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Positivity, vanishing theorems and rigidity of Codimension one Holomorphic Foliations

O. Calvo-Andrade[(1)]

To José Manuel Aroca in his 60 birthday

Résumé. — Il est connu que l'espace des feuilletages holomorphes de codimension 1 dont les singularités ont un fibré normal donné a la structure d'une variété algébrique. Le but de cet article est de décrire ses composantes irréductibles. Pour ceci, nous nous intéressons au problème de l'existence d'un facteur intégral pour une 1-forme différentielle tordue sur une variété projective. Nous ferons une analyse géométrique du feuilletage de codimension 1 associé à cette forme. Le point essentiel de cet article consiste en la compréhension du rôle joué par une condition de positivité sur un objet associé au feuilletage.

Abstract. — It is a known fact that the space of codimension one holomorphic foliations with singularities with a given 'normal bundle' has a natural structure of an algebraic variety. The aim of this paper is to consider the problem of the description of its irreducible components. To do this, we are interested in the problem of the existence of an integral factor of a twisted integrable differential 1–form defined on a projective manifold. We are going to do a geometrical analysis of the codimension one foliation associated to this form. The essential point of this paper consists in understanding the role played by a positive condition on some object associated to the foliation.

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Contents

0. Introduction

The present paper is essentially a summary of the papers [5], [6], [7], [8], [9], [12] and [10], which mainly study deformations of codimension one singular holomorphic foliations with a meromorphic first integral, represented by logarithmic forms or foliations arising by actions of the affine group, and the geometric properties of the Kupka singular set of a foliation. The main point in this note, is to explain the role played by a positivity condition of some object associated with the foliation, in order to get some stability properties under deformations of the foliation, with these ideas, we are going to give new proofs of some results of the papers cited above.

Let M be a complex manifold, let $E \to M$ be a holomorphic vector bundle over M and let L be a holomorphic line bundle. Recall that the vector space of holomorphic sections of the vector bundle $E \otimes L$, is in one to one correspondence with the space of meromorphic sections of E with a pole along a hypersurface $D \subset M$ which is the zero locus of a holomorphic section of the line bundle L.

A codimension one holomorphic foliation with singularities on a manifold M, may be defined by a holomorphic section ω of $T^*M \otimes L$, where L is a holomorphic line bundle, satisfying the integrability condition $\omega \wedge d\omega = 0$ as a section of $\bigwedge^3 T^*M \otimes L^{\otimes 2}$. Two sections ω and ω_1 define the same foliation if there exists a nowhere vanishing holomorphic function φ on M, such that $\omega = \varphi \cdot \omega_1$, in this case, we say that the sections ω and ω_1 are equivalent. We denote by $\mathcal{F}(M, L)$, the set of equivalence classes of integrable, holomorphic sections of $T^*M \otimes L$, the cotangent bundle of M twisted by a holomorphic line bundle L. If M is a compact complex manifold, it is a well known fact, that $\mathcal{F}(M, L)$ is an algebraic variety, with singularities in general. The first natural problem is the enumeration and description of the irreducible components of the space $\mathcal{F}(M, L)$. We will give a partial solution to this problem when the line bundle L is **positive.**

When trying to determine an irreducible component of $\mathcal{F}(M, L)$, one can proceed in two steps :

1. Describe some irreducible subvariety $\mathcal{A}(\mathbf{Q}, L) \subset \mathcal{F}(M, L)$, where $\mathbf{Q}$ denotes a discrete invariant, which is related to the main discrete invariant of the foliation, $c = c_1(L) \in \mathrm{H}^2(M, \mathbb{Z})$ the Chern class of the line bundle L

2. Study a neighborhood $\mathcal{U} \subset \mathcal{F}(M, L)$ of a generic member $\mathcal{F}$ of the family $\mathcal{A}(\mathbf{Q}, L)$.

In some cases, one can show that all such foliations on the neighborhood $\mathcal{U} \subset \mathcal{F}(M, L)$, also belong to the family $\mathcal{A}(\mathbf{Q}, L)$. In this case, the closure $\overline{\mathcal{A}(\mathbf{Q}, L)}$ will be an irreducible component of the space $\mathcal{F}(M, L)$.

One of the simplest families of foliations are those which are defined by a meromorphic closed 1–form. Theorem (2.5) gives us a normal form of such families of foliations. One of the first problems is

Problem 0.1. — *Under which conditions a foliation $\omega \in \mathcal{F}(M, L)$ may be represented by a meromorphic, closed one form?*

The local, non–singular case is well understood: An integrating factor of a holomorphic 1–form ω is a holomorphic function g, such that

$$\Omega = \frac{\omega}{g} \quad \text{is closed.}$$

If the function g does not vanish, then

$$\omega = g \cdot df \quad \text{for some} \quad f \in \mathcal{O}, \tag{0.1}$$

and in such a case the function f is called a **holomorphic first integral** of the 1–form ω. The leaves of the foliation are the level surfaces of the function $f = c$.

The classical Frobenius theorem states that a germ of a nowhere vanishing 1–form ω has a nowhere vanishing integrating factor if and only if it satisfies the following integrability condition :

$$\omega \wedge d\omega = 0 \Longleftrightarrow \omega = g \cdot df \quad g \in \mathcal{O}^* \quad \text{and} \quad f \in \mathcal{O}. \tag{0.2}$$

B. Malgrange has generalized this result to the singular case [28]. He showed that if the singular set $S_\omega = \{x \mid \omega(x) = 0\}$ of a germ of an integrable holomorphic 1–form ω has codimension at least three, then ω has a nowhere vanishing integrating factor, i. e. $\omega = g \cdot df, \quad g \in \mathcal{O}^*$.

Also in the local case, Cerveau and Mattei in [17] have obtained numerous results relating to this problem, even when the singular set has codimension two, and the integrating factor vanishes on a hypersurface.

The global case has been considered by Poincaré. Let ω be a polynomial 1–form in the complex projective plane $\mathbb{P}^2$:

Under which conditions has ω a rational first integral?

The following result is known: A foliation ω has a rational first integral if and only if the closure of any leaf is an algebraic hypersurface. Therefore, the above question is equivalent to :

Under which conditions the closure of any solution of the differential equation $\omega = 0$ is an algebraic hypersurface?

The Poincaré problem can be divided in two parts:

1. Under which conditions has ω an integrating factor?
2. If ω has an integrating factor, by integration along paths we get a multi valuate function, then the question is when it is a rational function.

We are going to consider the first problem. Our viewpoint will be the following:

Some hypothesis on the foliation and positivity condition $\Longrightarrow$ Existence of an integrating factor.

The notion of positivity as well as many of its consequences has been recently generalized, and we believe that many of the results of this work, can be proved in other cases.

The paper is organized as follows:

We begin with the notion of positivity, and we will give generalizations of three classical theorems: Lefschetz's theorem on hyperplane sections, Hartogs extension theorem and Zariski's theorem on fundamental group of quasi–projective manifolds. Those results will be used later.

In section 2 we define (2.1) **codimension one holomorphic foliations** as an equivalence class of sections ω of the holomorphic vector bundle $T^*M \otimes L$, where L is a holomorphic line bundle. In this section, we will also introduce the notions of **families of foliations** (2.3) and **integrating factors** (2.4).

Section 3 is dedicated to the study of the **Kupka singular set** (definition 3.1), and we introduce the Zariski open set $\mathrm{K}(M, L)$, of those foliations having a single compact, connected component of the Kupka singular set.

In section 4 we prove that holomorphic foliations arising from the fibers of a generic rational map $\phi : M \to \mathbb{P}^1$, describe some irreducible components of $\mathrm{K}(M, L)$. Furthermore, a generic element of this component is structurally stable, and on the other hand, the Kupka component is precisely the set of **base points** of the pencil.

In section (5) we describe some geometric properties of the Kupka set. The main point in this section is the description of the irreducible components of $\mathrm{K}(M, L)$.

In section (6) we describe other irreducible components of $\mathcal{F}(M, L)$: the logarithmic foliations. Finally, in the last section (7) we consider foliations on projective spaces and describe several irreducible components.

1. Positive Vector Bundles

In this section we will discuss the notion of positivity of holomorphic vector bundles in complex manifolds and some of its consequences.

Let M be a compact, complex manifold. Set $\mathcal{O}_M$ or simply $\mathcal{O}$ the sheaf of holomorphic functions on M. Let $E \to M$ be a holomorphic vector bundle. In what follows we are going to denote by $\mathcal{O}(E)$ the sheaf of holomorphic sections of the vector bundle E. It is well known that $\mathcal{O}(E)$ is a locally free sheaf of $\mathcal{O}$–modules and conversely, any locally free sheaf $\mathcal{S}$ of $\mathcal{O}$ modules over M corresponds to a holomorphic vector bundle : i. e. there exists a holomorphic vector bundle $E \to M$ such that $\mathcal{S} = \mathcal{O}(E)$. In the cases $E = TM$ or $E = T^*M$ we are going to denote the sheaf of its holomorphic sections by

Θ_M and Ω^1_M respectively. We will denote by $\Gamma(E)$, the (finite dimensional) complex vector space of holomorphic sections of the bundle $E \to M$. Recall that $\Gamma(E)$ corresponds to the cohomology group $\mathrm{H}^0\big(M, \mathcal{O}(E)\big)$.

Now, we are going to present three classical results in complex algebraic geometry, these theorems will be generalized and we are going to use later.

1. **Lefschetz's Theorem of hyperplane sections:** Let $M \subset \mathbb{P}^m$ be smooth complex submanifold of $\dim(M) = n$, and let X be a generic hyperplane section $X = P^{m-1} \cap M$. Denote by $\jmath : X \hookrightarrow M$ the inclusion map. Then the natural map

$$\mathrm{H}^q(\jmath) : \mathrm{H}^q(M, \mathbb{Q}) \to \mathrm{H}^q(X; \mathbb{Q})$$

   induced by the inclusion, is an isomorphism for $q < n-1$ and injective for $q = n-1$

2. **Hartog's extension theorem:** Let M be a Stein manifold of dimension $m \geqslant 2$ and $K \subset M$ be a compact subset such that $M^m - K$ is connected. Then every holomorphic function

$$f : (M - K) \to \mathbb{C}$$

   can be extended to M.

3. **Zariski's Theorem:** Let $C \subset \mathbb{P}^2$ be a nodal curve, then the fundamental group of $\mathbb{P}^2 - C$ is Abelian.

We will give some generalizations of each of these results. In order to do this we are going to introduce the notion of **positivity** as in [23].

Definition 1.1. — *Let M be a compact complex manifold, and let $E \to M$ be a holomorphic vector bundle. We say that E is* **positive** *if there exists a hermitian metric $\langle \cdot\,, \cdot \rangle$ in E whose* **curvature tensor** $\mathbf{\Theta} = (\Theta^\rho_{\sigma i j})$ *has the property that the hermitian quadratic form*

$$\mathbf{\Theta}(\zeta, \eta) = \sum_{\rho,\sigma,i,j} \Theta^\rho_{\sigma i j} \zeta^\sigma \overline{\zeta}^\rho \eta^i \overline{\eta}^j$$

is positive definite in the variables ζ, η.

Recall that a divisor D on a projective manifold M is called **very ample**, if there exists an embedding in the projective space $\imath : M \hookrightarrow \mathbb{P}^N$ such that $M \cap \mathbb{P}^{N-1} = D$. A holomorphic line bundle $L \to M$ is **very ample** if it is the line bundle associated with a very ample divisor ($L = [D]$); in other words, the holomorphic line bundle L induces an embedding into the projective space given by

$$\begin{aligned} \imath_L : \quad M & \hookrightarrow \mathbb{P}^N \simeq \mathbb{P}_{\Gamma(L)}. \\ x & \longmapsto [\sigma_0(x) : \ldots : \sigma_N(x)], \end{aligned}$$

where $\{\sigma_0, \ldots, \sigma_N\}$ is a basis of the vector space $\Gamma(L) \simeq \mathrm{H}^0\big(M, \mathcal{O}(L)\big)$, in particular, for each point $p \in M$ the fiber of L at p is generated by the sections.

Kodaira embedding Theorem states that if $L \to M$ is a positive line bundle, there exists $n_0 \in \mathbb{N}$, such that for all $n > n_0$, the holomorphic line bundle $L^{\otimes n}$ is very ample, see for instance [24, pag. 181].

Now, we can state in terms of line bundles the Lefschetz's theorem as follows: Let $L \to M$ be a positive line bundle, and let $\sigma \in \mathrm{H}^0\big(M, \mathcal{O}(L)\big)$ be a holomorphic section whose zero set $X \subset M$ is a smooth submanifold of M. Then, the natural map

$$\mathrm{H}^q(M, \mathbb{Q}) \to \mathrm{H}^q(X, \mathbb{Q})$$

induced by the inclusion is an isomorphism for $q \leqslant n-1$ and injective for $q = n-1$.

Let L be a very ample line bundle and let σ be a holomorphic section of L with zero set $\{\sigma = 0\} = X$. Consider the embedding $\imath_L : M \hookrightarrow \mathbb{P}^N$. Then X is a hyperplane section i.e. $X = M \cap \mathbb{P}^{N-1}$. It is a well known fact that a closed analytic subset (in this case $M - X$) of a Stein manifold ($\mathbb{C}^N = \mathbb{P}^N - \mathbb{P}^{N-1}$), is a Stein manifold, in which case an open neighborhood U of X is the complement of a compact subset of a Stein manifold. Furthermore, if we put a Hermitian metric on L compatible with the complex structure, then the function

$$\log\langle \sigma ; \sigma \rangle : M - X \to \mathbb{R},$$

is plurisubharmonic, and this result generalizes as follows:

Let $E \to M$ be a rank r holomorphic vector bundle with a holomorphic section σ. Assume that E is positive respect to the Hermitian metric $\langle \cdot , \cdot \rangle$, and the zero set $X = \{\sigma = 0\}$ is a r–codimensional smooth submanifold of M. Consider the $\mathcal{C}^\infty$ function $\phi(z) = \langle \sigma(z) ; \sigma(z) \rangle$. Then the function

$$\begin{aligned} \psi : M - X & \quad\to \quad \mathbb{R} \\ z & \longmapsto \quad \log \phi(z), \end{aligned}$$

is an **exhaustion function** on $M - X$, that is,

$$M - X = \bigcup_{a \in \mathbb{R}} \psi^{-1}(a, \infty) \quad \text{and} \quad \psi^{-1}(-\infty) = X,$$

moreover, the Levi form

$$L(\psi) = \left(\frac{\partial^2 \psi}{\partial \overline{z}_i \partial z_j} \right)$$

has at least $n - r + 1$ positive eigenvalues, in particular, when the rank of the vector bundle E is 1, this function has n positive eigenvalues, and the function is **plurisubharmonic**, and hence the complement of X is a Stein manifold. [23, pag. 205].

The calculation of the eigenvalues of the Levi form of the function ψ implies, by a Morse theoretic argument, the generalization of Lefschetz's Theorem. On the other hand, since the function $\log \phi(z)$ is an exhaustion function, we have the generalization of Hartog's Theorem, because, when the rank of the vector bundle E is 1, we have seen that $M - X$ is a Stein manifold.

Now, Lefschetz's and Hartogs Theorems may be generalized as follows:

THEOREM 1.2. — *Let $E \to M$ be a rank–k, positive vector bundle, and let σ be a holomorphic section whose zero set $X = \{\sigma = 0\}$ is a smooth, codimension k submanifold. Then*

- *Lefschetz: The natural map induced by the inclusion*

$$\mathrm{H}^q(M, \mathbb{Q}) \to \mathrm{H}^q(X, \mathbb{Q})$$

is an isomorphism for $q < n - k$ and injective for $q = n - k$.

- *Hartogs: Let $\mathcal{G}$ be a locally free analytic sheaf, and let $X \subset U$ be an open, connected neighborhood of X, such that $M - U$ is compact. Then the map*

$$\Gamma(M, \mathcal{G}) \to \Gamma(U, \mathcal{G}),$$

is surjective.

The proof of this theorem appears in [23].

Remark 1.3. — In the extension of the Hartogs theorem above, when the vector bundle E has rank $k > 1$ the manifold $M - X$ is not a Stein manifold.

A generalization of Zariski's theorem is the following result [31, p. 314–315]:

THEOREM 1.4. — *Let V be an algebraic manifold and let $D \subset V$ be a divisor with normal crossings. If any irreducible component C_i of D is positive, then the kernel of the inclusion $\imath_* : \pi_1(V-D,p) \to \pi_1(V,p)$ consists of central elements.*

Theorem (1.4) implies that, if V is a simply connected surface, the fundamental group of the complement of a normal crossing divisor with positive irreducible components is abelian, in particular when $M = \mathbb{P}^2$ and D is a nodal curve we have Zariski's theorem.

2. The Space of Holomorphic Foliations

A codimension one holomorphic foliation $\mathcal{F}$ with singularities on a complex manifold M, may be given by a family (ω_α) of holomorphic non–identically vanishing 1–forms defined in an open cover $\mathfrak{U} := \{U_\alpha\}$ of M, satisfying for any α the integrability condition $\omega_\alpha \wedge d\omega_\alpha = 0$, and the co-cycle condition $\omega_\alpha = \lambda_{\alpha\beta}\omega_\beta$, where $\{\lambda_{\alpha\beta}\}$ are never vanishing holomorphic functions in the open sets $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ satisfying $\lambda_{\alpha\beta} \cdot \lambda_{\beta\gamma} \cdot \lambda_{\gamma\alpha} = 1$ when $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$. Let (ω'_α) be another family of holomorphic 1–forms satisfying the integrability and the co–cycle condition $\omega'_\alpha = \lambda'_{\alpha\beta}\omega'_\beta$, where $\lambda'_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$. The families (ω_α) and (ω'_α) are *equivalent* if there exists a family of holomorphic functions $\{\rho_\alpha\} \in \mathcal{O}^*(U_\alpha)$ such that $\omega_\alpha = \rho_\alpha \cdot \omega'_\alpha$. In this case, it is not difficult to see that the cocycles $\{\lambda_{\alpha\beta}\}$ and $\{\lambda'_{\alpha\beta}\}$ are cohomologous in $\mathrm{H}^1(\mathfrak{U}, \mathcal{O}^*)$. Thus, if L denotes the line bundle represented by the cocycle $\{\lambda_{\alpha\beta}\}$, the families $\{\omega_\alpha\}$ and $\{\omega'_\alpha\}$ define holomorphic sections of the bundle $T^*M \otimes L$. We are going to denote by $\Omega^1(L)$ the sheaf of holomorphic sections of the bundle $T^*M \otimes L$.

DEFINITION 2.1. — *A* ***codimension one holomorphic foliation with singularities*** *in a complex manifold M is an equivalence class of sections*

$$\omega \in \mathrm{H}^0\big(M, \Omega^1(L)\big),$$

where L is a holomorphic line bundle and $\omega \wedge d\omega = 0$.

Given a codimension one foliation $\mathcal{F} = \mathcal{F}_\omega$ represented by an integrable section $\omega \in \mathrm{H}^0\big(M, \Omega^1(L)\big)$, we define the **tangent sheaf of the foliation** as the subsheaf $T_{\mathcal{F}} \subset \Theta_M$ of the germs of holomorphic vector fields that vanishes the section ω, that is

$$T_{\mathcal{F}} := \{\mathbf{X} \in \Theta_M | \imath_{\mathbf{X}}\omega = 0\}, \tag{2.1}$$

The sheaf $T_{\mathcal{F}}$ is a coherent subsheaf of Θ, but in general it is not locally free. Foliations with locally free tangent sheaf are considered in [10], [14], [18] and in chapter 7 in section (7.3). On the other hand, the subsheaf $T_{\mathcal{F}}$ is closed under the Lie bracket of holomorphic vector fields, and we have the exact sequence

$$0 \to T_{\mathcal{F}} \to \Theta \to Q \to 0 \quad Q \simeq \Theta/T_{\mathcal{F}} \tag{2.2}$$

where Q is a rank one coherent and not locally free sheaf.

The set of singularities $S_\omega = \{p \in M | \, \omega(p) = 0\}$ is an analytic subset of M. In what follows, we are going to assume that the singular set of a codimension one holomorphic foliation has codimension greater or equal than two, and it corresponds to the set of points where the quotient sheaf Q is not locally free. In fact, if $\mathcal{J}_{S_\omega}$ denotes the ideal sheaf of the singular set S_ω, we have

$$Q \simeq \mathcal{O}(L) \otimes \mathcal{J}_{S_\omega}. \tag{2.3}$$

Now, in the complement of the singular set $M - S_\omega$, by Frobenius theorem, we get a codimension one holomorphic foliation in the usual sense.

The **leaves** of the foliation with singularities $\mathcal{F}_\omega$ are the leaves of the foliation defined in $M - S_\omega$. If $V \subset M$ is a compact hypersurface invariant by a foliation $\mathcal{F}_\omega$, then $V - V \cap S_\omega$ is a leaf of the codimension one foliation defined in $M - S_\omega$. In general, we have that $V \cap S_\omega \neq \emptyset$. In any case by abuse of language, we will say that V is a **compact leaf** of the foliation $\mathcal{F}_\omega$.

We will denote by $\mathcal{F}(M, L)$ the set of codimension one foliations, represented by an equivalence class $[\omega]$ of an integrable section $\omega \in \mathrm{H}^0\big(M, \Omega^1(L)\big)$. If M is compact and has complex dimension $\geqslant 3$, the set $\mathcal{F}(M, L)$ is an algebraic subvariety of the projective space $\mathbb{P}_{\mathrm{H}^0(M,\Omega^1(L))}$ in general with singularities and with several irreducibles components [22, pag. 133]. It is also defined by quadratic equations, A first natural problem is the following:

Problem 2.2. — *Let M be a compact complex manifold with complex dimension $\geqslant 3$ and let L be a holomorphic line bundle over M. Describe and enumerate the irreducible components of the space $\mathcal{F}(M, L)$.*

This problem remains open even when M is a projective space $\mathbb{P}^n$. The Picard group $Pic(\mathbb{P}^n) \simeq \mathbb{Z}$ and it is generated by the hyperplane bundle $\mathbb{H}$, with associated sheaf denoted by $\mathcal{O}(1)$. In what follows, let us denote by $\mathcal{F}(n, \nu)$ instead $\mathcal{F}(\mathbb{P}^n, \mathbb{H}^\nu)$, then it is known that :

- For $n = 2$ the set of foliations $\mathcal{F}(2, \nu)$ is a Zariski open subset of the projective space associated to the vector space $\mathrm{H}^0\big(\mathbb{P}^2, \Omega^1(\nu)\big)$ [20].

- For $n \geqslant 3$
  - $\mathcal{F}(n,\nu) = \emptyset$ for $\nu < 2$.
  - $\mathcal{F}(n,2)$ is smooth and irreducible.
  - For $\nu > 2$ the set $\mathcal{F}(n,\nu)$ has several irreducible components that are described only for $\nu \leqslant 4$ [16].

One of the objectives of this paper is to consider this problem when the line bundle L is positive. In order to do this, we will need the following notion.

Definition 2.3. — *A* ***holomorphic family*** $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ *of* ***codimension one holomorphic foliations*** *with singularities parameterized by a complex analytic space* $\mathcal{T}$ *consists of :*

- *A holomorphic family of complex manifolds* $\{M_t\}$ *given by a smooth map* $\pi : \mathcal{M} \to \mathcal{T}$ *between complex spaces with* $M_t = \pi^{-1}(t)$*.*
- *A holomorphic foliation* $\widetilde{\mathcal{F}}$ *on* $\mathcal{M}$ *such that its leaves are contained on the t–fibers, and the restriction* $\widetilde{\mathcal{F}}|_{M_t} = \mathcal{F}_t$ *is a codimension one holomorphic foliation (with singularities) on* M_t*.*

Usually, we are going to consider $\mathcal{M} = \mathcal{T}\times M \overset{\pi}{\longrightarrow} \mathcal{T}$, where $0 \in \mathcal{T} \subset \mathbb{C}^N$, is an open set, and $\pi(t,z) = t$.

Let $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ be a family of foliations given as a foliation $\widetilde{\mathcal{F}}$ on $\mathcal{M} \to \mathcal{T}$. The **perturbed holonomy** of a leaf $\mathcal{L}$ of the foliation $\mathcal{F}_0$ is the holonomy of $\mathcal{L}$ as a leaf of foliation $\widetilde{\mathcal{F}}$.

Let Σ be a transversal section to the foliation $\mathcal{F}_0$ at the point $p \in \mathcal{L}$, i. e. $\Sigma_p\mathcal{L}$. Now, let $U \subset \mathcal{T}$ a neighborhood of $0 \in \mathcal{T}$. The set $U \times \Sigma$, is a transversal section to the foliation $\widetilde{\mathcal{F}}$ at the point $(p,0)$. It is not difficult to see, that the perturbed holonomy has the form :

$$H_\delta(t,z) = \big(t, h_\delta(t,z)\big) : U\times\Sigma \longrightarrow U\times\Sigma, \quad \delta \in \pi_1(\mathcal{L},p),$$

where the map $h_\delta(z,0) : (\Sigma,p) \to (\Sigma,p)$, is the holonomy associated to the loop $\delta \in \pi_1(\mathcal{L},p)$, when we consider $\mathcal{L}$ as a leaf of the foliation $\mathcal{F}_0$.

A compact leaf V of the foliation $\mathcal{F}$ is **stable** for the family $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ if for any tubular neighborhood $\pi : \mathcal{V} \to V$ of V, there exists a neighborhood $\mathcal{N}$ of $0 \in \mathcal{T}$ such that for $t \in \mathcal{N}$ there exists a compact leaf $V_t \subset \mathcal{V}$ of $\mathcal{F}_t$, such that the map $\pi|_{V_t} : V_t \to V$ is a diffeomorphism.

A necessary condition for the stability of a compact leaf V, is the existence of a fixed point for the perturbed holonomy, that is, let $V = \overline{\mathcal{L}}$, then $H_\delta(t,y) = \big(t, h_\delta(t,y)\big)$, and we must find an analytic function $t \mapsto y_t$ satisfying:

$$y_0 = 0 \quad h_\delta(t,y_t) = y_t \quad \forall\, t \quad \text{and} \quad \forall \delta \in \pi_1(\mathcal{L}).$$

The closure of leaf $\mathcal{L}_{y_t}$ of the foliation $\mathcal{F}_t$ through the point y_t, is diffeomorphic to V.

The other important notion that we need is the following :

Definition 2.4. — *A* ***holomorphic integrating factor*** *of a foliation $\mathcal{F}_\omega \in \mathcal{F}(M,L)$ represented by a section $\omega \in \mathrm{H}^0(M,\Omega^1(L))$ is a holomorphic section $\varphi \in \mathrm{H}^0\big(M,\mathcal{O}(L)\big)$ such that the meromorphic 1–form*

$$\Omega = \frac{\omega}{\varphi}$$

is closed.

Let $\omega \in \mathcal{F}(M,L)$ and $\varphi \in \mathrm{H}^0(M,\mathcal{O}(L))$ be a holomorphic integrating factor of the foliation represented by ω. Assume that the decomposition of the divisor

$$D = \{\varphi = 0\} = \sum_{i=1}^{k} r_i \cdot D_i, \quad D_i \in Div(M) \quad r_i \in \mathbb{N}.$$

Now, for all $i = 1,\dots,k$, let $L_i = [D_i]$ be the holomorphic line bundle associated with the divisor D_i, and let $\varphi_i \in \mathrm{H}^0\big(M,\mathcal{O}(L_i)\big)$ be a holomorphic section of the line bundle L_i such that $\{\varphi_i = 0\} = D_i$. With this notation, the line bundle $L = L_1^{r_1} \otimes \cdots \otimes L_k^{r_k}$, and the integrating factor φ may be written as $\varphi = c \cdot \varphi_1^{r_1} \cdots \varphi_k^{r_k}$, for some $c \in \mathbb{C}^*$. In [8] may be found the following result [8].

Theorem 2.5. — *Let M be a projective manifold with $\mathrm{H}^1(M,\mathbb{C}) = 0$, and let $\varphi = \varphi_1^{r_1} \cdots \varphi_k^{r_k} \quad r_i \in \mathbb{N}$ be an integrating factor of $\omega \in \mathcal{F}(M,L)$, where φ_i are as above, then:*

$$\Omega = \sum_{i=1}^{k} \lambda_i \frac{d\varphi_i}{\varphi_i} + d\left(\frac{\psi}{\varphi_1^{r_1-1}\cdots\varphi_k^{r_k-1}}\right),$$

where $\lambda_i \in \mathbb{C}$ and ψ is a holomorphic section of the line bundle

$$\left[\sum_{i=1}^{k}(r_i - 1)\{\varphi_i = 0\}\right],$$

therefore

$$\omega = \varphi_1^{r_1} \cdots \varphi_k^{r_k} \left[\sum_{i=1}^{k} \lambda_i \frac{d\varphi_i}{\varphi_i} + d\left(\frac{\psi}{\varphi_1^{r_1 - 1} \cdots \varphi_k^{r_k - 1}} \right) \right].$$

The residue Theorem implies the relation:

$$\sum_{i=1}^{k} \lambda_i \cdot [\{\varphi_i = 0\}] = \sum_{i=1}^{k} \lambda_i \cdot c_1(L_i) = 0 \in \mathrm{H}^2(M, \mathbb{C}),$$

where $[X]$ denotes the fundamental class of the submanifold $X \subset M$, and $c_1(L) \in \mathrm{H}^2(M, \mathbb{Z})$ denotes the first Chern class of a line bundle L.

We say that an integrating factor is **reduced** when for all $i = 1, \ldots, k$, the numbers $r_i = 1$, and in this case, Theorem 2.5 shows that

$$\omega = \varphi_1 \cdots \varphi_k \left(\sum_{i=1}^{k} \lambda_i \frac{d\varphi_i}{\varphi_i} \right),$$

and the foliation is represented by a closed logarithmic 1–form, in this case, the foliation is called **logarithmic**.

Another important case for us is when the integrating factor has the form $\varphi = \varphi_2^{p+1} \quad p \in \mathbb{N}$, and $\{\varphi_2 = 0\}$ is an irreducible positive divisor. In this case, the foliation is represented by a section with the expression

$$\omega = \varphi_2^{p+1} \cdot d\left(\frac{\varphi_1}{\varphi_2^{p}} \right) = \varphi_2 d\varphi_1 - p\varphi_1 d\varphi_2,$$

and the foliation has the meromorphic first integral $\varphi = \varphi_1/\varphi_2^p : M \longrightarrow \mathbb{P}^1$.

The existence of two linearly independent integrating factors φ_1 and φ_2, implies that the foliation has a **meromorphic first integral**, namely, the leaves of the foliation are the level hypersurfaces of the rational map

$$\frac{\varphi_1}{\varphi_2} : M - \{\varphi_1 = 0\} \cap \{\varphi_2 = 0\} \to \mathbb{P}^1.$$

In fact an easy calculation shows that if ω is a section which defines the foliation, then

$$\omega \wedge (\varphi_1\, d\varphi_2 - \varphi_2\, d\varphi_1) = 0.$$

Let $\varphi \in \mathrm{H}^0\big(M, \mathcal{O}(L)\big)$ be an integrating factor of the foliation represented by $\omega \in \mathcal{F}(M, L)$. Suppose that in an open covering $\{U_\alpha\}$ of M

such that the sections ω and φ are locally defined by $\omega_\alpha \in \Omega^1(U_\alpha)$ and $\varphi_\alpha \in \mathcal{O}(U_\alpha)$, then

$$d\omega_\alpha = \theta_\alpha \wedge \omega_\alpha \quad \text{where} \quad \theta_\alpha = \frac{d\varphi_\alpha}{\varphi_\alpha}.$$

The meromorphic 1–forms $\{\theta_\alpha\}$ are closed and have a simple pole along an invariant divisor of the foliation. Moreover, they satisfy the relation

$$\theta_\beta - \theta_\alpha = \frac{d\lambda_{\alpha\beta}}{\lambda_{\alpha\beta}}, \quad \text{in} \quad U_\alpha \cap U_\beta$$

hence, they define a flat connection of the line bundle $L|_{M-\{\varphi=0\}}$.

Also observe that, if a foliation ω has a holomorphic integrating factor $\varphi = \varphi_1^{r_1} \cdots \varphi_k^{r_k}$, the hypersurfaces $\{\varphi_i = 0\}_{i=1}^k$ are compact leaves of the foliation, and any other leaf has trivial holonomy, furthermore, depending on the residues, and the periods of the closed holomorphic 1–form η, there will be no more compact leaves.

Finally, we suppose that when $\mathrm{H}^1(M, \mathbb{C}) \neq 0$, under suitable positivity conditions on the divisors $\{\varphi_i = 0\}$, the meromorphic 1–form Ω has the expression:

$$\Omega = \sum_{i=1}^k \lambda_i \frac{d\varphi_i}{\varphi_i} + d\left(\frac{\psi}{\varphi_1^{r_1-1} \cdots \varphi_k^{r_k-1}}\right) + \eta,$$

where η is a closed, holomorphic 1–form on M.

3. Kupka Singularities

In this section, we are going to consider the Kupka set of a foliation. It is a class of singularities of codimension one holomorphic foliations, which have stability properties under deformations, therefore, they appear on an open subset of $\mathcal{F}(M, L)$.

Definition 3.1. — *Let $\omega \in \mathcal{F}(M, L)$. The* ***Kupka singular set*** *of the foliation ω consists of the points:*

$$K_\omega = \{p \in M \,|\, \omega(p) = 0;\, d\omega(p) \neq 0\,\}.$$

As the reader may check, the Kupka set is well defined, it does not depend on the section ω which represents the foliation. The main properties of the Kupka singular set are summarized in the following result, the proof of which may be found in [29].

Theorem 3.2. — *Let M be a complex manifold of dimension $n \geqslant 3$, and let ω be an integrable 1–form on M. The Kupka set K_ω is a codimension two smooth submanifold of M, and moreover :*

1. *The Kupka set a the* ***local product structure****: for every connected component $K \subset K_\omega$, there exists a holomorphic 1–form*

$$\eta = A(x,y)\,dx + B(x,y)\,dy$$

called the ***transversal type at*** *K, defined on a neighborhood V of $0 \in \mathbb{C}^2$ and vanishing only at 0, an open cover $\{U_\alpha\}$ of a neighborhood of K in M and a family of submersions $\varphi_\alpha =: U_\alpha \to \mathbb{C}^2$, such that:*

$$\varphi_\alpha^{-1}(0) = K \cap U_\alpha, \quad \text{and :} \quad \omega_\alpha = \varphi_\alpha^* \eta$$

where ω_α defines $\mathcal{F}$ in U_α.

2. *K_ω is persistent under variation of ω; namely, for $p \in K_\omega$ with transversal type η as above, and for any foliation ω' sufficiently close to ω, there is a holomorphic 1-form η', close to η and defined on a neighborhood of $0 \in \mathbb{C}^2$ and a submersion φ' close to φ, such that ω' is defined by $(\varphi')^*\eta'$ on a neighborhood of p.*

We are going to do the proof of the first part of the Theorem (3.2) in the three dimensional case.

Since the problem is local, we are going to assume that $M \subset \mathbb{C}^3$ is an open neighborhood of $p = 0 \in \mathbb{C}^3$. Let $\mathbf{Z}$ be a holomorphic vector field defined by the equation

$$\imath_{\mathbf{Z}} dz_1 \wedge dz_2 \wedge dz_3 = d\omega. \tag{3.4}$$

The formula for the Lie derivative $L_{\mathbf{X}}\omega = \imath_{\mathbf{X}} d\omega + d(\imath_{\mathbf{X}}\omega) = 0$ and the integrability condition on $\omega \wedge d\omega = 0$ implies the equations

$$\imath_{\mathbf{Z}}\omega = 0 \quad \text{and} \quad L_{\mathbf{Z}}\omega = 0 \tag{3.5}$$

Now, since $\mathbf{Z}(0) \neq 0$, there exits a holomorphic change of coordinates $\psi(t,x,y) = (z_1,z_2,z_3)$ defined on a neighborhood of $0 \in \mathbb{C}^3$ such that

$$\psi_*\left(\frac{\partial}{\partial t}\right) = \mathbf{Z}\big(\psi(t,x,y)\big) \tag{3.6}$$

and then, the equations (3.5) and (3.6) implies that

$$\psi^*\omega(t,x,y) = A(x,y)dx + B(x,y)dy = \eta.$$

Now $\varphi(t,x,y)=(x,y)$ is the desired submersion, the Kupka set is the set of the points $(t,0,0)$ and corresponds to the orbit through $p=0$ of the local one parameter group defined by the vector field $\mathbf{Z}$. □

The germ at $0\in\mathbb{C}^2$ of η is well defined up to biholomorphism and multiplication by non–vanishing holomorphic functions. We will call it the **transversal type** of the foliation at K. Let $\mathbf{X}$ be the dual vector field of η. Since $d\omega\neq 0$, we have that $\mathrm{Div}\mathbf{X}(0)\neq 0$, thus the linear part $\mathbf{D}=D\mathbf{X}(0)$, which is well defined up to linear conjugation and multiplication by scalars, has at least one non–zero eigenvalue. We will say that $\mathbf{D}$ is the **linear type** of K. After normalizing, we may assume that the eigenvalues are 1 and μ. We will distinguish three possible types of Kupka type singularities:

- Saddle–node : $\mu=0$. In this case, the transversal type has the normal form :
$$\eta(x,y)=\big(x(1+\lambda y^p)+yR(x,y)\big)\,dy-y^{p+1}\,dx,$$
where $\lambda\in\mathbb{C}\quad p\in\mathbb{N}\quad\text{and}\quad R\in\mathcal{O}$.

- Semi-simple: if $\mu\neq 0$ and $\mathbf{D}$ is semi-simple.

- Non–semisimple : when $\mu=1$ and the matrix $\mathbf{D}$ is not semisimple. In this case, the transversal type is in a suitable coordinate system equal to
$$\eta(x,y)=(x+y)\,dx-x\,dy.$$

Given a point p of the Kupka set, as a consequence of the local product structure, we have the following result concerning with the tangent sheaf of the foliation at p

Theorem 3.3. — *Let $\omega\in\mathcal{F}(M,L)$ and let $p\in K_\omega$. Then the tangent sheaf is locally free at p.*

Proof. — In fact, let be $p\in K_\omega$, and let
$$\varphi:U\to\mathbb{C}^2\quad\text{and}\quad\eta=a(x,y)\,dx+b(x,y)\,dy,$$
be as in theorem (3.2). The tangent sheaf of the foliation, on a coordinate neighborhood U of p is generated by
$$T_{\mathcal{F}_\omega}(U)=\left\{\mathbf{X}(x,y)=b(x,y)\frac{\partial}{\partial x}-a(x,y)\frac{\partial}{\partial y},\frac{\partial}{\partial z_3},\dots,\frac{\partial}{\partial z_n}\right\}.\qquad\square$$

Now, consider a compact connected component of the Kupka set. The topological properties of the embedding $\jmath : K \hookrightarrow M$, which can be measured in terms of the Chern classes of the normal bundle $\nu_K(M)$, and the linear transversal type are strongly related.

Let K be a compact connected component of the Kupka singular set of the foliation represented by the section ω. If the linear type of the component K has eigenvalues 1 and $\mu \neq 1$, then it is shown in [21] that the normal bundle $\nu_K(M)$ splits in a direct sum of holomorphic line bundles L_1 and L_μ, corresponding to the eigenvectors of the linear transversal type. Moreover, the Chern classes of these line bundles satisfy the relation

$$\mu \cdot c_1(L_1) - c_1(L_\mu) = 0 \in \mathrm{H}^2(K, \mathbb{C}).$$

This relation implies that, if the first Chern class of the normal bundle does not vanish, then the eigenvalue μ must be a rational number.

When $\mu = 1$, there exists the following exact sequence of holomorphic line bundles

$$0 \to L_1 \to \nu_K \to L_1 \to 0,$$

which are classified by the group $\mathrm{H}^1\big(K, \mathcal{O}(L_1 \otimes L_1^{-1})\big) \simeq \mathrm{H}^1(K, \mathcal{O})$.

Recall that a linear vector field on $\mathbf{D}$ with eigenvalues 1 and μ belongs to the **Poincaré Domain** if μ is not a negative real number. In this case, the Poincaré–Dulac's theorem, shows that a holomorphic vector field $\mathbf{X}$ with linear part $\mathbf{D}$, is analytically linearizable when $\mu \notin \{2, 3, \ldots, 1/2, 1/3, \ldots\}$, and in case $\mu \in \mathbb{N}$, the holomorphic normal form is

$$\mathbf{X}(x, y) = (x + ay^\mu)\frac{\partial}{\partial x} + \mu \cdot y\frac{\partial}{\partial y} \quad a \in \mathbb{C}.$$

When the Linear type of a compact connected component of the Kupka set belongs to the Poincaré Domain, it is shown in [21] the following result.

Theorem 3.4. — *Let K be a compact connected component of K_ω such that the first Chern class of the normal bundle of K in M is non–zero and has linear type in the Poincaré domain. Then the transversal type is linearizable and semisimple with eigenvalues 1, $\mu \in \mathbb{Q}$. Moreover, for any deformation, the transversal type is constant through the deformation.*

We say that a foliation ω has a **Kupka component** when K_ω contains a compact, connected component K. We denote by $\mathrm{K}(M, L) \subset \mathcal{F}(M, L)$, the Zariski open set of those foliations having a single Kupka component, and by $\mathrm{K}(M,L,\eta)\subset \mathrm{K}(M,L)$, the set of foliations whose Kupka component has transversal type η.

Problem 3.5. — *Determine the decomposition into irreducible components of the sets* $\overline{\mathrm{K}(M,L)}$ *and* $\overline{\mathrm{K}(M,L,\eta)}$.

The next two sections are devoted to give some partial solutions on the above problem.

4. Rational Maps

Let $L \longrightarrow M$ be a holomorphic line bundle on the complex manifold M. Given a $(k+1)$–dimensional subspace $V\subset \Gamma(L)$ generated by holomorphic sections $\{\varphi_i\}_{i=0}^k$, we may define a meromorphic map

$$\begin{array}{rccl} \jmath_V: & M & \longrightarrow & \mathbb{P}^k \simeq \mathbb{P}(V). \\ & x & \longmapsto & [\varphi_0(x):\ldots:\varphi_k(x)], \end{array}$$

observe that the map is not well defined on the set of the **base points** of $\jmath_V$, the set defined by

$$\{\varphi_1=0\}\cap\cdots\cap\{\varphi_k=0\}=\{x\in M|\varphi(x)=0 \quad \text{for all} \quad \varphi\in V\}.$$

In particular, when the subspace V has dimension two, we obtain a meromorphic map to $\mathbb{P}^1$ given by $x\mapsto \varphi(x):=[\varphi_1(x):\varphi_2(x)]$.

Given a map $\varphi=[\varphi_1:\varphi_2]$, we are able to define the holomorphic section

$$\omega=\varphi_1\,d\varphi_2-\varphi_2\,d\varphi_1\in \mathrm{H}^0\big(M,\Omega^1(L^{\otimes 2})\big),$$

which is integrable, but in general, it may not define a codimension one foliation, this is because its singular locus could be of codimension one. This case holds for example, if one of the sections $\{\varphi_i\}_{i=1,2}$ is a power. To solve this problem, we proceed as follows.

Let L_1 and L_2 be holomorphic line bundles on M such that $L_1^p=L_2^q$, where p and q are relatively prime positive integers. Given φ_1 and φ_2 holomorphic sections of the line bundles L_1 and L_2 respectively, the holomorphic section:

$$\omega=p\varphi_1\,d\varphi_2-q\varphi_2\,d\varphi_1\in\Gamma(M,T^*M\otimes L_1\otimes L_2),$$

has the integrating factor $\varphi_1\cdot\varphi_2$, and thus, it is integrable. Moreover the leaves of the foliation represented by ω, are the fibers of the meromorphic map $\varphi=\varphi_1^p/\varphi_2^q$. In what follows, we will say that the map φ is a **meromorphic first integral** of the foliation represented by ω.

Through this section, we will assume the following generic conditions:

1. The holomorphic line bundles L_1 and L_2 are positive.

2. The hypersurfaces $\{\varphi_1 = 0\}$ and $\{\varphi_2 = 0\}$ are irreducible, reduced, smooth, and meet transversely along a codimension two submanifold K.

3. The subvarieties defined by $\lambda\varphi_1^p - \mu\varphi_2^q = 0$ with $[\lambda : \mu] \in \mathbb{P}^1$, are smooth on $M - K$, except for a finite set of points $\{[\lambda_i : \mu_i]\}_{\{i=1,\dots,k\}}$, where they have just a non–degenerate critical point.

A meromorphic map satisfying conditions (1) (2) and (3) is called a **Lefschetz Pencil**, if $p = q = 1$, and a **branched Lefschetz pencil** otherwise.

Observe that the second generic condition, implies that the Kupka set of the foliation is precisely the set where the rational map φ is not defined, i. e., the Kupka set consists of **base points** of the map φ. Therefore, the Kupka set is the intersection of two positive divisors

$$K_\omega = \{\varphi_1 = 0\} \cap \{\varphi_2 = 0\},$$

and the foliation $\omega \in \mathrm{K}(M, L)$.

Also observe that the normal bundle of the Kupka set has non–vanishing first Chern class, since

$$\nu_K = [\{\varphi_1 = 0\}]|_K \oplus [\{\varphi_2 = 0\}]|_K,$$

and by hypothesis both divisors are positive.

The transversal type is given by the linear 1–form $\eta = px\,dy - qy\,dx$, that belongs to the Poincaré domain. Theorem (3.4) implies that the transversal η type is fixed under deformations of the foliation.

Now, Lefschetz's theorem implies that the first cohomology group with a complex coefficients of a generic leaf of the foliation represented by ω, is isomorphic to the first cohomology group of the ambient space M. On the other hand, the topological behavior of the fibers of a generic rational map looks like a **Seifert fibration**. Thus, if $\mathrm{H}^1(M, \mathbb{C}) = 0$, the generic leaf has a vanishing first cohomology group and the foliation may be stable, just as in the case of foliations without singularities [27], [19].

Let $\omega = p\varphi_2\,d\varphi_1 - q\varphi_1\,d\varphi_2$ be a section satisfying only the first and second generic conditions. The leaves $\{\varphi_i = 0\}_{i=1,2}$ are the only ones without (a priori) trivial holonomy, because $\varphi_1 \cdot \varphi_2$ is an integrating factor. Let γ_K^i be the generator of the kernel of $\imath_* : \pi_1(\{\varphi_i = 0\} - K) \to \pi_1(\{\varphi_i = 0\})$, where $K = \{\varphi_1 = 0\} \cap \{\varphi_2 = 0\}$, Now, by Theorem (1.4), this element is central in $\pi_1(\{\varphi_i = 0\} - K; *)$. Moreover, the holonomy map is given by:

$$\begin{aligned} h_{\gamma_K^1}(y) &= \exp\left(2\pi i \frac{q}{p}\right) \cdot y \\ h_{\gamma_K^2}(y) &= \exp\left(2\pi i \frac{p}{q}\right) \cdot y. \end{aligned}$$

Thus, if $p/q \notin \mathbb{N}$ or $q/p \notin \mathbb{N}$, at least one compact leaf has finite and non–trivial (linear) holonomy, and since γ_K^i is central in $\pi_1(\{\varphi_i = 0\} - K; *)$, the leaves without trivial linear holonomy are stable under deformations, that means, *the branching locus of the foliation* must be stable under deformations of the foliation. This fact is proved in [7] and [8].

The following results describe some irreducible components of $\mathrm{K}(M, L)$, when the line bundle L is positive.

THEOREM 4.1. — *Let M be a projective manifold of complex dimension $\geqslant 3$ and $\mathrm{H}^1(M, \mathbb{C}) = 0$. Let $\mathcal{F}$ be a foliation arising from the fibers of a Lefschetz or a Branched Lefschetz Pencil. Then any deformation of a foliation $\mathcal{F}$, has a meromorphic first integral.*

Proof. — (Idea) The Kupka set of the foliation $\mathcal{F}$, are the base points of the map. Moreover, the transversal type is fixed under the deformations [21]. Now, we have two cases.

In the branched case, we first prove that the branching locus is stable, the main steps of the proof are the following (see [8] for details).

1. One of the leaves $\{\varphi_i = 0\} \quad i = 1, 2$ is stable.
2. The existence of such a compact leaf implies the existence of an integrating factor, which can be calculated explicitly.

The first point comes from the discusion above and the conclusion follows now from theorem (2.5).

To prove the unbranched case, we proceed as follows [21]: The transversal type of the Kupka set is $x\,dy - y\,dx$, and it remains constant under deformations. Now, let $\{\mathcal{F}_t\}$, be a family of foliations, and the foliation $\mathcal{F}_0 = \mathcal{F}$, arise from the fibers of a Lefschetz Pencil.

It is possible to obtain a desingularization of the family, after a blowing–up along the Kupka set, and then we obtain a family of foliations which are fibrations over a Zariski open subset of the projective line $\mathbb{P}^1$ such that its fibers have vanishing $\mathrm{H}^1(M,\mathbb{C})=0$. As a consequence, the strict transformed of the foliations $\mathcal{F}_t$ are fibrations. After a blowing–down, we see that the foliations $\mathcal{F}_t$ have a meromorphic first integral. □

Another proof on the unbranched case using infinitesimal methods, may be found in [29].

Corollary 4.2. — *Let M be a projective manifold of complex dimension $\geqslant 3$ and $\mathrm{H}^1(M,\mathbb{C})=0$. Then, there exist irreducible components of $\mathcal{F}(M,L)$ such that a generic element has a meromorphic first integral and its generic foliation is structurally stable.*

These irreducible components, may be parameterized as follows. For the branched case, consider the map:

$$\begin{array}{rcl}\Phi:\mathbb{P}_{\Gamma(L_1)}\times\mathbb{P}_{\Gamma(L_2)} & \to & \mathcal{F}(M,L),\quad L=L_1\otimes L_2\\ (\,[\varphi_1],[\varphi_2]\,) & \mapsto & p\varphi_1\,d\varphi_2-q\varphi_2\,d\varphi_1,\end{array}$$

where $\Gamma(L_i)$ denotes the vector space of holomorphic sections of the line bundle L_i.

For the unbranched case, let $L=L_1\otimes L_1$. Now, we consider $\mathbb{G}(2,\Gamma(L_1))$ the Grassmanian manifold of two planes in the vector space $\Gamma(L_1)$, and let $\langle\varphi_1,\varphi_2\rangle$ be the two plane generated by the sections φ_1 and φ_2. The map :

$$\begin{array}{rcl}\mathbb{G}(2,\Gamma(L_1)) & \longrightarrow & \mathcal{F}(M,L)\\ \langle\varphi_1,\varphi_2\rangle & \longmapsto & \varphi_1\,d\varphi_2-\varphi_2\,d\varphi_1,\end{array}$$

gives a parameterization of an irreducible component. Moreover, the generic element is a Lefschetz Pencil [21].

Remark 4.3. — Theorem (4.1) and its Corollary (4.2) are not true without the hypothesis $\mathrm{H}^1(M,\mathbb{C})=0$.

In fact, if $\mathrm{H}^1(M,\mathbb{C})\neq 0$, we can find a non–zero holomorphic closed 1–form θ on M and consider the one parameter family of foliations

$$\omega_t:=p\varphi_1\,d\varphi_2-q\varphi_2\,d\varphi_1+t\cdot\varphi_1\varphi_2\cdot\theta\in\mathrm{H}^0\big(M,\Omega^1(L_1\otimes L_2)\big),\quad t\in\mathbb{C}.$$

The foliation represented by ω_t has an integrating factor $\varphi_1\cdot\varphi_2$, but it has a meromorphic first integral if and only if $t=0$.

When we study the deformations of an unbranched rational function, we are able to make the following construction involving the fundamental group of the manifold M, in order to get deformations of the holonomy of the foliation.

Let L_1 be a very ample line bundle. Consider the plane generated in the vector space $\mathrm{H}^0\big(M, \mathcal{O}(L_1))\big)$ by two linearly independent sections $\langle \varphi_1, \varphi_2 \rangle$. Such sections define a meromorphic map $\varphi := \varphi_1/\varphi_2$ that we assume to be a Lefschetz Pencil. Observe that $\sigma = (\varphi_1, \varphi_2)$ defines a holomorphic section of the rank two vector bundle $E = L_1 \oplus L_1$, and its zero locus is the Kupka set of the foliation $\omega = \varphi_1\, d\varphi_2 - \varphi_2\, d\varphi_1$.

Let $\mathrm{Hom}\big(\pi_1(M), \mathbf{PSL}_2(\mathbb{C})\big)$ be the analytic set of representations of the fundamental group $\pi_1(M,*)$ of M to $\mathbf{PSL}_2(\mathbb{C})$. We are going to denote by $\mathbf{e} : \pi_1(M) \to \mathbf{PSL}_2(\mathbb{C})$ the trivial representation $\mathbf{e}(\gamma) = id \in \mathbf{PSL}_2(\mathbb{C})$ for each $\gamma \in \pi_1(M,*)$. Finally we consider the holomorphic line bundle $L = L_1 \otimes L_1 = det(E)$.

Now, let $\mathbf{p} : \widetilde{M} \to M$ be the universal covering space of M. For any representation $\rho \in U \subset \mathrm{Hom}\big(\pi_1(M), \mathbf{PSL}_2(\mathbb{C})\big)$, where U is a neighborhood of the trivial representation, consider $\mathbb{P}^1 \subset P_\rho \to M$, the holomorphic $\mathbb{P}^1$ flat bundle over M defined by the representation $\rho \in U$, that is, the quotient space

$$\mathbb{P}^1 \times \widetilde{M} \to \mathbb{P}^1 \times \widetilde{M}/\simeq \qquad (\mathbf{z}, m) \simeq \big(\rho(\gamma)[\mathbf{z}], \gamma^{-1} \cdot m\big) \tag{4.1}$$

These flat bundles are a deformation of the trivial $\mathbb{P}^1$ bundle over M, and it is the projectivization of the holomorphic vector bundle $E = L_1 \oplus L_1$.

In this way, we have constructed a family of $\mathbb{P}^1$ flat bundles parameterized by the neighborhood $U \subset \mathrm{Hom}\big(\pi_1(M), \mathbf{PSL}_2(\mathbb{C})\big)$. This family is a deformation of the bundle $\mathbb{P}(L_1 \oplus L_1) \simeq \mathbb{P}^1 \times M$, that is, we have constructed a $\mathbb{P}^1$ bundle $\mathcal{P} \to U \times M$ with $\mathcal{P}|_{\{\mathbf{e}\}\times M} = \mathbb{P}(L_1 \oplus L_1)$. Then there exists a family of holomorphic vector bundles $\mathcal{E} \to U \times M$ such that $\mathbb{P}(\mathcal{E}) = \mathcal{P}$. Moreover, we are able to choose $\mathcal{E}$ such that $det(\mathcal{E})$ defines a trivial deformation of the bundle $det(E) = L$. With this additional assumption, the holomorphic vector bundle $\mathcal{E}$ is uniquely determined (see [4, page 121]).

Now, consider the family of holomorphic vector bundles constructed above, and satisfying the condition of $\det(\mathcal{E}) = L_1 \otimes L_1 = L$. Since by hypotheses, the holomorphic line bundle L_1 is very ample, Kodaira vanishing theorem implies that, $\mathrm{H}^i\big(M, \mathcal{O}(L_1)\big) = 0$ for all $i > 0$, this implies that the same holds for the vector bundle $E_{\mathbf{e}} = L_1 \oplus L_1$, in particular $\mathrm{H}^1\big(M, \mathcal{O}(E_{\mathbf{e}})\big) = 0$, then, for the analytic family of holomorphic vector bundles E_ρ parameterized by an analytic set $U \subset \mathrm{Hom}\big(\pi_1(M), \mathbf{PSL}_2(\mathbb{C})\big)$, any holomorphic section σ of E_0 may be extended to a holomorphic section σ_ρ of E_ρ, that is, there exists a holomorphic section $\widetilde{\sigma}$ of the vector bundle $\mathcal{E}$ such that $\widetilde{\sigma}(\mathbf{e},\cdot) = (\varphi_1, \varphi_2)$.

Consider now the section $\sigma_\rho = \widetilde{\sigma}(\rho,\cdot)$ of the vector bundle E_ρ for $\rho \in U$, and consider it as a meromorphic section of its projectivization the $\mathbb{P}^1$ bundle P_ρ. The section $[\sigma_\rho]$ has its base points along a codimension two smooth submanifold $K_\rho = \{\sigma_\rho = 0\}$.

Finally, let $\mathcal{H}_\rho$ be the horizontal foliation on the $\mathbb{P}^1$ bundle P_ρ, then we get a codimension one foliation on M by $\mathcal{F}_\rho := [\sigma_\rho]^*(\mathcal{H}_\rho)$, this foliation has a Kupka component of radial transversal type along the submanifold K_ρ with radial transversal type, and moreover, each foliation belongs to the set $K(M, L, x\,dy - y\,dx)$.

Observe that, if we consider the trivial representation $\mathbf{e}$, the foliation $\sigma^*_{\varphi,\mathbf{e}}(\mathcal{H}_\mathbf{e})$ is the foliation defined by the fibers of the map φ. The important point is that this construction gives all deformations of a Lefschetz Pencil as a foliation, and any representation close to the trivial representation $\mathbf{e}$ may be realized as the deformation of the holonomy of a Lefschetz Pencil. In other words, we have shown the following result :

Theorem 4.4. — *Let L_1 be a very ample holomorphic line bundle over M with $h^i\big(M, \mathcal{O}(L_1)\big) = 0$ for all $i > 0$ and let $\mathcal{F} \in \mathcal{F}(M, L)$ $\quad L = L_1 \otimes L_1$ be a holomorphic foliation arising by the fibers of a Lefshetz Pencil then any representation $\rho : \pi_1(M, *) \to \mathbf{PSL}_2(\mathbb{C})$ sufficiently close to the identity may be realized as the deformation of the holonomy of the foliation $\mathcal{F}$.*

We observe, that the family of projective flat bundles with the meromorphic section $(\mathcal{P}, [\sigma_\rho])$ described in the proof of the theorem (4.4) above, defines the developing map of the family of transversely projective foliations.

Question 4.5. — *Assume that* $\mathrm{H}^1(M, \mathbb{C}) = 0$. *Does any irreducible component of the algebraic set $\overline{\mathrm{K}(M, L)}$ for L positive, consists of foliations with a meromorphic first integral?.*

This question will be considered in the next section.

5. Geometric Properties of the Kupka Set

In this section, we discuss some ideas concerning the irreducible components of the set $\overline{\mathrm{K}(M, L)}$. We will assume that the line bundle L is positive or very ample and M is embedded in the projective space $\mathbb{P}_{\Gamma(L)}$.

Let $\omega \in \mathrm{K}(M, L)$ with compact and connected Kupka set $K_\omega = K$. The Kupka set K has the following important geometric property: It is **sub-canonically embedded**, i.e, the canonical bundle of a Kupka component K can be extended to a globally defined holomorphic line bundle [12].

In fact, an easy calculation involving the local product structure (Theorem (3.2) part (1)), and the exact sequence

$$0 \to TK \to TM|_K \to \nu_K(M) \to 0, \tag{5.1}$$

gives the adjunction formula for the Kupka singular set [12]

$$\Omega_K^{n-2} = \Omega_M^n \otimes L|_K, \quad \text{and} \quad \wedge^2 \nu_K(M) = L|_K \tag{5.2}$$

where Ω_K^{n-2} and Ω_M^n denote the canonical bundles of the Kupka set K and M respectively.

The formula above implies that when the line bundle L is positive, the first Chern class of the normal bundle ν_K does not vanish.

Let $\jmath : K \hookrightarrow M$ be the inclusion map, then $c_1(\nu_K) = \jmath^*(c_1(L)) \in \mathrm{H}^2(K, \mathbb{Z})$, since the line bundle L is positive, it has a non–zero first Chern class and the same is true for the restriction to K.

Hence, by Theorem (3.4), the linear transversal type is equivalent to the 1–form $\eta = px\,dy - qy\,dx$, where p, q are relatively prime integers. Moreover, by a Serre's construction ([32], or [12, appendix]), the normal bundle of the Kupka set can be extended to a rank–2 holomorphic vector bundle $E \to M$, and K can be viewed as the zero locus of a global holomorphic section σ of E. Many properties of K are strictly related to the properties of E. The section σ defines the exact sequence :

$$0 \to \mathcal{O}_M \xrightarrow{\cdot\sigma} E \to \mathcal{J}_K \otimes L \to 0, \tag{5.3}$$

which gives the Koszul resolution of the sheaf of ideals $\mathcal{J}_K$ of the Kupka set. The total Chern class of the vector bundle E is given by

$$c(E) = 1 + c_1(L) + [K] \in \mathrm{H}^*(M, \mathbb{C}), \tag{5.4}$$

where $[K]$ denotes the fundamental class of K in $\mathrm{H}^4(M, \mathbb{Z})$.

The following result is a generalization of the Cerveau–Lins theorem [15].

Theorem 5.1. — *Let $\omega \in \mathrm{K}(M, L)$ be a foliation with L a positive line bundle. If the rank–2 vector bundle E associated with the Kupka set is positive then:*

1. *If* $\mathrm{H}^1(M,\mathbb{C})=0$*, the foliation has a meromorphic first integral.*

2. *If the transversal type is not the radial vector field, then the foliation has at least one compact leaf, and has an* ***affine transversal structure****.*

3. *If the transversal type is the radial vector field, then the foliation has a* ***transversal projective structure****.*

The main point in the proof of this theorem, lies in the fact that in the complement $M-K$, we have the Hartog's theorem 1.2, the argument of proof may be found in [9].

In [15], Cerveau and Lins use the hypothesis that the Kupka set is a **complete intersection**. When the transversal type is different to the radial vector field, they use the fact that the complement of a codimension two complete intersection is $n-2+1$ plurisubharmonic, so the hypothesis of the second part of the Theorem (1.2) is satisfied.

Question 5.2. — *Let* $\omega\in\mathrm{K}(M,L)$ *where the line bundle* L *is positive. Is the vector bundle* E *associated with the Kupka component positive?.*

The main result towards an affirmative answer to this question is the following Theorem [12]:

Theorem 5.3. — *Let* $\omega\in\mathrm{K}(M,L)$*, with* L *a very ample line bundle. If the linear part of the transversal type of the Kupka component is given by the form* $\eta_{pq}=px\,dy-qy\,dy$*, then the total Chern class of* E *is :*

$$c(E)=\Big(1+\frac{p}{p+q}c_1(L)\Big)\wedge\Big(1+\frac{q}{p+q}c_1(L)\Big)\in\mathrm{H}^*(M,\mathbb{C}).$$

Moreover the linear transversal type η_{pq} *belongs to the Poincaré Domain and the transversal type is linearizable.*

Proof. — We have seen that the linear transversal type is η_{pq} for some relatively prime integers p,q.

The second Chern class $c_2(E)=[K]$ of the vector bundle associated with the Kupka set, is computed with the Baum–Bott formula [1]. Since the linear transversal type is η_{pq} we have

$$\frac{(p+q)^2}{pq}[K]=c_1(L)^2,$$

and then
$$c_2(E) = [K] = \frac{pc_1(L)}{p+q} \wedge \frac{qc_1(L)}{p+q},$$
and we get the desired formula.

We are going to prove that the transversal type belongs to the Poincaré Domain. Consider the embedding of $\jmath : M \hookrightarrow \mathbb{P}\big(\mathrm{H}^0(M, \mathcal{O}(L))\big)$; the degree of the Kupka set, which is a positive integer, is given by:

$$\begin{aligned} \left(\frac{1}{2\pi i}\right)^n \int_M [K] \wedge c_1(L)^{n-2} &= \left(\frac{1}{2\pi i}\right)^n \int_M \frac{p \cdot q}{(p+q)^2} c_1(L)^n \\ &= \frac{p \cdot q}{(p+q)^2} \cdot \mathrm{vol}(M), \end{aligned}$$

hence, $p \cdot q > 0$, the linear transversal type belongs to the Poincaré domain and by theorem (3.4) the transversal type is linearizable. $\square$

As a consequence, whenever L is a very ample holomorphic line bundle, we obtain a decomposition

$$\overline{\mathrm{K}(M, L)} = \bigcup_{(p,q)} \overline{\mathrm{K}(M, L, \eta_{pq})} \tag{5.5}$$

where p and q are positive integers such that the cohomology classes

$$\frac{p \cdot c_1(L))}{p+q} \quad \frac{q \cdot c_1(L))}{p+q} \in H^2(M, \mathbb{Z})$$

If the transversal type is different to the radial vector field, Lübke inequality [26] implies that the rank two vector bundle E, associated to the Kupka set, can not be stable or semistable with respect to the Kähler form $c_1(L))$, which is given by the embedding induced by the line bundle L.

On the other hand, if the transversal type is the radial vector field, the total Chern class of the bundle E is given by

$$c(E) = \left(1 + \frac{c_1(L)}{2}\right)^2 \in \mathrm{H}^*(M, \mathbb{C}),$$

which is compatible with the existence of a **projectively flat structure** on the vector bundle E.

If the vector bundle E is projectively flat, then the $\mathbb{P}^1$ bundle $\mathbb{P}(E)$ is flat, and it is defined by a representation $\rho : \pi_1(M) \to \mathbf{PSL}(2, \mathbb{C})$. In this case the foliation is the pull–back by the meromorphic section $[\sigma]$ of the horizontal foliation $\mathcal{H}$ of the flat bundle $\mathbb{P}(E)$. The section $[\sigma]$ has its base points in the Kupka set, and the foliation has a **projective transversal structure** just as in Theorem (4.4).

In the last section, we will state some consequences of Theorem (5.3) for foliations in projective spaces.

6. Logarithmic Foliations

In this section, we are going to describe other irreducible components of the set $\mathcal{F}(M, L)$ for some positive line bundle L over M. We will assume through this section that M has complex dimension $n \geqslant 3$ and $\mathrm{H}^1(M, \mathbb{C}) = 0$.

Recall [24, page 449] that a meromorphic 1–form Ω over M is called **logarithmic**, if for any local defining equation φ of its polar divisor Ω_∞, the forms $\varphi \cdot \Omega$ and $\varphi \cdot d\Omega$ are holomorphic.

Now, let Ω be a closed logarithmic 1–form with polar divisor Ω_∞. Given a section $\varphi \in \mathrm{H}^0\big(M, \mathcal{O}([\Omega_\infty])\big)$, whose zero set is Ω_∞, by theorem (2.5), the twisted 1–form $\omega = \varphi \cdot \Omega$ is an integrable section of the bundle $T^*M \otimes [\Omega_\infty]$. Now, let

$$\Omega_\infty = \sum_{i=1}^{k} D_i \in Div(M)$$

be the decomposition in irreducible components of the polar divisor Ω_∞, then

$$\Omega = \sum_{i=1}^{k} \lambda_i \, \frac{d\varphi_i}{\varphi_i}, \quad \lambda_i = \frac{1}{2\pi i} \int_{\gamma_i} \Omega,$$

where $\varphi_i \in \mathrm{H}^0\big(M, \mathcal{O}([D_i])\big)$, is a holomorphic sections that vanishes on the divisor D_i, and γ_i denotes the generator of the kernel of the inclusion $\imath_* : \pi_1(M - D_i) \to \pi_1(M)$. Moreover, by the residue theorem, the following relation holds :

$$\sum_{i=1}^{k} \lambda_i \cdot c_1([D_i]) = 0 \in \mathrm{H}^2(M; \mathbb{Z}).$$

The singular set of the integrable 1–form

$$\omega = \varphi_1 \cdots \varphi_k \sum_{i=1}^{k} \lambda_i \, \frac{d\varphi_i}{\varphi_i},$$

has codimension $\geqslant 2$ whenever the complex numbers $\lambda_i \neq 0$ for all $i = 1, \ldots, k$.

From now on we will assume that $k \geqslant 3$. We are going to suppose that the following generic conditions are verified :

- The line bundles $L_i := [D_i] \quad i = 1, \ldots, k$ are ample.
- The hypersurfaces $\{\varphi_i = 0\}_{1=1}^{k}$ are irreducible, reduced and the divisor $\{\varphi_1 \cdots \varphi_k = 0\}$ is a divisor with normal crossings.
- Hyperbolicity: $\lambda_i/\lambda_j \notin \mathbb{R} \quad$ for $\quad i \neq j$.

We can state our other result [6] :

THEOREM 6.1. — *Let M be a projective manifold such that*

$$\dim_{\mathbb{C}} M \geqslant 3 \quad \text{and} \quad \mathrm{H}^1(M, \mathbb{C}) = 0.$$

Then any deformation of a generic logarithmic foliation is a logarithmic foliation.

As a corollary, we get :

COROLLARY 6.2. — *Let M be a projective manifold such that*

$$\dim_{\mathbb{C}} M \geqslant 3 \quad \text{and} \quad \mathrm{H}^1(M, \mathbb{C}) = 0.$$

Then there are irreducible components of $\mathcal{F}(M, L)$ such that a generic element is represented by a logarithmic 1–form.

The proof of the theorem (6.1) begin with the following remarks.

Consider the 1–form

$$\omega = \varphi_1 \cdots \varphi_k \sum_{i=1}^{k} \lambda_i \frac{d\varphi_i}{\varphi_i},$$

the hypersurfaces $\{\varphi_i = 0\}_{i=1}^{k}$ are the compact leaves of the foliation defined by ω and they have linearizable holonomy. Moreover, any other leaf has trivial holonomy.

In fact, given a point $p \in \mathcal{L}_1 = \{\varphi_1 = 0\} - S(\omega)$ and a transversal $\Sigma \pitchfork_p \mathcal{L}_1$ at p, which we identify with an open disk $\Delta \subset \mathbb{C}$, let $\gamma \in \pi_1(\mathcal{L}_1, p)$ and $\widetilde{\gamma}_y$ be a lift of the curve γ, tangent to the leaves and beginning at $y \in \Sigma$. The holonomy map associated to γ is, by definition, $h_\gamma(y) = \widetilde{\gamma}_y(1)$.

Take $\beta : [0,1] \to \Sigma - \{p\}$ a differentiable curve with $\beta(0) = \widetilde{\gamma}_y(1)$ and $\beta(1) = \widetilde{\gamma}_y(0)$. The curves γ and $\beta * \widetilde{\gamma}_y$ are homotopic in the manifold M.

Consider the meromorphic 1–form

$$\Omega = \sum_{i=1}^{k} \lambda_i \frac{d\varphi_i}{\varphi_i}.$$

It is possible to find a holomorphic coordinate system $\varphi : \Delta \times \mathbb{C}^{n-1} \to V$ satisfying the conditions

- $\varphi(0,0) = p \in V$
- $\varphi(0,w) = V \cap \mathcal{L}_1$
- $\varphi(z,0) = \Sigma$.
- $\varphi^*(\Omega|_V) = \lambda_1 \dfrac{dz}{z}$.

In this way we get:

$$\int_{\beta * \widetilde{\gamma}_y} \Omega = 2i\pi \sum_{i=1}^{k} n_i \lambda_i$$

for some $n_i \in \mathbb{Z}$.

Now, since $\widetilde{\gamma}_y$ is tangent to the foliation and $\beta * \widetilde{\gamma}_y$ is free homotopic to γ, we have:

$$\int_{\beta * \widetilde{\gamma}_y} \Omega = \int_{\beta} \Omega = \lambda_1 \log\left(\frac{\widetilde{\gamma}_y(1)}{y}\right),$$

and we conclude that:

$$h_\gamma(y) = \left(\prod_{i=2}^{k} \mu_{i1}^{n_i}\right) \mu_{\gamma 1} \cdot y,$$

where

$$\mu_{i1} := \exp\left(2i\pi \frac{\lambda_i}{\lambda_1}\right), \quad \text{and} \quad \mu_{\gamma 1} := \exp\left(\frac{1}{\lambda_1}\right).$$

On the other hand, any leaf in $M - \{\varphi_1 \cdots \varphi_k = 0\}$ has trivial holonomy since the foliation is defined by a holomorphic closed 1–form there. □

We now have a partial converse of the last statement:

Theorem 6.3. — *Let M be a projective manifold, $\omega \in \mathcal{F}(M,L)$. Let V be a smooth, ample compact leaf of ω which has linearizable holonomy and $V \cap S_\omega = \cup_{i=1}^{k} \overline{K_i}$, where $K_i \subset K_\omega$ are connected components with transversal type $\eta_i = x\,dy + \lambda_i y\,dx, \quad \lambda_i \notin \mathbb{Q}$. Then, there exists a reduced integrating factor whose zero locus contains V.*

In [17], it is proved that on a neighborhood of the hypersurface V, the foliation has a reduced integrating factor φ, since V is an ample hypersurface. By the second part of the Theorem (1.2), this integrating factor may be extended to M. The details may be found in [5], [6] and [13]. □

Now, let

$$\omega = \varphi_1 \cdots \varphi_k \sum_{i=1}^{k} \lambda_i \frac{d\varphi_i}{\varphi_i},$$

be a generic logarithmic 1–form. For any subset $\mathbf{I} \subset \{1,\ldots,k\}$ the analytic sets

$$S_{\mathbf{I}} = \{p \in M \,|\, p \in \{\varphi_i = 0\}_{i\in\mathbf{I}}\},$$

define a Whitney stratification for the divisor $\{\varphi_1 \cdots \varphi_k = 0\}$. We will denote by I the number of elements of $\mathbf{I}$. Note that $S_{\mathbf{I}}$ has codimension I. We set:

$$\Omega_{\mathbf{I}} = \sum_{j=1}^{I} \lambda_{i_j} \frac{dz_j}{z_j} \quad \mathbf{I} = \{i_1,\ldots,i_I\}.$$

In [33, page 399] it is demonstrated that if ω is like above, for any $\mathbf{I} \subset \{1,\ldots,k\}$ there exists an open covering $\mathcal{U}_{\mathbf{I}} = \{U_\alpha\}$ of $S_{\mathbf{I}}$ and a family of submersions $\{\varphi_\alpha = (y_\alpha^1 \ldots y_\alpha^I) : U_\alpha \to \mathbb{C}^I\}$ such that:

- $\{\varphi_{j_i} = 0\} \cap U_\alpha = \{y_\alpha^i = 0\}$.
- $\varphi_\alpha^*(y_\alpha^1 \cdots y_\alpha^I \Omega_{\mathbf{I}}) = \omega|_{U_\alpha}$.

Let $K_{\mathbf{I}} \subset S_{\mathbf{I}}$ be a compact subset and $W = \cup_\alpha U_\alpha$, where $\{U_\alpha\}$ is as in the above theorem. The following result may be found in [6] or [5].

Proposition 6.4. — *Let ω be as above. If $\lambda_i \neq \lambda_j$ for all $i,j \in \mathbf{I}$, and there is a pair (λ_k,λ_l) with $\lambda_k/\lambda_l \notin \mathbb{R}$, then for any compact $K_{\mathbf{I}} \subset S_{\mathbf{I}}$, and any deformation ω_t of ω, there exists a family of embeddings $\Phi_t : K_{\mathbf{I}} \to W$, a family of submersions $\varphi_{\alpha,t} : U_\alpha \to \mathbb{C}^I$, and a family of logarithmic 1–forms $\Omega_{\mathbf{I},t}$ such that :*

- $\varphi_{\alpha,t}^{-1}(0) = \Phi_t(K_{\mathbf{I}}) \cap U_\alpha$.

- *The 1–form* $\varphi^*_{\alpha,t}(y^1_{\alpha,t}\cdots y^I_{\alpha,t}\Omega_{\mathbf{I},t})$ *defines the foliation on the open set* U_α.

This result means that the singular set of a deformation of a generic logarithmic 1–form, still looks like the singular set of a logarithmic 1–form.

We are now in position to prove Theorem 6.1:

Proof. — Let ω_t be a family of foliations with $\omega_0 = \omega$ with

$$\omega = \varphi_1 \ldots \varphi_k \sum_{i=1}^{k} \lambda_i \, \frac{d\varphi_i}{\varphi_i}.$$

The proof consists on two steps [6]:

- By our generic conditions, we show that all the leaves $\{\varphi_i = 0\}$, are stable, thus there exist holomorphic line bundles L_{it} with holomorphic sections $\{\varphi_{it}\}$ such that $\{\varphi_{it} = 0\}$ are compact leaves of the foliation ω_t.
- Theorem 6.4 implies that $\varphi_{1t}\cdots\varphi_{kt}$ is a reduced integrating factor of the foliation represented by the section ω_t.

Remark 6.5. — In Theorem 6.1 it is necessary that the line bundles L_i for $i = 1, \ldots, k$, and hence $L = L_1 \otimes \cdots \otimes L_k$, to be ample line bundles as the following example shows :

Let $M = \mathbb{P}^1 \times \mathbb{P}^2$, then we have that $\mathrm{H}^1(M, \mathbb{C}) = 0$ and $\mathrm{H}^2(M, \mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$. Now, we are going to consider a holomorphic line bundle L over M with Chern class $(0, n)$ where $n > 1$.

In this case, the set $\mathcal{F}(M, L)$ is an open set of the projective space associated to $\mathrm{H}^0(M, \Omega(L)) \simeq \mathrm{H}^0(\mathbb{P}^2, \Omega^1(n))$. If ω is an integrable holomorphic section of $T^*M \otimes L$, the induced foliation $\mathcal{F}_\omega$ has the form $\mathbb{P}^1 \times \mathcal{F}$, where $\mathcal{F}$ is a foliation with singularities of Chern class n in $\mathbb{P}^2$; thus the set $\mathrm{Log}(\{L_i\})$ does not contains an open subset of $\mathcal{F}(M, L)$.

7. Foliations in Projective Spaces

In this section, we are going to study the set $\mathcal{F}(n, c)$ of codimension one holomorphic foliations on the projective space $\mathbb{P}^n$ and having Chern class $c \in \mathtt{H}^2(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}$.

We recall the following well known facts on the geometry of the projective space.

1. The Picard group $Pic(\mathbb{P}^n) \simeq \mathbb{Z}$ and it is generated by the hyperplane bundle denoted $\mathcal{O}(1)$, and for each $c \in \mathbb{Z}$ we denote $\mathcal{O}(c) = \mathcal{O}(1)^{\otimes c}$ for $c \geqslant 0$ and $(\mathcal{O}(1)^*)^{\otimes(-c)}$ for $c < 0$.

2. Given $c \in \mathbb{N}$, the vector space of holomorphic sections of the bundle $\mathcal{O}(c)$ over $\mathbb{P}^n$ is isomorphic to the space of homogeneous polynomials of degree c on $\mathbb{C}^{n+1}$ [24].

3. The Euler sequence.

$$0 \to \mathcal{O}_{\mathbb{P}^n} \longrightarrow \bigoplus_{n+1} \mathcal{O}_{\mathbb{P}^n}(1) \stackrel{\mathfrak{E}}{\to} T\mathbb{P}^n \to 0, \tag{7.1}$$

4. After dualizing and twisting the above sequence by the line bundle $\mathcal{O}(c)$ we get

$$0 \to T\mathbb{P}^{n*} \otimes \mathcal{O}(c) \to \bigoplus_{n+1} \mathcal{O}(c-1) \to \mathcal{O}(c) \to 0, \tag{7.2}$$

and from the exact long cohomology sequence,

$$0 \to \mathrm{H}^0\big(\mathbb{P}^n, \Omega^1(c)\big) \to \bigoplus_{n+1} \mathrm{H}^0\big(\mathbb{P}^n, \mathcal{O}(c-1)\big) \to \mathrm{H}^0\big(\mathbb{P}^n, \mathcal{O}(c)\big) \to 0$$

As a consequence, we have that the holomorphic sections of the sheaf $\Omega^1_{\mathbb{P}^n}(c)$ is identify with the set of dicritical 1–forms in $\mathbb{C}^{n+1}$ of degree $c-1$, that is

$$\begin{aligned} \omega(z_0, \dots, z_n) &= \sum_{i=0}^{n} A_i(z_0, \dots, z_n)\, dz_i \quad (z_0 \dots, z_n) \in \mathbb{C}^{n+1} \\ 0 &= \sum_{i=0}^{n} z_i \cdot A_i(z_0, \dots, z_n) \\ A_i & \quad \text{homogeneous polynomials of degree } c-1 \end{aligned}$$

Therefore, the space of codimension one holomorphic foliations $\mathcal{F}(n, c)$ with Chern class c may be identified with 1–forms ω in $\mathbb{C}^{n+1}$ as above such that

1. $\omega \wedge d\omega = 0$.

2. $Cod(S_\omega) \geqslant 2$, where $S_\omega = \{p \in \mathbb{P}^n | \omega(p) = 0\}$ is the singular set.

7.1. Logarithmic and Rational Components

Logarithmic and rational foliations on the space $\mathcal{F}(n,c)$ are obtained as follows.

First observe that a holomorphic integrating factor φ of a foliation $\omega \in \mathcal{F}(n,c)$ is a holomorphic section of the bundle $\mathcal{O}_{\mathbb{P}^n}(c)$, i. e. a homogeneous polynomial of degree c in the variables $(z_0,\dots,z_n) \in \mathbb{C}^{n+1}$.

Given a partition $\mathbf{P} = (n_1,\dots,n_p) \quad p \geqslant 2$ of $c \in \mathbb{N}, \quad c \geqslant 2$, i. e. n_i are positive integers such that

$$\sum_{i=1}^{p} n_i = c,$$

the $(c-1)$–homogeneous 1–form in $\mathbb{C}^{n+1}$ defined by

$$\omega = \varphi_1 \cdots \varphi_p \sum_{i=1}^{p} \lambda_i \frac{d\varphi_i}{\varphi_i},$$

where $\varphi_i \in \mathrm{H}^0\big(\mathbb{P}^n, \mathcal{O}(n_i - 1)\big)$, i. e. φ_i are homogeneous polynomials of degree n_i in $\mathbb{C}^{n+1}$, and $\lambda_i \in \mathbb{C}$.

By the Euler formula, the 1–form ω is dicritical, if and only if the residues $\lambda = (\lambda_1,\dots,\lambda_p)$ satisfy the relation

$$\sum_{i=1}^{p} \lambda_i \cdot n_i = 0.$$

We denote by

$$\Lambda_{\mathbf{P}} = \left\{ (\lambda_1 : \cdots : \lambda_p) \in \mathbb{C}^p \quad \text{such that} \quad \sum_{i=1}^{p} \lambda_i \cdot n_i = 0 \right\}.$$

Observe that when $p = 2$, we obtain $\lambda_1 \cdot n_1 + \lambda_2 \cdot n_2 = 0$, and then, $\lambda_1/\lambda_2 = -n_2/n_1$, we are able to choose p_1 and p_2 positive integers with $(p_1,p_2) = 1$ or $p_1 = p_2 = 1$ such that $\lambda_1/\lambda_2 = p_1/p_2$ foliation represented by ω has the meromorphic first integral $\varphi_1^{p_2}/\varphi_2^{p_1}$.

Now, we consider the projective space $\mathbb{P}(\Lambda_{\mathbf{P}})$ associated with the hyperplane $\Lambda_{\mathbf{P}} \subset \mathbb{C}^p$, and we define the algebraic map

$$\begin{aligned} \Phi_{\mathbf{P}} : \mathbb{P}(\Lambda_{\mathbf{P}}) \times \prod_{i=1}^{p} \mathbb{P}_{\mathrm{H}^0(\mathbb{P}^m, \mathcal{O}(n_i))} \quad &\longrightarrow \quad \mathcal{F}(m,c) \\ ([\lambda_1 : \ldots : \lambda_p], [\varphi_1], \ldots, [\varphi_p]) \quad &\mapsto \quad \varphi_1 \cdots \varphi_p \sum_{i=1}^{p} \lambda_i \frac{d\varphi_i}{\varphi_i}. \end{aligned}$$

Observe that if $\mathbf{P}(n_1,\ldots,n_p)$ and $\mathbf{Q}=(m_1,\ldots,m_q)$ are partitions of the same number c, the images under the maps $\Phi_{\mathbf{P}}$ and $\Phi_{\mathbf{Q}}$ may have non–empty intersection. In fact, assume for instance that $q=p-1\geqslant 2$, and that $m_1=n_1+n_2$ and $m_i=n_i$, for $i=2,\ldots q$, then the 1–form:

$$([\lambda_1:\ldots:\lambda_p],[\varphi_1],\ldots,[\varphi_p])\mapsto \varphi_1\cdots\varphi_p\sum_{i=1}^{p}\lambda_i\frac{d\varphi_i}{\varphi_i}\quad \text{where}\quad \lambda_1=\lambda_2,$$

belongs to the intersection.

Theorem 7.1 *Assume* $n\geqq 3$, and $k\geqq 2$. *Let*

$$\omega=\varphi_1\cdots\varphi_k\sum_{i=1}^{k}\lambda_i\,\frac{d\varphi_i}{\varphi_i}\in\mathcal{F}(n,c),$$

be generic. If ω' *is a deformation of* ω*, then there exists homogeneous polynomials* φ_i' *where* $deg(\varphi_i')=deg(\varphi_i)$ *for all* $i=1,\ldots,k$*, and* λ_i' *such that*

$$\omega'=\varphi_1'\cdots\varphi_k'\sum_{i=1}^{k}\lambda_i'\frac{d\varphi_i'}{\varphi_i'}.$$

Proof. — It follows from Theorem (4.1) if $k=2$, and from corollary (6.2) when $k\geqq 3$. □

Let $\mathrm{L}(n,\mathbf{P},c)\subset\mathcal{F}(n,c)$ be the image of the map $\Phi_{\mathbf{P}}$. Then the set of logarithmic foliations $\mathrm{L}(n,\mathbf{P},c)$ is parameterized by a product of projective spaces and this implies the result which follows.

Corollary 7.2. — *For* $n\geqslant 3$, $p\geqslant 3$*, and for any partition* $\mathbf{P}=(n_1,\ldots,n_p)$ *of* $c>1$ *the set* $\mathrm{L}(n,\mathbf{P},c)$ *is an irreducible component of* $\mathcal{F}(n,c)$*. Furthermore, the dimension of this component is:*

$$\sum_{i=1}^{p}\binom{n+n_i}{n_i}-2.$$

Proof. — It follows from corollary (6.2) and the fact that

$$\dim_{\mathbb{C}}\mathrm{H}^0\big(\mathbb{P}^n,\mathcal{O}(n_i)\big)=\mathrm{h}^0\big(\mathbb{P}^m,\mathcal{O}(n_i)\big)=\binom{n+n_i}{n_i}.$$

Remark 7.3. — Observe that if $p = 2$ *in the above theorem, the set*

$$\mathrm{L}\big(n, (n_1, n_2), c\big),$$

coincide with the set of foliations with a meromorphic first integral, we are going to denote by $R(m, n_1, n_2) \subset \mathcal{F}(m, n_1 + n_2)$.

Observe that the generic element of $R(c, n_1, n_2)$ *belongs to* $\mathrm{K}(n, n_1 + n_2)$.

7.2. Foliations with Kupka component on the projective space

In this section, we want to describe the set $\mathrm{K}(n, c) \subset \mathcal{F}(n, c)$. Towards this description, we have the following remarkable result due to Cerveau and Lins [15].

Theorem 7.4. — *[15] Let* $\omega \in K(n, c)$ *with Kupka component* $K_\omega = K$ *then* $\omega = pf\,dg - qg\,df$ *if and only if* K *is a complete intersection.*

The main conjecture is that the Kupka set K is always a complete intersection. In this direction, the following results are known.

Let $\omega \in \mathrm{K}(n, c)$ with Kupka component $K_\omega = K$. As we have seen in section 5, the Kupka set K is subcanonically embedded, the formula (5.2) of the canonical bundle Ω^{m-2}_K of K and the second exterior power of the normal bundle of the Kupka set $\wedge^2\nu_K(\mathbb{P}^n)$ are given by:

$$\Omega^{n-2}_K = \big(\Omega^n_{\mathbb{P}^n} \otimes \mathcal{O}(c)\big)|_K = \mathcal{O}_K(c - n - 1), \quad \wedge^2\nu_K(\mathbb{P}^n) = \mathcal{O}_K(c).$$

This property, by a Serre contruction implies that there exists a couple (E, σ), where $E \to \mathbb{P}^n$ is a rank two holomorphic vector bundle and σ a holomorphic section of E inducing the exact sequence

$$0 \to \mathcal{O} \xrightarrow{\sigma} E \to \mathcal{J}_K(c) \to 0 \tag{7.3}$$

and the Kupka set is a complete intersection if and only if E splits as a direct sum of line bundles.

By using this properties of the Kupka set, Theorem (4.1) in the case of foliations in projective space, may be shown as follows.

Theorem 7.5. — *Let* $\mathcal{F}$ *be a foliation arising from the fibers of a Lefschetz or a Branched Lefschetz Pencil on the projective space* $\mathbb{P}^m$ $\quad m \geqslant 3$. *Then any deformation of a foliation* $\mathcal{F}$, *has a meromorphic first integral.*

Proof. — Let $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ be a family of foliations parameterized by the analytic set $\mathcal{T}$, such that $\mathcal{F}_0$ correspond to a foliation arising from the fibers of a Lefschetz or a Branched Lefschetz pencil on $\mathbb{P}^n \quad n \geqslant 3$.

Since any foliation $\mathcal{F}_t$ has a compact Kupka component K_t, we obtain a holomorphic family of holomorphic vector bundles with a section (E_t, σ_t).

Since $E_0 = L_1 \oplus L_2$ and L_1, L_2 are holomorphic line bundles implies that $\mathrm{H}^1(\mathbb{P}^n, End(E_0))$ vanishes, it follows that the family of vector bundles E_t is rigid, i. e. $E_t \simeq E_0$ [4], the conclusion follows from theorem (5.1). ☐

Let $\omega \in \mathrm{K}(m, c)$ with Kupka component $K_\omega = K$. The linear the transversal type is of the type $\eta_{pq} = px\,dy - qy\,dx, \quad p, q \in \mathbb{Z}$ therefore, the associated vector bundle E, has total Chern class given by

$$c(E) = 1 + \nu \cdot \mathbf{h} + \left(\frac{p\cdot c}{p+q}\frac{q\cdot c}{p+q}\right)\cdot \mathbf{h}^2, \quad \left(\frac{p\cdot c}{p+q}\frac{q\cdot c}{p+q}\right) = \deg K_\omega,$$

and may be written as

$$c(E) = c(\mathcal{O}(a) \oplus \mathcal{O}(b)) \quad \text{where} \quad a = \left(\frac{p\cdot c}{p+q}\right), \quad b = \left(\frac{q\cdot c}{p+q}\right),$$

moreover, the numbers $a, b \in \mathbb{N}$. In particular, the linear transversal type belongs to the Poincaré domain, and the transversal type is linearizable.

In this way, we obtain the decomposition of $\mathrm{K}(n, c)$ in terms of their admissible transversal type

$$\mathrm{K}(n, c) = \bigcup_{(p,q)\in C(c)} \mathrm{K}(n, c, \eta_{pq}), \tag{7.4}$$

with

$$C(c) = \left\{(p, q) \in \mathbb{N} \,\middle|\, \left(\frac{pc}{p+q}, \frac{qc}{p+q}\right) \in \mathbb{N}\times\mathbb{N}\right\}.$$

The main consequence is the following result [12].

Theorem 7.6. — *Let $\omega \in \mathrm{K}(m, c)$ with a Kupka component $K = K_\omega$, then K is a numerically complete intersection.*

The question if any foliation $\omega \in \mathrm{K}(n, c) \quad n \geqslant 3$ has a meromorphic first integral, is related to the question on rank–two holomorphic vector bundles over $\mathbb{P}^n$:

Question 7.7. — *Does a rank–two holomorphic vector bundle on $\mathbb{P}^n$, which has the same Chern classes as a split bundle, split?*

The answer of this question is not true in dimension 3 [32]. On the other hand, Theorem (7.6) shows that the vector bundle associated to a Kupka component is not stable in the sense of Mumford–Takemoto [32], furthermore, there is a conjecture postulated by Grauert–Schneider which states the following :

Conjecture 7.8. — *Any holomorphic vector bundle of rank–two on $\mathbb{P}^n$ with $n \geqslant 5$, which is not stable, splits.*

Let $\mathrm{K}(n, c, \eta) \subset \mathrm{K}(n, c)$ be the set of those foliations with a Kupka component with transversal type η. The affirmative solution of the Grauert–Schneider conjecture, implies that $\mathrm{K}(n, c, \eta) \quad n \geqslant 4$ is an irreducible component of $\mathcal{F}(n, c)$ and the generic element is structurally stable. [8]

In dimension 3, any smooth, non–complete intersection and projectively normal curve cannot be the Kupka set of a foliation [12]. This fact gives many examples of smooth curves, which are not complete intersections and cannot be the Kupka set of any foliation. In particular, this shows that the twisted cubic cannot be the Kupka set of a foliation in $\mathbb{P}^3$.

Finally, since the vector bundle E associated with a Kupka set can not be stable, the result in [25] and theorem (7.6) shows that the Kupka set cannot be an abelian surface in $\mathbb{P}^4$.

Finally, a positive solution of this problem, is given in the following results [2, 3, 9].

Theorem 7.9. — *Let $\omega \in K(n, c)$ with $m \geqslant 6$ or $n \geqslant 3$ and the transversal type $\eta \neq \eta_{11}$, then the vector bundle E associated to the Kupka component splits in a direct sum of holomorphic line bundles, and the foliation ω has a meromorphic first integral.*

If the transversal type η is not the radial foliation, the same conclusion holds for any $n \geqslant 3$ [9], the proof is a consequence of the fact that the normal bundle of the Kupka component K of a foliation with positive normal bundle, and with transversal type $\eta_{pq} = px\,dy - qy\,dx, \quad 1 \leqslant p < q$ is positive, and then, Hartogs extension theorem (1.2) also holds in this situation.

7.3. Polynomial Representations of the Affine Lie Algebra

Other irreducible components of the space of codimension one holomorphic foliations on the projective space $\mathbb{P}^3$, which are defined by a polynomial action of the affine group $\mathbf{Aff}(\mathbb{C}) \times \mathbb{C}^3 \longrightarrow \mathbb{C}^3$.

For each $1 \leqslant \nu \in \mathbb{N}$, consider the couple of polynomial vector fields on the affine 3–space $\mathbb{A}_{\mathbf{z}} = \{(1 : z_1 : z_2 : z_3)\} \subset \mathbb{P}^3$:

$$\begin{aligned} \mathbf{T}_\nu(z_1, z_2, z_3) &= \frac{\partial}{\partial z_1} + \nu \cdot z_1^{\nu-1}\frac{\partial}{\partial z_2} + k \cdot z_2^{\nu-1}\frac{\partial}{\partial z_3} \\ \mathbf{L}_\nu(z_1, z_2, z_3) &= z_1\frac{\partial}{\partial z} + \nu \cdot z_2\frac{\partial}{\partial z_2} + k \cdot z_3\frac{\partial}{\partial z_3} \end{aligned}$$

where $k := \nu(\nu - 1) + 1$.

The main properties of these vector fields that we are going to use, as the reader may easily check, are the following :

1. The vector fields $\mathbf{T}_\nu$ and $\mathbf{L}_\nu$ are complete.
2. $[\mathbf{T}_\nu, \mathbf{L}_\nu] = \mathbf{T}_\nu$.
3. They are linearly independent outside the affine rational curve
$$\Gamma_\nu(t) = (t, t^\nu, t^k).$$

The first two conditions, implies that the vector fields $\mathbf{T}_\nu$ and $\mathbf{L}_\nu$, are the infinitesimal generators of an action

$$\varphi_\nu : \mathbf{Aff}(\mathbb{C}) \times \mathbb{C}^3 \to \mathbb{C}^3$$

of the affine group on $\mathbb{C}^3$, the tangent space at the orbit trough the point $\mathbf{z} \in \mathbb{C}^3$, is generated by the vectors $\mathbf{T}_\nu(\mathbf{z})$ and $\mathbf{L}_\nu(\mathbf{z})$ in $T_{\mathbf{z}}\mathbb{C}^3$.

The third condition implies that for each point $\mathbf{z} \in \mathbb{C}^3 - \Gamma_\nu$, the φ_ν–orbit through $\mathbf{z}$ has dimension two, and the curve Γ_ν is one orbit of this action.

The orbits of the action φ_ν, are the leaves of the foliation represented by the polynomial 1–form $\omega_\nu = \imath_{\mathbf{T}_\nu}\imath_{\mathbf{L}_\nu} dz_1 \wedge dz_2 \wedge dz_3$, explicitly, the 1–form ω_ν has the expresion

$$\omega_\nu = k\nu(z_1^{\nu-1}z_3 - z_2^\nu)dz_1 + k(z_1 z_2^{\nu-1} - z_3)dz_2 + \nu(z_2 - z_1^\nu)dz_3.$$

A direct calculation gives $d\omega_\nu = (\nu + k)\imath_{\mathbf{T}_\nu} dz_1 \wedge dz_2 \wedge dz_3$, consequently, the diffeomorphism

$$\Phi_\nu(t, u, v) = \left(t, u + t^\nu, v + k\int_0^t [u + s^\nu]^{\nu-1}\, ds\right) = (z_1, z_2, z_3)$$

which is the time t of the flow of the vector field $\mathbf{T}_\nu$ with initial conditions $(0,u,v) \in \mathbb{C}^3$, satisfies the equation

$$(\Phi_\nu)_* \left(\frac{\partial}{\partial t}\right)(z_1,z_2,z_3) = \mathbf{T}_\nu(z_1,z_2,z_3),$$

and then, by using equation (3.5), we get

$$\Phi_\nu^*(\omega_\nu) = -kv\,du + \nu u\,dv,$$

therefore, the leaves of the foliation are diffeomorphic to $\mathbb{C}^* \times \mathbb{C}$, and the foliation has a rational first integral given by

$$(z_2 - z_1^\nu)^k \Big/ \left(z_3 - \sum_{j=0}^{\nu-1} \binom{\nu-1}{j} \frac{1}{(\nu j+1)\cdot \nu^j} (z_2 - z_1^\nu)^{(\nu-1-j)} z_1^{(\nu j+1)} \right)^\nu .$$

The homogeneous 1–form

$$\begin{aligned} \overline{\omega_\nu} \;\; := \;\; & \big((\nu-1)z_1 z_2^\nu + (k-\nu)z_0^{\nu-1}z_2z_3 - (k-1)z_1^\nu z_3\big)dz_0 \\ & + k\nu z_0(z_1^{\nu-1}z_3 - z_2^\nu)dz_1 + kz_0(z_1z_2^{\nu-1} - z_0^{\nu-1}z_3)dz_2 \\ & + \nu z_0(z_0^{\nu-1}z_2 - z_1^\nu)dz_3 \end{aligned}$$

defines a foliation $\mathcal{F}_\nu \in \mathcal{F}(3,\nu+2)$. From this expression, it follows that the hyperplane $\{z_0 = 0\}$ is invariant by the foliation.

The singular set of the foliation, consists of three curves defined by the equations :

$$\begin{aligned} \Gamma_\nu &= \{z_1^{\nu-1}z_3 - z_2^\nu = z_1z_2^{\nu-1} - z_0^{\nu-1}z_3 = z_0^{\nu-1}z_2 - z_1^\nu = 0\} \\ \Lambda &= \{z_0 = z_1 = 0\} \\ \Xi_\nu &= \{z_0 = z_2^\nu - \nu z_1^{\nu-1}z_3 = 0\}. \end{aligned}$$

It is important to observe, that these three curves are invariant by a holomorphic vector field $\overline{\mathbf{L}}_\nu$ on the projective 3 space $\mathbb{P}^3$, which is an extension of the linear vector field $\mathbf{L}_\nu$ on $\mathbb{C}^3$, and then, they are rational and moreover, they are fixed by a one parameter group of automorphisms of $\mathbb{P}^3$; the flow of the vector field $\overline{\mathbf{L}}_\nu$. These curves meet at the point $[0:0:0:1]$. Moreover, the Kupka set of the foliation, is

$$K(\overline{\omega_\nu}) = \Gamma_\nu \cup \Lambda \cup \Xi_\nu - \{[0:0:0:1]\},$$

and at the point $[0:0:0:1]$ the 2–form $d\omega_\nu$ has an isolated singularity. As has been pointed in [16] and [10], this property of the singular set of the foliation, is stable under deformations.

Finally, observe that all leaves of these foliations are biholomorphic to $\mathbb{C}^* \times \mathbb{C}$.

It is possible to prove that, if $\mathcal{F} \in \mathcal{F}(3, \nu+2)$ is any foliation sufficiently close to $\mathcal{F}_\nu$, then it has an invariant hyperplane $\mathbb{H}_\mathcal{F}$, and moreover, in the affine space $\mathbb{P}^3 - \mathbb{H}_\mathcal{F}$ the foliation is defined by the orbits of an action

$$\varphi_\mathcal{F} : \mathbf{Aff}(\mathbb{C}) \times (\mathbb{P}^3 - \mathbb{H}_\mathcal{F}) \to (\mathbb{P}^3 - \mathbb{H}_\mathcal{F})$$

This implies the following result

Theorem 7.10. — *For any $2 \leqslant \nu \in \mathbb{N}$ there exists at least an irreducible component of $\mathcal{A}_\nu \subset \mathcal{F}(3, \nu+2)$ with the following properties:*

- *The generic element $\mathcal{F} \in \mathcal{A}_\nu$, has an invariant hyperplane $\mathbb{H}_\mathcal{F}$.*
- *In the affine space $\mathbb{P}^3 - \mathbb{H}_\mathcal{F}$, the leaves of the foliation are the orbits of an action of the affine group $\mathbf{Aff}(\mathbb{C})$.*

The proof of the theorem is based in the following remarks :

The foliation is defined by a linear vector field $\mathbf{L}_\nu$, and it extends to a global vector field $\overline{\mathbf{L}}_\nu \in \mathrm{H}^0(\mathbb{P}^3, \Theta_{\mathbb{P}^3})$, and the polynomial vector field $\mathbf{T}_\nu$, defines a bundle map $\overline{\mathbf{T}_\nu} : \mathcal{O}(2-\nu) \longrightarrow \Theta_{\mathbb{P}^3}$, therefore, the tangent sheaf $T_\mathcal{F}$ of the foliation is locally free and isomorphic to $\mathcal{O} \oplus \mathcal{O}(2-\nu)$, and defined by the linear map

$$\overline{\mathbf{L}}_\nu \oplus \overline{\mathbf{T}}_\nu : T_\mathcal{F} \simeq \mathcal{O} \oplus \mathcal{O}(2-\nu) \longrightarrow \Theta^3_\mathbb{P}.$$

The main idea lies in the following two facts

1. Any deformation of a generic foliation $\mathcal{F} \in \mathcal{A}_\nu$ has a locally free tangent sheaf.
2. Since $\mathrm{H}^1\big(\mathbb{P}^3, End(T_\mathcal{F})\big) \simeq 0$ any deformation of a generic foliation $\mathcal{F} \in \mathcal{A}_\nu$ has tangent sheaf isomorphic to $\mathcal{O} \oplus \mathcal{O}(2-\nu)$.

The first point follows from the stability of the singular set, and the second by well known results on deformation of holomorphic vector bundles [4].

The details of the proof may be found in [16] for $\nu = 2$. The general case is proved in [10] and [11].

The irreducible component $\mathcal{A}_\nu \subset \mathcal{F}(3, \nu+2)$ is rigid, therefore, it is the orbit of the natural action of $\mathbf{PGL}(4, \mathbb{C}) \times \mathcal{F}(3, \nu+2) \to \mathcal{F}(\nu, 3)$ on the space of foliations of the foliation $\mathcal{F}$. This implies that the dimension of the irreducible component

$$\dim_{\mathbb{C}}(\mathcal{A}_\nu) = \dim_{\mathbb{C}}\mathbf{PGL}(4,\mathbb{C}) - h^0(\mathbb{P}^3, \mathcal{O}(T_{\mathcal{F}})) = \begin{cases} 13 & \text{if} \quad \nu = 2 \\ 14 & \text{if} \quad \nu > 2 \end{cases}.$$

The family defined above may be extended to a foliation on $\mathbb{P}^n \quad n \geqslant 4$ by considering a fixed linear projection $\ell : \mathbb{P}^n \to \mathbb{P}^3$, and defining the set of foliations

$$\mathcal{A}_\nu(n) = \ell^*(\mathcal{A}_\nu) := \{\mathcal{G} \in \mathcal{F}(n, \nu+2) | \mathcal{G} = \ell^*(\mathcal{F}) \quad \mathcal{F} \in \mathcal{A}_\nu\},$$

it is shown in [10], that the set of foliations $\mathcal{A}_\nu(n)$, is an irreducible component of the space $\mathcal{F}(n, \nu+2)$.

There are many other examples of foliations of $\mathbb{P}^3$ which are defined by actions of the affine group $\mathbf{Aff}(\mathbb{C})$, or more generally, by polynomial representations of the affine Lie algebra $\mathfrak{aff}(\mathbb{C})$ on polynomial vector fields of an affine subspace $\mathbb{A} \simeq \mathbb{C}^3 \subset \mathbb{P}^3$, but in many cases, these families not define irreducible components of the space of foliations [10].

One example of this situation, is the one parameter family of foliations of degree 3 defined by the vector fields

$$\begin{aligned} \mathbf{X}_\sigma &= \frac{\partial}{\partial z_1} + 2 \cdot z_1 \frac{\partial}{\partial z_2} + 3 \cdot ((1-\sigma) \cdot z_2 + \sigma z_1^2) \frac{\partial}{\partial z_3} \\ \mathbf{L} &= z_1 \frac{\partial}{\partial z_1} + 2 \cdot z_2 \frac{\partial}{\partial z_2} 3 z_3 \frac{\partial}{\partial z_3} \end{aligned}$$

on the affine space $\mathbb{A} = \{[1 : z_1 : z_2 : z_3]\}$.

The extension $\overline{\omega}_\sigma$ of this one parameter family of foliations to the projective 3–space, has the meromorphic first integral

$$H_\sigma(z_0, z_1, z_2, z_3) = \frac{(z_0 z_2 - z_2^2)^3}{(z_0^2 z_3 - 3\sigma \cdot z_0 z_1 z_2 + (3\sigma - 1) z_1^3)^2} = \frac{F^3}{G_\sigma^2}, \qquad (7.5)$$

and for any value of $\sigma \neq 0$, the family is contained in the irreducible component $R(3, 2, 3) \subset \mathcal{F}(3, 5)$, but the value $\sigma = 0$ corresponds to a member of the family $\mathcal{A}_2 \subset \mathcal{F}(3, 4)$. As the reader can check, the 1–form $\omega = 3G_0\, dF - 2F\, dG_0$, is singular along the hyperplane $z_0 = 0$.

The tangent sheaf $T_{\mathcal{F}_\sigma}$ for any member of the family $\{\mathcal{F}_\sigma | \mathbb{C} \ni \sigma \neq 0\}$ of the one parameter family defined above, is locally free and isomorphic to $T_{\mathcal{F}_\sigma} \simeq \mathcal{O} \oplus \mathcal{O}(-1)$, in particular, the foliation is preserved by a one parameter group of automorphism of $\mathbb{P}^3$, this property does not hold for a generic member of the irreducible component $R(3,2,3)$.

Other examples of foliations on projective spaces are given by rational pull–backs of foliations on the plane, that is, let $n \geqslant 2$ and let $\varphi : \mathbb{P}^n \to (\mathbb{P}^2, \mathcal{F})$ be a rational map and $\mathcal{F} \in \mathcal{F}(2,c)$. Then we get a foliation $\mathcal{G} = \varphi^* \mathcal{F}$.

When the map φ is a linear projection, we have that $\mathcal{G} \in \mathcal{F}(n,c)$, the tangent sheaf is locally free and isomorphic to $\mathcal{O}(3-c) \oplus \mathcal{O}(1)^{\oplus n-2}$.

We denote by

$$PB(1,n,c) = \ell^* \mathcal{F}(2,c), \quad \ell : \mathbb{P}^n \to \mathbb{P}^2 \text{ a fixed linear projection} \tag{7.6}$$

It is proved in [15] the following theorem:

THEOREM 7.11. — *The sets $PB(1,n,c)$ are irreducible components of $\mathcal{F}(n,c)$.*

There is a systematic study of foliations with locally free tangent sheaf, the main references are [14] and [18].

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