Rend. Sem. Mat. Univ. Padova, Vol. 144 (2020), 271–279
DOI 10.4171/RSMUP/69

Linear forms in a playful universe

Burkhard Wald (∗)

Abstract – Instead of the axiom of choice, we assume that every set of reals has the Baire property. It is shown that under this condition the concept of slenderness known from the theory of abelian groups becomes meaningful for vector spaces.

Mathematics Subject Classification (2010). 03E60, 03E75, 20K25, 54E52.

Keywords. Slendernes for abelian groups, rings and modules, linear forms, infinite direct sums and cartesian products, set theory without axiom of choice, determinacy of infinite games.

1. Introduction

Usually in mathematics the axiom of choice AC is assumed. One consequence of this axiom is that every vector space has a base. This means that each vector space is a direct sum of a number of copies of the underlying field. Thus all vector spaces have the same algebraic structure. This is the reason why the special case of vector spaces is uninteresting from a module-theoretical point of view.

In the theory of abelian groups there is a lot of structural diversity instead. An example is that for abelian groups infinite cartesian products on the one hand and infinite direct sums on the other hand are two fundamentally different structures. In particular, infinite cartesian products of abelian groups are far from being free, i.e. from having a basis. This is related to the concept of slenderness for abelian groups (cf. [3, 159ff]) and the fact that the ring $\mathbb{Z}$ is a slender abelian group, discovered by Specker [12]. Slenderness for modules over arbitrary rings was studied by Lady [7]. A ring R is called *slender* if R is slender as R-module. We give the definition for slenderness in Section 2.

(∗) *Indirizzo dell'A.*: Universität Duisburg Essen, Schützenbahn 70, 45127 Essen, Germany
E-mail: burkhard.wald@uni-due.de

The basis theorem for vector spaces is the reason why fields fail to be slender. One can ask what the situation is in a set theory in which the axiom of choice does not apply. As an alternative to the axiom of choice, we focus our attention on the axiom of determinacy AD or its weaker variants. This was inspired by a paper by Felgner and Schulz [1]. The axiom AD was introduced by Mycielski and Steinhaus [8]. A well-known conclusion is that every set of reals has the Baire property. Shelah proved that the theory ZF + DC + BP is equiconsistent to ZFC [11]. Here ZF is the Zermelo–Fränkel set theory, ZFC is the Zermelo–Fränkel set theory together with the axiom of choice, DC is the axiom of dependent choice and BP means that every set of reals has the Baire property. AD is more stringent than BP because AD implies the existence of measurable cardinals. For our purposes it is sufficient to assume BP. Our result is that under this assumption, fields like $\mathbb{R}$, $\mathbb{Q}$ and also at most countable fields are slender rings.

The fact that a set of reals has the Baire property is related to the determinacy of so-called "Bannach–Masur games." The definition of this infinite games was first given by Oxtoby [10]. We give a short outline in Section 3 and refer to the explanations in the book of Jech [5, pp. 553–555].

The last section deals with the subgroup B of the Bear–Specker group, which consists of all bounded functions. Nöbeling has shown that B is a free abelian group [9], while Specker previously came to the same result by assuming the continuum hypothesis [12]. In both articles, ZFC is assumed as usual. Felgner and Schulz showed that under the assumption of AD the group B fails to be free [1]. In Section 5 we give results which indicate that under ZF + DC + BP the structure of B is more similar to that of the complete Bear–Specker group than that of a direct sum.

We use notations that are common in set theory. The set of natural numbers is denoted by ω.

2. On slenderness

Let R be a ring. Then R^ω is the R-module consisting of all functions from ω into R. For $i \in \omega$ let e_i be the element of R^ω for which $e_i(j) = 1$ if $i = j$ and $e_i(j) = 0$ otherwise. It's common to write some $a \in R^\omega$ as the infinite sum $\sum_{i\in\omega} a(i)e_i$. A R-module M is called *slender* if for every homomorphism φ from R^ω into M it is $\varphi(e_i) = 0$ for almost all $i \in \omega$. It is not difficult to see that $\varphi = 0$ in the case that $\varphi(e_i) = 0$ for all $i \in \omega$. This implies that every homomorphism φ from R^ω into a slender module M is induced by a finite sequence $x_0, \ldots, x_{n-1}$ in M such that $\varphi(a) = \sum_{i<n} a(i)x_i$. Hence $\hom(R^\omega, M) \cong M^{<\omega}$, where $M^{<\omega}$ is the direct sum of countable many copies of M. For a module M the dual module

$M^\star$ is the module hom(M, R). Is is easy to see that $(R^{<\omega})^\star \cong R^\omega$. If R is slender we also have the converse $(R^\omega)^\star \cong R^{<\omega}$.

If $\prod_{i\in\omega} A_i$ is an arbitrary product of R-modules A_i and φ a homomorphism from $\prod_{i\in\omega} A_i$ into a slender R-module M, then there exists some $n \in \omega$ and homomorphisms φ_i from A_i into M for $i < n$ such that $\varphi(a) = \sum_{i<n} \varphi_i(a_i)$ for all $a \in \prod_{i\in\omega} A_i$. It is a conclusion that a R-module is slender as a R-module, if it is slender as an abelian group. In Section 4 we show that under our set theoretical assumption $\mathbb{Q}$ and $\mathbb{R}$ are slender abelian groups. Hence $\mathbb{Q}$ and $\mathbb{R}$ are also slender as rings.

3. On games

For a set X and natural number $n \in \omega$ we denote by X^n the set of all functions from n to X. Furthermore $X^{<\omega}$ is the union $\bigcup_{n\in\omega} X^n$. Functions are identified with their graphs and hence for functions f and g notations like $f \subset g$ and $f \cup g$ make sense. An element s of $X^{<\omega}$ is also viewed as the sequence $s = \langle s(0), \ldots, s(n-1)\rangle$. For two sequences s and t the concatenation $s^\frown t$ is defined in the natural way. If we have a sequence $s_n \in X^{<\omega}$ for each $n \in \omega$ we can build an infinite concatenation

$$a = s_0{}^\frown s_1{}^\frown s_2{}^\frown \ldots \tag{1}$$

Here $a \in X^\omega$ is the union $\bigcup_{n\in\omega} a_n$ where $a_0 = s_0$ and $a_n = a_{n-1}{}^\frown s_n$.

It is common to regard X^ω also from a topological point of view. To do this, take the discrete topology on X and build the product topology on X^ω. In this topology the set $\{U_s : s \in X^{<\omega} \wedge s \subset a\}$ build an neighbour basis for an element $a \in X^\omega$, where U_s is defined by $\{b \in X^\omega : s \subset b\}$ for $s \in X^{<\omega}$. Let us look again at the infinite concatenation a we built above. In this situation we have $\{a\} = \bigcap_{n\in\omega} U_{a_n}$ where a_n is defined as before.

From now on we assume that X has at least two elements. For a set $A \subset X^\omega$ we define an infinite game $G^{**}_X(A)$ for two players as follows. The two players alternately choose finite sequences s_n from $X^{<\omega}$ at step $n \in \omega$. Player I wins the game if the concatenation a build like (1) is an element of A. Otherwise player II wins the game. Of course, if A is small in a certain way, player II has a good change to win the game. For example, it is not difficult to see that there is a winning strategy for player II if A is at most countable. The game is determined when there is a winning strategy either for player I or for player II.

You can easily modify a winning strategy for player I in the game $G^{**}_X(A)$ and get a winning strategy for player II in game $G^{**}_X(U_s \setminus A)$ when s is the starting sequence of player I in the first game. Here U_s is defined by $\{a \in X^\omega : s \subset a\}$.

Similarly, a winning strategy for player II for a game $G_X^{**}(U_s \setminus A)$ results in a winning strategy for player I for the game $G_X^{**}(A)$.

In a topological space a subset A is called *nowhere dense* if the interior of the closure of the set is empty. A is called *meager* if A is a countable union of nowhere dense subsets. We say that *A has the Baire property* if there is an open set B such that $A \triangle B = (A \setminus B) \cup (B \setminus A)$ is meager. Of course meager sets has the Baire property. If A has the Baire property and is not meager, then there is an s such that $U_s \setminus A$ is meager.

It is known that player II has a winning strategy for the game $G_X^{**}(A)$ exactly when A is meager and Player I has a winning strategie for $G_X^{**}(A)$ if $U_s \setminus A$ is meager. Hence if A has the Baire property, then $G_X^{**}(A)$ is determined.

We are especially interested in the two cases in which X is $\{0, 1\}$ or ω. Then X^ω is the Cantor space respectively Baire space. Both cases can be viewed as topological subspaces of the real line. So if we assume that every set of reals has the Baire property, all games $G_X^{**}(A)$ for our two cases of X are determined. Our proofs in the further course of this paper follow the same pattern. We begin with a decomposition $\bigcup_{n\in\omega} A_n$ of the base set X^ω. Then one of the subsets A_n fails to be meager. Then for this A_n, player I has a winning strategy.

4. The main result

Our first Theorem is about slenderness of abelian groups.

THEOREM 4.1. *In the set theoretical setting* ZF + DC + BP *the following holds*:

a. *every lineary ordered abelian group is slender*;
b. *every at most countable abelian group is slender.*

PROOF. (a) We start with a homomorphism φ from $\mathbb{Z}^\omega$ to a lineary ordered abelian group G. The domain $\mathbb{Z}^\omega$ of φ is divided in the subsets

$$A_1 = \{x \in \mathbb{Z}^\omega : \varphi(x) < 0\},$$
$$A_2 = \{x \in \mathbb{Z}^\omega : \varphi(x) = 0\},$$
$$A_3 = \{x \in \mathbb{Z}^\omega : \phi(x) > 0\}.$$

By Lemma 2.1 player I has a winning strategy for one of the three sets. Let us start with a winning strategy for A_1. The strategy gives the first move $s = \langle x_0, \ldots, x_k\rangle$ for player I. We will show that $\varphi(e_i) = 0$ for all $i > k$. Assuming $\varphi(e_m) \neq 0$ for some $m > k$ we choose $c \in \mathbb{Z}$ such that

$$\sum_{i=0}^{k} s_i \varphi(e_i) < c\varphi(e_m) \tag{2}$$

Now we play two games, both opened by player I with move s_0. In the first game we take as the first move of player II the sequence $s_1 = \langle 0, \dots, 0, -c \rangle$ just with length $m-k$. After this player I makes his second move s_2 in the first game concordantly with the strategy. Unlike in the first game, player II responds in the second game with the concatenation of s_1 and $-s_2$, where $-s_2$ is formed from s_2 by changing the sign. From now on player II always plays the sequence $-s$ if player I has played s in the other game before. So as a result of the two games we get two elements of $\mathbb{Z}^\omega$

$$a = s_0{}^\frown s_1{}^\frown s_2{}^\frown s_3{}^\frown -s_4{}^\frown \dots,$$
$$b = s_0{}^\frown (s_1{}^\frown s_2)^\frown -s_3{}^\frown s_4{}^\frown \dots,$$

which are in A_1 because player I used his winning strategy for A_1. Thus

$$0 > \varphi(a) + \varphi(b) = \varphi(a+b) = \sum_{i=0}^{k} s_i \varphi(e_i) - c\varphi(e_m)$$

This is a contradiction to equation (2).

The proof for the cases that player I has a winning strategy for A_2 or for A_3 works analogously with the correct choice of c.

(b) Let φ be an homomorphism from $\mathbb{Z}^\omega$ to a abelian group G and G is at most countable. For one of the subsets $A_g = \{x \in \mathbb{Z}^\omega : \varphi(x) = g\}$ player I must have a winning strategy. As above we construct two sequences

$$a = s_0{}^\frown s_1{}^\frown s_2{}^\frown s_3{}^\frown s_4{}^\frown \dots,$$
$$b = s_0{}^\frown (s_1'{}^\frown s_2)^\frown s_3{}^\frown s_4{}^\frown \dots,$$

where $s_1 = \langle 0, \dots, 0, 1 \rangle$ and $s_1' = \langle 0, \dots, 0, 0 \rangle$ are of the same length $m+1$. In the construction of a the part s_1 is the move of player II and s_2 the second move of player I. In the construction of b the first move of player II is $s_1'{}^\frown s_2$ and s_3 is the second move of player I. We get

$$0 = g - g = \varphi(a) - \varphi(b) = \varphi(a-b) = \varphi(e_{k+m})$$

where k is the length of s_0. Because we can choose m arbitrarily $\varphi(e_i) = 0$ for almost all $i \in \omega$. □

Because subgroups of slender groups are slender and because exact sums of slender groups are slender, the arguments we have noted in 3 results in the following corollary.

Corollary 4.2. *In the set theoretical setting* $ZF + DC + BP$ *the following holds*:

a. *every subfield of the field of complex numbers is a slender ring*;

b. *every at most countable field is a slender ring.*

5. A further result

Let B be the subgroup of $\mathbb{Z}^\omega$ which contains all functions $a: \omega \to \mathbb{Z}$ for which $a(\omega)$ is finite. An abelian group is called *B-slender* if for every homomorphism from B into G we have $\varphi(e_i) = 0$ for almost all $i \in \omega$.

Theorem 5.1. *The assumption* ZF + DC + BP *implies that every at most countable abelian group is B-slender.*

Proof. Unlike in the proof of Theorem 4.1(b), we define the sets A_g by

$$\{x \in \{0,1\}^\omega : \varphi(x) = g\}.$$

We can then proceed in the same way, since the construction of a and b does not leave the set $\{0,1\}^\omega$. □

The subgroup B is a special case of the so-called "monotonic subgroups" of $\mathbb{Z}^\omega$. M-slenderness for monotonic subgroups M of $\mathbb{Z}^\omega$ has been studied in detail by Fuchs and the author together with Göbel and Kolman [2, 4, 6, 13]. However, ZFC has always been assumed and consequently, because of Nöbeling's theorem, the trivial group 0 was the only B-slender group.

Now we ask whether we can extend Theorem 5.1 analogous to Theorem 4.1(a) to linearly ordered groups. The answer is no. For this we define a homomorphism from B to $\mathbb{R}$ as follows. We take a absolute convergent series $\sum_{i\in\omega} c_i$ in $\mathbb{R}$ and define φ from B to $\mathbb{R}$ by $\varphi(a) = \sum_{i\in\omega} a(i)c_i$ for all $a \in B$.

Theorem 5.2. *We assume* ZF + DC + BP. *If G is a B-slender abelian group an φ an homomorphism from B into G with $\varphi(e_i) = 0$ for all $i \in \omega$, the $\varphi = 0$.*

Proof. Note that the simple proof of the analogous statement for slenderness does not work for B-slenderness. Nevertheless the theorem is valid in our universe. We use the known fact that under $ZF + DC + BP$ every ultrafilter of ω is principal. Because this can be proven with the same method we have applied here, we will give a sketch of the proof. In a well-known way, we understand subsets of ω as elements of $\{0,1\}^\omega$. Hence an ultrafilter $\mathcal{F}$ can viewed as a subset of $\{0,1\}^\omega$ and thus player I has a winning strategy either for the game $G_X^{**}(\mathcal{F})$ or for the game $G_X^{**}(\{0,1\}^\omega \setminus \mathcal{F})$. As in the previous proofs, we can use the winning strategy to construct two functions a and b in $\{0,1\}^\omega$ that differ in almost all places. This defines two sets A and B for which $A \cap B$ and $(\omega \setminus A) \cap (\omega \setminus B)$ are finite. Because A and B are based on the winning strategy for the same of the two games, either A and B belong to $\mathcal{F}$ or A and B belong not to $\mathcal{F}$. But if A and B are not in $\mathcal{F}$ then $\omega \setminus A$ and $\omega \setminus B$ are in $\mathcal{F}$. Hence in both cases $\mathcal{F}$ contains a finite element. This shows that $\mathcal{F}$ must be a principal filter.

Now we start with the proof of the theorem and assume that $\varphi(a) \neq 0$ for some $a \in B$. For any $W \subseteq \omega$ we define $a|_W = \sum_{i \in W} a(i)e_i$. First we assume

$$(\forall W \subseteq \omega)(\varphi(a|_W) \neq 0 \\ \implies (\exists U, V \subseteq W)(U \cap V = \emptyset \wedge \varphi(a|_U) \neq 0 \wedge \varphi(a|_V) \neq 0))$$

and lead this to a contradiction.

From this assumption we can use DC to get a family $(U_j, V_j)_{j \in \omega}$ of pairs of subsets of ω such that $U_j \cap V_j = \emptyset$, $\varphi(a|_{U_j}) \neq 0$, $\varphi(a|_{V_j}) \neq 0$ and furthermore $U_{j+1} \subset U_j$ and $V_{j+1} \subset U_j$ for all $j \in \omega$. Notice that the family $(V_j)_{j \in \omega}$ is pairwise disjoint. Hereby we can construct a new homomorphism ψ from B into G by defining $\psi(x) = \varphi\big(\sum_{j \in \omega} x(i)a|_{V_j}\big)$ for $x \in B$. Because all $a|_{V_j}$ are parts of the same bounded function a and x also is a bounded function, the construction $\sum_{j \in \omega} x(i)a|_{V_j}$ is bounded as well. But now we have a contradiction to the B-slenderness of G, because $\psi(e_j) = \varphi(a|_{V_j}) \neq 0$ for all $j \in \omega$.

Consequently there must be some $W \subseteq \omega$ with $\varphi(a|_W) \neq 0$ and

$$(\forall U, V \subseteq W)(U \cap V = \emptyset \implies (\varphi(a|_U) = 0 \vee \varphi(a|_V) = 0)). \tag{3}$$

Next we show that the set

$$\mathcal{F} = \{U \subseteq \omega : \varphi(a|_{W \cap U}) \neq 0\}$$

is an ultrafilter. Therefore we have to prove:

a. $\emptyset \notin \mathcal{F}$;
b. if $U \subseteq V \subseteq \omega$ and $U \in \mathcal{F}$, then $V \in \mathcal{F}$;
c. if $U \in \mathcal{F}$ and $V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$;
d. for every $U \subseteq \omega$ either $U \in \mathcal{F}$ or $\omega \setminus U \in \mathcal{F}$.

(a) is obvious.

For (b), notice that $(W \cap U) \cap (W \cap (V \setminus U)) = \emptyset$ and $\varphi(a|_{W \cap U}) \neq 0$. Hence $\varphi(a|_{W \cap (V \setminus U)}) = 0$ by (3). Because

$$a|_{W \cap V} = a|_{W \cap U} + a|_{W \cap (V \setminus U)}$$

we get $\varphi(a|_{W \cap V}) \neq 0$ and therefore $V \in \mathcal{F}$.

For (c), we get $\varphi(a|_{W\cap(V\setminus U)}) = 0$ in the same way. Now we have

$$a|_{W\cap V} = a|_{W\cap U\cap V} + a|_{W\cap(V\setminus U)}$$

and hence $\varphi(a|_{W\cap U\cap V}) = \varphi(a|_{W\cap V}) \neq 0$. This means that $U \cap V \in \mathcal{F}$.

At least (d) is a direct consequence of $\varphi(a|_W) \neq 0$.

The assumption that $\varphi(e_i) = 0$ for all $i \in \omega$ implies that $\mathcal{F}$ fails to principal. □

A consequence of this theorem is that images of homomorphisms of B into B-slender abelian groups are finitely generated as we know it from slenderness too. This is required for the next corollary.

Corollary 5.3. *Direct sums of B-slender groups are B-slender.*

Proof. The proof is identical to that given in [3, p. 160] for slenderness. Let φ be a homomorphism into a sum $\bigoplus_{i\in I} G_i$ of B-slender abelian groups G_i. Then the image of φ is contained in a subgroup $\bigoplus_{i\in I'} G_i'$ of the full sum, where G_i' is a finitely generated subgroup of G_i and I' is at most countable. Thus the image of φ is countable and therefore also B-slender. Hence $\varphi(e_i) = 0$ for almost all $i \in \omega$. □

In conclusion, it remains the open question whether any of the statements we have proved with the axiom BP is equivalent to BP.

References

[1] U. Felgner – K. Schulz, *Algebraische Konsequenzen des Determiniertheits-Axioms,* Arch. Math. (Basel) 42 (1984), no. 6, pp. 557–563.

[2] L. Fuchs, *A note on certain subgroups of products of infinite cyclic groups,* Comment. Math. Univ. St. Paul 19 (1970), pp. 51–54.

[3] L. Fuchs, *Infinite abelian groups,* two volumes, Pure and Applied Mathematics, 36, Academic Press, New York and London, 1970/1973.

[4] R. Göbel – B. Wald, *Wachstumstypen und schlanke Gruppen,* in *Abelian Groups and their Relationship to the Theory of Modules* (INDAM, Rome, 1977), Symposia Mathematica, XXIII, Academic Press, New York and London, 1979, pp. 201–239.

[5] Th. Jech, *Set theory,* Pure and Applied Mathematics, Academic Press, New York and London, 1978.

[6] O. Kolman – B. Wald, *M-slenderness*, Israel J. Math. 217 (2017), pp. 303–312.

[7] E. L. Lady, *Slender rings and modules,* Pacific J. Math. 49 (1973), pp. 397–406.

[8] J. Mycielski – H. Steinhaus, *A mathematical axiom contradicting the axiom of choise,* Bull. Acad. Polon 10 (1962), pp. 1–3.

[9] G. Nöbeling, *Verallgemeinerung eines Satzes von Herrn E. Specker,* Invent. Math. 6 (1968), pp. 41–55.

[10] J. C. Oxtoby, *The Banach–Mazur game and Banach category theorem,* in *Contributions to the theory of games,* Vol. III, Annals of Mathematics Studies, 39, Princeton University Press, Princeton, N.J., 1957, pp. 159–163.

[11] S. Shelah, *Can you take Soloway's inaccessible away*?, Israel J. Math. 48 (1984), no. 1, pp. 1–47.

[12] E. Specker, *Additive Gruppen von Folgen ganzer Zahlen,* Portugal. Math. 9 (1950), pp. 131–140.

[13] B. Wald, *Schlankheitsgrade kotorsionsfreier Gruppen*, Ph.D. thesis, Universität Essen, Essen, 1979.

Manoscritto pervenuto in redazione il 29 febbraio 2020.