L$. In particular, we have an isomorphism ρ_L from F_L to TL . It follows that the tangent map $T\rho_L : TF_L \rightarrow T(TL)$ is also an isomorphism. On the other hand, according to the decomposition $E_L = K_L \oplus F_L$, we have also a decomposition $TE_L = TK_L \oplus TF_L$.

Now, assume that there exists a non linear connection on E and let $D : TE \rightarrow E$ be the associated map connection. The map $\hat{D}_L = D \circ \hat{\iota} \circ (T\rho_L)^{-1}$ is smooth and maps the fiber of $T_{(x,u)}(TL)$ over $(x, u) \in TL$ into the fiber of $(E_L)_x$ over $x \in L$. As $\hat{\iota}$ is an isomorphism from $(E_L)_x$ to the fiber E_x over $\iota(x)$, it follows that \hat{D}_L is a linear continuous map between these fibers. In particular, we can consider \hat{D}_L as a map from $T(TL)$ into E_L . Now if Π_L is the projection of E_L on F_L parallel to K_L , the map $D_L = \rho_L \circ \Pi_L \circ \hat{D}_L$ defines a Koszul connection on TL . \square

7.2. Criterion of integrability for local direct limits of local Koszul Banach bundles

Let M be a n.n.H. convenient manifold and denote by TM its dynamical tangent bundle. In the same way, a distribution Δ on M is again an assignment $\Delta : x \mapsto \Delta_x \subset T_x M$ on M where Δ_x is a subspace of $T_x M$. The notion of integrability and involutivity of a distribution recalled in Subsection 7.1 can be clearly adapted to the convenient context.

We will now give a criterion of integrability for direct limit of local Koszul Banach bundles. More precisely we have:

DEFINITION 7.3. — *A distribution Δ on a n.n.H. convenient manifold M is called a local direct limit of local Koszul Banach bundles if the following property is satisfied:*

- (*) *for any $x \in M$, there exists an open neighbourhood U of x and a strong ascending sequence of anchored Banach bundles $(E_n, \pi_n, U_n, \rho_n)_{n \in \mathbb{N}^*}$ endowed with a Koszul connection ∇^n such that $U = \varinjlim U_n$, $\varinjlim \rho_n(E_n) = \Delta|_U$ and such that E_n is a complemented subbundle of E_{n+1} .*

Remark 7.4. — In the context of paracompact finite dimensional manifolds or Hilbert manifolds, the condition of the existence of a Koszul connection ∇^n and E_n complemented in E_{n+1} are automatically satisfied.

We then have the following criterion of integrability:

THEOREM 7.5. — *Let Δ be a local direct limit of local Koszul Banach bundles. Assume that in the property (*) there exists an almost Lie bracket $[\cdot, \cdot]_n$ on $(E_n, \pi_n, U_n, \rho_n)$ such that $(E_n, \pi_n, U_n, \rho_n, [\cdot, \cdot]_n)$ is a Banach algebroid, and over each point $y_n \in U_n$ the kernel of ρ_n is complemented in the fiber $\pi_n^{-1}(y_n)$ and the range of ρ_n is closed.*

Then the distribution Δ is integrable and the maximal integral manifold N through $x = \varinjlim x_n$ is a weak n.n.H. convenient submanifold of M which is a direct limit of the set of maximal leaves N_n of $\rho_n(E_n)$ through x_n in M_n . Moreover, each maximal leaf has the limit chart property at any point and if M is Hausdorff so is each leaf.

Proof. — At first, for each $n \in \mathbb{N}^*$, we can apply the first part of Theorem 7.2. Therefore, with the notation of property (*), if we fix some $x = \varinjlim x_n$, there exists a maximal integral manifold N_n of $\rho_n(E_n)$ through x_n in U_n . Recall that we have $U_n \subset U_{n+1}$ and $E_n \subset E_{n+1}$ over U_n . Therefore, according to Property (2) of Definition 6.9, for any $y \in N_n$, we have

$T_y N_n \subset T_y N_{n+1}$ on $N_n \cap N_{n+1}$. Since N_{n+1} is a maximal integral manifold of $\rho_{n+1}(E_{n+1})$ in U_{n+1} and $U_n \subset U_{n+1}$, if y belongs to N_n , we have a smooth curve in N_n which joins x_n to y and since $E_n \subset E_{n+1}$ over U_n this curve must be contained in N_{n+1} and so N_n must be contained in N_{n+1} . Now, on the one hand, over each point of U_n the kernel of ρ_n is complemented in each fiber and, on the other hand, over N_n the kernel of ρ_n is a subbundle of $E_n|_{N_n}$. The same property is true for $E_{n+1}|_{N_{n+1}}$. But over $N_n \subset N_{n+1}$, we have

$$\rho_n(E_n|_{N_n}) = TN_n \subset (TN_{n+1})|_{N_n} = \rho_{n+1}(E_{n+1}|_{N_n}).$$

Therefore $(\ker \rho_{n+1})|_{N_n} \subset (\ker \rho_n)|_{N_n}$. But, from our assumption, we have the following Whitney decomposition:

$$E_{n+1}|_{N_n} = F_{n+1} \oplus (\ker \rho_{n+1})|_{N_n} \quad \text{and} \quad E_n|_{N_n} = F_n \oplus (\ker \rho_n)|_{N_n}.$$

Therefore

$$(\ker \rho_n)|_{N_n} = (\ker \rho_{n+1})|_{N_n} \oplus F_{n+1} \cap (\ker \rho_n)|_{N_n}$$

Finally we obtain:

$$(TN_{n+1})|_{N_n} = TN_n \oplus \rho_{n+1}(F_{n+1} \cap (\ker \rho_n)|_{N_n}).$$

Now, from property (*) and the second part of Theorem 7.2, we have a linear connection on TN_n . Thus the ascending sequence (N_n) satisfies the assumption of Corollary 4.9, $N = \varinjlim N_n$ has a structure of convenient manifold modelled on an LB-space. Moreover, by construction, we have $TN = \Delta|_N$. This means that Δ is an integral manifold of Δ through x . Moreover, N satisfies the direct limit chart property.

Take any maximal integral manifold L of Δ and choose some $x = \varinjlim x_n$ in L . From our previous construction we have a sequence of Banach integral manifolds (N_n) such that $N = \varinjlim N_n$ is an integral manifold of Δ through x . Therefore N is open in L . Since N has the direct limit chart property, the same is true of L .

Now as the intersection of an open set in M with any leaf L is an open set of L , thus, if M is an Hausdorff topological space, L inherits of this property. \square

From this result we easily obtain:

COROLLARY 7.6. — *Let $(E_n, \pi_n, M_n, \rho_n, [\cdot, \cdot]_n)_{n \in \mathbb{N}^*}$ be a strong ascending sequence of Banach algebroids provided with a Koszul connection on each E_n such that over each point $x_n \in M_n$ the kernel of ρ_n is complemented in the fiber $\pi_n^{-1}(x_n)$ and the range of ρ_n is closed. Then $\Delta = \varinjlim \rho_n(E_n)$ is an integrable distribution on $M = \varinjlim M_n$. Moreover, for any $x = \varinjlim x_n$, the*

maximal leaf through x is a weak *n.n.H.* convenient submanifold of M and there exists a leaf N_n of $\rho_n(E_n)$ in M_n through x_n such that the sequence $(N_n)_{n \in \mathbb{N}^*}$ is an ascending sequence of Banach manifolds whose direct limit $N = \varinjlim N_n$ is an integral manifold of Δ through x such that N has the direct limit chart property at x . Moreover, if M is Hausdorff so is each leaf.

Now according to Remark 7.4 we also easily obtain:

COROLLARY 7.7. — *Let Δ be a distribution on a direct limit $M = \varinjlim M_n$ of finite dimensional (resp. Hilbert) paracompact manifolds. Assume that, for any $x = \varinjlim x_n$, there exists a sequence of finite rank (resp. Hilbert) algebroids $(E_n, \pi_n, U_n, \rho_n)_{n \in \mathbb{N}^*}$ such that $U = \varinjlim U_n$, $\varinjlim \rho_n(E_n) = \Delta|_U$. Then Δ is integrable and the maximal integral manifold N through $x = \varinjlim x_n$ is a weak convenient submanifold of M which is the direct limit of the set of maximal leaves N_n of $\rho_n(E_n)$ through x_n in M_n . Moreover, each maximal leaf has the limit chart property at any point and is a Hausdorff convenient manifold.*

7.3. Application

Consider a direct sequence of Banach Lie groups $G_1 \subset G_2 \subset \dots \subset G_n \subset \dots$ such that the Lie algebra \mathcal{G}_n is complemented in the Lie algebra \mathcal{G}_{n+1} for all $n \in \mathbb{N}^*$.

Note that this situation always occurs if for all $n \in \mathbb{N}^*$, each Lie group G_n is finite dimensional or is a Hilbert Lie group. This assumption is also valid for the sequence $G_n = GL(E_n)$ where E_n is a direct sequence of Banach spaces such that each E_n is closed and complemented in E_{n+1} .

According to Example 6.6, assume that for each $n \in \mathbb{N}^*$, we have a smooth right action $\psi_n : M_n \times G_n \rightarrow M_n$ of G_n over a Banach manifold M_n where $M_1 \subset M_2 \subset \dots \subset M_n \subset \dots$ is an ascending sequence such that M_n is a Banach submanifold of M_{n+1} . We get a strong ascending sequence Lie Banach algebroids $(M_n \times \mathcal{G}_n, \pi_n, M_n, \Psi_n, [\cdot, \cdot]_{\mathcal{G}_n})$. Since each Banach bundle $M_n \times \mathcal{G}_n$ is trivial, we obtain a sequence of Banach Lie algebroids with anchors

$$\begin{aligned} \Psi_n : M_n \times \mathcal{G}_n &\longrightarrow TM_n \\ (x_n, X_n) &\longmapsto T_{(x_n, e_n)} \psi_n(0, X_n) \end{aligned}$$

Because these bundles are trivial, we get a sequence of compatible trivial Koszul connections ∇^n on $M_n \times \mathcal{G}_n$. Now, from the Corollary 7.6, we obtain:

THEOREM 7.8. — *In the previous context, we obtain a smooth right action $\psi = \varinjlim \psi_n$ of $G = \varinjlim G_n$ on the convenient manifold $M = \varinjlim M_n$. Moreover, if the kernel of Ψ_n is complemented in each fiber $\pi_n^{-1}(x_n)$ and the range of Ψ_n is closed, then the orbit $\psi(x, G)$ of this action through $x = \varinjlim x_n$ is a weak n.n.H. convenient submanifold of M which is the direct limit of the set of G_n -orbits $\{\psi_n(x_n, G_n)\}_{n \in \mathbb{N}^*}$. If M is Hausdorff, so is each orbit.*

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