

Iterated blowups of two-dimensional regular local rings

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ABSTRACT – We explore sequences of iterated blowups of two-dimensional regular local rings. Classical results of Zariski and Abhyankar show that the directed union of blowups of this type is a valuation ring. We show that the value group of such a valuation ring is determined by the irrational number γ that is the value of an infinite continued fraction associated to the sequence of blowups.

MATHEMATICS SUBJECT CLASSIFICATION (2010). Primary: 13A18; Secondary: 13H05.

KEYWORDS. Blowup, quadratic transform, regular local ring, valuation ring.

1. Introduction

The study of two-dimensional iterated blowups of algebraic surfaces has a long history, going back to Zariski’s 1939 paper [15] and Abhyankar’s 1956 paper [1]. Their work showed that the directed union of these blowups is a valuation domain [1, Lemma 12]. For higher dimensions, this is not necessarily the case: see Shannon, Granja, and Cutkosky [13, 6, 3]. On the other hand, Heinzer, Rotthaus, and S Wiegand show that many rank-one discrete valuation domains can be realized as directed unions of regular local rings of various dimensions [12].

In the past few years there has been renewed interest in the two-dimensional case, yielding new results by Guerrieri, Heinzer, Kim, Loper, Olberding, Schoutens, Toeniskoetter, and others. See, for example, [5, 8, 9, 10].

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The results in this note stem from conversations in 1996, when Karen Smith visited the University of Nebraska. We thank her for her insights and interest, which led to consideration of iterated blowups.

Our results will not be surprising to the experts. In particular, papers by Spivakovsky [14] and by Bruce, Logue, and Walker [2] discuss the connections among quadratic transforms, continued fractions, and valuations. Our purpose in this note is to describe a concrete situation where these connections become quite transparent. It is possible that one could obtain some of our results from the much more general constructions in these papers. Indeed, the valuation tree of [2, Figure 1] is another way of encoding the procedure we describe in Sections 2 and 3 of this paper.

1.1 – Definitions and notation

Let k be an algebraically closed field, and consider the sequence

$$X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} X_2 \xleftarrow{\pi_3} X_3 \xleftarrow{\quad} \dots,$$

where $X_0 = \mathbb{P}_k^2$, each π_i is the blowup (quadratic transform) at a point $x_i \in X_i$, and $\pi_i(x_i) = x_{i-1}$ for each $i \geq 1$. The local rings $R_i := \mathcal{O}_{X_i, x_i}$ form an increasing chain, and we put $V = \bigcup_i R_i$. All of the domains R_i and V have the same quotient field, namely $k(x, y)$, where x and y are indeterminates. Then V is a valuation domain [1, Lemma 12]. See also [7, II, Exercise 4.12 and V, Exercise 5.6]. We are interested here in the structure of the valuation ring V , in particular its value group.

NOTATION 1.1. If a and b are algebraically independent elements over k , we write $k[a, b]_{(-)}$ for the local ring $k[a, b]_{(a, b)}$. This notation is convenient, particularly when a and b are complicated expressions. The local rings R_i are all of the form $k[a, b]_{(-)}$.

In X_0 , choose affine coordinates x and y in an affine neighborhood \mathbb{A}_k^2 of x_0 , and assume x_0 is at the origin of \mathbb{A}^2 . The blowup of this affine neighborhood at the point x_0 is then

$$S := \{(x, y, u, v) \in \mathbb{A}_k^2 \times \mathbb{P}_k^1 \mid xv = yu\}.$$

Here u, v are homogeneous coordinates, and $\pi_1(x, y, u, v) = (x, y)$. The exceptional fiber, from which the point x_1 must be chosen, is

$$E = \pi_1^{-1}(0, 0) = \{(0, 0, u, v)\} \cong \mathbb{P}_k^1.$$

Points of E are of the form $(0, 0, 1, \alpha)$, where $\alpha \in k$, together with the “point at infinity” $(0, 0, 0, 1)$. In this paper, we consider only two choices:

1. Choice A: $x_1 = (0, 0, 1, 0)$;
2. Choice B: $x_1 = (0, 0, 0, 1)$.

In Choice A, choose the affine neighborhood

$$\{(x, y, 1, v) \mid xv = y\} \cong \mathbb{A}_k^2,$$

and then

$$R_1 = k\left[x, \frac{y}{x}\right]_{(-)}.$$

In Choice B, take the affine neighborhood

$$\{(x, y, u, 1) \mid yu = x\} \cong \mathbb{A}_k^2,$$

and then

$$R_1 = k\left[\frac{x}{y}, y\right]_{(-)}.$$

The same two choices are available at each stage. If $R_n = k[a, b]_{(-)}$, then $R_{n+1} = k[a, \frac{b}{a}]_{(-)}$ with Choice A, and $R_{n+1} = k[\frac{a}{b}, b]_{(-)}$ with Choice B.

2. Examples

Suppose we do Choice A three times, indicated here by “AAA” or “ A^3 ”:

$$(A^3) \quad R_1 = k\left[x, \frac{y}{x}\right]_{(-)} \subset R_2 = k\left[x, \frac{y}{x^2}\right]_{(-)} \subset R_3 = k\left[x, \frac{y}{x^3}\right]_{(-)}.$$

More generally, A^m changes $k[a, b]_{(-)}$ to $k[a, \frac{b}{a^m}]_{(-)}$, and B^m changes $k[a, b]_{(-)}$ to $k[\frac{a}{b^m}, b]_{(-)}$. For another example, do A twice, then B thrice, and then A once. The inclusions $R_0 \subset R_2 \subset R_5 \subset R_6$ look like this:

$$(A^2B^3A) \quad k[x, y]_{(-)} \subset k\left[x, \frac{y}{x^2}\right]_{(-)} \subset k\left[\frac{x^7}{y^3}, \frac{y}{x^2}\right]_{(-)} \subset k\left[\frac{x^7}{y^3}, \frac{y^4}{x^9}\right]_{(-)}.$$

Now consider infinite sequences of A’s and B’s. If the sequence consists entirely of A’s, the valuation ring $V = \bigcup_n k\left[x, \frac{y}{x^n}\right]_{(-)}$ is not Archimedean. To see this, let v be a valuation of $k(x, y)$, whose valuation ring is V . For every $n \geq 1$, one has $v\left(\frac{y}{x^n}\right) > 0$, and hence $v(y) > nv(x)$ for all $n \geq 1$. Similarly, if the sequence has all A’s *eventually*, or if the sequence has all B’s eventually, the resulting valuation ring is non-Archimedean.

3. Main results

Consider an infinite sequence of A's and B's in some order. We want to describe the valuation ring $V = \bigcup_n R_n$. The valuation ring is non-Archimedean if A occurs only finitely many times, or if B occurs only finitely many times. Therefore assume henceforth that each occurs infinitely often. In this case V is Archimedean, and we describe the value group explicitly. By interchanging x and y at each stage, one sees that the infinite sequences

$$A^{a_0}B^{a_1}A^{a_2}\dots \quad \text{and} \quad B^{a_0}A^{a_1}B^{a_2}\dots$$

result in isomorphic valuation rings. Therefore we may assume that A is the first operation. We set things up so that we can apply the results on continued fractions in the number theory book [11] by Hardy and Wright, using notation in keeping with that in Chapter 10 of that book. Consider the sequence

$$A^{a_0}B^{a_1}A^{a_2}B^{a_3}\dots,$$

where each a_i is a positive integer. We encode the process with the sequence

$$\mathbf{a} = [a_0, a_1, a_2, \dots].$$

Let $V_{\mathbf{a}}$ denote the valuation ring obtained as the union of the resulting R_i , and let $\Gamma_{\mathbf{a}}$ be its value group. Choose a valuation $v_{\mathbf{a}}$ of $k(x, y)$ whose valuation ring is $V_{\mathbf{a}}$.

LEMMA 3.1. *Given a sequence $\mathbf{a} = [a_0, a_1, a_2, \dots]$ of positive integers, let $\mathbf{b} = [a_n, a_{n+1}, a_{n+2}, \dots]$, the sequence obtained by deleting terms a_0, \dots, a_{n-1} . Then $V_{\mathbf{a}}$ and $V_{\mathbf{b}}$ are isomorphic valuation rings.*

PROOF. Let $k[a, b]_{(-)}$ be the ring resulting from applying the sequence

$$A^{a_0}B^{a_1}A^{a_2}B^{a_3}\dots B^{a_{n-1}}$$

to the ring $k[x, y]_{(-)}$. The map $x \mapsto a, y \mapsto b$ yields an isomorphism from $k[x, y]_{(-)}$ onto $k[a, b]_{(-)}$. Assuming n is even, we apply $A^{a_n}B^{a_{n+1}}A^{a_{n+1}}\dots$ to both rings, take unions, and obtain an isomorphism from $V_{\mathbf{b}}$ onto $V_{\mathbf{a}}$. If n is odd, we apply $B^{a_n}A^{a_{n+1}}B^{a_{n+2}}\dots$ to both rings, to get an isomorphism from V' onto $V_{\mathbf{a}}$, where V' is the valuation ring obtained by applying the sequence

$$B^{a_n}A^{a_{n+1}}B^{a_{n+2}}\dots$$

to $k[x, y]_{(-)}$. But, as pointed out above, V and V' are isomorphic valuation rings. \square

NOTATION 3.2. Two sequences $\mathbf{a} = [a_0, a_1, a_2, \dots]$ and $\mathbf{b} = [b_0, b_1, b_2, \dots]$ agree eventually provided there are positive integers m and n such that $a_{m+r} = b_{n+r}$ for all $r \geq 0$. Recall that two positive irrational numbers γ and δ are said to be equivalent provided there is an integer matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with determinant ± 1 such that $\delta = \frac{a\gamma+b}{c\gamma+d}$.

The next three theorems are the main results of this note:

THEOREM 3.3. Let $\mathbf{a} = [a_0, a_1, a_2, \dots]$ be a sequence of positive integers. Then the value group $\Gamma_{\mathbf{a}}$ is order-isomorphic to $\mathbb{Z} \oplus \mathbb{Z}\gamma \subset \mathbb{R}$, where the irrational number γ is the value of the infinite continued fraction

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}}.$$

PROOF. For each $n \geq 1$ let $c_n = [a_0, a_1, \dots, a_n]$ be the value of the following finite continued fraction

$$(1) \quad a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots + \cfrac{1}{a_n}}}}$$

By [11, Theorem 149], one has

$$(2) \quad c_n = \frac{p_n}{q_n},$$

where the p_i and q_i are defined recursively by

$$(3) \quad \begin{aligned} p_{-2} &= 0, & p_{-1} &= 1, & p_i &= a_i p_{i-1} + p_{i-2} & \text{for } 0 \leq i \leq n, \\ q_{-1} &= 0, & q_0 &= 1, & q_i &= a_i q_{i-1} + q_{i-2} & \text{for } 1 \leq i \leq n, \end{aligned}$$

We claim that

$$(4) \quad R_{a_n} = \begin{cases} k \left[\frac{x^{p_{n-1}}}{y^{q_{n-1}}}, \frac{y^{q_n}}{x^{p_n}} \right]_{(-)} & \text{if } n \text{ is even,} \\ k \left[\frac{x^{p_n}}{y^{q_n}}, \frac{y^{q_{n-1}}}{x^{p_{n-1}}} \right]_{(-)} & \text{if } n \text{ is odd.} \end{cases}$$

A direct check, using (3), shows that the right-hand side is $k[x, \frac{y}{x^{a_0}}]_{(-)}$ when $n = 0$ and is $k[\frac{x^{a_0 a_1 + 1}}{y^{a_1}}, \frac{y}{x^{a_0}}]_{(-)}$ when $n = 1$, in agreement with the rings R_{a_0} and R_{a_1} , respectively. Suppose now that the claimed formula is valid for some integer $n \geq 1$. If n is even, move to $R_{a_{n+1}}$ by performing the operation $B^{a_{n+1}}$. This operation converts the top expression on the right-hand side of (4) to

$$k\left[\frac{x^{p_{n-1}}}{y^{q_{n-1}}}\left(\frac{x^{p_n}}{y^{q_n}}\right)^{a_{n+1}}, \frac{y^{q_n}}{x^{p_n}}\right]_{(-)} = k\left[\frac{x^{p_{n-1}+a_{n+1}p_n}}{y^{q_{n-1}+a_{n+1}q_n}}, \frac{y^{q_n}}{x^{p_n}}\right]_{(-)}.$$

By the recursion in (3), the last expression above amounts to $k[\frac{x^{p_{n+1}}}{y^{q_{n+1}}}, \frac{y^{q_n}}{x^{p_n}}]_{(-)}$, which is indeed the desired formula for $R_{a_{n+1}}$, since $n + 1$ is odd. A similar calculation, using the operation $A^{a_{n+1}}$, verifies the claim in the case when n is odd.

The real number γ is the limit of the c_n and is irrational (see the comment after [11, Theorem 170]). Moreover, by [11, Theorem 167], γ is the unique real number δ satisfying

$$c_n < \delta < c_{n-1} \quad \text{for every even positive integer } n.$$

Put $\alpha = v_{\mathbf{a}}(x)$ and $\beta = v_{\mathbf{a}}(y)$. Let n be an arbitrary even positive integer. From (4) we see that $\frac{x^{p_{n-1}}}{y^{q_{n-1}}}$ and $\frac{y^{q_n}}{x^{p_n}}$ both have positive value, and hence $p_{n-1}\alpha - q_{n-1}\beta$ and $q_n\beta - p_n\alpha$ are both positive. Combining this observation with equation (2), one obtains the inequalities

$$c_n < \frac{\beta}{\alpha} < c_{n-1}.$$

Since this holds for every even positive integer n , we must have $\frac{\beta}{\alpha} = \gamma$.

Finally, we show that $\Gamma_{\mathbf{a}} = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$. (The sum, inside \mathbb{R} , is direct, since $\frac{\beta}{\alpha}$ is irrational.) Since this group is order-isomorphic to $\mathbb{Z} \oplus \mathbb{Z}\gamma$ (via multiplication by the positive element $\frac{1}{\alpha}$), this will complete the proof. Since $\Gamma_{\mathbf{a}}$ contains α and β , we have $\mathbb{Z}\alpha \oplus \mathbb{Z}\beta \subseteq \Gamma_{\mathbf{a}}$. For the reverse inclusion, let h be an arbitrary non-zero element of $k(x, y)$. To show $v_{\mathbf{a}}(h)$ belongs to $\mathbb{Z} \oplus \mathbb{Z}$, assume harmlessly that $h \in k[x, y]$. The value of a non-zero term $cx^i y^j$ of h , where $c \in k \setminus \{0\}$, is $i\alpha + j\beta$. The irrationality of $\frac{\beta}{\alpha}$ now implies that distinct terms have distinct values, and now [16, Chapter VI, §8, item (5)] shows that $v_{\mathbf{a}}(h)$ is the smallest of the values of the terms. \square

THEOREM 3.4. *Let γ be a positive irrational number. Then there is a sequence \mathbf{a} of positive integers such that $\Gamma(\mathbf{a})$ is order-isomorphic to $\mathbb{Z} \oplus \mathbb{Z}\gamma$.*

PROOF. By [11, Theorem 170] there is a sequence $\mathbf{a} = [a_0, a_1, a_2, \dots]$ for which the infinite continued fraction in Theorem 3.3 has value γ . Now Theorem 3.3 guarantees that the value group of $\Gamma(\mathbf{a})$ is order-isomorphic to $\mathbb{Z} \oplus \mathbb{Z}\gamma$. \square

THEOREM 3.5. *Let \mathbf{a} and \mathbf{b} be sequences of positive integers, and write $\Gamma_{\mathbf{a}} = \mathbb{Z} \oplus \mathbb{Z}\gamma$ and $\Gamma_{\mathbf{b}} = \mathbb{Z} \oplus \mathbb{Z}\delta$, where γ and δ are the positive irrational numbers from Theorem 3.3. Consider the following four statements:*

- i. $V_{\mathbf{a}}$ and $V_{\mathbf{b}}$ are isomorphic rings;
- ii. $\Gamma_{\mathbf{a}}$ and $\Gamma_{\mathbf{b}}$ are isomorphic as ordered groups;
- iii. \mathbf{a} and \mathbf{b} agree eventually;
- iv. the irrational numbers γ and δ are equivalent.

These implications hold: (iv) \iff (iii) \implies (i) \implies (ii).

PROOF. The implication (iii) \implies (i) follows from Lemma 3.1, and clearly (i) \implies (ii). The equivalence of (iii) and (iv) is [11, Theorem 175]. \square

Theorem 3.5 raises an obvious question: Does (ii) imply (iv), that is, are the four conditions equivalent? Another question is whether one can find a direct proof of the implication (iv) \implies (ii). A direct proof might well give a clue to proving the converse.

Acknowledgments. R. Wiegand's research was supported by a Simons Collaboration Grant. S. Wiegand's research was supported by a UNL Emeriti and Retiree Association Wisherd Award.

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Manoscritto pervenuto in redazione il 29 febbraio 2020.