ANNALES
DE LA FACULTÉ
DES SCIENCES
DE
TOULOUSE
Mathématiques

MARIUS VAN DER PUT, MARC REVERSAT

Galois theory of q-difference equations

Tome XVI, n° 3 (2007), p. 665-718.

<http://afst.cedram.org/item?id=AFST_2007_6_16_3_665_0>

Annales de la Faculté des Sciences de Toulouse Vol. XVI, n° 3, 2007
pp. 665–718

Galois theory of q-difference equations(*)

Marius van der Put(1), Marc Reversat(2)

Abstract. — Choose $q \in \mathbb{C}$ with $0 < |q| < 1$. The main theme of this paper is the study of linear q-difference equations over the field K of germs of meromorphic functions at 0. A systematic treatment of classification and moduli is developed. It turns out that a difference module M over K induces in a functorial way a vector bundle $v(M)$ on the Tate curve $E_q := \mathbb{C}^*/q^{\mathbb{Z}}$ that was known for modules with "integer slopes", [Saul, 2]). As a corollary one rediscovers Atiyah's classification ([At]) of the indecomposable vector bundles on the complex Tate curve. Linear q-difference equations are also studied in positive characteristic p in order to derive Atiyah's results for elliptic curves for which the j-invariant is not algebraic over $\mathbb{F}_p$.

Résumé. — Soit q un nombre complexe, $0 < |q| < 1$. On procède pour l'essentiel à une étude systématique des équations aux q-différences sur le corps K des fonctions méromorphes au voisinage de 0 (classifications, problèmes de modules). Cela conduit à associer à tout module aux différences M un fibré vectoriel $v(M)$ sur la courbe de Tate $E_q := \mathbb{C}^*/q^{\mathbb{Z}}$ (c'était déjà connu pour les modules « à pentes entières », [Saul, 2]), ce qui amène à retrouver la classification donnée par Atiyah des fibrés vectoriels indécomposables sur la courbe de Tate complexe ([At]). Dans le dernier paragraphe nous étudions les équations linéaires aux q-différences en caractéristique positive p, nous en déduisons les résultats d'Atiyah pour les courbes elliptiques dont le j-invariant est transcendant sur $\mathbb{F}_p$.

(*) Reçu le 14 juillet 2005, accepté le 6 mars 2006

(1) Department of Mathematics, University of Groningen, P.O.Box 800, 9700 AV Groningen, The Netherlands.
mvdput@math.rug.nl

(2) Institut de Mathématiques, Université Paul Sabatier, 31062 Toulouse cedex 9. 31062 Toulouse cedex 9, France,
marc.reversat@math.ups-tlse.fr

Introduction

Choose $q \in \mathbb{C}$ with $0 < |q| < 1$. The main theme of this paper is the study of linear q-difference equations over the field K of germs of meromorphic functions at 0. A more detailed and systematic treatment of classification and moduli is developed as a continuation of [vdP-S1] (Chapter 12), [vdP-R] and [vdP]. It turns out that a difference module M over K induces in a functorial way a vector bundle $v(M)$ on the Tate curve $E_q := \mathbb{C}^*/q^{\mathbb{Z}}$ (this is done here for all slopes, the case of integral slopes has been treated in [Sau1] and [Sau2]). As a corollary one rediscovers Atiyah's classification ([At]) of the indecomposable vector bundles on the complex Tate curve. Linear q-difference equations are also studied in positive characteristic in order to derive Atiyah's results for elliptic curves for which the j-invariant is not algebraic over $\mathbb{F}_p$.

A universal difference ring and a universal formal difference Galois group is introduced. For pure difference modules this ring provides an explicit expression of the difference Galois group. If the difference module has more than one slope, then part of the difference Galois group has an interpretation as 'Stokes matrices', related to a summation method for divergent solutions. We do not make any hypothesis on the slopes, when they are integers see [R-S-Z, Sau2]. The above moduli space is the algebraic tool to compute this part of the difference Galois group.

It is possible to provide the vector bundle $v(M)$ on E_q, corresponding to a difference module M over K, with a connection ∇_M. If M is regular singular, then ∇_M is essentially determined by the absense of singularities and 'unit circle monodromy'. More precisely, the monodromy of the connection $(v(M), \nabla_M)$ coincides with the action of two topological generators of the universal regular singular difference Galois group ([vdP-S1, Sau1]). For irregular difference modules, ∇_M will have singularities and there are various Tannakian choices for $M \mapsto (v(M), \nabla_M)$. Explicit computations are difficult, especially for the case of non integer slopes.

The case of modules with integer slopes, has been studied in [R-S-Z]. This answers a question of G.D. Birkhoff and follows ideas of G.D. Birkhoff, P.E. Guenther, C.R. Adams (see [Bir]).

1. Classification of q-difference equations

1.1. Some notation and formulas

A difference ring is a commutative R with a given automorphism ϕ. The skew ring of difference operators $R[\Phi, \Phi^{-1}]$, consists of the finite formal sums $\sum_{n\in\mathbb{Z}} a_n\Phi^n$ with all $a_n \in R$. The multiplication is defined by $\Phi r = \phi(r)\Phi$. A difference module M is a left $R[\Phi, \Phi^{-1}]$-module which is free and finitely generated as R-module. The action Φ_M of Φ on M is an additive bijection satisfying $\Phi_M(rm) = \phi(r)\Phi_M(m)$. Thus we may describe a difference module as a pair (M, F), with F an additive bijective map and such that $F(fm) = \phi(m)F(m)$ holds.

As before we choose $q \in \mathbb{C}$ with $0 < |q| < 1$. Further we fix τ in the complex upper half plane with $e^{2\pi i\tau} = q$. The fields $K = \mathbb{C}(\{z\})$ and $\widehat{K} = \mathbb{C}((z))$ are made into difference fields by the automorphisms ϕ given by $\phi(z) = qz$. These automorphisms are extended to the finite extensions K_n and $\widehat{K}_n$ of degrees n and to the algebraic closures K_∞ and $\widehat{K}_\infty$ of K and $\widehat{K}$, by $\phi(z^\lambda) = q^\lambda z^\lambda$ where $q^\lambda := e^{2\pi i\tau\lambda}$ for all $\lambda \in \mathbb{Q}$.

The formula $\phi(z^\lambda) = q^\lambda z^\lambda$ makes $\mathbb{C}[z, z^{-1}]$ into a difference ring. A difference module over this ring will be called a *global difference module*. As we will prove later on, any difference module M over K is obtained from a unique global module M_{global} as a tensor product, i.e., $M \cong K \otimes_{\mathbb{C}[z,z^{-1}]} M_{global}$.

Other difference rings that we will use are $\mathbb{C}[z^{1/n}, z^{-1/n}]$ and $O = O(\mathbb{C}^*)$, the ring of the holomorphic functions on $\mathbb{C}^*$.

Closely related to q-difference equations is the *complex Tate curve* $E_q := \mathbb{C}^*/q^{\mathbb{Z}}$. We write $pr : \mathbb{C}^* \to E_q$ for the natural map. *Theta functions* are related to both E_q and q-difference equations. Put $\Theta := \sum_{n\in\mathbb{Z}} q^{n(n-1)/2}(-z)^n$. Then

$$\Theta = d\prod_{n>0}(1 - q^n z^{-1}) \cdot \prod_{n\geqslant 0}(1 - q^n z) \text{ for some constant } d \neq 0.$$

The divisor of Θ on $\mathbb{C}^*$ is $\sum_{n\in\mathbb{Z}}[q^n]$. Further $-z\Theta(qz) = \Theta(z)$ or $-z\phi(\Theta) = \Theta$. The latter implies $\frac{dz}{z} + \phi(\frac{d\Theta}{\Theta}) = \frac{d\Theta}{\Theta}$. Moreover, the poles of $\frac{d\Theta}{\Theta}$ form the set $q^{\mathbb{Z}}$. Each pole is simple and has residue 1.

For $c \in \mathbb{C}^*$ one defines $\theta_c := \frac{\Theta(cz)}{\Theta(z)}$. This function has the property: $c \cdot \theta_c(qz) = \theta_c(z)$. Moreover, the differential form $\omega_c := \frac{d\theta_c}{\theta_c}$ is ϕ-invariant

and defines a differential form on E_q. If $c \notin q^{\mathbb{Z}}$, then ω_c has simple poles in $pr(c^{-1})$ and $1 = pr(1)$ with residues 1 and -1. Further $\omega_{qc} = \omega_c - \frac{dz}{z}$.

1.2. Regular singular difference modules

We recall some classical results (a modern proof is given in [vdP-S1]). The classification of regular singular modules over K and $\widehat{K}$ are similar and we restrict our attention to q-difference modules M over K. A difference module over K is called *regular singular* if there exists a lattice $M^0 \subset M$ over $\mathbb{C}\{z\}$ (i.e., $M^0 = \mathbb{C}\{z\}e_1 \oplus \cdots \oplus \mathbb{C}\{z\}e_m$ for some K-basis $\{e_1, \ldots, e_m\}$ of M) which is invariant under Φ and Φ^{-1} (or in later terminology, M is pure of slope 0). Then M can uniquely be written as $K \otimes_{\mathbb{C}} V$ where V is a finite dimensional vector space over $\mathbb{C}$ provided with a linear map $A : V \to V$ such that all its eigenvalues α satisfy $|q| < |\alpha| \leqslant 1$. The action of Φ on this tensor product is given by $\Phi(a \otimes v) = \phi(a) \otimes A(v)$, for $a \in K$ and $v \in V$.

For a regular singular M we define $M_{global} \subset M$ as the set of elements m such that the $\mathbb{C}$-vector space generated by $\{\Phi^n m |\ n \geqslant 0\}$ has finite dimension. Equivalently, $m \in M_{global}$, if and only if there exists a non zero $L \in \mathbb{C}[\Phi]$ such that $L(m) = 0$. Clearly M_{global} is a $\mathbb{C}[z, z^{-1}]$-submodule. More precisely,

LEMMA 1.1. — *$M_{global} = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} V$ and consequently the natural morphism $K \otimes_{\mathbb{C}[z,z^{-1}]} M_{global} \to M$ is an isomorphism.*

Proof. — Suppose that $m \in M$ (or even $m \in \widehat{K} \otimes_K M$) satisfies $L(m) = 0$ with $L = \Phi^d + c_{d-1}\Phi^{d-1} + \cdots + c_0 \in \mathbb{C}[\Phi]$ and $c_0 \neq 0$. Write $m = \sum_{n>>-\infty} z^n \otimes v_n$. Then $L(m) = \sum_n z^n \otimes (q^{nd}A^d + c_{d-1}q^{n(d-1)}A^{d-1} + \cdots + c_0)v_n$. One provides V with some norm. For large $|n|$, the linear map $q^{nd}A^d + c_{d-1}q^{n(d-1)}A^{d-1} + \cdots + c_0$ is invertible since the norm of either c_0 or $q^{nd}A^d$ is larger than the norm of the remaining part of the linear map. Thus $M_{global} \subset \mathbb{C}[z, z^{-1}] \otimes V$. The other inclusion is obvious. □

Remarks 1.2. — (1) The *unipotent difference module* U_m over K (or over $\widehat{K}$) is $U_m := K \otimes_{\mathbb{C}} \mathbb{C}^m$ with $\Phi(f \otimes v) = \phi(f) \otimes A(v)$, where $A : \mathbb{C}^m \to \mathbb{C}^m$ is the unipotent map which has a unique Jordan block. Any 1-dimensional regular singular difference module has the form $E(c) := Ke$ with $\Phi(e) = ce,\ c \in \mathbb{C}^*$ and one may normalize c such that $|q| < |c| \leqslant 1$. From the modules $\{E(c)\}_{|q|<|c|\leqslant 1}$ and U_2 one constructs every regular singular module by taking tensor products and direct sums.

(2) Let M be a regular singular module. For any $c \in \mathbb{C}^*$, $Eigen(\Phi, c) \subset M$ denotes the generalized eigenspace for the eigenvalue c. In other words,

$Eigen(\Phi, c)$ consists of the elements $m \in M$ such that there exists an integer $N > 0$ with $(\Phi - c)^N(m) = 0$. From the above one easily concludes that each $Eigen(\Phi, c)$ has finite dimension. Further $V = \oplus_{|q|<|c|\leqslant 1} Eigen(\Phi, c)$ and $M_{global} = \oplus_{c\neq 0} Eigen(\Phi, c)$.

1.3. The slope filtration

We describe here the slope filtration and give references for more details and proofs. It is well known that any difference module M over K contains a cyclic vector. This means that there exists an element $e \in M$ such that the homomorphism $K[\Phi, \Phi^{-1}] \to M$, given by $\sum a_n \Phi^n \mapsto \sum a_n \Phi^n(e)$, is surjective (compare Lemma 4.1). Thus M is isomorphic to $K[\Phi, \Phi^{-1}]/K[\Phi, \Phi^{-1}]L$ for some L of the form $\Phi^d + a_{d-1}\Phi^{d-1} + \cdots + a_0$, with all $a_i \in K$ and $a_0 \neq 0$. The difference operator L has a Newton polygon. For completeness we recall its definition. Let $ord : \widehat{K} \to \mathbb{Z} \cup \{+\infty\}$ denote the order function on $\widehat{K}^*$ extended by $ord(0) = +\infty$. In $\mathbb{R}^2$ one considers the convex hull of $\bigcup_{i=0}^{d} \{(i, -ord(a_i) + x_2) |\ x_2 \leqslant 0\}$. The finite part of the boundary of this convex set is the Newton polygon of L.

The module M (over K or over $\widehat{K}$) is called *pure* if this Newton polygon has only one slope. As in the case of differential operators, one can factorize L, viewed as an element of $\widehat{K}[\Phi, \Phi^{-1}]$, according to the slopes in any order that one chooses. This gives a unique decomposition of $\widehat{K} \otimes_K M$ as direct sum $N_1 \oplus N_2 \oplus \cdots \oplus N_r$ of pure difference modules over $\widehat{K}$ with slopes $\lambda_1 < \lambda_2 < \cdots < \lambda_r$. The rule $\Phi z^n = q^n z^n \Phi$ and $|q| < 1$ imply that the slope factorization $L_1 \cdot L_2 \cdots L_r$ of L where L_i has slope λ_i for $i = 1, \ldots, r$ is convergent, i.e., all L_i are in $K[\Phi, \Phi^{-1}]$.

One deduces from this the ascending *slope filtration* of M by submodules $0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_r = M$ such that each M_i/M_{i-1} is pure of slope λ_i and moreover $\widehat{K} \otimes M_i/M_{i-1} \cong N_i$. The slope filtration is unique. The *graded module* $gr(M)$ associated to M is $\oplus_{i=1}^{r} M_i/M_{i-1}$. We note that the above facts on slope filtration are already present in the work of G.D. Birkhoff, P.E. Guenther and C.R. Adams, see [Bir]. A modern proof is provided in [Sau3]. The difference module M over K is called *split* if M is isomorphic to $gr(M)$ (in other words, M is a direct sum of pure modules). Fix a direct sum $A = \oplus_{i=1}^{r} A_i$ of pure modules with slopes $\lambda_1 < \cdots < \lambda_r$. In section 3 we will construct a fine moduli space for the equivalence classes of the pairs (M, f), consisting of a difference module over K and an isomorphism $f : gr(M) \to A$.

1.4. Classification of pure modules over $\widehat{K}$ and K

Let $F \subset G$ be a finite extension of difference fields. Let M be a difference module over G. Then $Res(M)$ (the restriction of M to F) denotes M, considered as a difference module over F. One observes that
$\dim_F Res(M) = [G : F] \cdot \dim_G M$.

Put $\widehat{K}_n = \widehat{K}(z^{1/n})$ for any integer $n \geqslant 1$. We apply the above restriction to the extension $\widehat{K} \subset \widehat{K}_n$ in order to construct all irreducible modules over $\widehat{K}$. Consider integers t, n with $n \geqslant 1$ and $g.c.d.(t, n) = 1$ and $c \in \mathbb{C}^*$ with $|q|^{1/n} < |c| \leqslant 1$. Let $E(cz^{t/n}) := \widehat{K}_n e$ denote the difference module over $\widehat{K}_n$ given by $\Phi(e) = cz^{t/n}e$. Put $E := Res(E(cz^{t/n}))$.

Proposition 1.3 (The irreducible modules over $\widehat{K}$). —
(1) *E depends only on $t,\ n,\ c^n$.*
(2) *E is irreducible of dimension n and has slope t/n. The algebra of the $\widehat{K}$-linear endomorphisms of E, commuting with Φ, is $\mathbb{C}$.*
(3) *For any irreducible difference module I there are unique t, n and c^n with $n \geqslant 1$, $g.c.d.(t, n) = 1$ and $|q| < |c^n| \leqslant 1$, such that $I \cong Res(E(cz^{t/n}))$.*

Proof. — (1) and (2). E has basis $e, \Phi e, \cdots, \Phi^{n-1}e$ over $\widehat{K}$ and thus e is a cyclic vector for E. The minimal monic polynomial $L \in \widehat{K}[\Phi]$ with $Le = 0$ is $L = \Phi^n - q^{t(n-1)/2}c^n z^t$. Thus $E \cong \widehat{K}[\Phi, \Phi^{-1}]/\widehat{K}[\Phi, \Phi^{-1}]L$ and depends only on $t,\ n,\ c^n$. The operator L has slope t/n and degree n. If L has a non trivial decomposition $L_1 L_2$, then the Newton polygon of L is the sum of the Newton polygons of L_1 and L_2. In particular the Newton polygon of L contains (at least) three points with integral coordinates. Since $g.c.d.(t, n) = 1$, this is not the case and hence L and E are irreducible. We note that every non zero endomorphism ($\widehat{K}$-linear and commuting with Φ) of E is bijective. Since $\mathbb{C}$ is algebraically closed, this implies that the algebra of the endomorphisms ($\widehat{K}$-linear and commuting with Φ) of E is $\mathbb{C}$.
(3) Let I be an irreducible difference module, then I is pure and has a slope t/n with $n \geqslant 1$ and $g.c.d.(t, n) = 1$. Take a cyclic vector and let $L = \Phi^d + a_1\Phi^{d-1} + \cdots + a_{d-1}\Phi + a_d$ be its minimal polynomial (with $d = \dim_{\widehat{K}} I$). Then $\frac{ord(a_i)}{i} \geqslant t/n$ for all i and $\frac{ord(a_d)}{d} = t/n$. It follows that d is a multiple of n. Now $L \in \widehat{K}[\Phi, \Phi^{-1}] \subset \widehat{K}_n[\Phi, \Phi^{-1}] = \widehat{K}_n[\Psi, \Psi^{-1}]$ with $\Psi = \Phi z^{t/n}$.
Then L, as operator in Ψ, has slope 0. Hence L has a right hand factor of degree 1 in Ψ (or in Φ). This means that we have a morphism of q-difference modules over $\widehat{K}$

$$I = \widehat{K}[\Phi, \Phi^{-1}]/\widehat{K}[\Phi, \Phi^{-1}]L \to \widehat{K}_n[\Phi, \Phi^{-1}]/\widehat{K}_n[\Phi, \Phi^{-1}](\Phi - a)\ ,$$

for a suitable $a \in \widehat{K}_n$. This morphism is injective since I is irreducible. Counting the dimensions over $\widehat{K}$, one finds that $d = n$ and that the morphism is bijective. The right hand side is a one-dimensional difference module over $\widehat{K}_n$ and hence is isomorphic to $E(cz^{t/n})$ for some $c \in \mathbb{C}^*$ with $|q|^{1/n} < |c| \leqslant 1$ (compare [vdP-S1], p. 149-150).

We have to show that an isomorphism between $E_1 := Res(E(c_1 z^{\lambda_1}))$ and $E_2 := Res(E(c_2 z^{\lambda_2}))$ implies that $\lambda_1 = \lambda_2$ and $c_1^n = c_2^n$. The first statement is obvious since λ_i is the slope of E_i. We write $\lambda_1 = \lambda_2 = t/n$ with $n \geqslant 1$, $(t, n) = 1$. Let $F : E_1 \to E_2$ be an isomorphism. Then F is unique up to multiplication by a scalar in $\mathbb{C}^*$. Both modules have the structure of a difference module over $\widehat{K}_n$. Consider the map $z^{-1/n} \circ F \circ z^{1/n}$. This is also an isomorphism between the two difference modules over $\widehat{K}$. Hence $z^{-1/n} \circ F \circ z^{1/n} = cF$ for some $c \in \mathbb{C}^*$. Clearly $c^n = 1$. We change the $\widehat{K}_n$ structure of the module $E(c_2 z^{t/n})$ by applying a suitable automorphism of $\widehat{K}_n$ over $\widehat{K}$. Now $E(c_2 z^{t/n})$ is changed into $E(c_3 z^{t/n})$ with $c_3 = \zeta c_2$ for some ζ with $\zeta^n = 1$. Moreover, we have now $z^{-1/n} \circ F \circ z^{1/n} = F$. Thus $E(c_1 z^{t/n})$ and $E(c_3 z^{t/n})$ are isomorphic as difference modules over $\widehat{K}_n$. This implies $c_1 = c_3$ since $|q|^{1/n} < |c_1|, |c_3| \leqslant 1$. □

Remarks 1.4. — (1) We note that Proposition 1.3 extends to the case where the field $\mathbb{C}$ is replaced by any field C (of characteristic 0, with $q \in C^*$ not a root of unity and C not necessarily equal to its algebraic closure $\overline{C}$). This can be formulated as follows. One extends the action of ϕ on $K := C((z))$ to the field $\bigcup \overline{C}((z^{1/n!}))$ in the obvious way. This field contains the algebraic closure $\overline{K}$ of K. Take a non zero element $\alpha \in \overline{K}$ of degree m over K and consider the difference module $K(\alpha)e$ given by $\Phi(e) = \alpha e$. Then $K(\alpha)e$, viewed as a difference module over K, has dimension m and is irreducible. It depends only on the Galois orbit of α. Every irreducible difference module over K is obtained in this way.

(2) Put $K_n = K(z^{1/n})$. The difference module over K obtained by viewing $K_n e$ with $\Phi e = cz^{t/n} e$ as a difference module over K, will also be denoted by $Res(E(cz^{t/n}))$.

Corollary 1.5. — *Proposition* 1.3 *remains valid if* $\widehat{K}$ *is replaced by* K.

Proof. — From the slope filtration it follows that an irreducible difference module over K is pure of some slope t/n. Let K_n denote $K(z^{1/n})$. The factorization of L as element of $K_n[\Psi, \Psi^{-1}]$ is valid over K_n, because L is in this context a regular singular difference operator. □

Corollary 1.6 (Indecomposable modules). —
(1) *Let M be an indecomposable difference module over $\widehat{K}$. Then there are unique integers $t,\ n,\ m$ and $c^n \in \mathbb{C}^*$ with $n, m \geqslant 1$, g.c.d.$(t,n) = 1$, $|q| < |c^n| \leqslant 1$ such that M is isomorphic with $Res(E(cz^{t/n})) \otimes_{\widehat{K}} U_m$.*

(2) *Let M be an indecomposable pure difference module over K, then there are unique $t,\ n,\ m,\ c^n$ as above such that $M \cong Res(E(cz^{t/n})) \otimes_K U_m$.*

Proof. — (1) If the difference module M over $\widehat{K}$ is indecomposable, then M is pure. The proof of (2) that we will produce can be copied verbatim to complete the proof of (1).

(2) Let M/K be pure with slope t/n. We will concentrate on the non trivial case where $n > 1$. We consider now $K_n \otimes_K M$. This difference module over K_n has an action of the generator σ of the Galois group of K_n/K defined by $\sigma z^{1/n} = \zeta z^{1/n}$ with $\zeta = e^{2\pi i/n}$. One writes $K_n \otimes_K M$ as $K_n e \otimes_{\mathbb{C}} V$, where the action of Φ is given by $\Phi(e \otimes v) = z^{t/n} e \otimes Av$ and where $A : V \to V$ is a $\mathbb{C}$-linear map such that all its eigenvalues α satisfy $|q|^{1/n} < |\alpha| \leqslant 1$. We note that this presentation of $K_n \otimes_K M$ is unique. Moreover, the subset $\mathbb{C}[z^{1/n}, z^{-1/n}]e \otimes V$ consists of the elements f in $K_n \otimes_K M$ such that the $\mathbb{C}$-vector space generated by $\{(z^{-t/n}\Phi)^m f|\ m \in \mathbb{Z}\}$ has finite dimension. The vector space $e \otimes V$ consists of the elements $f \in K_n \otimes_K M$ such that there is a monic polynomial $L \in \mathbb{C}[(z^{-t/n}\Phi)]$ with $L(f) = 0$ and all the roots α of L satisfy $|q|^{1/n} < |\alpha| \leqslant 1$. Since σ commutes with Φ on $K_n \otimes_K M$ one has that $e \otimes V$ is invariant under σ. Hence we can write $\sigma(e \otimes v) = e \otimes B(v)$, where $B : V \to V$ is a linear map satisfying $B^n = 1$. The fact that σ and Φ commute translates into $BAB^{-1} = \zeta^t A$. This induces a decomposition $V = V_0 \oplus V_1 \oplus \cdots \oplus V_{n-1}$ into A-invariant subspaces with the property $B(V_i) = V_{i+1}$ (where we use the cyclic notation $V_n = V_0$).

The submodules of M are in bijection with the submodules of $K_n \otimes M$ that are invariant under σ. The latter are in bijection with the A-invariant subspaces W_0 of V_0. This bijection associates to W_0 the σ-invariant submodule $K_n e \otimes_{\mathbb{C}} (\oplus_{i=0}^{n-1} B^i W_0)$. In particular, M is indecomposable if and only if the action of A on V_0 has only one Jordan block. Suppose that A has this form and let c be the eigenvalue of A on V_0, then one has $N \cong Res(E(cz^{t/n})) \otimes_K U_m$ with $m = \dim V_0$. □

We note that there are indecomposable difference modules over K not described in part (2) of Corollary 1.6.

Corollary 1.7. — *Let M be a pure difference module over K with slope $\frac{t}{n}$ where g.c.d.$(t,n) = 1$ and $n > 1$. There exists a difference module N*

over K_n such that $Res(N) \cong M$. The module N is not unique. A similar statement holds for pure q-difference modules over $\widehat{K}$.

Definition 1.8. — *M_{global} for a pure module M over K.*
Suppose that the slope λ of the pure module M over K is an integer. Then $Kf \otimes_K M$, where Kf is the module defined by $\Phi f = z^{-\lambda} f$, is pure of slope 0. It follows that M has a unique finite dimensional $\mathbb{C}$-linear subspace W, such that W is invariant under the operator $z^{-\lambda}\Phi$ and the restriction $A \in \mathrm{GL}(W)$ has the property that every eigenvalue c of A satisfies $|q| < |c| \leqslant 1$. Moreover, the canonical K-linear map $K \otimes W \to M$ is a bijection.

For any $\mathbb{C}$-linear operator L on M and any $c \in \mathbb{C}$, one writes $Eigen(L, c)$ for the generalized eigenspace of L for the eigenvalue c. In other words $Eigen(L, c) = \bigcup_{s \geqslant 1} \ker((L - c)^s, M)$. With this terminology one has that

$$W = \oplus_{c,\ |q|<|c|\leqslant 1} Eigen(z^{-\lambda}\Phi, c)\ .$$

One defines $M_{global} := \mathbb{C}[z, z^{-1}] \otimes W = \oplus_{c \in \mathbb{C}^*} Eigen(z^{-\lambda}\Phi, c)$. This is a free $\mathbb{C}[z, z^{-1}]$-submodule of M, invariant under Φ and Φ^{-1}. Thus M_{global} is a global difference module. Further, the canonical map $K \otimes_{\mathbb{C}[z,z^{-1}]} M_{global} \to M$ is a bijection.

Now we consider a pure difference module M with slope $\lambda = t/n$, where $n \geqslant 1,\ (t, n) = 1$. By Corollary 1.7, there exists a module N over K_n such that $M = Res(N)$. As above, N has a unique finite dimensional $\mathbb{C}$-linear subspace W invariant under $z^{-\lambda}\Phi$, such that all eigenvalues c of the restriction A of $z^{-\lambda}\Phi$ to W satisfy $|q|^{1/n} < |c| \leqslant 1$. One defines $M_{global} := N_{global} = \mathbb{C}[z^{1/n}, z^{-1/n}] \otimes W$. Thus $M_{global} = \oplus_{c \in \mathbb{C}^*} Eigen(z^{-\lambda}\Phi, c)$. As before, M_{global} is a global difference module and the canonical map $K \otimes_{\mathbb{C}[z,z^{-1}]} M_{global} \to M$ is an isomorphism.

In order to see that the definition of M_{global} does not depend on the choice of N one considers the operator $(z^{-\lambda}\Phi)^n = q^{\alpha} z^{-t}\Phi^n$, where α is some rational number. It follows that M_{global} is also equal to $\oplus_{c \in \mathbb{C}^*} Eigen(z^{-t}\Phi^n, c)$. This expression is clearly independent of the choice of N. Thus we can formulate the definition of $M_{global} \subset M$ for a pure module over K of slope $\lambda = t/n$ with $n \geqslant 1,\ g.c.d.(t, n) = 1$ by the statement:
The following properties of $m \in M$ are equivalent.
(1) $m \in M_{global}$.
(2) *The $\mathbb{C}$-vector space generated by $\{(z^{-t}\Phi^n)^s m \mid s \geqslant 0\}$ has finite dimension.*
(3) *There exists a $L \in \mathbb{C}[T],\ L \neq 0$ such that $L(z^{-t}\Phi^n)(m) = 0$.*

The main technical difficulties in this paper arise from pure modules M with non integer slope $\lambda = t/n$. There are two methods to handle these. The first one (Corollary 1.7) is to write $M = Res(N)$ for some difference module N over K_n. The second one, used in the proof of Corollary 1.6, replaces M by $K_n \otimes M$ provided with the action of the Galois group of K_n/K. Both methods have their good and weak points. Now we develope the second method in more detail. The main idea is to replace a pure differential module N over K by $M = K_\infty \otimes_K N$ with decent data D. Here K_∞ denotes the algebraic closure of K. With this method one can more easily describe tensor products of pure modules over K.

1.4.1. Pure difference modules over K_∞ with descent data

K_∞ denotes the algebraic closure of K and Gal denotes the Galois group of K_∞/K. Let M be a difference module over K_∞. *Descent data* D for M means a map $\sigma \in Gal \mapsto D(\sigma)$ satisfying:
$D(\sigma)$ is a σ-linear bijection on M, $D(\sigma)$ commutes with Φ,
$D(\sigma_1)D(\sigma_2) = D(\sigma_1\sigma_2)$ and the stabilizer of any $m \in M$, i.e., the group $\{\sigma \in Gal \mid D(\sigma)m = m\}$, is an open subgroup of Gal.

One associates to a difference module N over K the module $M := K_\infty \otimes N$ with descent data given by $D(\sigma)(f \otimes n) = \sigma(f) \otimes n$ for all $f \in K_\infty$ and $n \in N$. This induces a functor from the category of the difference modules over K to the category of the difference modules over K_∞ provided with descent data.

Proposition 1.9. — *$N \mapsto (K_\infty \otimes N, D)$ is an equivalence of Tannakian categories.*

Proof. — The essential thing to prove is that any pair (M, D) is isomorphic to $(K_\infty \otimes N, D)$ for some difference module N over K.

Take a basis $e_1, \ldots, e_r$ of M over K_∞. Let the open subgroup $H := \{\sigma \in Gal \mid \sigma(e_j) = e_j \text{ for all } j\}$ have index m in Gal. Then $K_\infty^H = K_m$ and $M^H = K_m e_1 + \cdots + K_m e_r$. The cyclic group $Gal/H = Gal(K_m/K)$ acts on M^H. This action induces an element of $H^1(Gal(K_m/K), \mathrm{GL}(r, K_m))$. By Hilbert 90, this cohomology set is trivial. It follows that M^H contains a basis $f_1, \ldots, f_r$ over K_m, consisting of Gal-invariant elements. Now $N := Kf_1 \oplus \cdots \oplus Kf_r$ is equal to M^{Gal} and the natural map $K_\infty \otimes_K N \to M$ is an isomorphism. Since Φ commutes with the action of Gal, one has $\Phi(N) = N$. Thus N is a difference module over K and clearly induces the pair (M, D).

The tensor product of two pairs (M_1, D_1), (M_2, D_2) is defined as $(M_1 \otimes_{K_\infty} M_2, D_1 \otimes D_2)$. We note that the tensor product $D_1(\sigma) \otimes D_2(\sigma)$ of two σ-linear maps makes sense. It is easily seen that the above equivalence respects tensor products. $\square$

For a pure difference module N over K of slope λ, the module $M = K_\infty \otimes N$ is also pure with slope λ and has the form $K_\infty \otimes_{\mathbb{C}} V$, where V is a finite dimensional $\mathbb{C}$-vector space provided with an element $A \in \mathrm{GL}(V)$. The action of Φ on M is given by $\Phi(f \otimes v) = z^\lambda \phi(f) \otimes A(v)$.

The subspace V is not unique. By changing V, the eigenvalues of A are multiplied by arbitrary, rational powers of q. We normalize A and V as follows.

Choose a $\mathbb{Q}$-linear subspace $L \subset \mathbb{C}$ such that $L \oplus \mathbb{Q} = \mathbb{C}$. One requires that every eigenvalue c of A has the form $e^{2\pi i(a_0(c)+a_1(c)\tau)}$ with $a_0(c), a_1(c) \in \mathbb{R}$ and $a_1(c) \in L$.

After this normalization the subspace V of M is unique. Indeed, V is the direct sum of the kernels of $(z^{-\lambda}\Phi - c)^s$ with $s >> 0$ and $c \in \mathbb{C}^*$ with $a_1(c) \in L$.

We note that the use of this subspace L is somewhat artificial. It can be avoided at the cost of verifying that formulas that we will produce are independent of certain choices.

One observes that V is invariant under $D(\sigma)$ for all $\sigma \in Gal$. The group Gal is identified with $\widehat{\mathbb{Z}}$ and the action of Gal is expressed by $\sigma(z^\lambda) = e^{2\pi i\lambda\sigma} z^\lambda$. For the operators A and $D(\sigma)$, restricted to V, one finds the equality $AD(\sigma) = e^{2\pi i\lambda\sigma} D(\sigma)A$. Thus we have associated to a pure difference module N over K a tuple $data(N) := (\lambda, V, A, \{D(\sigma)\})$ with

- $\lambda \in \mathbb{Q}$.
- V a vector space over $\mathbb{C}$ of finite dimension.
- $A \in \mathrm{GL}(V)$ with eigenvalues in the subgroup $\{c = e^{2\pi i(a_0(c)+a_1(c)\tau)} | a_0(c) \in \mathbb{R},\ a_1(c) \in L\}$ of $\mathbb{C}^*$.
- a homomorphism $\sigma \in Gal \cong \widehat{\mathbb{Z}} \mapsto D(\sigma) \in \mathrm{GL}(V)$ satisfying $AD(\sigma) = e^{2\pi i\lambda\sigma} D(\sigma)A$.

On the other hand an object $(\lambda, V, A, \{D(\sigma)\})$ as above defines a pure module N over K of slope λ in the following way. Consider $M := K_\infty \otimes V$ with Φ given by $\Phi(f \otimes v) = z^\lambda \phi(f) \otimes A(v)$ and with descent data given by $D(\sigma)(f \otimes v) = \sigma(f) \otimes D(\sigma)v$. Then $N := M^{Gal}$.

Consider a morphism $f : N_1 \to N_2,\ f \neq 0$ between pure modules. Then N_1, N_2 have the same slope λ and f induces a morphism from $data(N_1)$ to $data(N_2)$, i.e., a linear map F between the two $\mathbb{C}$-vector spaces equivariant for the maps of the data. On the other hand, a $\mathbb{C}$-linear map F, equivariant for the maps of the data, comes from a unique morphism $f : N_1 \to N_2$.

Thus $N \mapsto data(N)$ is an equivalence between the category of the pure modules over K and the category of tuples $(\lambda, V, A, \{D(\sigma)\})$ defined above. One observes the following useful properties.

For pure difference modules N_i with $data(N_i) = (\lambda_i, V_i, A_i, \{D_i(\sigma)\})$ for $i = 1, 2$ one has the nice formula
$data(N_1 \otimes N_2) = (\lambda_1 + \lambda_2, V_1 \otimes V_2, A_1 \otimes A_2, \{D_1(\sigma) \otimes D_2(\sigma)\})$.

Let the pure modules N have $data(N) = (\lambda, V, A, \{D(\sigma)\})$. Then the dual module N^* has data $(-\lambda, V^*, B, \{E(\sigma)\})$, where V^* is the dual of V; $B = (A^{-1})^*$ and $E(\sigma) = (D(\sigma)^{-1})^*$.

The Φ-equivariant pairing $N \times N^* \to K$, given by $(n, \ell) \mapsto \ell(n) \in K$ translates for the data of N and N^* into the usual pairing $V \times V^* \to \mathbb{C}$, given by $(v, \ell) \mapsto \ell(v) \in \mathbb{C}$. This pairing is equivariant with respect to the prescribed actions on V and V^*.

2. Vector bundles and q-difference modules

We recall that O denotes the algebra of the holomorphic functions on $\mathbb{C}^*$ and that a difference module M over O is a left module over the ring $O[\Phi, \Phi^{-1}]$, free of some rank $m < \infty$ over O. Further $pr : \mathbb{C}^* \to E_q := \mathbb{C}^*/q^{\mathbb{Z}}$ denotes the canonical map. One associates to M the vector bundle $v(M)$ of rank m on E_q given by $v(M)(U) = \{f \in O(pr^{-1}U) \otimes_O M |\ \Phi(f) = f\}$, where, for any open $V \subset \mathbb{C}^*$, $O(V)$ is the algebra of the holomorphic functions on V.

On the other hand, let a vector bundle $\mathcal{M}$ of rank m on E_q be given. Then $\mathcal{N} := pr^*\mathcal{M}$ is a vector bundle on $\mathbb{C}^*$ provided with a natural isomorphism $\sigma_q^*\mathcal{N} \to \mathcal{N}$, where σ_q is the map $\sigma_q(z) = qz$. One knows that $\mathcal{N}$ is in fact a free (or trivial) vector bundle of rank m on $\mathbb{C}^*$ (see [For], p. 204). Therefore, M, the collection of the global sections of $\mathcal{N}$, is a free O-module of rank m provided with an invertible action Φ satisfying $\Phi(fm) = \phi(f)\Phi(m)$ for $f \in O$ and $m \in M$. It is easily verified that the above describes an equivalence v of tensor categories.

The equivalence v extends to an equivalence between the left $O[\Phi, \Phi^{-1}]$-modules which are finitely generated as O-module and the coherent sheaves

on E_q. This is an equivalence of Tannakian categories.

By an *admissible difference module over* O we will mean a left $O[\Phi, \Phi^{-1}]$-module which is a direct limit of left $O[\Phi, \Phi^{-1}]$-modules of finite type over O. The equivalence v extends to a Tannakian equivalence between category of the admissible difference modules over O and the category of the quasi-coherent sheaves on E_q.

LEMMA 2.1. — *There are isomorphisms* $\ker(\Phi - 1, M) \to H^0(E_q, v(M))$ *and* $\operatorname{coker}(\Phi - 1, M) \to H^1(E_q, v(M))$ *between these functors defined on the category of the admissible difference modules* M *over* O.

Proof. — The isomorphism $\ker(\Phi - 1, M) \to H^0(E_q, v(M))$ follows from the definition of v. Let v^{-1} denote the 'inverse' of the functor v. Then $\mathcal{M} \mapsto \ker(\Phi - 1, v^{-1}(\mathcal{M}))$ is canonically isomorphic to $\mathcal{M} \mapsto H^0(E_q, \mathcal{M})$. One observes that an exact sequence $0 \to M_1 \to M_2 \to M_3 \to 0$ of admissible difference modules over O induces (by the snake lemma) an exact sequence

$$0 \to \ker(\Phi - 1, M_1) \to \ker(\Phi - 1, M_2) \to \ker(\Phi - 1, M_3) \to$$

$$\operatorname{coker}(\Phi - 1, M_1) \to \operatorname{coker}(\Phi - 1, M_2) \to \operatorname{coker}(\Phi - 1, M_3) \to 0 \ .$$

From this it easily follows that the first right derived functor of the functor $M \mapsto \ker(\Phi - 1, M)$, on the category of admissible modules over O, is equal to $\operatorname{coker}(\Phi - 1, M)$. Now the second isomorphism of functors follows. □

EXAMPLES 2.2. — (1) Consider $M = Oe$ with $\Phi(e) = e$. Then $v(M)$ is the structure sheaf O_{E_q} of E_q. Any element in $m \in M$ can be written uniquely as $m = \sum_{n \in \mathbb{Z}} a_n z^n e$. Then $(\Phi - 1)m = \sum_{n \in \mathbb{Z}} (q^n - 1) a_n z^n e$. One observes that $\ker(\Phi - 1, M) = \mathbb{C}e$ and that $\operatorname{coker}(\Phi - 1, M)$ is represented by $\mathbb{C}e$. This illustrates Lemma 2.1.

(2) Consider difference module $M = Oe$ with $\Phi e = ce$ and $c \in \mathbb{C}^*$, $|q| < |c| \leqslant 1$. If $c \neq 1$, then $\theta_c e$ is a meromorphic section of $v(M)$ with divisor $-pr(c^{-1}) + pr(1)$. One concludes that $v(M) \cong O_{E_q}(pr(c^{-1}) - pr(1))$. Thus one finds all line bundles of degree 0 on E_q in this way.

(3) Consider the difference module $M := Oe$ with $\Phi e = (-z)e$. There is a Φ-invariant element, namely Θe. This is a global section of $v(M)$. The cokernel of the morphism $O_{E_q} \to v(M)$, given by $1 \mapsto \Theta e$, is a skyscraper sheaf with support $\{1\}$ and stalk $\mathbb{C}$ at that point. Indeed, the function Θ has simple zeros at $q^{\mathbb{Z}}$. One concludes that $v(M) \cong O_{E_q}([1])$. Using tensor products one obtains that the line bundle $v(M)$, with $M = Oe$, $\Phi e = cz^t e$, has degree t. Moreover every line bundle on E_q is obtained in this way.

(4) Let $M = O/O\Theta$ with the Φ-action induced by the usual one of O. Then $v(M)$ is the skyscraper sheaf on E_q with support $\{1\}$ and with stalk $\mathbb{C}$ at that point.

We recall that a *split difference module M over K* is a direct sum of pure modules M_i. The global module M_{global} over $\mathbb{C}[z, z^{-1}]$ associated to M is by definition the direct sum of the global modules $(M_i)_{global}$. A morphism $f : M \to N$ between split modules is easily seen to be the direct sum of morphisms between the pure components of M and N. In particular f maps M_{global} to N_{global}.

One associates to a split difference module M the difference module $O \otimes_{\mathbb{C}[z,z^{-1}]} M_{global}$ and, by Lemma 2.1, a vector bundle on E_q. For notational convenience we write again $v(M)$ for this vector bundle. In this way we obtain a functor v from the category of the split difference modules over K to the category of vector bundles on E_q. One observes that $\mathrm{Hom}(M_1, M_2) \to \mathrm{Hom}(v(M_1), v(M_2))$ is $\mathbb{C}$-linear and injective. Moreover, one easily sees that v preserves tensor products.

THEOREM 2.3. — *The functor v from the category of the split difference modules over K to the category of the vector bundles on E_q is bijective on isomorphy classes of objects. This bijection respects tensor products.*

Proof. — We have to show that v induces a bijection between the isomorphy classes of the indecomposable objects in the two categories. We start by proving that for an indecomposable pure difference module M the corresponding vector bundle $v(M)$ is indecomposable.

From Examples 2.2 one concludes that v provides a bijection between the isomorphy classes of the difference modules of dimension 1 over K and the isomorphy classes of all line bundles on E_q.

By Corollary 1.6, an indecomposable pure difference module has the form $M = Res(E(cz^{t/n})) \otimes U_m$, with unique $n \geqslant 1,\ (t, n) = 1,\ m \geqslant 1$ and c^n such that $|q| < |c^n| \leqslant 1$. The vector bundle $v(M)$ has clearly rank nm. The exterior product $\Lambda^{nm} M$ is equal to Kf with $\Phi f = sz^{tm} f$ for some $s \in \mathbb{C}^*$. Thus $v(M)$ has degree tm. The case $nm = 1$ has been treated above and we suppose now $nm > 1$.

One can present M_{global} as $\mathbb{C}[z^{1/n}, z^{-1/n}] \otimes W$, with W a $\mathbb{C}$-linear space of dimension m and Φ given by $\Phi(1 \otimes w) = cz^{t/n} \otimes U(w)$ where U is a unipotent map with minimal polynomial $(U - 1)^m = 0$. Then $O \otimes_{\mathbb{C}[z,z^{-1}]} M_{global}$ can be represented as $H_n \otimes W$, where H_n consists of the convergent Laurent series in $z^{1/n}$. Thus H_n consists of the expressions $\sum_{k=-\infty}^{+\infty} a_k z^{k/n}$ with

$\lim_{|k|\to\infty} |a_k|^{1/|k|} = 0$. One provides W with some norm $\| \ \|$. The elements of $H_n \otimes W$ have the form $\sum_{k=-\infty}^{+\infty} z^{k/n} \otimes w_k$ with $\lim_{|k|\to\infty} \|w_k\|^{1/|k|} = 0$. Then Φ acts on $H_n \otimes W$ by

$$\Phi(\sum z^{k/n} \otimes w_k) = \sum q^{k/n} z^{k/n} c z^{t/n} \otimes U(w_k) \ .$$

Write $\Psi = z^{-t}\Phi^n$. Then

$$\Psi(\sum z^{k/n} \otimes w_k) = \sum z^{k/n} \otimes dq^k U^n(w_k), \text{ with } d = c^n q^{t(n-1)/2} \ .$$

For each k, the vector space $z^{k/n} \otimes W$ is invariant under Ψ and this operator has eigenvalues $q^k d$ on this vector space. One concludes from this that the subset $M_{global} \subset H_n \otimes W$ consists of the elements fe such that there exists a non zero polynomial $L \in \mathbb{C}[T]$ with $L(\Psi)(fe) = 0$. This has as consequence that every O-linear endomorphism A of $O \otimes_{\mathbb{C}[z,z^{-1}]} M_{global}$, commuting with Φ, is the O-linear extension of a unique $\mathbb{C}[z, z^{-1}]$-linear endomorphism B of M_{global} commuting with Φ.

A direct sum decomposition of $v(M)$ induces a O-linear endomorphism A of $O \otimes_{\mathbb{C}[z,z^{-1}]} M_{global}$ commuting with Φ and such that $A^2 = A$. The corresponding B induces a direct sum decomposition of M_{global}, contradicting that M is indecomposable. Thus $v(M)$ is indecomposable. A similar reasoning proves that for indecomposable M_1, M_2 the relation $v(M_1) \cong v(M_2)$ implies that $M_1 \cong M_2$. In this way we have found a collection of indecomposable vector bundles on E_q. That we have found all of them follows at once from the classification given in [At], Theorem 10. Indeed, Atiyah constructs a certain indecomposable vector bundle of rank r and degree d, called $E_A(r, d)$. Let $h = (r, d)$. Then every indecomposable vector bundle of rank r and degree d has the form $L \otimes E_A(r, d)$ with L a line bundle of degree 0. This L is unique up to multiplication with a line bundle N such that $N^{\otimes r/h}$ is the trivial line bundle. $\square$

The final part of the proof of Theorem 2.3 depends on [At]. We present now a proof which only uses a simple result of this paper, namely Lemma 11, formulated as follows:
Let W be an indecomposable vector bundle of rank m and degree 0 on an elliptic curve E, then $W = L \otimes W'$, with L a line bundle of degree 0 and such that the indecomposable W' has a sequence of subbundles $0 = W'_0 \subset W'_1 \subset \cdots \subset W'_m = W'$ such that each quotient W'_{i+1}/W'_i is isomorphic to O_E.

Proof. — V is an indecomposable vector bundle on E_q, rank nm and degree tm with $n, m \geqslant 1$, $g.c.d.(t, n) = 1$. As before we consider $pr : \mathbb{C}^*_z \to E_q = \mathbb{C}^*_z/q^{\mathbb{Z}}$. The index z means that we use z as variable on this copy

of $\mathbb{C}^*$. Write $M := H^0(\mathbb{C}^*, pr^*(V))$. It suffices to produce a $\mathbb{C}[z, z^{-1}]$-submodule $M_0 \subset M$, invariant under Φ and Φ^{-1}, such that the natural map $O \otimes_{\mathbb{C}[z,z^{-1}]} M_0 \to M$ is bijective and $K \otimes_{\mathbb{C}[z,z^{-1}]} M_0$ is a pure module.

Let $\beta : \mathbb{C}_s^* \to \mathbb{C}_z^*$ be given by $s \mapsto s^n = z$ (or by $s = z^{1/n}$). Define the elliptic curve E by $\gamma : \mathbb{C}_s^* \to E := \mathbb{C}_s^*/(q^{1/n})^{\mathbb{Z}}$. There is an induced morphism of $\alpha : E \to E_q$, of degree n, such that $\alpha \circ \gamma = pr \circ \beta$.
The map α is an unramified cyclic covering of degree n. Let σ denote a generator of the automorphism group of this covering. One can take for σ the automorphism of $\mathbb{C}_s^* \to \mathbb{C}_z^*$, given by $s \mapsto e^{2\pi i/n} s$. The last map will also be called σ.

The vector bundle $\alpha^* V$ on E has rank nm and degree tnm. Then $\alpha^* V$ is a direct sum of indecomposable vector bundles $W_1 \oplus \cdots \oplus W_r$ on E. Let $W_1, W_2, \ldots, W_{r'}$ with $r' \leqslant r$ be all the W_i which have the same rank and degree as W_1. The direct sum $W_1 \oplus \cdots \oplus W_{r'}$ is invariant under the action of σ^*. Since V is indecomposable, one has $r' = r$. Thus all W_i have the same rank and degree. It follows that $O_E(-t[1_E]) \otimes \alpha^* V$ is the direct sum of indecomposable vector bundles W_i' on E of degree 0 and rank nm/r.

Using Lemma 11 of [At], we conclude that the difference module $H := H^0(\mathbb{C}_s^*, \gamma^* W_i')$ over $O(\mathbb{C}_s^*)$ has a sequence of submodules $0 = H_0 \subset \cdots \subset H_{nm/r} = H$ such that each quotient has the form $O(\mathbb{C}_s^*)e$ with $\Phi e = e$. Thus H has a basis $e_1, \ldots, e_{mn/r}$ over $O(\mathbb{C}_s^*)$ such that the matrix of Φ w.r.t. this basis is upper triangular and all its diagonal entries are 1. One easily verifies that a base change turns this matrix into a matrix with constant coefficients.

The difference module over $O(\mathbb{C}_s^*)$ associated to $O_E(t[1_E])$ has the form $O(\mathbb{C}_s^*)e$ with $\Phi e = cs^t e$ for some constant c. By taking the tensor product with this module and direct sums, we conclude that the difference module N over $O(\mathbb{C}_s^*)$, associated to $\gamma^* \alpha^* V = \beta^* pr^* V$, has a basis for which the matrices of Φ and Φ^{-1} have coefficients in $\mathbb{C} \cdot s^{\mathbb{Z}}$. In particular, there is a $\mathbb{C}[s, s^{-1}]$-module N_0 of N, invariant under Φ and Φ^{-1}, such that $O(\mathbb{C}_s^*) \otimes_{\mathbb{C}[s,s^{-1}]} N_0 = N$. This submodule is, by construction, invariant under the action of σ. Therefore the $\mathbb{C}[z, z^{-1}]$-module $M_0 := N_0^{\sigma}$ is finitely generated and invariant under the actions of Φ and Φ^{-1}. Moreover $O(\mathbb{C}_z^*) \otimes_{\mathbb{C}[z,z^{-1}]} M_0 = N^{\sigma} = M$ and $K \otimes_{\mathbb{C}[z,z^{-1}]} M_0$ is pure. □

Remarks 2.4. — Tensor products of pure difference modules over K.
One rediscovers Part III of [At] (for the base field $\mathbb{C}$) by using Theorem 2.3 and some calculations for difference modules. We will give some results.

(1) The indecomposable vector bundle corresponding to U_m is called F_m in [At]. The tensor product $U_a \otimes U_b$ corresponds to the tensor product of two

unipotent operators A, B on vector spaces V, W of dimensions a, b, having each only one Jordan block. One can find a decomposition of the unipotent operator $A \otimes B \in \mathrm{GL}(V \otimes W)$ as a direct sum of Jordan blocks. This will produce the formulas in [At], Theorem 8 and our method is close to the remarks [At], p. 438-439.

(2) Consider the module $(K_n e, \Phi e = cz^{t_1/n_1} e)$ with $(t_1, n_1) = 1$ and $n = dn_1$ with $d > 1$. We note that c is determined up to an element of $\mu_{n_1} \times q^{\mathbb{Z}/n}$, where μ_k denotes the group of the kth roots of unity. This module decomposes as the direct sum of the irreducible modules $(K_{n_1} e_j, \Phi e_j = q^{j/n} c z^{t_1/n_1} e_j)$ for $j = 0, \ldots, d-1$. A change of c will permute these modules.

(3) $M_i := (K_{n_i} e_i, \Phi e_i = c_i z^{t_i/n_i} e_i)$ for $i = 1, 2$ with $(t_i, n_i) = 1$. Let $d = (n_1, n_2)$ and $n = \frac{n_1 n_2}{d}$. We note that c_i is determined up to an element in $\mu_{n_i} \times q^{\mathbb{Z}/n_i}$. Write $t_1/n_1 + t_2/n_2 = t_3/n_3$ with $(t_3, n_3) = 1$ and $n = kn_3$. Then $M_1 \otimes M_2$ is the direct sum of $(K_n f_j, \Phi f_j = \zeta_j c_1 c_2 z^{t_3/n_3} f_j)$ for $j = 0, \ldots, d-1$ and suitable nth roots of unity ζ_j. Each term has a further decomposition, according to (2), if $k > 1$.

(4) $M = (K_n e, \Phi e = cz^{t/n} e)$ with $(t, n) = 1$. The dual M^* is equal to $(K_n e^*, \Phi e^* = c^{-1} z^{-t/n} e^*)$. Further $M \otimes M^*$ is the direct sum of the n^2 difference modules $(K e_{s,t}, \Phi e_{s,t} = q^{s/n} e^{2\pi i t/n} e_{s,t})$ with $0 \leqslant s, t < n$. This corresponds to the direct sum of all line bundles L of order dividing n.

(5) The homomorphism $\mathrm{Hom}(M_1, M_2) \to \mathrm{Hom}(v(M_1), v(M_2))$*, where* M_1, M_2 *are split modules over* K*, is in general not surjective. Indeed, the category of the split difference modules over* K *is Tannakian and the category of the vector bundles on* E_q *is not even an abelian category.*

THEOREM 2.5. — *Let* M *be a pure difference module over* K *with slope* $\lambda < 0$*. The maps*
(1) $\mathrm{coker}(\Phi - 1, M_{global}) \to \mathrm{coker}(\Phi - 1, O \otimes_{\mathbb{C}[z, z^{-1}]} M_{global})$
(2) $\mathrm{coker}(\Phi - 1, M_{global}) \to \mathrm{coker}(\Phi - 1, K \otimes_{\mathbb{C}[z, z^{-1}]} M_{global})$
are isomorphisms. Moreover, $\mathrm{coker}(\Phi - 1, M_{global})$ *is canonical isomorphic to* $H^1(E_q, v(M))$ *and has dimension* $-\lambda \cdot dim_K M$ *over* $\mathbb{C}$*.*

Before starting the technical proof we study the simplest example:

$$M_{global} = \mathbb{C}[z, z^{-1}]e \text{ with } \Phi e = z^{-1} e \ .$$

An element of $O \otimes M_{global} = Oe$ has the form $\sum a_n z^n e$. By Examples 2.2, $v(O \otimes M_{global}) = O_{E_q}(-[1])$ and by Lemma 2.1, one has $\ker(\Phi - 1, O \otimes M_{global}) = H^0(E_q, O_{E_q}(-[1])) = 0$. Consider the equation

$$(\Phi - 1) \sum a_n z^n e = \sum (a_{n+1} q^{n+1} - a_n) z^n e = \sum b_n z^n e \ ,$$

for a given $\sum b_n z^n e \in Oe$. There is at most one solution. If there is a solution then its coefficients satisfy the recurrence $a_{n+1}q^{n+1} - a_n = b_n$ which can also be written as

$$a_{n+1}q^{(n+2)(n+1)/2} - a_n q^{(n+1)n/2} = b_n q^{(n+1)n/2} \ .$$

Summation of these recurrences over all $n \in \mathbb{Z}$ implies $\sum b_n q^{(n+1)n/2} = 0$.

On the other hand, the condition $\sum b_n q^{(n+1)n/2} = 0$ leads to a formula $a_n := \sum_{s=n}^{\infty} -b_s q^{(s+1)s/2-(n+1)n/2}$. One easily verifies that this infinite sum converges, that the series $\sum a_n z^n e$ belongs to Oe and solves the equation. If, moreover, $\sum b_n z^n e \in \mathbb{C}[z, z^{-1}]e = M_{global}$, then also $\sum a_n z^n e$ lies in M_{global}. This proves part (1) of Theorem 2.5 for this example.

Proof. — (1) Let $-t/n < 0$, with $t, n \geqslant 1,\ g.c.d.(t, n) = 1$, be the slope of M. Then M_{global} can be written as $\mathbb{C}[z^{1/n}, z^{-1/n}] \otimes_{\mathbb{C}} W$ with Φ action given by $\Phi(f \otimes w) = z^{-t/n}\phi(f) \otimes A(w)$ for some $A \in \mathrm{GL}(W)$. Then $O \otimes M_{global}$ can be written as $O_n \otimes_{\mathbb{C}} W$, where O_n consists of the convergent Laurent series in $z^{1/n}$. The action of Φ is again given by $\Phi(f \otimes w) = z^{-t/n}\phi(f) \otimes A(w)$. Consider the equation

$$(\Phi - 1)(\sum_{k\in\mathbb{Z}} z^{k/n} \otimes x_k) = \sum_{k\in\mathbb{Z}} z^{k/n} \otimes (q^{(k+t)/n}A(x_{k+t}) - x_k) = \sum_{k\in\mathbb{Z}} z^{k/n} \otimes w_k,$$

where the given series $\sum z^{k/n} \otimes w_k$ is either finite or convergent. This yields recurrence relations for the x_k. Write $k = k_0 + st$ with $k_0 \in \{0, 1, \ldots, t-1\}$ and $s \in \mathbb{Z}$. The recurrence relations are

$$q^{(k_0+(s+1)t)/n}A(x_{k_0+(s+1)t}) - x_{k_0+st} = w_{k_0+st}$$

for $k_0 \in \{0, 1, \ldots, t-1\},\ s \in \mathbb{Z}$.

Put $m(k_0, s) = \frac{sk_0}{n} + \frac{s(s+1)t}{2n}$. Then the recurrences can be rewritten as

$$q^{m(k_0,s+1)}A^{s+1}(x_{k_0+(s+1)t}) - q^{m(k_0,s)}A^s(x_{k_0+st}) = q^{m(k_0,s)}A^s(w_{k_0+st}) \ .$$

Suppose that there exists a convergent solution $\sum z^{k/n} \otimes x_k$, then, for each k_0, the sum over all $s \in \mathbb{Z}$ of the left hand side is 0. Thus a necessary condition for the existence of a convergent solution is

$$\sum_{s\in\mathbb{Z}} q^{m(k_0,s)}A^s(w_{k_0+st}) = 0 \text{ for } k_0 = 0, 1, \ldots, t-1 \ .$$

This condition is also sufficient for the existence of a convergent solution. Indeed, the recurrences and the above condition on the coefficients $\{w_{k_0+st}\}$

lead to the formula

$$x_{k_0+st} = \sum_{a=s}^{\infty} -q^{m(k_0,a)-m(k_0,s)} A^{a-s}(w_{k_0+st}) \ .$$

The possibly infinite sum in this formula converges and one can verify that the resulting series $\sum z^{k/n} \otimes x_k$ lies in $O_n \otimes W$ and solves the above equation.

If there are ony finitely many $w_k \neq 0$ and the above condition is satisfied, then the unique solution $\sum_k z^k \otimes x_k$ has only finitely many $x_k \neq 0$. From this observation statement (1) follows.
The proof of (2) is similar. Lemma 2.1 and the above computation yield a proof of the last statement of the theorem. □

2.6 (Canonical representatives for $\mathrm{coker}(\Phi - 1, M_{global})$). — Let M be a pure difference module over K with slope $-t/n$ and $n, t \geqslant 1$, $g.c.d.(t,n) = 1$. As in the proof of Theorem 2.5, one writes $M_{global} = \mathbb{C}[z^{1/n}, z^{-1/n}] \otimes W$ with Φ action given by $\Phi(f \otimes w) = z^{-t/n}\phi(f) \otimes A(w)$. One requires that $|q|^{1/n} < |c| \leqslant 1$ for every eigenvalue of A. This makes the presentation unique.

Define $W^+ := (\mathbb{C}1 + \mathbb{C}z^{1/n} + \cdots + \mathbb{C}z^{(t-1)/n}) \otimes W$. We claim that

$$W^+ \to \mathrm{coker}(\Phi - 1, M_{global}) \text{ is a bijection.}$$

Indeed, write $w^+ \in W^+$ as $w^+ = \sum_{i=0}^{t-1} z^{i/n} \otimes w_i$. Then w^+ lies in the image of $\Phi - 1$ if and only if for all $k_0 \in \{0, 1, \ldots, t-1\}$ one has $\sum_{s \in \mathbb{Z}} q^{m(k_0,s)} w_{k_0+st} = 0$. This implies that $w^+ = 0$. Hence $W^+ \to \mathrm{coker}(\Phi - 1, M_{global})$ is injective. Since both spaces have dimension $t \cdot \dim W$, the map is a bijection.

For any $\mathbb{C}$-linear operator T on M, one writes $Eigen(T, c)$ for the generalized eigenspace of T for the eigenvalue $c \in \mathbb{C}$. Thus $Eigen(T, c) = \bigcup_{s \geqslant 1} \ker((T - c)^s, M)$. With this notation one observes that

$$W^+ = \oplus_{|q|^{t/n} < |c| \leqslant 1} Eigen(z^{t/n}\Phi, c) \text{ and } W^+ = \oplus_{|q|^t < |c| \leqslant 1} Eigen((z^{t/n}\Phi)^n, c) \, .$$

We write $Repr(M_{global})$ or $Repr(M)$ for the vector space W^+ of representative of $\mathrm{coker}(\Phi - 1, M)$. We note that $Repr(M)$ does not depend on the choice of the module N over K_n with $M = Res(N)$.

$M \mapsto Repr(M)$ has some functorial properties, namely:

A morphism $f : M_1 \to M_2$ between pure modules of the same negative slope maps $Repr(M_1)$ to $Repr(M_2)$.
This follows easily from the second description of W^+.

Let M_1, M_2 denote pure modules of slopes $\lambda_1, \lambda_2 < 0$. Then $Repr(M_1) \otimes_{\mathbb{C}} Repr(M_2)$ is mapped to $Repr(M_1 \otimes M_2) \subset M_1 \otimes M_2$.

The proof goes as follows. The pure module $M_3 := M_1 \otimes_K M_2$ has slope $\lambda_3 = \lambda_1 + \lambda_2$. Choose an integer $n \geqslant 1$ such that $n\lambda_1, n\lambda_2 \in \mathbb{Z}$. For each $i = 1, 2, 3$, the vector space $Repr(M_i)$ is equal to $\oplus_{|q|^{-n\lambda_i} < |c| \leqslant 1} Eigen((z^{-\lambda_i}\Phi)^n, c)$. The statement that we want to prove follows from $(z^{\lambda_i}\Phi)^n = q^{-\lambda_i n(n-1)/2} z^{-n\lambda_i}\Phi^n$ and the observation that $q^{-\lambda_3 n(n-1)/2} z^{-n\lambda_3}\Phi^n(w_1 \otimes w_2)$ equals

$$q^{-\lambda_1 n(n-1)/2} z^{-n\lambda_1}\Phi^n(w_1) \otimes q^{-\lambda_2 n(n-1)/2} z^{-n\lambda_2}\Phi^n(w_2)$$

for $w_1 \in Repr(M_1),\ w_2 \in Repr(M_2)$.

There is a second method, based on subsection 1.4.1, to define a subspace of representatives for $\mathrm{coker}(\Phi - 1, N)$ with N pure of slope $\lambda < 0$. Let $data(N) = (\lambda, V, A, \{D(\sigma)\})$ and $K_\infty \otimes N = K_\infty \otimes_{\mathbb{C}} V$. Then

$$V^+ := \sum_{s \in \mathbb{Q},\ 0 \leqslant s < -\lambda} z^s \otimes V$$

can be seen to be a vector space of representative for $\mathrm{coker}(\Phi - 1, K_\infty \otimes V)$. This set of representatives is invariant under the action of Gal and therefore $Repr^*(N) := (V^+)^{Gal}$ (depending on the choice of $L \subset \mathbb{C}$) is a $\mathbb{C}$-subspace of N_{global} representing $\mathrm{coker}(\Phi - 1, N)$. Again $N \mapsto Repr^*(N)$ has the same functorial properties as $M \mapsto Repr(M)$. However the behaviour of $Repr^*(N)$ with respect to tensor products is more transparent.

3. Moduli spaces for q-difference equations

For a difference module M over K we consider again the slope filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r$. One associates to M the graded module $gr(M) := \oplus_{i=1}^r M_i/M_{i-1}$. The aim is to produce a moduli space for the collection of all q-difference modules over K with a fixed $gr(M)$. This problem is the theme of [R-S-Z]. As Ch. Zhang has remarked, the problem is already present in a paper of G. Birkhoff and P.E. Guenther, see [Bir]. Here we treat the general case (i.e., arbitrary, rational slopes).

Fix a split difference module $S = P_1 \oplus P_2 \oplus \cdots \oplus P_r$ (with $r \geqslant 2$) over K such that each P_i is pure of slope λ_i and $\lambda_1 < \lambda_2 < \cdots < \lambda_r$. One considers the pairs (M, f) consisting of a difference module over K and an isomorphism $f : gr(M) \to S$. Two pairs $(M(i), f(i)),\ i = 1, 2$ are called equivalent if there exists an isomorphism $f : M(1) \to M(2)$ such

that $gr(M(1)) \stackrel{gr(f)}{\to} gr(M(2)) \stackrel{f(2)}{\to} S$ coincides with $f(1)$. One wants to give the collection of equivalence classes a structure of algebraic variety over $\mathbb{C}$. A naive, but useful way is to produce in every equivalence class a unique representative (M, f). Our aim is to define a moduli functor $\mathcal{F}$ and to find a fine moduli space for $\mathcal{F}$.

It is useful to give another formulation for the pairs (M, f). Let U be the group of the K-linear maps $u : S \to S$ such that u respects the filtration $P_1 \subset P_1 \oplus P_2 \subset \cdots \subset S$ of S and moreover $gr(u) = id$. Let $\Phi = \Phi_S$ denote the given action on S. For any $u \in U$ one defines the q-difference module M by $M := S$ provided with a new Φ-action, namely $u\Phi$. Let $\mathcal{F}^+(\mathbb{C})$ denote the set of all actions $\{u\Phi\}$. Two actions $u_1\Phi$ and $u_2\Phi$ are equivalent if there exists an $a \in U$ such that $u_1\Phi a = au_2\Phi$. Let $\mathcal{F}(\mathbb{C})$ denote the set of equivalence classes. We note that this coincides with the set of equivalence classes for the pairs $\{(M, f)\}$.

For any $\mathbb{C}$-algebra R (i.e., commutative and with a unit element) one defines $U(R)$ as the set of the $R \otimes_{\mathbb{C}} K$-linear maps u from $R \otimes_{\mathbb{C}} S$ to itself such u respects the filtration $R \otimes_{\mathbb{C}} P_1 \subset R \otimes_{\mathbb{C}} (P_1 \oplus P_2) \cdots \subset R \otimes_{\mathbb{C}} S$ and moreover $gr(u) = id$. Further $\mathcal{F}^+(R)$ denotes the set of all actions $\{u\Phi\}$ with $u \in U(R)$ on $R \otimes_{\mathbb{C}} S$. As before, two actions $u_1\Phi$ and $u_2\Phi$ are equivalent if there exists an $a \in U(R)$ such that $a^{-1}u_1\Phi a\Phi^{-1} = u_2$. The set of equivalence classes is denoted by $\mathcal{F}(R)$. Thus we obtain a covariant functor $\mathcal{F}$ defined on the category of the $\mathbb{C}$-algebras. One can view $\mathcal{F}$ as a contravariant functor on affine schemes over $\mathbb{C}$ and extend $\mathcal{F}$ to a contravariant functor from $\mathbb{C}$-schemes to the category of sets. Our aim is to show that $\mathcal{F}$ is representable by a certain $\mathbb{C}$-algebra, or in other terms by an affine scheme over $\mathbb{C}$. It will turn out that the scheme representing $\mathcal{F}$ is in fact $\mathbb{A}^N_{\mathbb{C}}$ for some N.

Example 3.1. — *$S = P_1 \oplus P_2$, with pure modules P_1, P_2 of slopes $\lambda_1 < \lambda_2$.*
Any element in U has the form $u = 1 + u_{1,2}$ with $u_{1,2} : P_2 \to P_1$ a K-linear map. Further $v := a^{-1}u\Phi a\Phi^{-1}$ satisfies $v_{1,2} = u_{1,2} - a_{1,2} + \Phi(a_{1,2})$. Here $\Phi(a_{1,2})$ denotes the action of Φ on the element $a_{1,2}$ of the pure module $B := \mathrm{Hom}_K(P_2, P_1) = P_2^* \otimes P_1$. The map $B \to \mathcal{F}(\mathbb{C})$ yields an isomorphism $\mathrm{coker}(\Phi - 1, B) \to \mathcal{F}(\mathbb{C})$.

For any $\mathbb{C}$-algebra R, one obtains in the same way one isomorphism $\mathrm{coker}(\Phi - 1, R \otimes_{\mathbb{C}} B) \to \mathcal{F}(R)$. In 2.6, one has defined the $\mathbb{C}$-vector space $Repr(B) \subset B_{global} \subset B$ of representatives for $\mathrm{coker}(\Phi - 1, B)$. One introduces the functor $\mathcal{F}^o$ by $\mathcal{F}^o(R) = R \otimes_{\mathbb{C}} Repr(B)$. One obtains an isomorphism of functors $\mathcal{F}^o(R) \to \mathcal{F}(R)$.

The finite dimensional $\mathbb{C}$-vector space $Repr(B)$ is seen as complex alge-

braic variety. Its algebra of regular functions $O(Repr(B))$ is the symmetric algebra of $Repr(B)^*$. Then

$$\mathrm{Hom}_{\mathbb{C}-\mathrm{algebra}}(O(Repr(B)), R) = \mathrm{Hom}_{\mathbb{C}-\mathrm{vectorspace}}(Repr(B)^*, R) = Repr(B)\otimes R\,.$$

Thus $\mathcal{F}^o$ and $\mathcal{F}$ are represented by $O(Repr(B))$. In terms of schemes, $\mathcal{F}$ is represented by $Repr(B)$, seen as affine space over $\mathbb{C}$. We note that $Repr(B)$ is in fact the $\mathbb{C}$-vector space $Ext^1(P_2, P_1)$, where P_1, P_2 are seen as left $K[\Phi, \Phi^{-1}]$-modules. The universal family is made explicit in the following examples.

(1) $P_i = Ke_i$ for $i = 1, 2$ and $\Phi e_1 = e_1,\ \Phi e_2 = z^t e_2$. Then the universal family is $K[x_0, \ldots, x_{t-1}]e_1 + K[x_0, \ldots, x_{t-1}]e_2$ with Φ given by $\Phi e_1 = e_1$ and $\Phi e_2 = z^t e_2 + (x_0 + x_1 z + \cdots + x_{t-1} z^{t-1})e_1$.

(2) $P_1 = Res(K_n e_1)$ with $\Phi e_1 = z^{-t/n} e_1,\ t, n \geqslant 1,\ g.c.d.(t, n) = 1$ and $P_2 = Ke_2$ with $\Phi e_2 = e_2$. The universal family is:
$K_n[x_0, \ldots, x_{t-1}]e_1 + K[x_0, \ldots, x_{t-1}]e_2$ with Φ given by $\Phi e_1 = z^{-t/n} e_1$ and $\Phi e_2 = e_2 + (x_0 + x_1 z^{1/n} + \cdots + x_{t-1} z^{(t-1)/n})e_1$.

THEOREM 3.2. — *Let $S = P_1 \oplus \cdots \oplus P_r$ be a direct sum of pure modules with slopes $\lambda_1 < \cdots < \lambda_r$. The functor $\mathcal{F}$ associated to S is represented by a free polynomial ring over $\mathbb{C}$ in $N := \sum_{i<j}(\lambda_j - \lambda_i) \cdot \dim_K P_i \cdot \dim_K P_j$ variables. Equivalently, $\mathcal{F}$, seen as a contravariant functor from $\mathbb{C}$-schemes to sets, is represented by the affine space $\mathbb{A}^N_{\mathbb{C}}$.*

Proof. — We will use induction with respect to r. The case $r = 2$ is dealt with in Example 3.1. Take $r = 3$. Write $\mathcal{F}_{1,2,3}$ for the functor associated to $P_1 \oplus P_2 \oplus P_3$. Further $\mathcal{F}_{1,2},\ \mathcal{F}_{2,3}, \mathcal{F}_{1,3}$ are the functors associated to $P_1 \oplus P_2,\ P_2 \oplus P_3,\ P_1 \oplus P_3$. There is a morphism of functors $T : \mathcal{F}_{1,2,3} \to \mathcal{F}_{1,2} \times \mathcal{F}_{2,3}$. An element of $\mathcal{F}_{1,2,3}(R)$, represented by a filtration $M_1 \subset M_2 \subset M_3$, is mapped to the pair of equivalence classes $([M_1 \subset M_2], [M_2/M_1 \subset M_3/M_1])$ in $\mathcal{F}_{1,2}(R) \times \mathcal{F}_{2,3}(R)$. We *claim* that $\mathcal{F}_{1,2,3}$ is a *trivial torsor* over $\mathcal{G} := \mathcal{F}_{1,2} \times \mathcal{F}_{2,3}$ for the algebraic group $\mathcal{F}_{1,3}$. In other words, we will produce an action m, functorial in R,

$$m : \mathcal{F}_{1,3}(R) \times \mathcal{F}_{1,2,3}(R) \to \mathcal{F}_{1,2,3}(R)\ ,$$

such that the map $(g, h) \mapsto (m(g, h), h)$ from $\mathcal{F}_{1,3}(R) \times \mathcal{F}_{1,2,3}(R)$ to the fibre product $\mathcal{F}_{1,2,3}(R) \times_{\mathcal{G}(R)} \mathcal{F}_{1,2,3}(R)$ is a bijection. We note that this definition becomes the usual one, after introducing the algebraic group $(\mathcal{F}_{1,3})_{\mathcal{G}}$ over $\mathcal{G}$, by the formula $(\mathcal{F}_{1,3})_{\mathcal{G}}(R) = \mathcal{F}_{1,3}(R) \times \mathcal{G}(R)$.

The triviality of the torsor means that there is an isomorphism of functors $\mathcal{F}_{1,3} \times (\mathcal{F}_{1,2} \times \mathcal{F}_{2,3}) \to \mathcal{F}_{1,2,3}$ compatible with T. The last statement and the case $r = 2$ imply the theorem for the case $r = 3$.

We represent, as before, an element of $\mathcal{F}_{1,2,3}(R)$ as an equivalence class of actions $u\Phi$ on S with $u \in U(R)$. Let $\sim$ denote the equivalence relation on $U(R)$, given by $u_1 \sim u_2$ if there exists an $a \in U(R)$ with $a^{-1}u_1\Phi a\Phi^{-1} = u_2$. Then $U(R)/\sim$ identifies with $\mathcal{F}_{1,2,3}(R)$. Let $U'(R)$ denote the subgroup of $U(R)$ consisting of the elements u such that $u - 1$ is 0 on $R \otimes_{\mathbb{C}} (P_1 \oplus P_2)$ and $R \otimes_{\mathbb{C}} P_3$ is mapped to $R \otimes_{\mathbb{C}} P_1$. On $U'(R)$ we introduce the equivalence relation $\sim$ by $u_1 \sim u_2$ is there exists an $a \in U'(R)$ with $a^{-1}u_1\Phi a\Phi^{-1} = u_2$. Now $U'(R)/\sim$ identifies in an obvious way with $\mathcal{F}_{1,3}(R)$.

A small calculation, based on the observation that $U'(R)$ lies in the center of $U(R)$, shows that the map $U'(R) \times U(R) \to U(R)$, given by $(u', u) \mapsto u'u$, has the property that $u_1' \sim u_2'$ and $u_1 \sim u_2$ imply $u_1'u_1 \sim u_2'u_2$. Thus there is an induced action $m : (U'(R)/\sim) \times (U(R)/\sim) \to (U(R)/\sim)$. One can verify that m defines a group action, that m has the torsor property over $\mathcal{G}(R)$, and that the construction is functorially in R. Finally, we want to show that the torsor is trivial. Thus we need to define a functorial isomorphism $\mathcal{F}_{1,3}(R) \times \mathcal{G}(R) \to \mathcal{F}_{1,2,3}(R)$. This is done as follows. One considers the map

$$\mathcal{F}^o_{1,3}(R) \times \mathcal{F}^o_{1,2}(R) \times \mathcal{F}^o_{2,3}(R) \to \mathcal{F}^+_{1,2,3}(R) \text{ (see Example 3.1)} ,$$

given by $(u_{1,3}, u_{1,2}, u_{2,3}) \mapsto \begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ 0 & 1 & u_{2,3} \\ 0 & 0 & 1 \end{pmatrix} \cdot \Phi$. One easly verifies that the resulting map

$$\mathcal{F}^o_{1,3}(R) \times \mathcal{F}^o_{1,2}(R) \times \mathcal{F}^o_{2,3}(R) \to \mathcal{F}_{1,2,3}(R),$$

is bijective and depends functorially on R. Since the maps $\mathcal{F}^o_{i,j}(R) \to \mathcal{F}_{i,j}(R)$ are isomorphisms functorial in R, we have found a trivialization of the torsor. This ends the proof for the case $r = 3$.

For $r = 4$ and $S = P_1 \oplus \cdots \oplus P_4$, one defines in a similar way functors $\mathcal{F}_{1,2,3,4}$, $\mathcal{F}_{2,3,4}$ etc. With the same methods one proves that the morphism of functors $\mathcal{F}_{1,2,3,4} \to \mathcal{G}$, where $\mathcal{G} := \mathcal{F}_{1,2,3} \times_{\mathcal{F}_{2,3}} \mathcal{F}_{2,3,4}$, is a trivial torsor for the algebraic group $\mathcal{F}_{1,4}$. It is clear how to extend this to any $r > 4$. □

Remarks 3.3. — (1) Let $S = P_1 \oplus \cdots \oplus P_r$ with corresponding functor $\mathcal{F}$. The method of the proof of Theorem 3.2 yields maps $\prod_{i<j} \mathcal{F}^o_{i,j}(R) \to \mathcal{F}^+(R)$, functorial in R. Let $\mathcal{F}^o(R)$ denote the image of the this map. Then

$\mathcal{F}^o$ is a functor and the obvious morphism $\mathcal{F}^o \to \mathcal{F}$ is an isomorphism. In particular one finds an isomorphism of functors $\prod_{i<j} \mathcal{F}_{i,j} \to \mathcal{F}$ which is obtained by trivializing the sequence of torsors involved in $\mathcal{F}$.

(2) The case of Theorem 3.2 where the slopes are integers is present in a paper of G. Birkhoff and P.E. Guenther, see [Bir], where they present normal forms for these equations. This case is also treated in [R-S-Z, Sau2, Sau3].

(3) Each $\mathcal{F}_{i,j}$ has the natural structure of algebraic group over $\mathbb{C}$ and is in fact equal to $Ext^1(P_i, P_j)$. However, $\mathcal{F}$ has no evident structure of (unipotent) algebraic group. In contrast with this, the obvious injective map $\prod_{i<j} \mathcal{F}^o_{i,j}(R) \to U(R)$ has as image a *subgroup* of $U(R)$.

In proving this, one has to consider indices $a < b < c$ and one has to show that the the obvious map $\mathcal{F}^o_{a,b}(R) \times \mathcal{F}^o_{b,c}(R)$ to $R \otimes \mathrm{Hom}(P_c, P_a)$ has image in $\mathcal{F}^o_{a,c}(R)$. This follows from the statement: $Repr(M_1) \otimes Repr(M_2)$ is mapped to $Repr(M_1 \otimes M_2) \subset M_1 \otimes M_2$, proved in 2.6.

The functorial isomorphism $\prod_{i<j} \mathcal{F}^o_{i,j} \to \mathcal{F}$ provides the latter with a structure of unipotent linear algebraic group.

4. Global difference modules

The skew ring $\mathbb{D} := \mathbb{C}[z, z^{-1}][\Phi, \Phi^{-1}]$ is defined by the relation $\Phi z = qz\Phi$. We recall that a *global difference module* is a left $\mathbb{D}$-module N, which is as $\mathbb{C}[z, z^{-1}]$-module, finitely generated and free.

It suffices in fact to assume that N is finitely generated as $\mathbb{C}[z, z^{-1}]$-module. Indeed, let N_0 denote the torsion submodule of N and let $I \subset \mathbb{C}[z, z^{-1}]$ denote the anihilator ideal of N_0. Then N_0 and I are invariant under the action of Φ and ϕ. This implies $I = (1)$, $N_0 = \{0\}$ and N is free.

Lemma 4.1. — *Any global difference module has a cyclic vector.*

Proof. — We imitate the proof of [vdP-S2], Proposition 2.9. Let the global difference module N have rank n over $\mathbb{C}[z, z^{-1}]$. It suffices to find an element $m \in N$ such that the $\mathbb{C}[z, z^{-1}]$-module N' generated by $\{\Phi^s m |\ s \geqslant 0\}$ has rank n. Indeed, this implies that the global difference module N/N' has rank 0 and is therefore equal to 0. This condition on m translates into $m \wedge \Phi m \wedge \Phi^2 m \wedge \cdots \wedge \Phi^{n-1} m \neq 0$ as element of the global difference module $\Lambda^n N$.

We recall that for any integer $d \geqslant 1$, the global difference module $\Lambda^d N$ is defined as the dth exterior product of N as $\mathbb{C}[z, z^{-1}]$-module with Φ-action given by $\Phi(m_1 \wedge \cdots \wedge m_d) = (\Phi m_1) \wedge \cdots \wedge (\Phi m_d)$.

Suppose that we have found an element $e \in N$ such that $\{\Phi^s e |\ s \geqslant 0\}$ generates a $\mathbb{C}[z, z^{-1}]$-module N' of rank $m < n$. Then $e \wedge \cdots \wedge \Phi^{m-1} e \neq 0$ and $e \wedge \cdots \wedge \Phi^m e = 0$. Take an element $f \in N \setminus N'$ and consider $\tilde{e} = e + \lambda z^s f$ with $\lambda \in \mathbb{Q}$ and $s \in \mathbb{Z}$. We claim that for suitable λ and s one has that $E(\lambda, s) := \tilde{e} \wedge \cdots \wedge \Phi^m \tilde{e} \neq 0$. From this claim the lemma follows. One can write $E(\lambda, s)$ as a sum of terms, which are wedge products of some $\Phi^u e$ and some $\Phi^v f$ (like $e \wedge \Phi e \wedge \Phi^2 f \wedge \cdots \wedge \Phi^m f$) with coefficients $(\lambda z^s)^a q^b$ for $a = 0, \ldots, m+1$ and suitable b. If $E(\lambda, s) = 0$ for all λ and s, then each of these wedge products is 0. This contradicts $e \wedge \Phi e \wedge \cdots \wedge \Phi^{m-1} e \wedge \Phi^m f \neq 0$. $\square$

A global module N will be called *pure* of slope λ if $\mathbb{C}(\{z\}) \otimes_{\mathbb{C}[z, z^{-1}]} N$ is pure with slope λ and $\mathbb{C}(\{z^{-1}\}) \otimes_{\mathbb{C}[z, z^{-1}]} N$ is pure with slope $-\lambda$. An example of a pure global module with slope t/n (and $n \geqslant 1$, $g.c.d.(t, n) = 1$) and rank n is $\mathbb{C}[z^{1/n}, z^{-1/n}]e$ with $\Phi e = c z^{t/n} e$ and $c \in \mathbb{C}^*$.

Lemma 4.2. — Pure global modules.

(1) *A global module N is pure with slope t/n (with $n \geqslant 1$ and $g.c.d.(t, n) = 1$) if and only if $N \cong \mathbb{D}/\mathbb{D}L(z^{-t}\Phi^n)$ for a monic $L \in \mathbb{C}[T]$ with $L(0) \neq 0$.*

(2) *Let $L \in \mathbb{C}[T]$ have the form $\prod_{j=1}^{s} (T - c_j)^{m_j}$ with distinct $c_1, \ldots c_s \in \mathbb{C}^*$. Then $\mathbb{D}/\mathbb{D}L(z^{-t}\Phi^n)$ is the direct sum of the indecomposable global modules $\mathbb{D}/\mathbb{D}(z^{-t}\Phi^n - c_j)^{m_j}$.*

(3) *Let N be a pure global module of slope t/n and $m \in N$. The operator S in $\mathbb{C}[z, z^{-1}][\Phi]$ of minimal degree in Φ, satisfying $S(m) = 0$, has the form $S = P(z^{-t}\Phi^n)$ with monic $P \in \mathbb{C}[T]$ and $P(0) \neq 0$.*

Proof. — (1) Suppose that N is pure with slope t/n and has rank $m \cdot n$. Consider a cyclic vector e of N and let $L \in \mathbb{C}[z, z^{-1}][\Phi]$ denote a non zero element of minimal degree (namely $m \cdot n$) in Φ such that $Le = 0$. Clearly the constant term $L(0) \in \mathbb{C}[z, z^{-1}]$ is different from 0. After multiplication by an invertible element of $\mathbb{C}[z, z^{-1}]$, one may suppose that $L(0) \in \mathbb{C}[z]$ and that the constant term of $L(0)$ is 1. Now Φ^{-1} acts bijectively on $N \cong \mathbb{D}/\mathbb{D}L$. This implies that $L(0) = 1$. The Newton polygon of L, considered over $\mathbb{C}(\{z\})$, contains only terms $z^a \Phi^b$ with $a + b\frac{t}{n} \geqslant 0$. The Newton polygon of L, considered over the field $\mathbb{C}(\{z^{-1}\})$ contains only terms $z^a \Phi^b$ with $a + b\frac{t}{n} \leqslant 0$. Thus $L \in \mathbb{C}[z^{-t}\Phi^n]$ and $L(0) = 1$. After multiplication by a constant, we may suppose that L is monic.

On the other hand, a global module, given as $\mathbb{D}/\mathbb{D}L$ with $L \in \mathbb{C}[\Psi]$ and $L(0) \neq 0$, is clearly a pure global module with slope t/n.

The proof of (2) is straightforward and (3) follows from the observation that $\mathbb{D}m$ is again a pure global module of slope t/n. □

A global difference module has, in general, no slope filtration! We consider the category of the global modules N that have *an ascending slope filtration*, i.e., a sequence of submodules $0 = N_0 \subset N_1 \subset \cdots \subset N_r = N$, such that each quotient N_i/N_{i-1} is pure with slope λ_i and $\lambda_1 < \cdots < \lambda_r$.

THEOREM 4.3. — *The functor T from the category of the global difference modules with ascending slope filtration to the category of the difference modules over K, given by $N \mapsto K \otimes_{\mathbb{C}[z,z^{-1}]} N$, is an equivalence of Tannakian categories.*

Remarks 4.4. — For a given module M over K, the global module with ascending slope filtration N with $T(N) = K \otimes N \cong M$ has the property that $\mathbb{C}(\{z^{-1}\}) \otimes N$ is a difference module over the field $\mathbb{C}(\{z^{-1}\})$ which is a direct sum of pure modules. Indeed, $\mathbb{C}(\{z^{-1}\}) \otimes N$ has both an ascending and a descending slope filtration. In other words, N is an algebraic vector bundle above $\mathbb{P}^1 - \{0, \infty\}$ with a Φ-action and a filtration according to the slopes at $z = 0$.

One can see Theorem 4.3 as an analogue of a theorem of G. Birkhoff which states that every differential module over K comes from a connection on $\mathbb{P}^1$ which has only singularities at 0 and ∞. Moreover the singularity at ∞ is regular singular.

The proof of Theorem 4.3 is given in the following series of observations

Observations 4.5. — (1) Let N be a global difference module with ascending slope filtration $0 = N_0 \subset N_1 \subset \cdots \subset N_r = N$ such that N_i/N_{i-1} is pure of slope $\lambda_i = t_i/n_i$ with $n_i \geqslant 1$, $g.c.d.(t_i, n_i) = 1$ and $\lambda_1 < \cdots < \lambda_r$.

Let $m \in N$. The difference operator $L \in \mathbb{C}[z, z^{-1}][\Phi] \subset \mathbb{D}$, monic and of minimal degree in Φ, satisfying $L(m) = 0$, has the form

$$L = q^a z^b L_1(z^{-t_1}\Phi^{n_1}) \cdots L_r(z^{-t_r}\Phi^{n_r})$$

with $L_1, \ldots, L_r$ monic elements of $\mathbb{C}[T]$ and suitable $a, b \in \mathbb{Z}$.

Proof. — L_r is the polynomial of Lemma 4.2, applied to the image of m in N_r/N_{r-1}. If m happens to be an element of N_{r-1}, then $L_r = 1$.

Lemma 4.2 applied to the image of $L_r(z^{-t_r}\Phi^{n_r})(m)$ in N_{r-1}/N_{r-2} produces L_{r-1}. Induction finishes the proof. One multiplies the above operator with a suitable term $q^a z^b$ to obtain a monic operator. $\square$

(2) Using (1) one deduces that any morphism between global difference modules with ascending slope filtrations respects the filtrations. Further the full subcategory of the category of all global modules, whose objects are the global modules with ascending filtration is closed under direct sums, tensor products, duals, submodules and quotients. In particular, this subcategory is a Tannakian category.

(3) The indecomposable difference module $M := Res(K_n e) \otimes_K U_m$ with $\Phi e = cz^{t/n}$ has a cyclic vector f such that $L = (z^{-t}\Phi^n - c^n q^{t(n-1)/2})^m$ is a polynomial of minimal degree in Φ with $L(f) = 0$. Clearly, $M = T(N)$ with $N = \mathbb{D}/\mathbb{D}(z^{-t}\Phi^n - c^n q^{t(n-1)/2})^m$ and M_{global} coincides, according to Definition 1.8, with N.

One concludes that for a split difference module M over K there exists a unique (up to isomorphism) global module N, direct sum of pure global ones, such that $T(N) \cong M$.

(4) Let N be as in (1). The associated graded global module $gr(N)$ is defined as $\sum_{j=1}^{r} N_j/N_{j-1}$.

Let a direct sum of pure global modules $U = R_1 \oplus \cdots \oplus R_r$ with slopes $\lambda_1 < \cdots < \lambda_r$ be given. As in section 3, one considers the set of equivalence classes of the pairs (N, f) consisting of a global module with an ascending slope filtration and an isomorphism $f : gr(N) \to U$. Two pairs (N_i, f_i) are equivalent if there exists an isomorphism $\alpha : N_1 \to N_2$ such that $gr(N_1) \stackrel{gr(\alpha)}{\to} gr(N_2) \stackrel{f_2}{\to} S$ coincides with f_1. As in section 3 one proves that the set of equivalence classes is in a natural way isomorphic to $\prod_{i<j} \operatorname{coker}(\Phi - 1, \operatorname{Hom}(R_j, R_i))$.

(5) We use the notation of (4) and put $S = T(U)$ and $P_j = T(Q_j)$. The functor T maps the set of equivalence classes for $U = R_1 \oplus \cdots \oplus R_r$ to the set of equivalence classes for the graded module $S = P_1 \oplus \cdots \oplus P_r$ over K, studied in section 3. From Theorem 2.5 one easily concludes that T defines an isomorphism between the two classes of objects.

This implies that the functor T induces a bijection between the isomorphy classes of global difference modules with an ascending filtration and the isomorphy classes of difference modules over K.

(6) The final part of the proof of Theorem 4.3 consists of verifying that the map $\mathrm{Hom}(N_1, N_2) \to \mathrm{Hom}(T(N_1), T(N_2))$ is bijective. Since T clearly commutes with tensor products we may suppose that $N_1 = \mathbb{C}[z, z^{-1}]e$ with $\Phi e = e$. Thus we have to show, for any global module N with ascending filtration, that $\ker(\Phi - 1, N) \to \ker(\Phi - 1, T(N))$ is bijective. The injectivity is obvious.

Let $0 = N_0 \subset N_1 \subset \cdots \subset N_r = N$ be the slope filtration. Then $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$ with $M_j = T(N_j)$ is the slope filtration of $M = T(N)$.

Let N' be the $\mathbb{C}[z, z^{-1}]$-submodule generated by $\{m \in N |\ \Phi(m) = m\}$. Then N' is a pure global submodule of N with slope 0. Define $s \leqslant r$ to be the smallest integer such that $N' \subset N_s$. The map $\alpha : N_s \to N_s/N_{s-1}$ is not zero on N'. Hence $\lambda_s = 0$. The restriction of α to N' is injective. Indeed, according to (1), a non zero element of N_{s-1} cannot have minimal polynomial $\Phi - 1$.

Let M' denote the K-subspace of M generated by $\{m \in M | \Phi(m) = m\}$. Then M' is a pure submodule of M with slope 0. Thus $M' \subset M_s$ and the restriction of $T(\alpha) : M_s \to M_s/M_{s-1}$ to M' is injective. The image $T(\alpha)(M')$ contains $T(\alpha)(N')$. Let $M^- \subset M_s$ denote the preimage $T(\alpha)^{-1}T(\alpha)(M')$. Then $M' \to M^-/M_{s-1}$ is a bijection and thus M^- is a direct sum $M_{s-1} \oplus M'$. The global module N^- with ascending filtration and T-image $M_{s-1} \oplus M'$ is clearly $N_{s-1} \oplus \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} V$ where V is a $\mathbb{C}$-vector space where Φ acts as the identity. Then $\ker(\Phi - 1, N) = \ker(\Phi - 1, N^-) = V = \ker(\Phi - 1, M^-) = \ker(\Phi - 1, M)$. □

Remarks 4.6. — (1) Let N be as in part (1) of Observation 4.5 and put $M = T(N) = K \otimes_{\mathbb{C}[z,z^{-1}]} N$. The elements m of $N = M_{global}$ are characterized by the condition that the monic polynomial L of minimal degree in Φ, satisfying $L(m) = 0$, has the form

$$L = q^a z^b L_1(z^{-t_1}\Phi^{n_1}) \cdots L_r(z^{-t_r}\Phi^{n_r})$$

with $L_1, \ldots, L_r$ monic elements of $\mathbb{C}[T]$ and suitable $a, b \in \mathbb{Z}$.

(2) Let N be a pure global module of slope $\lambda > 0$, then $\mathrm{coker}(\Phi - 1, K \otimes N)$ is zero. Moreover, $\mathrm{coker}(\Phi - 1, N)$ is a $\mathbb{C}$-vector space of dimension $\lambda \cdot \mathrm{rank} N$ and the natural map $\mathrm{coker}(\Phi - 1, N) \to \mathrm{coker}(\Phi - 1, \mathbb{C}(\{z^{-1}\}) \otimes N)$ is a bijection.

5. The Galois group of a q-difference module

5.1. Universal Picard-Vessiot extensions

We recall that a difference module M over K is *split* if M is a direct sum of pure modules. A split difference module M over K has a Picard-Vessiot extension $PV(M)$ and the (difference) Galois group is the group of the automorphisms of $PV(M)/K$ which commute with the action of ϕ on $PV(M)$. The *universal Picard-Vessiot extension* $Univ$ for the family of all split modules over K is the direct limit of all $PV(M)$. The universal difference Galois group $\mathbb{G}_{univ}$ is the group of all K-automorphisms of $Univ$ which commute with the action of ϕ on $Univ$. The restriction of $\mathbb{G}_{univ}$ to the subring $PV(M) \subset Univ$ is the Galois group of M.

We note that the Picard-Vessiot ring of the difference module $\widehat{K} \otimes M$ over $\widehat{K}$ is simply $\widehat{K} \otimes_K PV(M)$. In particular, M and $\widehat{K} \otimes_K M$ have the same Galois group. Moreover, $\widehat{K} \otimes_K Univ$ is the universal Picard-Vessiot extension for the collection of *all* difference modules over $\widehat{K}$ since every difference module over $\widehat{K}$ is split.

We give the explicit description of $Univ$, given in [vdP-S1], p.150, in a slightly changed form. $Univ := K[\{e(cz^\lambda)\}, \ell]$, with $c \in \mathbb{C}^*$ and $\lambda \in \mathbb{Q}$. The only relations are:

$$e(c_1 z^{\lambda_1}) e(c_2 z^{\lambda_2}) = e(c_1 c_2 z^{\lambda_1+\lambda_2}), \quad e(1) = 1, \quad e(q) = z^{-1}.$$

The algebraic closure of K embeds in $Univ$, by identifying z^λ with $e(e^{-2\pi i \tau \lambda})$ for all $\lambda \in \mathbb{Q}$. The ϕ-action on $Univ$ is given by

$$\phi(e(cz^\lambda)) = (cz^\lambda)^{-1} \cdot e(cz^\lambda), \quad \phi(\ell) = \ell + 1.$$

The group $\mathbb{G}_{univ}$ consists of elements $\sigma = (h, s, a)$ with $h : \mathbb{C}^* \to \mathbb{C}^*$ a homomorphism with $h(q) = 1$, $s : \mathbb{Q} \to \mathbb{C}^*$ a homomorphism and $a \in \mathbb{C}$. The action of σ is given by

$$\sigma(e(cz^\lambda)) = s(\lambda) \cdot h(c) \cdot e(h(e^{2\pi i \tau \lambda})) \cdot e(cz^\lambda), \quad \sigma(\ell) = \ell + a.$$

We will produce topological generators (for the Zariski topology) for this rather complicated group. Define the elements $\Gamma, \Delta \in \mathbb{G}_{univ}$ by

$$\Gamma(e(cz^\lambda)) = e^{2\pi i a_1} \cdot e(e^{2\pi i \lambda}) \cdot e(cz^\lambda) \text{ and } \Gamma(\ell) = \ell + \frac{1}{\tau}$$

$$\Delta(e(cz^\lambda)) = e^{-2\pi i a_0} \cdot e(cz^\lambda) \text{ and } \Delta(\ell) = \ell + 1,$$
$$\text{where } c = e^{2\pi i (a_0 + a_1 \tau)} \text{ with } a_0, a_1 \in \mathbb{R}.$$

For the commutator $Com := \Gamma\Delta\Gamma^{-1}\Delta^{-1}$ one calculates the formulas

$$Com(e(cz^\lambda)) = e^{2\pi i\lambda} \cdot e(cz^\lambda) \text{ and } Com(\ell) = \ell.$$

To any homomorphism $s : \mathbb{Q} \to \mathbb{C}^*$ one associates the element $\tilde{s} \in \mathbb{G}_{univ}$ by the formulas

$$\tilde{s}(e(cz^\lambda)) = s(\lambda) \cdot e(cz^\lambda) \text{ and } \tilde{s}(\ell) = \ell \ .$$

We note that (any) $\tilde{s}$ commutes with Γ and Δ. Write $D = \tilde{s}$ with s given by $s(\lambda) = e^{2\pi i\tau\lambda}$.

PROPOSITION 5.1. — *Γ, Δ, D are topological generators for $\mathbb{G}_{univ}$.*

Proof. — Let H be the group generated by Γ, Δ, D and let $Qt(Univ)$ denote the total quotient ring of $Univ$. We recall that $Univ$ is the direct limit of Picard-Vessiot rings $PV(M)$ of split modules M over K. The group $\mathbb{G}_{univ}$ is the projective limit of the automorphism groups G_M of these $PV(M)$. We have to show that the image of H in G_M is Zariski dense. This is equivalent to the statement that K is the set of the H-invariant elements of the total quotient ring of $PV(M)$. Thus the statement of the proposition is equivalent with the set of the H-invariant elements of $Qt(Univ)$ is K.

(1) *If $\xi \in K[\{e(c)\}, \ell]$ is invariant under Γ and Δ, then $\xi \in K$.*

Proof of (1). — Write $\xi = \sum_{i=0}^{m} \xi_i \ell^i$ and all $\xi_i \in K[\{e(c)\}]$ and we may suppose that $\xi_m \neq 0$. Now $R := \{e^{2\pi i(a_0+a_1\tau)} |\ a_0, a_1 \in \mathbb{R},\ 0 \leqslant a_0, a_1 < 1\}$ is a set of representatives of $\mathbb{C}^*/q^{\mathbb{Z}}$. Each ξ_i has uniquely the form $\sum_{c\in R} a_{i,c}e(c)$. One sees that ξ_m is invariant under Γ and Δ. From $\Gamma e(c) = e^{2\pi i a_1} \cdot e(c)$ and $\Delta e(c) = e^{-2\pi i a_0} \cdot e(c)$ it follows that $\xi_m \in K$. We may suppose that $\xi_m = 1$ and we have to prove that $m = 0$. Suppose that $m > 0$ and apply Γ to ξ. Comparing the coefficient of ℓ^{m-1}, one finds that $\Gamma(\xi_{m-1}) + \frac{m}{\tau} = \xi_{m-1}$. This is clearly impossible for $m > 0$ and an element in $K[\{e(c)\}]$. We conclude that $m = 0$ and $\xi \in K$. □

(2) *Let $\xi \in Univ^H$, then $\xi \in K$.*

Proof of (2). — One write $\xi = \sum_{\lambda\in\mathbb{Q}} a_\lambda e(z^\lambda)$ with all $a_\lambda \in K[\{e(c)\}, \ell]$. Then $D(\xi) = \sum a_\lambda e^{2\pi i\lambda\tau} e(z^\lambda)$. It follows that $a_\lambda = 0$ for $\lambda \neq 0$. By (1), $\xi \in K$. □

(3) *Let $\xi \in Qt(Univ)^H$. Then $I := \{a \in Univ \mid a\xi \in Univ\}$ is the unit ideal.*

(3) implies $\xi \in Univ$ and, by (2), $\xi \in K$. This finishes the proof of 5.1.

Proof of (3). — The ideal I is invariant under H. Write any element $a \in I$ as $\sum_{\lambda\in\mathbb{Q}} a_\lambda e(z^\lambda)$ with all $a_\lambda \in K[\{e(c)\}, \ell]$. Then $D^m(a) = \sum a_\lambda e^{2\pi i\tau m\lambda} \cdot e(z^\lambda) \in I$ for every $m \geqslant 0$. It follows that all $a_\lambda \in I$. Thus I is generated by an ideal $J \subset K[\{e(c)\}, \ell]$ that is also H-invariant. Let $m \geqslant 0$ be minimal such that J contains a non zero element of degree m in ℓ. Let $J_m \subset K[\{e(c)\}]$ denote the ideal of the coefficients of ℓ^m of the elements in J with degree $\leqslant m$. Then J_m is again H-invariant. Consider a non zero element $A = \sum_{c\in R} a_c e(c) \in J_m$ with all $a_c \in K$. From $\Gamma^n A \in J_m$ and $\Delta^n A \in J_m$ for all $n \geqslant 0$ one concludes that each $e(c)$ with $a_c \neq 0$ lies in J_m. Hence J_m is the unit ideal. If $m = 0$, then the proof of (2) is finished. If $m > 0$, then, by construction, J contains an element B of degree m in ℓ and with leading coefficient 1. Let B' be another element having these properties. Then $B - B' = 0$ since its degree in ℓ is $< m$. Thus B is unique and therefore invariant under Γ and Δ. Using (1), one obtains the contradiction $B \in K$. □

The proof of Corollary 5.2 is similar to that of Proposition 5.1.

Corollary 5.2. — *The universal ring $Univ_{rs}$ for the category of regular singular q-difference modules over K is $K[\{e(c)\}, \ell]$. The restrictions Γ_{rs}, Δ_{rs} of Γ, Δ to the universal ring $Univ_{rs}$ commute and are topological generators for the group $\mathbb{G}_{rs}$ of all automorphisms of the difference ring $Univ_{rs}$.*

We introduce some Tannakian categories. The first one $RegSing_K$ consists of the regular singular difference modules over K.

The second one, called $Tuples_1$, has as objects the tuples (V, Γ_V, Δ_V) where V is a finite dimensional vector space over $\mathbb{C}$ and Γ_V, Δ_V are commuting elements in $\mathrm{GL}(V)$ and such that all their eigenvalues have absolute value 1. A morphism $(V, \Gamma_V, \Delta_V) \to (W, \Gamma_W, \Delta_W)$ is a linear map $f : V \to W$ satisfying $f \circ \Gamma_V = \Gamma_W \circ f$ and $f \circ \Delta_V = \Delta_W \circ f$.

The third one, called $Unitcirc$, consists of the finite dimensional complex representations ρ of the group $\mathbb{Z}^2$ such that every element in the image of ρ has all its *eigenvalues on the unit circle* $S^1 := \{z \in \mathbb{C} |\ |z| = 1\}$.

Lemma 5.3. — *There are equivalences of Tannakian categories*

$$RegSing_K \to Tuples_1 \to Unitcirc\ .$$

Proof. — To a regular singular difference module M over K one associates its 'solution space' $V := \ker(\Phi - 1, Univ_{rs} \otimes_K M)$. Then V is a

complex vector space and the canonical map $Univ_{rs} \otimes_{\mathbb{C}} V \to Univ_{rs} \otimes_K M$ is bijective. The group $\mathbb{G}_{rs}$ acts on V. Let $\Gamma_V,\ \Delta_V$ denote the actions of $\Gamma,\ \Delta$ on V. The eigenvalues of $\Gamma_V,\ \Delta_V$ have absolute value 1, according to the definition of Γ and Δ. Then we associate to M the tuple (V, Γ_V, Δ_V).

To a tuple (V, Γ_V, Δ_V) we associate the representation $\rho : \mathbb{Z}^2 \to \mathrm{GL}(V)$ with $\rho(1,0) = \Gamma_V$ and $\rho(0,1) = \Delta_V$.

It is easily verified that the functors, defined above, are equivalences of tannakian categories. □

We recal that $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$ and consider the maps

$$can : \mathbb{C}_u \overset{u \mapsto e^{2\pi i u}}{\to} \mathbb{C}^*_z \overset{pr}{\to} E_q\ .$$

The indices u and z denote the global variables for these spaces. The map pr is the obvious map. The map can is the universal covering. Its kernel $\mathbb{Z} + \mathbb{Z}\tau$ can be identified with $\pi_1(E_q)$. Let a, b denote the generators of $\pi_1(E_q)$ corresponding to $1, \tau$ (or to the two circles $\mathbb{R}/\mathbb{Z}1$ and $\mathbb{R}/\mathbb{Z}\tau$ on E_q).

We identify the group $\mathbb{Z}^2$ (or equivalently the group generated by Γ_{rs}, Δ_{rs}) with $\pi_1(E_q)$, by $(1,0)$ (or Γ_{rs}) is mapped to a and $(0,1)$ (or Δ_{rs}) is mapped to b. In this way one finds a Tannakian equivalence between $RegSing_K$ and the category of the 'unit circle' representations of $\pi_1(E_q)$.

To a connection $(\mathcal{M}, \nabla)$ on E_q one associates its monodromy representation, i.e., a representation of $\pi_1(E_q)$. In this way the category of the connections on E_q is equivalent to the category of the representations of $\pi_1(E_q)$. Combining this with Lemma 5.3 one obtains

Corollary 5.4. — *The category of the regular singular difference modules over K is Tannakian equivalent to a full subcategory of connections on E_q.*

In section 6, this equivalence will be made explicit. This result is, in a different form, present in the work of J. Sauloy, see [R-S-Z].

5.2. Galois groups for split modules over K

Now we extend the above to larger categories. We introduce a category $Tuples_2$ as follows. The objects are tuples

$$(V, \{V_\lambda\}_{\lambda \in \mathbb{Q}}, \Gamma_V, \Delta_V),\ \text{satisfying :}$$

(a) V is a finite dimensional vector space over $\mathbb{C}$.
(b) V is the direct sum of the subspaces V_λ (and thus there are only finitely many λ with $V_\lambda \neq 0$).
(c) Γ_V, Δ_V are invertible operators on V respecting the direct sum decomposition and such that all their eigenvalues are on the unit circle $S^1 = \{z \in \mathbb{C} |\ |z| = 1\}$.
(d) $\Gamma_V \Delta_V \Gamma_V^{-1} \Delta^{-1}$ acts on each V_λ as multiplication by $e^{2\pi i \lambda}$.

A morphism $(V, \{V_\lambda\}, \Gamma_V, \Delta_V) \to (W, \{W_\lambda\}, \Gamma_W, \Delta_W)$ is a linear map $f : V \to W$ which respects the direct sum decompositions and satisfies $f \circ \Gamma_V = \Gamma_W \circ f,\ f \circ \Delta_V = \Delta_W \circ f$. The tensor product of two objects $(V, \{V_\lambda\}, \Gamma_V, \Delta_V)$ and $(W, \{W_\lambda\}, \Gamma_W, \Delta_W)$ is $(U, \{U_\lambda\}, \Gamma_U, \Delta_U)$, given by $U = V \otimes W,\ U_\lambda = \sum_{\mu_1+\mu_2=\lambda} V_{\mu_1} \otimes W_{\mu_2}$ and $\Gamma_U = \Gamma_V \otimes \Gamma_W,\ \Delta_U = \Delta_V \otimes \Delta_W$. It is easily seen that $Tuples_2$ is a Tannakian category. Further, the Galois group of an object $(V, \{V_\lambda\}, \Gamma_V, \Delta_V)$ is the algebraic subgroup of $\mathrm{GL}(V)$ generated by the maps Γ_V, Δ_V, D_V, where the last map is defined by: D_V is multiplication by $e^{2\pi i \tau \lambda}$ on the direct summands V_λ of V.

One defines a functor $\mathcal{F}$ from the category of the split q-difference modules over K to the category $Tuples_2$ as follows. For a module M, $\mathcal{F}(M) = (V, \{V_\lambda\}, \Gamma_V, \Delta_V)$, where
(i) $V := \ker(\Phi - 1,\ Univ \otimes_K M)$.
(ii) $V_\lambda := \ker(\Phi - 1,\ Univ_\lambda \otimes_K M)$, where $Univ_\lambda := K[\{e(c)\}, \ell] e(z^\lambda)$.
(iii) Γ_V, Δ_V are induced by the action of Γ, Δ on $Univ$ and $Univ \otimes_K M$.
Since $\Phi - 1$ commutes with Γ, Δ, the latter maps leave V invariant.
The definition of $\mathcal{F}(f)$, for a morphism f, is obvious. The verification that $\mathcal{F}$ is a functor between Tannakian categories is straightforward.

Proposition 5.5. — *$\mathcal{F}$ is an equivalence between the Tannakian categories of the split difference modules over K and $Tuples_2$.*

Proof. — One defines a functor $\mathcal{G}$ from the category $Tuples_2$ to the category of the split difference modules over K, in the following way. The image M of an object $(V, \{V_\lambda\}, \Gamma_V, \Delta_V)$ is the set of the $\mathbb{G}_{univ}$-invariant elements (or the elements invariant under Γ, Δ, D) of $Univ \otimes_{\mathbb{C}} V$. The action on the last object is defined by $\Gamma(u \otimes v) = \Gamma(u) \otimes \Gamma_V(v)$ for $u \in Univ,\ v \in V$ and similarly for Δ and D. The proof that $\mathcal{G}$ is the 'inverse' of $\mathcal{F}$ is similar to the proof of Proposition 5.1. □

The (difference) Galois group of a split module M over K coincides with that of the object $\mathcal{F}(M)$. We note that $Tuples_2$ is equivalent to the category of all difference modules over $\widehat{K}$, too.

EXAMPLE 5.6. — Let $M = Res(E(cz^{t/n}))$ with $n > 1$, $g.c.d.(t,n) = 1$ and $c = e^{2\pi i(a_0+a_1\tau)}$ with $a_0, a_1 \in \mathbb{R}$ and $0 \leqslant a_0, a_1 < 1$. Write M as $K_n e$ with $\Phi e = cz^{t/n}e$. Then $\{z^{j/n}e |\ j = 0, \ldots, n-1\}$ is a basis of M over K. To obtain a basis for $V = V_{t/n} = \ker(\Phi - 1, Univ_{t/n} \otimes M)$ one observes that V is in fact $\ker(\Phi - 1,\ K[e(e^{-2\pi i\tau/n}), e(e^{2\pi i/n})]e(cz^{t/n}) \otimes_K K_n e)$. After identifying $z^{1/n}$ with $e(e^{-2\pi i\tau/n})$, this space takes the form $K_n[e(e^{2\pi i/n})]e(cz^{t/n}) \otimes K_n e$ and the Φ-action is given by $\Phi(ae(cz^{t/n}) \otimes be) = \phi(a)z^{-t/n}e(cz^{t/n}) \otimes \phi(b)z^{t/n}e$. From this expression one finds a basis $\{v_j |\ j = 0, \ldots, n-1\}$ of V of the form

$$v_j = \sum_{s=0}^{n-1} \Phi^s\{e(e^{2\pi ij/n})e(cz^{t/n}) \otimes e\}\ .$$

We use the 'cyclic' notation $v_{j+n} = v_j$. The actions of Γ and Δ are given by

$$\Gamma v_j = e^{2\pi i a_1} v_{j+t} \text{ and } \Delta v_j = e^{-2\pi i(a_0+j/n)} v_j\ .$$

The difference Galois group G is topologically generated by Γ, Δ and D (the latter is here the multiplication by $e^{2\pi i\tau t/n}$). One finds the exact sequence $1 \to G^o \to G \to (\mathbb{Z}/n\mathbb{Z})^2 \to 0$ with $G^o = \mathbb{G}_m = \mathbb{C}^* id$. Further $\Gamma^n = e^{2\pi i n a_1} id$, $\Delta^n = e^{2\pi i n a_0} id$, $\Gamma\Delta\Gamma^{-1}\Delta^{-1} = e^{2\pi i t/n} id$ and G/G^o is generated by the images of Γ and Δ.

5.3. Galois groups for modules over K with two slopes

Let the q-difference module M over K be given by an exact sequence $0 \to P_1 \to M \to P_2 \to 0$, where P_1, P_2 are pure with slopes $\lambda_1 < \lambda_2$. The Picard-Vessiot ring $PV(M)$ of M contains $PV(P_1 \oplus P_2)$, the Picard-Vessiot ring of $P_1 \oplus P_2$. Thus there is an exact sequence $1 \to H \to G \to G' \to 1$, where G and G' are the difference Galois groups of M and $P_1 \oplus P_2$. The group G' has been calculated in subsections 5.1 and 5.2. Further, H is the difference Galois group of the extension $PV(P_1 \oplus P_2) \subset PV(M)$.

Let V_1, V_2, V denote the solution spaces for P_1, P_2, M. There is an obvious exact sequence $0 \to V_1 \to V \to V_2 \to 0$. The space V_1 is invariant under the action of G on V and the induced action of G on $gr(V) := V_1 \oplus V_2$ coincides with G'. Hence H can be identified with a $\mathbb{C}$-vector space of linear maps from V_2 to V_1. Indeed, for $h \in H$, the kernel of $h - 1$ contains V_1 and the image of $h - 1$ lies in V_1. In other words, H can be identified with a linear subspace W of $V_2^* \otimes_{\mathbb{C}} V_1$, which is the solution space of $P_2^* \otimes_K P_1$.

For any difference module N over K we write $\widehat{N}$ for the difference module $\widehat{K} \otimes_K N$ over $\widehat{K}$. Consider the split exact sequence

$$0 \to \widehat{P}_1 \to \widehat{M} \to \widehat{P}_2 \to 0.$$

The splitting $\widehat{P}_2 \to \widehat{M}$ is unique since the only morphism $\widehat{P}_2 \to \widehat{P}_1$ of q-difference modules over $\widehat{K}$ is the zero map. Moreover, the canonical morphism $\widehat{K} \otimes PV(P_1 \oplus P_2) \to PV(\widehat{M})$ is an isomorphism. The resulting embedding $PV(P_1 \oplus P_2) \subset PV(\widehat{M})$ extends in a unique way to an embedding $PV(M) \subset PV(\widehat{M})$. The solution space V of M can be defined as $\ker(\Phi - 1, \widehat{K} \otimes_K Univ \otimes_K M)$. Similar expressions are valid for V_1, V_2, the solution spaces of P_1, P_2. Write G'' for the difference Galois group of $PV(\widehat{M})$. The restriction map induces an isomorphism $G'' \to G'$. We conclude that the exact sequences

$$1 \to H \to G \to G' \to 1 \text{ and } 0 \to V_1 \to V \to V_2 \to 0$$

have canonical splittings. Write $V = V_1 \oplus V_2$ for this splitting, then G' (or G'') can be identified with the subgroup of G leaving both V_1 and V_2 invariant. Thus G is the semi-direct product $H \rtimes G'$. The exact sequence, defining M, is given by an element $\xi \in \operatorname{coker}(\Phi - 1, \operatorname{Hom}(P_2, P_1))$. The main issue is to derive H (or W) from the given ξ and to formulate ξ in terms of the solution spaces V_1, V_2.

Let N be a pure difference module with slope $\lambda < 0$ and solution space $V(N)$. Our first aim is to formulate $\operatorname{coker}(\Phi - 1, N)$ in terms of $V(N)$.

LEMMA 5.7. — *Let N be a pure module with slope $\lambda < 0$. The canonical map $a \mapsto 1 \otimes a$ from N to $Univ \otimes_K N$ induces an injective map*

$$\operatorname{coker}(\Phi - 1, N) \to \operatorname{coker}(\Phi - 1, Univ \otimes_K N)$$

whose image consists of the elements which are invariant under the universal group $\mathbb{G}_{univ}$.

Proof. — It suffices to consider a pure, indecomposable $N = Res(E(dz^{t/n})) \otimes U_m$. By writing N as an extension of $Res(E(dz^{t/n})) \otimes U_{m-1}$ by $Res(E(dz^{t/n}))$, one is reduced to verify the lemma for $Res(E(dz^{t/n}))$. For notational reasons we write $\mathbb{R} = L \oplus \mathbb{Q}$ for some $\mathbb{Q}$-vector space L. Let $C \subset \mathbb{C}^*$ denote the elements c of the form $e^{2\pi i(a_0 + a_1\tau)}$ with $a_1 \in L$. Then we can write $Univ = \oplus_{c \in C, \lambda \in \mathbb{Q}} K_\infty[\ell] e(cz^\lambda)$. Then $Univ \otimes_K N = \oplus K_\infty[\ell] e(cz^\lambda) \otimes_K N$. Each direct summand is invariant under the action of Φ and $\mathbb{G}_{univ}$. The cokernel of $\Phi - 1$ of each direct summand has a certain $\mathbb{G}_{univ}$-action. Using the topological generators Γ, Δ, D of this group, one finds that only the direct summand $K_\infty[\ell] \otimes_K N$ can produce $\mathbb{G}_{univ}$-invariant elements in the cokernel of $\Phi - 1$. A further inspection shows that this contribution comes from the subspace $K \otimes_K N$. Thus we find the required bijection. □

We observe that $Univ \otimes_K N = Univ \otimes_{\mathbb{C}} V(N)$, where the solution space $V(N)$ of N is defined as $\ker(\Phi - 1, Univ \otimes_K N)$. The $\mathbb{G}_{univ}$-invariant part of $\mathrm{coker}(\Phi - 1, Univ \otimes_{\mathbb{C}} V(N))$, comes from $K[\{e(c)\}, \ell]e(z^{-\lambda}) \otimes_{\mathbb{C}} V(N)$, where D acts as the identity. We conclude the following.

LEMMA 5.8. — *Let N be a pure module with slope $\lambda < 0$ and solution space $V(N)$. Then* $\mathrm{coker}(\Phi - 1, N)$ *can be identified with the $\mathbb{C}$-linear subspace of*

$$(\mathrm{coker}(\phi - 1, K[\{e(c)\}, \ell]e(z^{-\lambda})) \otimes_{\mathbb{C}} V(N),$$

consisting of the elements invariant under Γ and Δ.

LEMMA 5.9. — *Let $\lambda < 0$. The subspace $H(\lambda) := \{\oplus_{|q|^{-\lambda} < |c| \leqslant 1} \mathbb{C}e(cz^{-\lambda})\}[\ell]$ of $K[\{e(c)\}, \ell]e(z^{-\lambda})$ has the following properties:*
$H(\lambda) \to \mathrm{coker}(\phi - 1, K[\{e(c)\}, \ell]e(z^{-\lambda}))$ is bijective and
$H(\lambda)$ is invariant under the actions of Γ, Δ and D.

Proof. — From Theorem 2.5 it follows that we may, for the calculation of this cokernel, replace K by $\mathbb{C}[z, z^{-1}]$. After replacing z by $e(q^{-1})$, we have to compute the coker of $\phi - 1$ on the space $\{\oplus_{c \neq 0} \mathbb{C}e(cz^{-\lambda})\}[\ell]$. Using 2.6 one finds that $H(\lambda)$ has the first property. An inspection of the actions of Γ, Δ, D yields the second property. □

COROLLARY 5.10. — *The canonical map* $\mathrm{coker}(\Phi - 1, N) \to (H(\lambda) \otimes V(N))^{<\Gamma, \Delta>}$ *(i.e., the elements invariant under the group generated by Γ and Δ) is an isomorphism.*

COROLLARY 5.11. — *The exact sequences $0 \to P_1 \to M \to P_2 \to 0$ (with P_1, P_2 pure modules with slopes $\lambda_1 < \lambda_2$ and solution spaces V_1, V_2) are in bijection with the elements of $(H(\lambda_1 - \lambda_2) \otimes \mathrm{Hom}(V_2, V_1))^{<\Gamma, \Delta>}$.*

THEOREM 5.12. — *Let the difference module M be given by an exact sequence $0 \to P_1 \to M \to P_2 \to 0$ with P_1, P_2 pure modules with slopes $\lambda_1 < \lambda_2$. Let V_1, V_2 denote the solution spaces of P_1, P_2 and write $V = V_1 \oplus V_2$ for the solution space of M.*

Let $\xi \in \mathrm{coker}(\Phi - 1, \mathrm{Hom}(P_2, P_1))$ and its image
$\xi' \in (H(\lambda_1 - \lambda_2) \otimes \mathrm{Hom}(V_2, V_1))^{<\Gamma, \Delta>}$ which represents the exact sequence.

Define $W \subset \mathrm{Hom}(V_2, V_1)$ to be the smallest subspace such that
$\xi' \in H(\lambda_1 - \lambda_2) \otimes_{\mathbb{C}} W$. Then:

The difference Galois group $G \subset \mathrm{GL}(V)$ of M is the semi-direct product $G = H \rtimes G'$ with: $G' = \{g \in G | g(V_i) = V_i \text{ for } i = 1, 2\}$ is the difference

Galois group of $P_1 \oplus P_2$ and H consists of the linear maps of the form $id + \tilde{w}$ where $\tilde{w}$ is equal to $V \stackrel{pr_2}{\to} V_2 \stackrel{w}{\to} V_1 \subset V$ with $w \in W$.

Proof. — It suffices to prove the last statement. For any submodule N of $\mathrm{Hom}(P_2, P_1)$ one considers the exact sequence $0 \to N \to \mathrm{Hom}(P_2, P_1) \to N' \to 0$. This gives rise to an exact sequence

$$0 \to \mathrm{coker}(\Phi - 1, N) \to \mathrm{coker}(\Phi - 1, \mathrm{Hom}(P_2, P_1)) \to \mathrm{coker}(\Phi - 1, N') \to 0 \ .$$

It follows that there exists a smallest submodule N_0 such that ξ lies in $\mathrm{coker}(\Phi - 1, N_0)$. The solution space $V(N_0)$ lies in $\mathrm{Hom}(V_2, V_1)$ and ξ' belongs to $H(\lambda_1 - \lambda_2) \otimes V(N_0)$ (and is invariant under $< \Gamma, \Delta >$).

It is clear that H identifies with $\{id + \tilde{w} |\ w \in W_1\}$ for some $W_1 \subset \mathrm{Hom}(V_2, V_1)$ which is invariant under the action of the difference Galois group of $\mathrm{Hom}(P_2, P_1)$, acting upon its solution space $\mathrm{Hom}(V_2, V_1)$. In other words, W_1 is invariant under $< \Gamma, \Delta >$. By Tannakian correspondence, W_1 corresponds to a submodule N_1 of $\mathrm{Hom}(P_2, P_1)$. Moreover, ξ lies in $\mathrm{coker}(\Phi - 1, N_1)$.

One concludes that $N_0 = N_1$ and that $W = W_1$. $\square$

The elements $\{e(cz^{-\lambda})\ell^m\}_{|q|^{-\lambda} < |c| \leqslant 1;\ m \geqslant 0}$, with $\lambda := \lambda_1 - \lambda_2$, form a $\mathbb{C}$-basis of $H(\lambda_1 - \lambda_2)$. Thus ξ' can be written uniquely as $\sum e(cz^{-\lambda})\ell^m \otimes w(c, m)$ with all $w(c, m) \in \mathrm{Hom}(V_2, V_1)$. Then W is the subspace of $\mathrm{Hom}(V_2, V_1)$ generated by $\{w(c, m)\}$. This observation makes it possible to compute the difference Galois group for explicit examples.

5.4. Modules over K with more slopes

We describe here the difference Galois group for the general case. The proof follows straightforward from the case of two slopes. Consider a q-difference module M over K with slope filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$. Write $gr(M) = P_1 \oplus \cdots \oplus P_r$ where P_i is pure of slope λ_i and $\lambda_1 < \cdots < \lambda_r$. According to Remarks 2.5, part (1), M is given by an element $\xi = \{\xi_{i,j}\}$ in $\prod_{i<j} \mathcal{F}^o_{i,j}(\mathbb{C}) \cong \prod_{i<j} \mathrm{coker}(\Phi - 1, \mathrm{Hom}(P_j, P_i))$. Using the unique direct sum decomposition of $\widehat{K} \otimes M = \oplus \widehat{K} \otimes P_i$ one finds a canonical decomposition $V = V_1 \oplus \cdots \oplus V_r$ of the solution space V of M, where V_i stands for the solution space of P_i. Moreover the difference Galois group $G \subset \mathrm{GL}(V)$ of M is a semi-direct product $G = H \rtimes G'$. The group G' is the difference Galois group of $P_1 \oplus \cdots \oplus P_r$ and consists of the $g \in G$ that leave each subspace V_i invariant. The normal subgroup H is generated by subgroups $H_{i,j}$ with $i < j$. This subgroup $H_{i,j}$ consists of the maps $id + \tilde{w}_{i,j}$

with $\tilde{w}_{i,j} = V \overset{pr_j}{\to} V_j \overset{w_{i,j}}{\to} V_i \subset V$ with $w_{i,j} \in W_{i,j} \subset \mathrm{Hom}(V_j, V_i)$. Further $W_{i,j}$ is the smallest $\mathbb{C}$-linear subspace of $\mathrm{Hom}(V_j, V_i)$ such that the element $\xi_{i,j} \in \mathrm{coker}(\Phi - 1, \mathrm{Hom}(P_j, P_i))$ is represented by $\xi'_{i,j} \in H(\lambda_i - \lambda_j) \otimes W_{i,j}$. According to Remarks 3.3, part (3), H is in fact equal to set of elements $id + \sum_{i<j} \tilde{w}_{i,j}$ with all $w_{i,j} \in W_{i,j}$. Indeed, $w_{i,j} \circ w_{j,k} \in W_{i,k}$ for $i < j < k$ and any $w_{i,j} \in W_{i,j}$, $w_{j,k} \in W_{j,k}$.

The Tannakian category of difference modules over K has the following description in terms of solution spaces. One can attach to a difference module M the tuple $(V, \{V_\lambda\}, \Gamma_V, \Delta_V, \{\xi_{\lambda,\mu}\})$, where $(V, \{V_\lambda\}, \Gamma_V, \Delta_V)$ is the tuple associated to the split module $gr(M)$ and $\xi_{\lambda,\mu}$ is, for each $\lambda < \mu$, an element of $H(\lambda - \mu) \otimes \mathrm{Hom}(V_\mu, V_\lambda)$, invariant under the action of $< \Gamma_V, \Delta_V >$. The category of the above tuples is, in an obvious way, a Tannakian category. The data $\{\xi_{\lambda,\mu}\}$ come from divergent solutions (i.e., with coefficients in $\widehat{K}$). They can be seen as the equivalent of the Stokes matrices in the theory of irregular differential equations over $\mathbb{C}$.

6. Realizing q-difference modules over K as connections on the elliptic curve $E = \mathbb{C}^*/q^{\mathbb{Z}}$

As in section 2 we associate to any difference module M over K a vector bundle $v(M)$ on the elliptic curve E_q. The aim is to provide $v(M)$ with a suitable connection. The following theorem gives an explicit version of Corollary 5.4.

THEOREM 6.1 (REGULAR SINGULAR q-DIFFERENCE MODULES). — *Let a, b denote the generators of $\pi_1(E_q)$, corresponding to the shifts $u \mapsto u + 1$, $u \mapsto u + \tau$ on the universal covering $\mathbb{C}_u$ of E_q. Let $i :< \Gamma_{rs}, \Delta_{rs} > \to \pi_1(E_q)$ denote the isomorphism given by $\Gamma_{rs} \mapsto a$, $\Delta_{rs} \mapsto b$.*

For every regular singular difference module M over K, corresponding to a representation ρ of $< \Gamma_{rs}, \Delta_{rs} >$, there exists a unique 'unit circle' connection ∇_M on $v(M)$, corresponding to a representation ρ' of $\pi_1(E_q)$, such that $\rho = i \circ \rho'$.

This induces a Tannakian equivalence between the category of the regular singular difference modules over K and a full subcategory of the category of all 'unit circle' connections on E_q.

Proof. — We recall that a connection $\nabla : \mathcal{M} \to \Omega_{E_q} \otimes \mathcal{M}$ is called *unit circle* if the corresponding representation $\rho' : \pi_1(E_q) \to \mathrm{GL}(V)$ has the property that every eigenvalue of every $\rho'(\alpha)$ has absolute value 1.

(1) We start by giving an explicit construction for rank one difference modules M over K. Write $M = Ke$ with $\Phi e = ce$ and $c \in \mathbb{C}^*$. A connection ∇_M on $v(M)$ is equivalent with a connection $\nabla : Oe \to \frac{dz}{z} \otimes Oe$ commuting with Φ (we recall that O is the algebra of holomorphic functions on $\mathbb{C}^*$). One concludes that $\nabla(e) = a(c)\frac{dz}{z} \otimes e$, where $a : \mathbb{C}^* \to \mathbb{C}$ is a homomorphism. The differential equation $\nabla e = a(c)\frac{dz}{z} \otimes e$, considered on $\mathbb{C}$, reads $\nabla e = 2\pi i a(c) du \otimes e$ where u is the variable of $\mathbb{C}$ and $z = e^{2\pi i u}$. The basic solution of this equation is $e^{-2\pi i a(c) u} e$ and e can be interpreted as $\theta_{c^{-1}}(e^{2\pi i u})$. The shifts $u \mapsto u + 1$ and $u \mapsto u + \tau$ multiply this equation with $e^{-2\pi i a(c)}$ and $e^{-2\pi i a(c)\tau} c$. Write $c = e^{2\pi i (a_0 + a_1 \tau)}$ with $a_0 = a_0(c), a_1 = a_1(c) \in \mathbb{R}$. The 'unit circle' condition implies that $a(c) = a_1(c)$ and the basic solution is multiplied by $e^{-2\pi i a_1}$ and $e^{2\pi i a_0}$. Thus we found a unique unit circle connection on $v(Ke)$ and moreover the actions of Γ_{rs}, Δ_{rs} coincide with the action of the fundamental group $\pi_1(E_q)$.

(2) The next case to consider is $M = K \otimes W$ with $\Phi(f \otimes w) = \phi(f) \otimes A(w)$ and $A \in \mathrm{GL}(W)$ is unipotent. Define $\nabla : O \otimes W \to O\frac{dz}{z} \otimes W$ by $\nabla(1 \otimes w) = \frac{1}{2\pi i \tau} \frac{dz}{z} \otimes (1 \otimes \log(A)(w))$. Clearly ∇ commutes with Φ. One can verify that this definition of ∇ induces a regular connection on $v(M)$. The pullback of this connection to $\mathbb{C}$ has the fundamental matrix $A^{u/\tau}$. The shifts $u \mapsto u + 1,\ u \mapsto u + \tau$ multiply this matrix with $A^{1/\tau}$ and A, in accordance with the actions of Γ_{rs} and Δ_{rs}.

The general case is obtained from these two special cases by taking tensor products and direct sums.

(3) The map $\mathrm{Hom}(M, N) \to \mathrm{Hom}((v(M), \nabla_M), (v(N), \nabla_N))$ is a bijection since both groups translate into homomorphisms between representations. □

Remarks 6.2. — (1) We note that not every unit circle connection on E is isomorphic to some $(v(M), \nabla_M)$. E.g., the free vector bundle $O_E e_1 \oplus O_E e_2$, provided with the unit circle connection ∇ given by $\nabla e_1 = 0,\ \nabla e_2 = \frac{dz}{z} \otimes e_1$, is not isomorphic to some $(v(M), \nabla_M)$.

(2) Our aim is to extend Theorem 6.1 to the category of all difference modules over K. The first step is to extend Theorem 6.1 to the category of the split difference modules. Explicitly, one wants to construct for every split difference module M a connection ∇_M on $v(M)$ such that $\nabla_M : v(M) \to \Omega([1]) \otimes v(M)$ (i.e., ∇_M has a regular singularity at $1 \in E_q$). Moreover, one requires that $M \mapsto \nabla_M$ is functorial, commutes with direct sums and tensor products, and extends the construction in Theorem 6.1 of ∇_M for regular singular difference modules.

(3) The pure module $M = Ke$ with $\Phi e = (-z)e$ is the first candidate for a construction of $\nabla_M : v(M) \to \Omega([1]) \otimes v(M)$. We note that $v(M) = O_{E_q}([1])e$. Any connection ∇ on this line bundle with at most a regular singularity at 1 is given by $\nabla e = a\frac{dz}{z} \otimes e$ for some constant a. The assumption $a = 0$ is the natural choice for ∇_M. Thus ∇_M is the unique extension of the trivial connection on $O_{E_q}e$, given by $\nabla e = 0$ to a connection $\nabla_M : v(M) \to \Omega([1]) \otimes v(M)$. One can describe this connection on Oe by $\nabla e = -\frac{d\Theta}{\Theta} \otimes e$. This leads to a formula for ∇_M for the difference modules $M = Ke$ with $\Phi e = c(-z)^t e$, namely

$$\nabla e = (-t\frac{d\Theta}{\Theta} + a_1\frac{dz}{z}) \otimes e \text{ where } c = e^{2\pi i(a_0+a_1\tau)} \text{ and } a_0, a_1 \in \mathbb{R} .$$

One observes that for difference modules of rank one , the map $M \mapsto (v(M), \nabla_M)$ respects tensor products.

(4) Instead of continuing the method of (3), we will use subsubsection 1.4.1 to give a general construction of ∇_M for split difference modules over K.

THEOREM 6.3. — *There exists a functor $N \mapsto (v(N), \nabla_N)$ from the category of the split difference modules over K to the category of the connections on E_q with at most a regular singularity at $1 \in E_q$. This functor extends the one defined in Theorem 6.1. Moreover, the functor $N \mapsto (v(N), \nabla_N)$ commutes with tensor products and is faithful.*

Proof. — We use the notation and the results of subsubsection 1.4.1.

(1) *Construction of ∇_M for pure difference modules M over K_∞.*

Consider a pure difference module M over K_∞ with slope λ. Then $M = K_\infty \otimes_{\mathbb{C}} V$ with Φ given by $\Phi(f \otimes v) = z^\lambda \phi(f) \otimes A(v)$ where $A \in \mathrm{GL}(V)$ has the property that every eigenvalue c of A has the form $c = e^{2\pi i(a_0(c)+a_1(c)\tau)}$ with $a_0(c) \in \mathbb{R}$ and $a_1(c) \in L \subset \mathbb{R}$. Let $a_1(A_{ss}) \in \mathrm{End}(V)$ be obtained from A_{ss} by replacing every eigenvalue c of A_{ss} by $a_1(c)$. We introduce the notation: O_n is the algebra of the convergent Laurent series in the variable $z^{1/n}$ and $O_\infty = \bigcup O_n$. With these notation one defines the connection

$$\nabla_M : O_\infty \otimes_{\mathbb{C}} V \to \frac{1}{\Theta} O_\infty \frac{dz}{z} \otimes V \text{ by}$$

$$\nabla_M(v) = -\lambda\frac{d\Theta}{\Theta} \otimes v + \frac{dz}{z} \otimes (\ a_1(A_{ss}) + \frac{1}{2\pi i\tau}\log(A_u)\ )(v) .$$

The last formula is extended by $\nabla_M(f \otimes v) = df \otimes v + f\nabla_M(v)$ and by additivity to a ∇_M defined on $O_\infty \otimes_{\mathbb{C}} V$. By construction, ∇_M commutes with the action of Φ.

For two pure difference modules $M_i = K_\infty \otimes_{\mathbb{C}} V_i,\ i = 1, 2$ over K_∞ with Φ-actions given by the slopes λ_1, λ_2 and $A_i \in \mathrm{GL}(V_i),\ i = 1, 2$, the pure module $M_3 = M_1 \otimes_{K_\infty} M_2$ has the form $K_\infty \otimes_{\mathbb{C}} (V_1 \otimes V_2)$ with Φ-action given by the slope $\lambda_1 + \lambda_2$ and $A_3 := A_1 \otimes A_2 \in \mathrm{GL}(V_1 \otimes V_2)$. One observes that the eigenvalues c of A_3 satisfy again $a_1(c) \in L \subset \mathbb{R}$. Further $(A_3)_{ss} = (A_1)_{ss} \otimes (A_2)_{ss}$ and $a_1((A_1)_{ss} \otimes (A_2)_{ss})$ is equal to $(a_1((A_1)_{ss}) \otimes id) + (id \otimes a_1((A_2)_{ss}))$. There is a similar formula for $\frac{1}{2\pi i\tau} \log(A_3)_u$.

One concludes that ∇_{M_3} is the tensor product $\nabla = \nabla_{M_1} \otimes \nabla_{M_2}$. The latter is defined by $\nabla(m_1 \otimes m_2) = (\nabla_{M_1}(m_1) \otimes m_2) + (m_1 \otimes \nabla_{M_2} m_2)$ (for $m_i \in O_\infty \otimes V_i,\ i = 1, 2$).

A morphism $f : M_1 \to M_2$ between two difference modules $M_i = K_\infty \otimes_{\mathbb{C}} V_i,\ i = 1, 2$ with the same slope λ and Φ given by $A_i \in \mathrm{GL}(V_i),\ i = 1, 2$ is induced by a $\mathbb{C}$-linear map $F : V_1 \to V_2$ satisfying $F \circ A_1 = A_2 \circ F$. Therefore f induces a morphism between the two connections ∇_{M_i}.

(2) *Construction of* $(v(N), \nabla_N)$ *for a pure difference module* N *over* K.
The connection $\nabla_N : v(N) \to \Omega_{E_q}([1]) \otimes v(N)$, that we want to construct, translates into a connection with the same name

$$\nabla_N : O \otimes_{\mathbb{C}[z,z^{-1}]} N_{global} \to \frac{1}{\Theta} O \frac{dz}{z} \otimes_{\mathbb{C}[z,z^{-1}]} N_{global}\ ,$$

which commutes with the action of Φ.

Put $M = K_\infty \otimes_K N$. This difference module is equipped with the data of N, i.e., $data(N) = (\lambda, V, A, \{D(\sigma)\})$. We note that $O_\infty \otimes_{\mathbb{C}} V = O_\infty \otimes_{\mathbb{C}[z,z^{-1}]} N_{global}$ and that $(O_\infty \otimes_{\mathbb{C}} V)^{Gal}$ is equal to $O \otimes_{\mathbb{C}[z,z^{-1}]} N_{global}$. The ∇_M, constructed above, descends to a ∇_N for N if and only if ∇_M commutes with the $\{D(\sigma)\}$.

The formula $D(\sigma)^{-1} A D(\sigma) = e^{2\pi i\lambda\sigma} A$ implies that $D(\sigma)^{-1} A_u D(\sigma) = A_u$ and $D(\sigma)^{-1} A_{ss} D(\sigma) = e^{2\pi i\lambda\sigma} A_{ss}$. The eigenspace of A_{ss} for the eigenvalue c is mapped by $D(\sigma)$ to the eigenspace for the eigenvalue $ce^{2\pi i\lambda\sigma}$. From $a_1(ce^{2\pi i\lambda\sigma}) = a_1(c)$ it follows that $D(\sigma)$ leaves every eigenspace invariant of $a_1(A_{ss})$ for the eigenvalues of this map. Thus $D(\sigma)$ commutes with $a_1(A_{ss})$ and with A_u, too. This implies that ∇_M commutes the $\{D(\sigma)\}$.

(3) $(v(N), \nabla_N)$ *for a split difference module* N *over* K.
For $N = N_1 \oplus \cdots \oplus N_r$ with all N_i pure of slopes $\lambda_1 < \cdots < \lambda_r$ one defines $(v(N), \nabla_N) := \oplus_{i=1}^r (v(N_i), \nabla_{N_i})$.

A morphism between split modules over K is the sum of morphisms between pure modules with the same slope. The latter induces a morphism between the corresponding connections, according to (1). Thus $N \mapsto$

$(v(N), \nabla_N)$ is a functor. According to (1) and 1.4.1, this functor preserves tensor products.

For proving 'faithful' it suffices to show that, for a pure module N, the map $\ker(\Phi - 1, N) \to \{\xi \in H^0(E_q, v(N))|\ \nabla_N \xi = 0\}$ is injective. If the slope of N is not 0, then the left hand side is 0. If the slope is 0, then by Theorem 6.1, the above map is bijective. □

Remark. — The following example shows that the functor of Theorem 6.3 is not fully faithful. Put $N = Ke$ with $\Phi(e) = (-z)^t e$ and $t > 0$. Then $v(N) = O_{E_q}(t \cdot [1])$ and ∇_N is induced by $d : O_{E_q} \to \Omega_{E_q}$, using the inclusion $O_{E_q} \subset O_{E_q}(t \cdot [1])$. In this case, $\ker(\Phi - 1, N) = 0$ and $\{\xi \in H^0(E_q, v(N))|\ \nabla_N \xi = 0\}$ has dimension 1.

We want to extend the functor of Theorem 6.3 to the category of all difference modules over K. We start with an example.

Example 6.4. — $N = Ke_1 + Ke_2$ *with* $\Phi e_1 = (-z)^t e_1,\ \Phi e_2 = e_2 + pe_1$, $t \in \mathbb{Z},\ t > 0$ *and* $p \in K$.

We may suppose $p \in \mathbb{C}[z, z^{-1}]$. The aim is to produce a connection

$$\nabla : Oe_1 + Oe_2 \to \frac{1}{\Theta^t} O \frac{dz}{z} \otimes_O (Oe_1 + Oe_2)\ ,$$

such that ∇ commutes with Φ and ∇ induces for the pure module Ke_1 and Ke_2 the connections of Theorem 6.3.

Consider the inclusion $Oe_1 + Oe_2 \subset Of_1 + Of_2$ with $f_1 = \Theta^{-t} e_1,\ f_2 = e_2$. Then $\Phi f_1 = f_1$ and $\Phi f_2 = f_2 + p\Theta^t f_1$. We propose $\nabla f_1 = 0$ and $\nabla f_2 = \omega \otimes f_1$ with $\omega \in O\frac{dz}{z}$. The condition $\Phi\nabla = \nabla\Phi$ is equivalent to $(\phi - 1)(\omega) = d(p\Theta^t)$.

The equation is solved as follows. There exists $f \in O$ with $(\phi - 1)(f) = p\Theta^t - c$ where c is the constant term of $p\Theta^t$. This f is unique up to its constant term. Now $\omega := d(f)$ satisfies $(\phi - 1)(df) = d((\phi - 1)(f)) = d(p\Theta^t)$.

Thus $\nabla e_1 = t\frac{d\Theta}{\Theta} \otimes e_1$ and $\nabla e_2 = \Theta^{-t}\omega \otimes e_1$. Since ∇ commutes with Φ this induces a connection $\nabla : v(N) \to \Omega_{E_q}(t[1]) \otimes v(N)$ with a pole at $1 \in E_q$ of order at most t. Moreover ∇ induces on $v(Ke_1)$ and $v(Ke_2)$ the connections of Theorem 6.3.

In the next example we give a construction of ∇_N for difference modules N over K with two integer slopes.

EXAMPLE 6.5. — *$N = P_1 \oplus P_2$ with P_1, P_2 global modules with slopes $t_1 < t_2$, $t = t_2 - t_1 \in \mathbb{Z}$ and Φ given by $\Phi(p_1 + p_2) = \Phi_1 p_1 + \ell(\Phi_2 p_2) + \Phi_2 p_2$ for some $\mathbb{C}[z, z^{-1}]$-linear map $\ell : P_2 \to P_1$.*

Put $Q_1 = \mathbb{C}[z, z^{-1}]e \otimes P_1$ with $\Phi(e) = (-z)^t e$ and $t = t_2 - t_1$ and put $Q_2 = P_2$. Then $O \otimes P_1$ is embedded into $O \otimes Q_1$ by $p \mapsto \Theta^t e \otimes p$. Now Φ on $O \otimes Q_1 + O \otimes Q_2$ is given by

$$\Phi(e \otimes p_1 + p_2) = (-z)^t e \otimes \Phi_1(p_1) + \Theta^t e \otimes \ell(\Phi_2 p_2) + \Phi_2(p_2) \ .$$

∇ on $O \otimes Q_1 + O \otimes Q_2$ will be given by a formula of the type: $\nabla(e \otimes p_1) = -t\frac{d\Theta}{\Theta} e \otimes p_1 + e \otimes \nabla_1 p_1$, where ∇_i are the connections for P_i imposed by Theorem 6.3. Further $\nabla p_2 = \nabla_2 p_2 + m(p_2)$ where $m : O \otimes P_2 \to O\frac{dz}{z} \otimes Q_1$ is a yet unknown O-linear map. Thus m is an element of $O\frac{dz}{z} \otimes_O T$ with $T = O \otimes_{\mathbb{C}[z,z^{-1}]} (P_2^* \otimes Q_1)$. We note that $P_2^* \otimes Q_1$ is a pure global module of slope 0.

Write $\tilde{\ell} : O \otimes P_2 \xrightarrow{\ell} O \otimes P_1 \subset O \otimes Q_1$. Thus $\tilde{l}$ is an element of T. The condition $\Phi\nabla = \nabla\Phi$ is equivalent to $\Phi_T(m) - m = \nabla_T(\tilde{\ell})$, where Φ_T and ∇_T denote the Φ-action and the connection for T.

According to Lemma 6.6, there is a canonical solution m for this equation. The corresponding ∇ induces a connection $\nabla_N : v(N) \to \Omega_{E_q}(t[1]) \otimes v(N)$ with a pole at $1 \in E_q$ of order at most t. Further ∇_N induces the connections on $v(P_1)$ and $v(P_2)$ prescribed by Theorem 6.3.

The action of Φ on $O \otimes Q_1 + O \otimes Q_2$ can be changed into an equivalent one by adding to $\tilde{\ell}$ an expression $(\Phi - 1)(\xi)$ with $\xi \in T$. This is compatible with the construction of ∇_N, according to part (1) of Lemma 6.6. After such a change, one may suppose that $\tilde{\ell}$ maps Q_2 into Q_1. In this situation the connection on $Q_1 \oplus Q_2$ is the one prescribed by Theorem 6.3, according to part (2) of Lemma 6.6

LEMMA 6.6. — (1) *Let P be a pure global module of slope 0. There is a canonical $\mathbb{C}$-linear map $f \in O \otimes P \to \omega(f) \in O\frac{dz}{z} \otimes P$ satisfying $(\Phi - 1)\omega(f) = \nabla f$. Further $\omega(\Phi(f)) = \Phi(\omega(f))$.*
(2) *Let $(Q_1, \Phi_1), (Q_2, \Phi_2)$ denote two pure global modules of the same slope and let $\ell : Q_2 \to Q_1$ be a $\mathbb{C}[z, z^{-1}]$-linear map. Define the pure module $N = Q_1 \oplus Q_2$ by $\Phi(q_1 + q_2) = \Phi_1(q_1) + \Phi_2(q_2) + \ell(\Phi_2(q_2))$.*

The connection on $O \otimes N$, defined by Theorem 6.3 coincides with the connection ∇ given by the formula $\nabla(q_1 + q_2) = \nabla_1(q_1) + \nabla_2(q_2) + \omega(\ell)(q_2)$.

Proof. — Write $P = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} W$ with Φ defined by $\Phi(w) = A(w)$ and $\nabla(w) = \frac{dz}{z} \otimes (a_1(A_{ss}) + \frac{1}{2\pi i\tau} \log(A_u))(w)$. For convenience we suppose

that the eigenvalues c of A satisfy $|q| < |c| \leqslant 1$. Write $f = \sum_{n\in\mathbb{Z}} z^n \otimes f_n$ and $\omega(f) = \sum_{n\in\mathbb{Z}} z^n \frac{dz}{z} \otimes \omega_n$ with all $f_n, \omega_n \in W$.

$$\text{Then } f = \sum_n z^n \frac{dz}{z} \otimes \{(n + a_1(A_{ss}) + \frac{1}{2\pi i\tau} \log A_u)(f_n)\}$$

$$\text{and } (\Phi - 1)\omega(f) = \sum_n z^n \frac{dz}{z} \otimes (q^n A - 1)(\omega_n) \ .$$

This produces the equations $(q^n A - 1)\omega_n = (n + a_1(A_{ss}) + \frac{1}{2\pi i\tau} \log A_u) f_n$. For $n \neq 0$, the map $q^n A - 1$ is invertible and there is a unique solution ω_n. For the equation $(A - 1)\omega_0 = (a_1(A_{ss}) + \frac{1}{2\pi i\tau} \log A_u) f_0$ we write W as a direct sum $\oplus W_c$, where W_c is the generalized eigenspace for A and the eigenvalue c. For $c \neq 1$, the restriction of the equation to W_c has a unique solution since $A - 1$ is invertible on W_c. On the space W_1, the equation reads $(A - 1)\omega_0^1 = (\frac{1}{2\pi i\tau} \log A_u) f_0^1$, where ω_0^1, f_0^1 denote the components in W_1 of ω_0 and f_0. On W_1 we have $A = A_u$. The canonical solution, that we propose, is given by $\omega_0^1 = \sum_{j=0}^{\infty} (-1)^j \frac{(A_u - 1)^j}{2\pi i\tau (j+1)} (f_0^1)$.

It is clear that $f \mapsto \omega(f)$ is $\mathbb{C}$-linear. The formula $\Phi(\omega(f)) = \omega(\Phi f)$ follows from the explicit definition of $\omega(f)$. The expression 'canonical' means the following. Let $\alpha : P_1 \to P_2$ be a morphism between global modules of slope 0 and let $f \in O \otimes P_1$. Then α applied to $\omega(f)$ is equal to $\omega(\alpha(f))$.

A straightforward calculation shows (2). □

THEOREM 6.7. — *There exists a $\mathbb{C}$-linear functor $N \mapsto (v(N), \nabla_N)$ from the category of the difference modules N over K with integer slopes to the category of the connections on E_q with at most a pole at the point $1 \in E_q$. This functor extends the one of Theorem 6.3, is faithful and commutes with tensor products.*

Proof. — Consider pure global modules P_j, $j = 1, \ldots, r$ with integer slopes $\lambda_1 < \cdots < \lambda_r$. Let Φ_j denote the action of Φ on P_j. Let $\mathbb{C}[z, z^{-1}]$-linear maps $\ell_{i,j} : P_j \to P_i$ for $i < j$, be given. Define the global module N with ascending slope filtration by $N = P_1 \oplus \cdots \oplus P_r$ and $\Phi(p_1 + \cdots + p_r) = \sum_{j=1}^r (\Phi_j(p_j) + \sum_{i<j} \ell_{i,j}\Phi_j(p_j))$. We want to construct a canonical connection on $O \otimes N$ commuting with Φ.

Define $Q_j := \mathbb{C}[z, z^{-1}]e_j \otimes P_j$ with $\Phi e_j = (-z)^{\lambda_r - \lambda_j} e_j$ for $j = 1, \ldots, r$. Let Φ_j^* be the action of Φ on Q_j. One embeds $O \otimes P_j$ into $O \otimes Q_j$ by $p_j \mapsto \Theta^{\lambda_r - \lambda_j} e_j \otimes p_j$. Then $O \otimes N = \oplus O \otimes P_j$ embeds into $O \otimes Q$ with

$Q = \oplus_{j=1}^{r} Q_j$. Let $\tilde{\ell}_{i,j} : O \otimes Q_j \to O \otimes Q_i$ be the map derived from $\ell_{i,j}$. Then Φ on $O \otimes Q$ is given by

$$\Phi(q_1 + \cdots + q_r) = \sum_{j=1}^{r} (\Phi_j^*(q_j) + \sum_{i<j} \tilde{\ell}_{i,j} \Phi_j^*(q_j)) \ .$$

With this formula the embedding is Φ-equivariant. On $O \otimes Q$ one wants to define a connection ∇ of the form

$$\nabla(q_1 + \cdots + q_r) = \sum_{j=1}^{r} \nabla_j(q_j) + \sum_{j=1}^{r} \sum_{i<j} m_{i,j}(q_j) \ ,$$

with O-linear maps $m_{i,j} : O \otimes Q_j \to O\frac{dz}{z} \otimes_O (O \otimes Q_i)$. The condition $\Phi\nabla = \nabla\Phi$ translates into $(\Phi - 1)m_{i,j} = \nabla(\tilde{\ell}_{i,j})$ for all $i < j$. These equations are solved in the canonical way of Lemma 6.6. The restriction of this ∇ to $O \otimes N$ induce a connection $\nabla_N : v(N) \to \Omega_{E_q}((\lambda_r - \lambda_1)[1]) \otimes v(N)$.

The maps $\ell_{i,j}$ in the definition of N are not unique. They can be changed by adding maps $(\Phi - 1)r_{i,j}$ with $r_{i,j} : P_j \to P_i$. It is easily seen that ∇_N only depends on the equivalence classes of the $\ell_{i,j}$. Using Lemma 6.6, one shows that the above defines a $\mathbb{C}$-linear functor.

Let N_1, N_2 denote two global difference modules with ascending slope filtration and with integer slopes. The above constuction embeds $O \otimes N_i$ into $O \otimes M_i$ (for $i = 1, 2$) where $O \otimes M_i$ has only one slope. The connections on N_1, N_2 are the restrictions of the connections on $O \otimes M_i$ prescribed by Theorem 6.3. The above construction applied to $N_3 = N_1 \otimes N_2$ embeds $O \otimes N_3$ into the tensor product of the pure modules $O \otimes M_i$. According to Theorem 6.3, the connection on this tensor product is the tensor product of the connections on the $O \otimes M_i$. We conclude that the functor, construction above, respects tensor products. $\square$

Remarks 6.8. — (1) For general difference modules M over K it is also possible to define a connection ∇_N on $v(N)$ with at most a pole at $1 \in E_q$. However for non integer slopes there seems not be a canonical choice for ∇_N.

(2) In the situation of Example 6.5, the map $\ell : P_2 \to P_1$ is responsible for divergence, 'Stokes matrices' and the unipotent part of the difference Galois group of N. In general, the connection $\nabla_N : v(N) \to \Omega_{E_q}(t[1]) \otimes v(N)$ has a pole of order t. The irregularity of the connection ∇_N locally at $1 \in E_q$ will produce Stokes matrices (in the classical sense) and unipotent elements of the local analytic differential Galois group which depend on ℓ. The precise relation remains to be investigated.

(3) It is interesting to apply another method to Example 6.4. For any integer $t > 0$ one defines $G_t = \sum_{n\in\mathbb{Z}}(q^t)^{n(n-1)/2}(-z^t)^n$. One observes that $(-z^t)G_t(qz) = G_t(z)$ and that the set of the zeros of G_t is $\mu_t \times q^{\mathbb{Z}}$, where μ_t is the group of the t-th roots of unity.

Let again $N = Ke_1 + Ke_2$ with $\Phi(e_1) = (-z)^t e_1$, $\Phi(e_2) = e_2 + pe_1$ with $p \in \mathbb{C}[z, z^{-1}]$. Define now $f_1 = G_t^{-1}e_1$ and $f_2 = e_2$. Then $Oe_1 + Oe_2$ embeds into $Of_1 + Of_2$ and $\Phi(f_1) = (-1)^{t-1}f_1$ and $\Phi(f_2) = f_2 + pG_tf_1$. The connection ∇ is defined by $\nabla f_1 = 0$, $\nabla f_2 = \omega \otimes f_2$ where ω is the canonical solution of $(\Phi - 1)\omega = (-1)^{t-1}d(pG_t)$. The resulting connection ∇_N has at most simple poles at the image points of μ_t in E_q.

(4) The variation in (3) on Example 6.4 extends to a functor on the category of the difference modules over K with integer slopes to the category of the connections on E_q having at most simple poles in the images of $\bigcup_{t\geqslant 1}\mu_t$ in E_q. This functor is constructed as in the proof of Theorem 6.7 and it has again the properties: $\mathbb{C}$-linear, faithful and commuting with tensor products.

For this variation on Theorem 6.7 and in connection with Example 6.5, one observes that $\ell : P_2 \to P_1$ contributes to the poles of ∇_N on the image points of μ_t in E_q. Thus ℓ contributes to the monodromy group for the connection ∇_N.

7. Positive characteristic

Atiyah's paper makes some excursions to positive characteristic. Here, we do the same for q-difference equations. We replace the field $\mathbb{C}$ by a field $\mathbf{C}$ which is algebraically closed and complete for a non trivial non archimedean valuation. The case where $\mathbf{C}$ has characteristic 0 (i.e., $\mathbf{C} \supset \mathbb{Q}_p$ for some prime p) is not very interesting since most of the preceeding results can be copied from the complex case with the help of some rigid analysis.

In this section we consider an algebraically closed field $\mathbf{C}$ of characteristic $p > 0$, complete with respect to a non trivial valuation. Further we choose a $q \in \mathbf{C}$ with $0 < |q| < 1$.

Over $K = \mathbf{C}(\{z\})$, $\widehat{K} = \mathbf{C}((z))$ and $\mathbf{C}(z)$ one can define q-difference modules and study their properties. The elliptic curve associated to this is the Tate curve $E_q := \mathbf{C}^*/q^{\mathbb{Z}}$, constructed with the help of rigid analysis. We note that this curve is special in the sense that its j-invariant is transcendental over $\mathbb{F}_p$ and in particular E_q is ordinary. We make now a quick investigation of the main results of this paper in this new context.

LEMMA 7.1. — *There exists an explicit pair (F, ϕ) of an algebraically closed field F and an automorphism ϕ, such that $F \supset \widehat{K}$ and ϕ extends the given automorphism of $\widehat{K}$. The pair (F, ϕ) induces automorphisms of the algebraic closures of $\widehat{K}$ and K extending the given ϕ.*

Proof. — The algebraic closure of $\widehat{K}$ has no explicit description. However, there is an explicit algebraically closed field $F := \mathbf{C}((z^{\mathbb{Q}}))$ containing $\widehat{K}$. The elements of this field are expressions $\sum_{\lambda\in\mathbb{Q}} a_\lambda z^\lambda$ with all $a_\lambda \in \mathbf{C}$ and such that $\{\lambda|\ a_\lambda \neq 0\}$ is a well ordered subset of $\mathbb{Q}$. It is well known that F is a maximally complete field with residue field $\mathbf{C}$ and value group $\mathbb{Q}$. In particular, F is algebraically closed. Choose a homomorphism $\lambda \mapsto q^\lambda$ from $\mathbb{Q}$ to $\mathbf{C}^*$ with $q^1 = q$. One defines an automorphism ϕ of F by the formula $\phi(\sum_{\lambda\in\mathbb{Q}} a_\lambda z^\lambda) = \sum_{\lambda\in\mathbb{Q}} a_\lambda q^\lambda z^\lambda$. This extends the action of ϕ on $\widehat{K}$. The algebraic closures of $\widehat{K}$ and K can be seen as subfields of F. They are obviously invariant under ϕ. This proves the assertion. □

LEMMA 7.2. — *Let $\widehat{K} \subset L$ be an extension of degree $m < \infty$ such that ϕ extends to an automorphism of L. Then $L = \widehat{K}(z^{1/m})$. A similar statement holds for K replacing $\widehat{K}$.*

Proof. — Write $L = \mathbf{C}((t))$. Then $z = a_m t^m + a_{m+1}t^{m+1} + \cdots$ with $a_m \neq 0$. The action of ϕ on L has therefore the form $\phi(t) = q_1 t + \cdots$ with $q_1^m = q$. Since $0 < |q_1| < 1$, one can produce an element $s \in \mathbf{C}[[t]]$ such that $\mathbf{C}[[s]] = \mathbf{C}[[t]]$ and $\phi(s) = q_1 s$. Thus we may assume that $\phi(t) = q_1 t$. Then

$$q(a_m t^m + a_{m+1}t^{m+1} + \cdots) = qz = \phi(z) = a_m q_1^m t^m + a_{m+1} q_1^{m+1} t^{m+1} + \cdots$$

implies that $z = a_m t^m$. This proves the statement for the case $\widehat{K}$.

For the case K, one has to show that $L = \mathbf{C}\{t\}$ contains an element s with $\mathbf{C}\{s\} = \mathbf{C}\{t\}$ and $\phi(s) = s$. Write $\phi(t) = q_1 t + a_2 t^2 + \cdots$ and $s = t + b_2 t^2 + b_3 t^3 + \cdots$. Then $\phi(s) = q_1 s$ leads to a sequence of equations for the b_i. An inspection shows that the convergence of the series $q_1 t + \sum_{n\geqslant 2} a_n t^n$ implies the convergence of the series $t + \sum_{n\geqslant 2} b_n t^n$. □

Most of the preceeding sections remain valid, after a small adaptation, in the present context. We give now some details.

In section 1, the fields K_∞ and $\widehat{K}_\infty$ should be read here, not as the algebraic closures but as the fields $\cup_{n\geqslant 1} K_n$ with $K_n := \mathbf{C}(\{z^{1/n}\})$ and $\cup_{n\geqslant 1}\widehat{K}_n$ with $\widehat{K}_n := \mathbf{C}((z^{1/n}))$. The complex function theory for E_q is replaced by the rigid analytic theory (see for instance [Fr-vdP]) and the

formulas in subsection 1.1 remain valid. The only part of section 1 that has no (obvious) translation is subsubsection 1.4.1.

All of section 2 remains valid with the exception of Remarks 2.4. Indeed, the formulas for the decomposition of tensor products of indecomposable modules (or for indecomposable vector bundles on E_q) are different in positive characteristic. Especially, the decomposition of $U_a \otimes_K U_b$ poses a non trivial combinatorial problem, solved in [At] in the context of vector bundles on E_q.

No changes are needed for the results of sections 3 and 4. However we will rewrite Section 5 completely by developing a suitable Picard-Vessiot theory, calculating difference Galois groups and a universal Picard-Vessiot ring. We take [vdP-S 1,2] as guide for this.

It is *not* possible to attach, as in Theorem 6.1, to regular singular difference modules M over K, connections on $v(M)$. Indeed, for the module Ke with $\Phi e = ce$ and $c \in \mathbf{C}^*$, the connection must have the form $\nabla e = a(c)\frac{dz}{z} \otimes e$, where $a : \mathbf{C}^* \to \mathbf{C}$ is a homomorphism and satisfies $a(q) = 1$. However, $a(q) = p \cdot a(q^{1/p}) = 0$.

7.1. Picard-Vessiot theory and examples

Consider a field F provided with an automorphism ϕ of infinite order. The field of constants $C := \{f \in F |\ \phi(f) = f\}$ is supposed to be algebraically closed. A difference module M is a finite dimensional vector space over F, provided with a bijective additive map $\Phi : M \to M$ satisfying $\Phi(fm) = \phi(f)\Phi(m)$. After choosing a basis of M over F, the equation $\Phi(m) = m$ translates into a matrix difference equation $y = A\phi(y)$ with $A \in \mathrm{GL}_n(F)$.

A Picard-Vessiot ring PV for M (or $y = A\phi(y)$) is a commutative F-algebra with unit element satisfying

1. An extension of ϕ as automorphism of PV is given.
2. PV has no ϕ-invariant ideals except $\{0\}$ and PV.
3. There is a $U \in \mathrm{GL}_n(PV)$ with $U = A\phi(U)$.
4. PV is generated over F by the entries of U and U^{-1}.

With the methods of [vdP-S 1,2] one shows the existence and unicity of PV (up to ϕ-isomorphisms). One observes that PV is reduced and has in

general zero divisors. The field of constants of the total ring of fractions of PV is again C.

The naive definition of the difference Galois group G of M is: G is the 'abstract' group of the F-automorphisms of PV commuting with ϕ. This definition is insufficient in positive characteristic. A more precise definition is the following. One defines a covariant functor $\mathcal{G}$ from the category of the finitely generated commutative C-algebras R to the category of groups by $\mathcal{G}(R)$ is the group of the automorphisms of $R \otimes_C PV$ which are $R \otimes_C F$-linear and commute with the action of ϕ. It can be shown that $\mathcal{G}$ is represented by an affine group scheme of finite type over C. The difference Galois group G is by definition this group scheme.

If the field F has characteristic zero then G is a reduced linear algebraic group. In our case, where F has characteristic $p > 0$, the difference Galois group need not be reduced. In the following examples we calculate the Picard-Vessiot rings and their difference Galois groups for some typical equations. For all examples we take $F = \widehat{K} = \mathbf{C}((z))$ with $\mathbf{C}$ as before. Since the examples are pure modules, one may replace $\widehat{K}$ by $K = \mathbf{C}(\{z\})$ everywhere. For convenience, we calculate for modules M the 'contravariant solutions', i.e., $\ker(\Phi - 1, \mathrm{Hom}_(M, PV))$, in stead of the 'covariant solutions', i.e., $\ker(\Phi - 1, PV \otimes M)$.

Example 1. — The extension $\widehat{K} \subset \widehat{K}(z^{1/p})$ is the Picard-Vessiot extension for the difference equation $\phi(y) = q^{1/p}y$. Its difference Galois group is the group $\mu_{p,\mathbf{C}}$. More generally, $\widehat{K}(z^{1/n})$ is the Picard-Vessiot extension of an equation $\phi(y) = q^{t/n}y$ with $(t, n) = 1$ and its difference Galois group is $\mu_{n,\mathbf{C}}$. We recall that $\mu_{n,\mathbf{C}} = \mathrm{Spec}(\mathbf{C}[t]/(t^n - 1))$, with co-multiplication given by $t \mapsto t \otimes t$.

Example 2. — Equation $\phi(y) = cy$ with $c \in \mathbf{C}^*$. Suppose that for all $n \geqslant 1$ the only solution of $\phi(y) = c^n y$ in $\widehat{K}$ is $y = 0$, then the Picard-Vessiot extension is $\widehat{K}[Y, Y^{-1}]$ with $\phi(Y) = cY$ and the difference Galois group is $\mathbb{G}_{m,\mathbf{C}}$.
Suppose that there exists a non zero $y \in \widehat{K}$ such that $\phi(y) = c^a y$ for some $a \geqslant 1$. Then c has the form $\zeta q^{t/n}$ with ζ a primitive dth root of unity and $n \geqslant 1$, $(t, n) = 1$. We consider the two equations $\phi(y) = q^{t/n}y$ and $\phi(y) = \zeta y$ separately. The first equation is considered in example 1. The second equation has Picard-Vessiot ring $\widehat{K}[y]$ with equation $y^d = 1$ and $\phi(y) = \zeta y$. This ring has obviously zero divisors (if $d > 1$). Its difference Galois group is $\mu_{d,\mathbf{C}} \cong \mathbb{Z}/\mathbb{Z}d$ over $\mathbf{C}$ since d is not divisible by p. The Picard-Vessiot extension for $\phi(y) = cy$ is a subring of $\widehat{K}[z^{1/n}][y]$. The difference Galois group is therefore a quotient of the group $\mu_{n,\mathbf{C}} \times \mathbb{Z}/\mathbb{Z}d$.

Example 3. — $U_m = \widehat{K} \otimes V$ with $\dim V = m$, $\Phi(1 \otimes v) = 1 \otimes U(v)$ and $U \in \mathrm{GL}(V)$ the indecomposable unipotent operator. There exists an element $e \in V$ such that $e, (U-1)e, \ldots, (U-1)^{m-1}e$ is a basis of V. Hence $U_m \cong \widehat{K}[\Phi, \Phi^{-1}]/\widehat{K}[\Phi, \Phi^{-1}]((\Phi - 1)^m)$. Thus we have to find the Picard-Vessiot ring for the equation $(\phi - 1)^m(y) = 0$.

(a) Suppose that $1 < m \leqslant p$. The difference ring $A_1 := \widehat{K}[\ell]$, defined by $\ell^p - \ell = 0$ and $\phi(\ell) = \ell + 1$, has only trivial ϕ-invariant ideals. The elements $\binom{\ell}{i}$ for $i = 0, \ldots, m-1$ are $\mathbf{C}$-linear independent solutions of $(\phi - 1)^m(y) = 0$. Since A_1 is generated over $\widehat{K}$ by ℓ one finds that A_1 is the Picard-Vessiot extension for the equation $(\phi - 1)^m(y) = 0$. Let R be a $\mathbf{C}$-algebra and σ a $R \otimes_{\mathbf{C}}$ automorphism of $R \otimes_{\mathbf{C}} K[\ell]$, which commute with ϕ. Then σ is determined by $\sigma(\ell)$ and $\sigma(\ell) = \ell + a$ where a is any element in R with $a^p = a$. The difference Galois group is therefore the group $\mathbb{Z}/\mathbb{Z}p$ over $\mathbf{C}$. In view of further equations we write $\ell = \ell_1$.

(b) Suppose that $p < m \leqslant p^2$. The Picard-Vessiot ring for $(\phi - 1)^m(y) = 0$ is $A_2 := \widehat{K}[\ell_1, \ell_2]$, defined by $\ell_1^p - \ell_1 = 0$, $\ell_2^p - \ell_2 = 0$, $\phi(\ell_1) = \ell_1 + 1$, $\phi(\ell_2) = \ell_2 + \binom{\ell_1}{p-1}$. The set of maximal ideals of A_2 is $\{(\ell_1 - a, \ell_2 - b) | a, b \in \mathbb{F}_p\} \cong \mathbb{F}_p^2$. One calculates that ϕ acts transitively on this set and one concludes that A_2 has only trivial ϕ-invariant ideals. One observes that

$$(\phi-1)^p(\ell_2) = (\phi-1)^{p-1}\left(\binom{\ell_1}{p-1}\right) = 1 \text{ and } (\phi-1)^{p^2-1}\binom{\ell_2}{p-1}\binom{\ell_1}{p-1} = 1\,.$$

The conclusion is that $\mathbf{C}[\ell_1, \ell_2]$ is the kernel of $(\phi - 1)^{p^2}$, acting on A_2. This shows that A_2 is generated by the solutions of $(\phi - 1)^m y = 0$ and thus that A_2 is indeed the Picard-Vessiot ring.

We have to represent the functor $\mathcal{G}$, given by $\mathcal{G}(R)$ is the group of the difference automorphism of $R \otimes_{\mathbf{C}} A_2$ over $R \otimes_{\mathbf{C}} K$. For the calculation of $\mathcal{G}(R)$ we suppose for convenience that $\mathrm{Spec}(R)$ is connected. Then $a \in R$, $a^p = a$ implies $a \in \mathbb{F}_p$. Further $\mathbb{F}_p[\ell_1, \ell_2] = \{\xi \in R \otimes_{\mathbf{C}} A_2 |\ \xi^p = \xi \text{ and } (\phi - 1)^{p^2}\xi = 0\}$. Thus any $\sigma \in \mathcal{G}(R)$ induces an automorphism of $\mathbb{F}_p[\ell_1, \ell_2]$ commuting with ϕ. On the other hand, any automorphism of $\mathbb{F}_p[\ell_1, \ell_2]$ commuting with ϕ, extends uniquely to an element of $\mathcal{G}(R)$.

The algebra $\mathbb{F}_p[\ell_1, \ell_2]$ is seen as $\mathbb{F}_p[\phi]$-module. The element $\xi = \binom{\ell_2}{p-1} \cdot \binom{\ell_1}{p-1}$ is a generator and the module can be written as $\mathbb{F}_p[t]/(t^{p^2})$, where $t = \phi - 1$. We recall that $t^{p^2-1}\xi = 1$. Thus an automorphism σ satisfies $\sigma\xi = a\xi$ with $a \in \mathbb{F}_p[t]/(t^{p^2})$ and $a \equiv 1 \bmod (t)$. On the other hand any a, as above, produces a unique automorphism. The above group is the cyclic group of order p^2, generated by $a = 1 + t \bmod (t^{p^2})$. Thus the difference Galois group is equal to the group $\mathbb{Z}/\mathbb{Z}p^2$ over $\mathbf{C}$.

(c) In a similar way one obtains that the difference ring $A_k := \widehat{K}[\ell_1, \cdots, \ell_k]$, given by the equations $\ell_i^p - \ell_i = 0$ for $i = 1, \ldots, k$ and

$$(\phi - 1)(\ell_i) = \binom{\ell_{i-1}}{p-1}\binom{\ell_{i-2}}{p-1}\cdots\binom{\ell_1}{p-1}$$

for $i = 2, \ldots, k$ and $(\phi-1)\ell_1 = 1$, is the Picard-Vessiot ring for $(\phi-1)^m y = 0$ for m such that $p^{k-1} < m \leqslant p^k$. Its difference Galois group is the group $\mathbb{Z}/p^k\mathbb{Z}$ over $\mathbf{C}$. As in the case $k = 2$, the difference Galois group is identified with the group of the automorphisms of $Z := \mathbb{F}_p[\ell_1, \ldots, \ell_k]$ which commute with ϕ. Now Z, as a $\mathbb{F}_p[\phi] = \mathbb{F}_p[t]$-module (with $t = \phi - 1$), has $\xi = \binom{\ell_k}{p-1}\binom{\ell_{k-1}}{p-1}\cdots\binom{\ell_1}{p-1}$ as cyclic element, and is isomorphic to $\mathbb{F}_p[t]/(t^{p^k})$. The automorphisms are given by the elements $a \in \mathbb{F}_p[t]/(t^{p^k})$ with $a \equiv 1 \bmod (t)$. This group is cyclic of order p^k and has generator $a = 1 + t \bmod (t^{p^k})$.

Example 4. — $M = \widehat{K}(z^{1/n})e$, $\Phi e = z^{t/n}e$ with $n \geqslant 1$, $(t, n) = (p, n) = 1$. The corresponding scalar equation is $\phi^n(y) = q^{t(n-1)/2}z^t y$. By definition, PV contains an invertible element α satisfying the equation. Any other solution y has the form $\tilde{y}\alpha$ with $\phi^n(\tilde{y}) = \tilde{y}$. Hence PV contains $\widehat{K}[y_1]$ with $y_1^n = 1$ and $\phi(y_1) = \zeta_n y_1$ where ζ_n is a primitive nth root of unity. The invertible element $u := \phi(\alpha)\alpha^{-1}$ satisfies the equation $\phi^n(u) = q^t u$. All solutions of the latter equation have the form $z^{t/n}y$ with $\phi^n y = y$. Thus y is an invertible element of $\mathbf{C}[y_1]$. It follows that $z^{t/n}$ and $z^{1/n}$ are in PV. After changing α we may suppose that $u = z^{t/n}$. This leads to the assertion that the Picard-Vessiot ring is $\widehat{K}(z^{1/n})[y_1, \alpha, \alpha^{-1}]$ with the rules: $y_1^n = 1$, $\phi(y_1) = \zeta_n y_1$ (with ζ_n a primitive nth root of unity); α transcendental over $\widehat{K}$ and $\phi(\alpha) = z^{t/n}\alpha$. The elements $\{y_1^i\alpha |\ i = 0, \ldots, n-1\}$ form a $\mathbf{C}$-basis of solutions. The inclusion $\widehat{K}[z^{1/n}][y_1] \subset \widehat{K}[z^{1/n}][y_1, \alpha, \alpha^{-1}]$ induces an exact sequence for the difference Galois group, namely

$$1 \to \mathbb{G}_m \to G \to \mu_n \times \mu_n \to 1\ .$$

The term $\mu_n \times \mu_n$ is the difference Galois group of the Picard-Vessiot extension $\widehat{K}(z^{1/n})[y_1]$ of the equation $\phi^n(y) = qy$.

Example 5. One considers the difference module $M = \widehat{K}(z^{1/p})e$ with $\Phi e = z^{t/p}e$ (and $1 \leqslant t < p$), seen as difference module of dimension p over $\widehat{K}$. A corresponding scalar equation is $\phi^p(y) = q^{t(n-1)/2}z^t y$. The method of example 4 yields that the Picard-Vessiot extension for M is $PV = \widehat{K}(z^{1/p})[\ell_1, \alpha, \alpha^{-1}]$, satisfying the following rules: $\ell_1^p - \ell_1 = 0$ and $\phi(\ell_1) = \ell_1 + 1$, α is transcendental over $\widehat{K}$ and $\phi(\alpha) = z^{t/p}\alpha$. The inclusion

$\widehat{K}(z^{1/p})[\ell_1] \subset \widehat{K}(z^{1/p})[\ell_1, \alpha, \alpha^{-1}]$, induces an exact for the difference Galois group G

$$1 \to \mathbb{G}_m \to G \to \mathbb{Z}/p\mathbb{Z} \times \mu_p \to 1 \ .$$

The group $\mathbb{Z}/p\mathbb{Z} \times \mu_p$ is the difference Galois group of the Picard-Vessiot extension $\widehat{K}(z^{1/p})[\ell_1]$ of the equation $\phi^p(y) = qy$.

Example 6. — $M = \widehat{K}(z^{1/p^k})e$ with $\Phi e = z^{t/p^k}e$ and $(t,p) = 1$. A corresponding scalar equation is $\phi^{p^k}(y) = q^{t(p^k-1)/2}z^t y$. The Picard-Vessiot ring is $\widehat{K}(z^{1/p^k})[\ell_1, \ldots, \ell_k, \alpha, \alpha^{-1}]$ with $\phi(\alpha) = z^{t/p^k}\alpha$. The difference Galois group admits an exact sequence

$$1 \to \mathbb{G}_m \to G \to \mathbb{Z}/p^k\mathbb{Z} \times \mu_{p^k} \to 1 \ .$$

7.2. The universal difference ring over $\widehat{K}$

The above examples and the classification of the indecomposable difference modules over $\widehat{K}$ lead to the following description of the universal Picard-Vessiot ring $Univ$ of $\widehat{K}$. Let $\widehat{K}^+$ denote the union of the fields $\widehat{K}(z^{1/n})$ (for all $n \geqslant 1$). One introduces symbols $e(cz^\lambda)$ for $c \in \mathbf{C}^*$, $\lambda \in \mathbb{Q}$ and ℓ_k for $k \in \mathbb{Z}$, $k \geqslant 1$. Then $Univ = \widehat{K}^+[\{\ell_k\}_{k\geqslant 1}\{e(cz^\lambda)\}]$. The only relations are $e(c_1z^{\lambda_1}) \cdot e(c_2z^{\lambda_2}) = e(c_1c_2z^{\lambda_1+\lambda_2})$; $e(q^\lambda) = z^{-\lambda}$ for $\lambda \in \mathbb{Q}$ (including $e(1) = 1$); $\ell_k^p - \ell_k = 0$ for all $k \geqslant 1$. The action of ϕ on $Univ$ is given by:
ϕ acts on $\widehat{K}^+$ by $\phi(\sum a_\lambda z^\lambda) = \sum a_\lambda q^\lambda z^\lambda$, $\qquad cz^\lambda\phi(e(cz^\lambda)) = e(cz^\lambda)$,
$\phi\ell_1 = \ell_1 + 1$, $\quad (\phi - 1)\ell_k = \binom{\ell_{k-1}}{p-1}\binom{\ell_{k-2}}{p-1}\cdots\binom{\ell_1}{p-1}$ for $k \geqslant 2$.

$Univ$ can also be described as the algebra $\widehat{K}[\{\ell_k\}_{k\geqslant 1}, \{e(c)\}_{c\in\mathbf{C}^*}, \{e(z^\lambda)\}_{\lambda\in\mathbb{Q}}]$ with the relations: $\ell_k^p = \ell_k$ for all $k \geqslant 1$; $e(c_1)e(c_2) = e(c_1c_2)$; $e(1) = 1$, $e(q) = z^{-1}$; $e(z^{\lambda_1})e(z^{\lambda_2}) = e(z^{\lambda_1+\lambda_2})$. The action of ϕ is given by the above formulas for $\phi(\ell_k)$ and $\phi(e(c))$ and $e(q^{-\lambda})\phi(e(z^\lambda)) = e(z^\lambda)$.

The universal Picard-Vessiot ring $Univ_{rs}$ for the regular singular difference equations over $\widehat{K}$ is the subring $\widehat{K}[\{\ell_k\}_{k\geqslant 1}, \{e(c)\}_{c\in\mathbf{C}}]$.

The difference Galois group $\mathbb{G}_{rs}$ of $Univ_{rs}$ can be identified with $\mathbb{Z}_p \times \mathrm{Hom}(\mathbf{C}^*/q^{\mathbb{Z}}, \mathbf{C}^*)$. The factor $\mathbb{Z}_p$ is the difference Galois group of $\widehat{K}[\{\ell_k\}_{k\geqslant 1}]$. The notation $\mathrm{Hom}(\mathbf{C}^*/q^{\mathbb{Z}}, \mathbf{C}^*)$ for the second factor is an 'abus de langage'. It hides the non reduced part of the difference Galois group. One writes $E_q := \mathbf{C}^*/q^{\mathbb{Z}}$ (with brute force) as a product $D \times (E_q)_{tors}$, where D is a divisible, torsion free group. Since D is a vector space over $\mathbb{Q}$, the term $\mathrm{Hom}(D, \mathbf{C}^*)$ defines a reduced affine group scheme. More precisely, D is

a direct limit of its free finitely generated subgroups and the affine group $\mathrm{Hom}(D, \mathbf{C}^*)$ is the projective limit of algebraic tori over $\mathbf{C}$.

The group $(E_q)_{tors}$ is a product $\{a \in \mathbf{C}^* | a \text{ root of unity}\} \times q^{\mathbb{Q}}/q^{\mathbb{Z}}$. This torsion group is isomorphic to $(\mathbb{Q}/\mathbb{Z}[1/p])^2 \times \mathbb{Q}_p/\mathbb{Z}_p$. The first factor yields the reduced affine group scheme which is the projective limit of $\mu_n \times \mu_n$, taking over all $n \geqslant 1$ not divisible by p. The second factor yields the non reduced affine group scheme which is the projective limit of the groups μ_{p^k}.

The difference Galois group $\mathbb{G}_{univ}$ of $Univ$ admits an exact sequence

$$1 \to \mathrm{Hom}(\mathbb{Q}, \mathbf{C}^*) \to \mathbb{G}_{univ} \to \mathbb{G}_{rs} \to 1$$

The term $\mathrm{Hom}(\mathbb{Q}, \mathbf{C}^*)$ is the affine group scheme which represents the $Univ_{rs}$-linear automorphisms of $Univ$, commuting with ϕ. We note that the affine group scheme $\mathbb{G}_{univ}$ is not commutative. Unlike the complex case we do not find explicit topological generators (like Γ, Δ, D) for $\mathbb{G}_{univ}$. This is due to the complicated structure of the group $\mathbf{C}^*$.

In the above we established a reasonable Picard-Vessiot theory and explicit calculations for the difference Galois groups of difference modules over $\widehat{K}$ (or equivalently for split difference modules over K). The explicit determination of the difference Galois group of (non split) difference modules over K can be copied from Section 5.

Bibliography

[At] Atiyah (M.F.). — Vector bundles on elliptic curves, Proc.London.Math. Soc., 7, 414-452 (1957).

[Bir] Collected mathematical papers of George David Birkhoff, Volume 1. Dover publications, 1968.

[For] Forster (O.), Riemannsche Fläche, Heidelberger Taschenbücher, Springer Verlag (1977).

[Fr-vdP] Fresnel (J.), van der Put (M.), Rigid Analytic Geometry and its Applications, Progress in Math., 218, 2004.

[vdP] van der Put (M.), Skew differential fields, differential and difference equations, Astérisque, 296, p. 191-207 (2004).

[vdP-R] van der Put (M.), Reversat (M.), Krichever modules for difference and differential equations, Astérisque, 296, 197-225 (2004).

[vdP-S1] van der Put (M.), Singer (M.F.), Galois theory of difference equations, Lect. Notes in Math., vol 1666, Springer Verlag, 1997.

[vdP-S2] van der Put (M.), Singer (M.F.), Galois theory of linear differential equations, Grundlehren der mathematische Wissenschaften 328, Springer Verlag, 2003.

[R-S-Z] Ramis (J.-P.), Sauloy (J.), Zhang (C.), La variété des classes analytiques d'équations aux q-différences dans une classe formelle , C.R.Acad.Sci.Paris, Ser. I 338 (2004).

[Sau1] Sauloy (J.), Galois theory of fuchsian q-difference equations, Ann. Sci. Éc. Norm. Sup. 4^e série 36, no 6, p. 925-968 (2003).

[Sau2] Sauloy (J.), Algebraic construction of the Stokes sheaf for irregular linear q-difference equations, Astérisque 296, p. 227-251 (2004).

[Sau3] Sauloy (J.), La filtration canonique par les pentes d'un module aux q-différences et le gradué associé, Ann. Inst. Fourier 54, no. 1, p.181–210 (2004).