Rend. Sem. Mat. Univ. Padova, Vol. 144 (2020), 289–302
DOI 10.4171/RSMUP/71

Finitely generated mixed modules of Warfield type

Paolo Zanardo (∗)

Abstract – Let R be a local one-dimensional domain, with maximal ideal $\mathfrak{M}$, which is not a valuation domain. We investigate the class of the finitely generated mixed R-modules of Warfield type, so called since their construction goes back to R. B. Warfield. We prove that these R-modules have local endomorphism rings, hence they are indecomposable. We examine the torsion part $t(M)$ of a Warfield type module M, investigating the natural property $t(M) \subset \mathfrak{M}M$. This property is related to b/a being integral over R, where a and b are elements of R that define M. We also investigate $M/t(M)$ and determine its minimum number of generators.

Mathematics Subject Classification (2010). 13G05, 13A15, 13A17.

Keywords. Local one-dimensional domains, finitely generated mixed modules, torsion elements, minimal sets of generators.

Introduction

The main motivation and the starting point for the present paper is an important result proved by R. B. Warfield in [7]. Namely, Warfield in Theorem 2 of [7] proved that every local commutative ring R that is not a generalized valuation ring admits indecomposable finitely presented n-generated modules, for every $n \geq 2$; recall that a generalized valuation ring, often called also chain ring, is a ring whose ideals are totally ordered by inclusion. Somehow surprisingly, in the preliminary section we will see that some reductions made by Warfield in the proof may lead to an indecomposable R-module that fails to be finitely presented (Example 1.1). Thus the proof of this result is not strictly correct. Nonetheless, we will see (Corollary 2.2) that Warfield's idea was correct: eliminating the misleading reductions, the module he constructed enjoys the desired properties.

(∗) *Indirizzo dell'A.*: Dipartimento di Matematica "Tullio Levi-Civita", Via Trieste 63, 35121 Padova, Italy.
E-mail: pzanardo@math.unipd.it

The present paper mainly deals with a class of finitely generated mixed modules over a local one-dimensional domain R. Recall that every finitely generated mixed module over a valuation domain splits into the direct sum of its torsion part and a free module. So we assume that R is not a valuation domain, hence there exist nonzero elements $a, b \in R$ such that both b/a and a/b do not lie in R.

We use a and b in a similar way as in [7] to define a finitely presented n-generated R-module M. As a matter of fact, our definition is simpler than Warfield's one, since it avoids the reductions made in the proof of Theorem 2 of [7]. We say that M is a *mixed module of Warfield type*. We will see that M is indeed a mixed module, since R is one-dimensional. In Theorem 2.1 we prove that mixed modules of Warfield type have local endomorphism rings, hence, in particular, they are indecomposable. Since, when needed, the arguments work also for a general commutative local ring R, we get Theorem 2 of [7] as Corollary 2.2.

In the third section we investigate the torsion part $t(M)$ of a mixed module M of Warfield type. In Theorem 3.1 we give a characterization of torsion elements of M that is crucial for our discussion. Then we focus on the natural property $t(M) \subset \mathfrak{M}M$, where $\mathfrak{M}$ is the maximal ideal of R. Interestingly, this property is related to either b/a or a/b being integral over R, where a and b define M. We show that if M is an n-generated mixed module of Warfield type defined by a and b, and either b/a or a/b is an integral element over R of degree $\le n-1$, then $t(M)$ is not contained in $\mathfrak{M}M$. Further results are achieved using the fact that every one-dimensional local domain R is dominated by an Archimedean valuation domain V of its field of quotients (see [6] and [9]). For instance, in Proposition 3.4 and Corollary 3.5 we use properties of an Archimedean valuation domain V dominating R to ensure $t(M) \subset \mathfrak{M}M$. In general, when R is dominated by finitely many Archimedean valuation domains, Proposition 3.6 shows that all the R-modules of Warfield type satisfy $t(M) \subset \mathfrak{M}M$ if and only if R is integrally closed.

The knowledge of $t(M)$ allows one to get information on $M/t(M)$. In the final section we show that $M/t(M)$ is isomorphic to an ideal of R (Proposition 4.1). The minimum number of generators of M and of $M/t(M)$ coincide when $t(M) \subset \mathfrak{M}M$. Otherwise, if M is defined by a, b and either b/a or a/b is integral over R of degree μ, say, then μ is the minimum number of generators of $M/t(M)$ (Theorem 4.3).

1. Preliminaries

In what follows R will usually denote a local one-dimensional integral domain, with maximal ideal $\mathfrak{M}$. We denote by Q the field of fractions and by $R^\times$ the set of the units of R. We will assume that R is not a valuation domain. Then R contains two nonzero elements a and b such that both b/a and a/b do not lie in R. Note that necessarily $a, b \in \mathfrak{M}$. For the standard facts on Valuation Theory used throughout, we refer to [2] or [4, 5].

If $\mathcal{C}$ is a class of R-modules, we say that *the Krull–Schmidt theorem holds for* $\mathcal{C}$ if every $M \in \mathcal{C}$ decomposes into a direct sum of indecomposable objects of $\mathcal{C}$ in a unique way, up to isomorphism.

If M is a finitely generated R-module, with a slight abuse of language we will say that M is n-generated if n is the minimum number of generators of M; equivalently, $n = \dim_{R/\mathfrak{M}} M/\mathfrak{M}M$. We denote by $\mathrm{gen}(M)$ the minimum number of generators of M.

Warfield in Theorem 2 of [7] proved that every local commutative ring R that is not a valuation domain admits indecomposable finitely presented n-generated modules, for every $n \geq 2$. We will see in the next section that Warfield's idea was correct. However, some reductions made by Warfield in the proof (which, by the way, force the involved modules to be torsion) may unexpectedly lead to an R-module which, although indecomposable, fails to be finitely presented.

In fact, in the next example we show that, for a suitable choice of R, a strict application of Warfield's construction ends with a finitely generated module that is not finitely presented. Here $n = 2$ and R is a one-dimensional local domain.

Example 1.1. Let $L \subset K$ be an extension of fields of infinite degree. Let $R = L + XK[[X]]$ be the ring of formal power series with coefficients in K and constant term in L. It is straightforward to verify that R is local and one-dimensional, with maximal ideal $\mathfrak{M} = XK[[X]]$. Note that $\mathfrak{M}$ is not finitely generated, since $[K : L] = \infty$. Pick $k \in K \setminus L$ and let $a = X$, $b = kX$. Since $k \notin L$, it follows that $a \notin bR$ and $b \notin aR$, whence we also get $aR \cap bR \subseteq \mathfrak{M}^2$. It is readily seen that $a\mathfrak{M} = b\mathfrak{M} = X\mathfrak{M} = \mathfrak{M}^2$, and therefore $\mathfrak{M}^2$ is not finitely generated and $aR \cap bR = \mathfrak{M}^2$. Now we construct a 2-generated R-module, following Warfield's construction in [7] (see also Theorem 3.1, p. 157, of [5]). We consider the ring $S = R/(aR \cap bR + a\mathfrak{M} + b\mathfrak{M}) = R/\mathfrak{M}^2$. Let $F = Sz_1 \oplus Sz_2$ be a free S-module and let $H = (\bar{a}z_1 - \bar{b}z_2)S \subset F$ where $\bar{a} = a + \mathfrak{M}^2$, $\bar{b} = b + \mathfrak{M}^2$. The S-module $M = F/H$ is finitely presented, by definition, and it turns out to be indecomposable, by the proof of Theorem 2 of [7]. Then M is indecomposable, as well, when regarded as an R-module. But M is not finitely presented as an R-module (this is the missing link in Warfield's argument). In fact, regarding to F and H as R-modules, we get

$$F = \frac{Rz_1 \oplus Rz_2}{\mathfrak{M}^2 z_1 \oplus \mathfrak{M}^2 z_2}, \quad H = \frac{(az_1 - bz_2)R + \mathfrak{M}^2 z_1 + \mathfrak{M}^2 z_2}{\mathfrak{M}^2 z_1 \oplus \mathfrak{M}^2 z_2}.$$

Therefore we have the following isomorphism of R-modules

$$M \cong \frac{Rz_1 \oplus Rz_2}{(az_1 - bz_2)R + \mathfrak{M}^2 z_1 + \mathfrak{M}^2 z_2}.$$

To see that M is not finitely presented, it suffices to prove that the R-module $(az_1 - bz_2)R + \mathfrak{M}^2 z_1 + \mathfrak{M}^2 z_2$ is not finitely generated (see Proposition 2.1, p. 152, of [5]). Using $a\mathfrak{M} = b\mathfrak{M} = \mathfrak{M}^2$, we easily verify that

$$(az_1 - bz_2)R + \mathfrak{M}^2 z_1 + \mathfrak{M}^2 z_2 = (az_1 - bz_2)R \oplus \mathfrak{M}^2 z_2$$

fails to be finitely generated, since $\mathfrak{M}^2$ is not finitely generated.

2. Mixed modules of Warfield type

We give the definition of modules of Warfield type. We firstly recall the core of the construction given by Warfield in [7] (for our convenience we make some minor changes in the notation). The starting point is a local commutative ring R, with maximal ideal $\mathfrak{M}$, that is not a generalized valuation ring (so R could contain zero-divisors, and we do not make assumptions on its dimension). Then $\mathfrak{M}$ contains two nonzero elements a and b such that $a \notin bR$ and $b \notin aR$. For any $n \geq 2$ we define the R-module $M = F/H$, where $F = Rz_1 \oplus \cdots \oplus Rz_n$ is free and $H = \langle az_i + bz_{i+1} : 1 \leq i < n \rangle$. Then $M = \langle x_i : 1 \leq i \leq n \rangle$, where $x_i = z_i + H$. The module M is finitely presented by definition, and we readily get $\dim_{R/\mathfrak{M}} M/\mathfrak{M}M = n$, hence M is n-generated.

A module defined in this way will be called *n-generated module of Warfield type*. When it is convenient to emphasize the role of the elements a and b, we say that *M is defined by a, b*.

We remark again that, in spite of the similarities, the modules defined above do not coincide with those defined by Warfield in [7]. In fact the reductions made by Warfield when R is a domain forced his modules to be torsion, although not necessarily finitely presented, as shown in Example 1.1.

Let R be a local one-dimensional integral domain. This is the case we are mostly interested in. We consider the n-generated R-module M constructed above. We prove that $x_1, \ldots, x_n$ are torsion-free elements. Fix $j \leq n$ and assume that $rx_j = 0$, or, equivalently, $rz_j \in H$ for some $r \in R$. It follows that

$$rz_j = \sum_{i=2}^{n} s_i (az_{i-1} + bz_i)$$

for suitable $s_2, s_3, \ldots, s_n \in R$. We firstly examine the case where $1 < j < n$. From the preceding relation we get $0 = s_2 a = s_2 b + s_3 a = \cdots = s_{j-1} b + s_j a$,

$r = s_j b + s_{j+1} a$ and $s_{j+1} b + s_{j+2} a = \cdots = s_{n-1} b + s_n a = s_n b = 0$. Now from the first equality we get $s_2 = 0$, hence from the second $s_3 = 0$, and so on, finally getting $s_j = 0$. In a similar way from the last equality we get $s_n = 0$, whence $s_{n-1} = \cdots = s_{j+1} = 0$. It follows that $r = 0$, and so x_j is torsion-free. A simpler argument shows that $r = 0$ also when either $j = 1$ or $j = n$.

We shall examine the torsion elements in details later. Now we just show the existence in M of nonzero torsion elements. Since R is one-dimensional, the multiplicative set $\{a_n\}_{n\in\mathbb{N}}$ must intersect the non-zero ideal bR, therefore there exists a minimum $m \geq 2$ such that $a^m \in bR$, say $a^m = bt$, where, necessarily, $t \in \mathfrak{M}$ since $b/a \notin R$. We show that $u = tx_1 + a^{m-1}x_2 \in M$ is nonzero torsion. Since $au = 0$, it suffices to show that $u \neq 0$ or, equivalently, $tz_1 + a^{m-1}z_2 \notin H$. In fact, $tz_1 + a^{m-1}z_2 = \sum_{i=2}^n s_i(az_{i-1} + bz_i)$ implies $t = s_2 a$, hence $a^{m-1} = (t/a)b = s_2 b \in bR$, and this is impossible, since m was minimum.

We conclude that M is a mixed module. Then we will say that M is an n-generated mixed module of Warfield type.

THEOREM 2.1. *Let R be a local one-dimensional domain and let M be an n-generated R-module of Warfield type. Then the endomorphism ring of M is local. Consequently, M is indecomposable.*

PROOF. Throughout the proof we keep the same notation used above in the definition of $M = \langle x_1, \ldots, x_n\rangle$. Pick any $\phi \in \mathrm{End}_R(M)$. Then $y_i = \phi(x_i) = \sum_{j=1}^n a_{ij} x_j$, for a suitable $n \times n$ matrix $T = (a_{ij})$ with entries in R. We will show that T is necessarily congruent to a scalar matrix modulo $\mathfrak{M}$. Since R is local, it follows that either T or $T - I$ is invertible (where I is the identity matrix). Therefore either ϕ or $\phi - 1$ is a unit, and therefore $\mathrm{End}_R(M)$ is a local ring (see Proposition 15.15 in [1]).

Since ϕ is an endomorphism, we get $ay_i + by_{i+1} = 0$, for $1 \leq i < n$. Equivalently, for any assigned $1 \leq i < n$ there exist elements $s_2^i, s_3^i, \ldots, s_n^i \in R$ such that

$$a\sum_{j=1}^{n} a_{ij} z_j + b\sum_{j=1}^{n} a_{i+1,j} z_j = \sum_{j=1}^{n-1} s_{j+1}^i (az_j + bz_{j+1}) \in H.$$

Equating the coefficients of the z_j we see that $s_2^i, s_3^i, \ldots, s_n^i$ satisfy the following system of equalities

$$S_i = \begin{cases} aa_{i1} + ba_{i+1,1} = s_2^i a \\ aa_{ij} + ba_{i+1,j} = s_{j+1}^i a + s_j^i b \quad (2 \leq j \leq n-1). \\ aa_{in} + ba_{i+1,n} = s_n^i b \end{cases}$$

We denote by Σ_n the system of equalities obtained by the juxtaposition of the systems S_i, for $1 \le i < n$. With a little abuse of language, we say that the elements $s_2^i, s_3^i, \dots, s_n^i$, for $1 \le i < n$, form a solution of Σ_n.

Our aim is to show that Σ_n may have a solution only if $a_{ij} \equiv 0$ modulo $\mathfrak{M}$, for all $i \ne j$, and there exists $d \in R$ such that $a_{ii} \equiv d$ modulo $\mathfrak{M}$, for $1 \le i \le n$. For the remainder of the proof we assume that all the congruences are modulo $\mathfrak{M}$.

We examine the system S_i. The first equality yields

$$a_{i1} \equiv s_2^i, \quad a_{i+1,1} \equiv 0. \tag{1}$$

In fact, $a_{i+1,1} \in \mathfrak{M}$, since otherwise $b \in aR$, and $a_{i1} - s_2^i \in \mathfrak{M}$, since otherwise $a \in bR$. In a similar way, we readily see that the last equality implies

$$a_{in} \equiv 0, \quad a_{i+1,n} \equiv s_n^i, \tag{2}$$

while the intermediate equalities imply

$$a_{ij} \equiv s_{j+1}^i, \quad a_{i+1,j} \equiv s_j^i. \tag{3}$$

for $1 < j < n$. From (1) we get

$$a_{21} \equiv a_{31} \equiv \cdots \equiv a_{n1} \equiv 0, \tag{4}$$

and from (2) we get

$$a_{1n} \equiv a_{2n} \equiv \cdots \equiv a_{n-1,n} \equiv 0. \tag{5}$$

Now from the equalities (3) we get $a_{ij} \equiv s_{j+1}^i \equiv a_{i+1,j+1}$. Hence for any $k \ge 1$ we have $a_{i,i+k} \equiv a_{n-k,n} \equiv 0$ by (5), and $a_{k+i,i} \equiv a_{k+1,1} \equiv 0$ by (4). Therefore T is congruent to a diagonal matrix modulo $\mathfrak{M}$.

It remains to show that $a_{11} \equiv a_{22} \equiv \cdots \equiv a_{nn}$. Again using (3), we get $a_{ii} \equiv s_{i+1}^i \equiv a_{i+1,i+1}$, for $2 \le i \le n-2$. Moreover, (1) and (3) imply $a_{11} \equiv s_2^1 \equiv a_{22}$, and (2) and (3) imply $a_{nn} \equiv s_n^{n-1} \equiv a_{n-1,n-1}$. □

We point out that in the following corollary, which is Warfield's result in [7], the ring R is neither assumed to be an integral domain, nor to be one-dimensional.

Corollary 2.2 (Warfield [7]). *Let R be a commutative local ring which is not a valuation ring. Then for every $n > 0$ there exists a finitely presented R-module M which is n-generated and indecomposable.*

Proof. The proof of Theorem 2.1 works *verbatim* when R contains zero-divisors and has any dimension. □

Since the modules of Warfield type have local endomorphism rings, we get the following corollary, by Azumaya's theorem (see Theorem 9.8, p. 51, of [5]).

Corollary 2.3. *The Krull–Schmidt theorem holds for the class of finite direct sums of finitely generated modules of Warfield type.*

It is important to remark that, in general, the Krull–Schmidt theorem does not hold for the class of finitely generated R-modules. For results showing dramatic failure of the Krull–Schmidt theorem for finitely generated modules over one-dimensional Noetherian domains, we refer to Roger Wiegand's paper [8].

Example 2.4. It is worth noting that if R is a local domain that is not one-dimensional, the R-modules of Warfield type may be torsion-free. For instance, let R be a local unique factorization domain that contains two non-associate prime elements a and b (thus R cannot be one-dimensional). Then the 2-generated module M of Warfield type defined by a, b is torsion-free. In fact, in the above notation, suppose for a contradiction that $u = c_1x_1 + c_2x_2 \in M$ is a nonzero torsion element. Then $r(c_1x_1 + c_2x_2) = 0$ for some $0 \neq r \in R$. Equivalently, $r(c_1z_1 + c_2z_2) = s(az_1 + bz_2)$, for some $s \in R$. We get $rc_1 = sa$, $rc_2 = sb$, where we may assume r, s coprime. If r is a unit, we get $u = (s/r)(ax_1+bx_2) = 0$, a contradiction. Otherwise we get $rc_1 = sa$, $rc_2 = sb$, where $r \in \mathfrak{M}$ is coprime to s. Since a and b are prime elements, it follows that r is a prime associated to both a and b, another contradiction.

Mixed modules of Warfield type may be constructed over any local integral domain R which is not a valuation domain, independently of its dimension. Given such a domain R, it is clear that we may choose a, b such that $b/a, a/b \notin R$ an $a, b \in tR$, for some $t \in \mathfrak{M}$. Then we consider the n-generated module M of Warfield type, defined by the elements a and b. It is straightforward to check that $u = (a/t)x_1 + (b/t)x_2 \neq 0$. Since $tu = ax_1 + bx_2 = 0$, we conclude that u is a nonzero torsion element of M, which is therefore a mixed module.

However, we prefer to focus on one-dimensional domains, since in that case the class of Warfield modules is contained in the class of mixed R-modules.

3. The torsion part of a module of Warfield type

In this section we always assume that R is a one-dimensional local domain and we examine the torsion elements of a module M of Warfield type. We denote by $t(M)$ the torsion submodule of M. We will investigate the natural problem of establishing when M satisfies the property $t(M) \subset \mathfrak{M}M$. Of course, $t(M) \neq \mathfrak{M}M$, since $M/\mathfrak{M}M$ is torsion and $M/t(M)$ is torsion-free.

The following result gives a characterization of torsion elements that will be crucial for our discussion.

THEOREM 3.1. *Let $M = \langle x_1, \dots, x_n \rangle$ be an n-generated mixed module of Warfield type, defined by $a, b \in R$. Then a nonzero element $u = \sum_{i=1}^{n} \lambda_i x_i \in M$ is torsion if and only if b/a is a root of $\lambda_1 X^{n-1} - \lambda_2 X^{n-2} + \cdots \pm \lambda_{n-1} X \mp \lambda_n \in R[X]$.*

PROOF. Let $0 \neq u = \sum_{i=1}^{n} \lambda_i x_i$ be an element of M. Then u is torsion if and only if

$$r \sum_{i=1}^{n} \lambda_i z_i = \sum_{j=1}^{n-1} s_{j+1}(a z_j + b z_{j+1}) \in H,$$

for suitable $0 \neq r \in R$ and $s_2, \dots, s_n \in R$. Equating the coefficients of the z_i, we see that u is torsion if and only if the following system of equations

$$\begin{cases} -t_1\lambda_1 + t_2 a = 0 \\ -t_1\lambda_j + t_j b + t_{j+1} a = 0 \quad (2 \leq j \leq n-1) \\ -t_1\lambda_n + t_n b = 0 \end{cases} \tag{6}$$

has a nontrivial solution $0 \neq r = t_1, s_2 = t_2, \dots, s_n = t_n$ in R. The linear system (6) has a nontrivial solution in Q if and only if the determinant of the associate matrix is zero. In that case, it also has a nontrivial solution in R, since the system is homogeneous. It is easy to verify that the matrix of the system has determinant

$$-\lambda_1 b^{n-1} + \lambda_2 b^{n-2} a + \cdots \mp \lambda_{n-1} b a^{n-1} \pm \lambda_n a^n,$$

that is zero if and only if b/a is a root of $\lambda_1 X^{n-1} - \lambda_2 X^{n-2} + \cdots \pm \lambda_{n-1} X \mp \lambda_n$. □

The case where $n = 2$ is somehow special.

THEOREM 3.2. *Let M be a 2-generated mixed module of Warfield type. Then $t(M) \subset \mathfrak{M}M$.*

PROOF. Assume for a contradiction that $u = \lambda_1 x_1 + \lambda_2 x_2 \in t(M)$, where some λ_i is a unit of R. Condition of Theorem 3.1 presently becomes $\lambda_1(b/a) - \lambda_2 = 0$. Then if $\lambda_1 \in R^\times$ we get $b/a = \lambda_2/\lambda_1 \in R$, impossible. In a similar way, if $\lambda_2 \in R^\times$ we get $a/b \in R$, another contradiction. □

If $\eta \in Q$ is an integral element over R, we define the *degree* of η to be the minimum $k > 0$ that is the degree of a monic polynomial $f(X) \in R[X]$ such that $f(\eta) = 0$ (recall that $f(X)$ is not uniquely determined).

When $n \geq 3$ it may happen that either b/a or a/b is integral over R of degree $\leq n-1$. In such case we are provided with a sufficient condition for $t(M) \not\subset \mathfrak{M}M$.

THEOREM 3.3. *Let M be an n-generated mixed module of Warfield type, defined by a and b. If either b/a or a/b is an integral element over R of degree $\leq n-1$, then $t(M)$ is not contained in $\mathfrak{M}M$.*

PROOF. Assume that b/a is integral of degree $\leq n-1$ over R. Since $b/a \notin R$, we must have $n \geq 3$. Multiplying by a suitable power of X, we get a monic polynomial

$$f(X) = X^{n-1} - \lambda_2 X^{n-2} + \lambda_3 X^{n-3} - \cdots \pm \lambda_{n-1} X \mp \lambda_n \in R[X]$$

having b/a as a root. Then $u = x_1 + \sum_{i=2}^{n} \lambda_i x_i \in M \setminus \mathfrak{M}M$ is a torsion element of M, by Theorem 3.1.

When a/b is integral of degree $\leq n-1$ we may argue in a symmetric way. □

We remark that Theorem 3.3 is not reversible, not even for $n = 3$, as shown by technical examples.

Recall that a valuation domain V with maximal ideal P is said to be Archimedean if it is one-dimensional. If V is a valuation domain of the field of quotients of R, we say that V dominates R if $\mathfrak{M} = P \cap R$. We recall that every local one-dimensional domain R is dominated by at least one Archimedean valuation domain V (see [6]; see also Theorems 3.2 and 4.1 of [9]).

We give conditions that ensure $t(M) \subset \mathfrak{M}M$. They are based on the Archimedean valuation domains V dominating R. For V an Archimedean valuation domain dominating R, we introduce the notation $v(\mathfrak{M}) = \inf\{v(\eta) : \eta \in \mathfrak{M}\}$, where v is the valuation determined by V.

PROPOSITION 3.4. *Let V be an archimedean valuation domain that dominates R. Let M be an n-generated mixed R-module of Warfield type, defined by a and b. Then $t(M)$ is contained in $\mathfrak{M}M$ if either $0 < (n-1)v(b/a) < v(\mathfrak{M})$ or $0 < (n-1)v(a/b) < v(\mathfrak{M})$.*

PROOF. Pick any element $u = \sum_{i=1}^{n} \lambda_i x_i \in t(M)$. We know that

$$\lambda_1 (b/a)^{n-1} - \lambda_2 (b/a)^{n-2} + \cdots \pm \lambda_{n-1}(b/a) \mp \lambda_n = 0. \tag{7}$$

Say $0 < (n-1)v(b/a) < v(\mathfrak{M})$. Assume, for a contradiction, that $u \notin \mathfrak{M}M$, and let j be the maximum index such that λ_j is a unit of R. Note that $j < n$, otherwise λ_n is a unit of V, and $v(\lambda_i(b/a)^{n-i}) \geq v(b/a) > 0$ for all $i < n$ contradicts (7). Now, since $v(b/a) > 0$, for every $i < j$ we get $v(\lambda_j(b/a)^{n-j}) = (n-j)v(b/a) < (n-i)v(b/a) \leq v(\lambda_i(b/a)^{n-i})$. Moreover, for $j < i \leq n$ we get $\lambda_i \in \mathfrak{M}$, and therefore $v(\lambda_j(b/a)^{n-j}) \leq (n-1)v(b/a) < v(\mathfrak{M}) \leq v(\lambda_i) \leq v(\lambda_i(b/a)^{n-i})$. We conclude that $(n-j)v(b/a)$ is the value of the first member in the equality (7), contradicting $v(0) = \infty$. A symmetric argument works when $0 < (n-1)v(a/b) < v(\mathfrak{M})$. □

Corollary 3.5. *Let R be a local one-dimensional domain which is dominated by an Archimedean non-discrete valuation domain V such that $v(\mathfrak{M}) > 0$. Then for every $n \geq 2$ there exists an n-generated mixed R-module M of Warfield type such that $t(M) \subset \mathfrak{M}M$.*

Proof. Since $v(\mathfrak{M}) > 0$ and V is not discrete, there exists $\alpha > 0$ in its value group such that $\alpha < v(\mathfrak{M})/(n-1)$. Let $z \in V$ be such that $v(z) = \alpha$. We have $z = b/a$, for suitable $a, b \in \mathfrak{M}$. In view of Proposition 3.4, the n-generated mixed module M of Warfield type, defined by a and b, satisfies our requirement. □

Consider the valuation domain $V = K[[X]]$ (K a field, X an indeterminate), and let $R = K + X^n K[[X]]$, where $n \geq 2$. Then R is local and Noetherian, since $\mathfrak{M} = X^n K[[X]]$ is generated by $X^n, X^{n+1}, \ldots, X^{2n-1}$. If we set $a = X^n$, $b = X^{n+1}$, the hypothesis of Proposition 3.4 is satisfied.

Let V be a valuation domain with value group $\mathbb{Q}$ and containing a field K. Let us consider the non-Noetherian one-dimensional domain $R = K + \mathfrak{M}$ where $\mathfrak{M} = \{z \in V : v(z) > 1\}$. Then R satisfies the hypothesis of Corollary 3.5.

We remark that a local one-dimensional domain R can be dominated by exactly two Archimedean valuation domains V_1, V_2, where V_1 is discrete and V_2 is not. See, for instance, Example 4.1 of [9].

Our next result shows that a strong condition is required to ensure that $t(M) \subset \mathfrak{M}M$, for every mixed module M of Warfield type.

Proposition 3.6. *Let R be a local one-dimensional non-valuation domain, dominated by only finitely many distinct Archimedean valuation domains. Then every R-module of Warfield type satisfies $t(M) \subset \mathfrak{M}M$ if and only if R is integrally closed.*

Proof. If R is not integrally closed, then Theorem 3.3 implies that there exist $n \geq 3$ and $a, b \in R$ such that the n-generated mixed module M defined by a and b satisfies $t(M) \not\subseteq \mathfrak{M}M$.

Conversely, assume that R is integrally closed. Then, by Propositions 3.1 and 3.2 of [9], R is dominated by a unique Archimedean valuation domain V, and the maximal ideal $\mathfrak{M}$ of R coincides with the maximal ideal P of V. We assume, for a contradiction, that there exist $n \geq 3$ and $a, b \in R$ such that the n-generated mixed module M, defined by a and b, satisfies $t(M) \not\subseteq \mathfrak{M}M$. Then Theorem 3.1 shows that

$$\lambda_1(b/a)^{n-1} - \lambda_2(b/a)^{n-2} + \cdots \pm \lambda_{n-1}(b/a) \mp \lambda_n = 0,$$

for suitable $\lambda_1, \dots, \lambda_n \in R$, not all lying in $\mathfrak{M}$. We firstly examine the case where $b/a \in V$. Let j be the minimum index such that $\lambda_j \notin \mathfrak{M} = P$. Note that $\lambda_1 \in \mathfrak{M}$, otherwise b/a should be integral over R, so $b/a \in R$, impossible. Hence $j > 1$, and we have $\lambda_i(b/a)^{n-i} \in P = \mathfrak{M}$, for $1 \leq i \leq j-1$. It follows that $\beta = \sum_{i<j} \lambda_i(b/a)^{n-i} \in \mathfrak{M}$, and so $\lambda_j(b/a)^{n-j} - \lambda_{j+1}(b/a)^{n-j-1} + \cdots \pm \lambda_n + \beta = 0$. Since $\lambda_j \in R^{\times}$, the preceding equation shows that b/a is integral over R, hence $b/a \in R$, a contradiction. The case where $a/b \in V$ is symmetric. □

A typical example of a local integrally closed one-dimensional domain that is not a valuation domain, is provided by the ring $R = F + XK[[X]]$, where the field $K = F(z)$ is a purely transcendental extension of the field F (see the examples in [3]). Note that R is not Noetherian.

The situation described in the following proposition is the opposite to that in Proposition 3.6.

Proposition 3.7. *Let R be a local one-dimensional domain, and assume that the integral closure $\overline{R}$ of R is a valuation domain. If every element of $\overline{R}$ has degree ≤ 2, then every n-generated R-module of Warfield type, with $n \geq 3$, satisfies $t(M) \not\subseteq \mathfrak{M}M$.*

Proof. Let M be defined by a, b. Since $\overline{R}$ is a valuation domain, then either b/a or a/b lies in $\overline{R}$. Thus either b/a or a/b is an integral element of degree $2 \leq n-1$. The conclusion follows from Theorem 3.3. □

Let $R = K + X^2K[[X]]$ (see the above example). Then $\overline{R} = K[[X]]$ is a valuation domain, and it is easy to check that R and $\overline{R}$ satisfy the hypotheses of the preceding proposition.

4. $M/t(M)$ and its minimal number of generators

Also in this section we keep the standing assumption that R is a one-dimensional local domain.

The study of the torsion part $t(M)$ of a module of Warfield type M allows us to get information on $M/t(M)$, specifically on its minimal number of generators $\mathrm{gen}(M/t(M))$.

Proposition 4.1. *Let M be a mixed R-module of Warfield type. Then $M/t(M)$ is isomorphic to an ideal of R. If $t(M) \subset \mathfrak{M}M$ we also have* $\mathrm{gen}(M/t(M)) = \mathrm{gen}(M)$.

Proof. Let $M = \langle x_1, \ldots, x_n \rangle$. Since $M/t(M)$ is a torsion-free R-module, we know that $M/t(M)$ is isomorphic to an ideal of R if and only if it has rank one. We will show that $R(y + t(M)) \cap R(x_1 + t(M)) \neq 0$ for every $0 \neq y + t(M) \in M/t(M)$. It suffices to verify that $Ry \cap Rx_1 \neq 0$ for any $0 \neq y \notin t(M)$. Say $y = \sum_{i=1}^{n} r_i x_i$. Since $bx_i = -ax_{i-1}$ for $2 \leq i \leq n$ it readily follows that $b^n y \in Rx_1$. Moreover $b^n y \neq 0$, since $y \notin t(M)$.

If $t(M) \subset \mathfrak{M}M$ we have $\mathfrak{M}(M/t(M)) = (\mathfrak{M}M)/t(M)$. Therefore

$$\mathrm{gen}(M/t(M)) = \dim_{R/\mathfrak{M}} M/t(M)/\mathfrak{M}(M/t(M)) = \dim_{R/\mathfrak{M}} M/\mathfrak{M}M = n.$$

□

In general, to calculate $\mathrm{gen}(M/t(M))$ we need to know how many elements in $t(M)$ are linearly independent modulo $\mathfrak{M}M$. This information will be provided if either b/a or a/b is integral over R.

We will need the following lemma, whose proof is straightforward.

Lemma 4.2. *Let R be a local domain and let $\eta \in Q$ be integral over R of degree m. Let $g(X) \in R[X]$ be a polynomial of degree $< m$ such that $g(\eta) = 0$. Then all the coefficients of $g(X)$ lie in $\mathfrak{M}$.*

We introduce some terminology, useful in the proof of the next theorem. In view of Theorem 3.1, $0 \neq w = \sum_{i=1}^{n} \lambda_i x_i \in t(M)$ if and only if b/a is a root of the polynomial $g_w(X) = \lambda_1 X^{n-1} - \lambda_2 X^{n-2} + \cdots \pm \lambda_{n-1} X \mp \lambda_n$. We will say that the torsion element w and the polynomial $g_w(X)$ are associated; for convenience, we define $g_w = 0$ if $w = 0$. Obviously, if $g_{w_1}, g_{w_2} \in R[X]$ are associated to the torsion elements w_1, w_2, then $c_1 g_{w_1} + c_2 g_{w_2}$ is associated to $c_1 w_1 + c_2 w_2$ $(c_1, c_2 \in R)$.

THEOREM 4.3. *Let M be an n-generated mixed R-module of Warfield type, defined by $a, b \in R$, where either b/a or a/b is integral over R of degree $m - 1$.*

i. *If $m \leq n$, then* $\operatorname{gen}(M/t(M)) = m - 1$.

ii. *If $m > n$, then $t(M) \subset \mathfrak{M}M$ and* $\operatorname{gen}(M/t(M)) = n$.

PROOF. (i) Under the present circumstances, Theorem 3.3 shows that $t(M) \not\subset \mathfrak{M}M$. Also note that if b/a and a/b are both integral, then they have the same degree $m - 1$. We firstly assume that b/a is integral. Let $f(X) = X^{m-1} + a_{m-2}X^{m-2} + \cdots + a_0 \in R[X]$ be such that $f(b/a) = 0$. Say $n = m + k$, where $k \geq 0$. Then b/a is a root of the polynomial $X^i f(X)$, for $0 \leq i \leq k$. It follows that $X^i f(X)$ is associated to the torsion element

$$u_i = (-1)^{k-i}(x_{k-i+1} - a_{m-2}x_{k-i+2} + \cdots \pm a_0 x_{n-i}), \quad 0 \leq i \leq k.$$

(Note that $n - (m - 1 + i) = k - i + 1$.) Let us pick an arbitrary $w \in t(M)$. Then w is associated to a unique $g(X) \in R[X]$ of degree $\leq n - 1$ such that $g(b/a) = 0$. Since $f(X)$ is monic, we can make the division of polynomials $g(X) = f(X)q(X) + r(X)$ in $R[X]$. Then Lemma 4.2 implies that $r(X) \in \mathfrak{M}[X]$, since $\deg r(X) < \deg f(X)$. We conclude that $g(X) \equiv f(X)q(X)$ modulo $\mathfrak{M}$. Note that $\deg q(X) \leq k$, since $\deg f(X) = m-1$, $\deg g(X) \leq n-1$ and $n = m+k$.

Let $q(X) = c_k X^k + c_{k-1}X^{k-1} + \cdots + c_0$, where $c_i \in R$ (it is possible that $\deg q(X) < k$). Then $f(X)q(X) = \sum_{i=0}^{k} c_i X^i f(X)$ and therefore $f(X)q(X)$ is associated to $c_0u_0 + c_1u_1 + \cdots + c_ku_k \in t(M)$. Since $w \in t(M)$ is associated to $g(X)$, and $g(X) \equiv f(X)q(X)$ modulo $\mathfrak{M}$, we get

$$w = c_0u_0 + c_1u_1 + \cdots + c_ku_k + y,$$

for a suitable $y \in \mathfrak{M}M \cap t(M)$. It follows that the arbitrary vector $w + \mathfrak{M}M \in (t(M)+\mathfrak{M}M)/\mathfrak{M}M$ lies in the $R/\mathfrak{M}$-vector space W spanned by $u_0+\mathfrak{M}M, u_1+\mathfrak{M}M, \ldots, u_k + \mathfrak{M}M$. Moreover these vectors are readily seen to be linearly independent, hence W has dimension $k + 1$. Let us choose $y_1, \ldots, y_{m-1} \in M$ such that

$$M/\mathfrak{M}M = \langle u_0 + \mathfrak{M}M, u_1 + \mathfrak{M}M, \ldots, u_k + \mathfrak{M}M, y_1 + \mathfrak{M}M, \ldots, y_{m-1} + \mathfrak{M}M\rangle$$

(recall that $n = k + m$). Then $M = \langle u_0, u_1, \ldots, u_k, y_1, \ldots, y_{m-1}\rangle$ and therefore $M/t(M) = \langle y_1 + t(M), \ldots, y_{m-1} + t(M)\rangle$. It is straightforward to check that $y_1 + t(M), \ldots, y_{m-1} + t(M)$ are linearly independent modulo $\mathfrak{M}$, and so $\operatorname{gen}(M/t(M)) = m - 1$, as required. Note that the argument works in case $k = 0$, as well.

With a similar discussion we can treat the case when a/b is integral and b/a is not.

(ii) Let $0 \neq u = \sum_{i=1}^{n} \lambda_i x_i$ be an arbitrary element of $t(M)$. Then b/a is a root of $\lambda_1 X^{n-1} - \lambda_2 X^{n-2} + \cdots \pm \lambda_n$ and, symmetrically, a/b is a root of $\lambda_n X^{n-1} - \lambda_{n-1} X^{n-2} + \cdots \pm \lambda_1$. Since $n - 1 < m - 1$, from Lemma 4.2 we derive that all the λ_i are in $\mathfrak{M}$. We conclude that $t(M) \subset \mathfrak{M}M$, and therefore $\text{gen}(M/t(M)) = n$, by Proposition 4.1. □

Acknowledgment. Research supported by Dipartimento di Matematica "Tullio Levi-Civita", Università di Padova under Progetto DOR1690814 "Anelli e categorie di moduli".

References

[1] F. K. Anderson – K. R. Fuller, *Rings and categories of modules,* Graduate Texts in Mathematics, 13, Springer-Verlag, Berlin etc., 1974.

[2] N. Bourbaki, *Commutative algebra,* Chapters 1–7, Translated from the French, Reprint of the 1972 edition, Elements of Mathematics (Berlin), Springer-Verlag, Berlin etc., 1989.

[3] S. Bazzoni – L. Salce, *Almost perfect domains,* Colloq. Math. 95 (2003), no. 2, pp. 285–301.

[4] L. Fuchs – L. Salce, *Modules over valuation domains,* Lecture Notes in Pure and Applied Mathematics, 97, Marcel Dekker, New York, 1985.

[5] L. Fuchs – L. Salce, *Modules over non-Noetherian domains,* Mathematical Surveys and Monographs, 84, American Mathematical Society, Providence, R.I., 2001.

[6] J. R. Smith, *Local domains with topologically T-nilpotent radical,* Pacific J. Math. 30 (1969), pp. 233–245.

[7] R. Warfield, *Decomposability of finitely presented modules,* Proc. Amer. Math. Soc. 25 (1970), pp. 167–172.

[8] R. Wiegand, *Direct sum decompositions over local rings,* J. Algebra 240 (2001), no. 1, pp. 83–97.

[9] P. Zanardo, *Almost perfect local domains and their dominating archimedean valuation domains,* J. Algebra Appl. 1 (2002), no. 4, pp. 451–467.

Manoscritto pervenuto in redazione il 28 ottobre 2019.