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Torsion-free abelian groups revisited

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Dedicated to Professor László Fuchs on his 95^{th} birthday

Abstract – Let G be a torsion-free abelian group of finite rank. The orbits of the action of $\mathrm{Aut}(G)$ on the set of maximal independent subsets of G determine the indecomposable decompositions. G contains a direct sum of pure strongly indecomposable groups as a subgroup of finite index.

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1. Introduction

Let G be a finite rank subgroup of $V = \mathbb{Q}^{\mathbb{N}}$. In the first part of this paper, Sections 2–4, I study the action of $\mathrm{Aut}(G)$ on the maximal independent subsets of V contained in G. I show that the orbits of this action determine the isomorphism classes of indecomposable direct decompositions of G.

In the second part, Sections 5 and 6, I study the action of $\mathrm{Aut}(G)$ on the set of strongly indecomposable quasi-decompositions of G. Each strongly indecomposable quasi-decomposition determines an isomorphism class of subgroups of G of finite index which are direct sums of strongly indecomposable pure subgroups.

Finally, in Section 7, I initiate a programme to classify strongly indecomposable groups.

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Among other new results of this paper are a group theoretic proof of Lady's Theorem (Corollary 6.5) which states that G has only finitely many non-isomorphic summands, and an extension of the notion of regulating subgroup from acd groups to all finite rank groups, (Remark 6.11(2)).

2. Notation

Let $V = \mathbb{Q}^{\mathbb{N}}$ and denote by $\mathcal{V}$ the set of finite rank additive subgroups of V. Given $G \in \mathcal{V}$ and $S \subseteq V$, let $\langle S \rangle$ be the subgroup of V generated by S, $[S]$ the subspace of V generated by S and $S_* = [S] \cap G$. If $S \subseteq G$, S_* is just the pure subgroup of G generated by S, but in general we do not insist that $S \subseteq G$.

Note that $\langle S \rangle$ is the group of all finite integral combinations of S, and $[S]$ is the vector space of all finite rational combinations of S. Since $S \subseteq V$ is integrally independent if and only if S is rationally independent, we generally omit the adjective.

We identify endomorphisms and automorphisms of G with their unique extensions to the vector space $[G]$.

If $r \in \mathbb{Q}^*$, the non-zero rationals, the statement $r = a/b$ will imply that $a \in \mathbb{Z}^*$, the non-zero integers, $b \in \mathbb{N}$, the natural numbers and $\gcd(a, b) = 1$.

A *type* is a group τ satisfying $\mathbb{Z} \leqslant \tau \leqslant \mathbb{Q}$. Since $\tau/\mathbb{Z} \leqslant \mathbb{Q}/\mathbb{Z} \cong \prod_{p \in \mathbb{P}} \mathbb{Z}(p^\infty)$, $\tau/\mathbb{Z}$ is a torsion group of p-rank at most 1 for each prime p.

Let $a \in G \in \mathcal{V}$. The *type of* a in G,

$$\operatorname{type}_G(a) = \{r \in \mathbb{Q} : ra \in G\}$$

which is clearly a type.[1] When there is no ambiguity, we omit the subscript G.

A maximal independent subset of $G \in \mathcal{V}$ is called a *basis of* G. Let $\texttt{Bases}(G)$ denote the set of bases of G. In particular, $\texttt{Bases}(G) \subseteq \texttt{Bases}([G])$, the set of vector space bases of $[G]$. It is well known, see for example [2, Theorem 16.3], that $\operatorname{rank}(G)$ is the cardinality of any basis. so $\operatorname{rank}(G) = \dim([G])$. The following proposition shows how the groups $\langle B \rangle$, B_* and $[B]$, where $B \in \texttt{Bases}(G)$, are related.

Proposition 2.1. *Let $G \in \mathcal{V}$ with $\operatorname{rank}(G) = k$ and let $B \in \texttt{Bases}(G)$.*

1. *$\langle B \rangle$ is a free subgroup of G of rank k;*
2. *$[B] = [G]$ is a subspace of V of dimension k;*

[1] This definition of $\operatorname{type}_G(a)$ is not the standard one, [3, §85], but is equivalent to it, as shown in [8, §2.2].

3. $B_* = G$;

4. $G/\langle B\rangle$ *and* $[B]/G$ *are torsion groups, the latter being divisible.*

PROOF. (1) Each $a \in \langle B\rangle$ has a unique representation as $a = \sum_{b\in B} n_b b,\ n_b \in \mathbb{Z}$.

(2) B is maximally independent in $[G]$.

(3) By definition, $B_* \leqslant G$. Since $[G] = [B],\ G \leqslant [B] \cap G = B_*$.

(4) $[B]/\langle B\rangle \cong (\mathbb{Q}/\mathbb{Z})^k$ is a torsion group; $G/\langle B\rangle$ is a subgroup and $[B]/G$ a factor group. □

To justify the name basis, we note that bases of groups share several properties with bases of vector spaces. In particular, they are independent spanning sets in the following sense:

PROPOSITION 2.2. *Let* $G \in \mathcal{V}$ *and* $0 \neq H \in \mathtt{Subgroups}(G)$. *Then,*

1. *every basis of* G *is a basis of* $[G]$;

2. *for every* $B \in \mathtt{Bases}([G])$, *there is a minimum* $m \in \mathbb{N}$ *such that* $mB = \{mb : b \in B\} \in \mathtt{Bases}(G)$;

3. H *has a basis of cardinality* $\ell \leqslant \operatorname{rank}(G)$;

4. *every basis of* H *extends to a basis of* G;

5. *if* $B \in \mathtt{Bases}(G)$, *then every* $a \in G$ *has a unique representation as* $k^{-1}\sum_{b\in B} n_b b$ *where* $n_b \in \mathbb{Z},\ k \in \mathbb{N}$ *and* $\gcd\{k, n_b : b \in B\} = 1$.

PROOF. (1) If $B \in \mathtt{Bases}(G)$, then B is a maximal independent subset of $[G]$.

(2) Since $[G]/G$ is torsion, each $b \in B$ has finite order, say m_b, modulo G. Let $m = \operatorname{lcm}\{m_b : b \in B\}$. Then m is minimal such that $mB \in \mathtt{Bases}(G)$.

(3) $[H] \leqslant [G]$ and (1) imply $\ell \leqslant \operatorname{rank}(G)$.

(4) Let C be a basis of H. Then C is independent in G and hence extends to a basis $B = C \cup D$. Hence there is a least $m \in \mathbb{N}$ such that $B = C \cup mD$ is a basis of G.

(5) Let $a \in G$. Since $B \cup \{a\}$ is integrally dependent, there exists a least $k \in \mathbb{N}$ such that $ka = \sum_{b\in B} n_b b : n_b \in \mathbb{Z}$. Hence $a = k^{-1}\sum_{b\in B} n_b b \in V$ and $\gcd\{k, n_b : b \in B\} = 1$. □

We call the expression $k^{-1}\sum_{b\in B} n_b b$ the *B-representation of* a.

3. Bases and decompositions

To simplify the notation, from now on 'decomposition' of a group means non-trivial direct decomposition and 'partition' of a set means partition into non-empty subsets.

Let $B \in \mathtt{Bases}(G)$ and let $C \dot\cup D$ be a partition of B. We say that $C \dot\cup D$ is a *splitting partition of* B, and B is a *splitting basis*, if $G = C_* \oplus D_*$, while B is an *indecomposable basis* if B has no splitting partition.

For clarification, note that G can have both splitting and non-splitting bases. For example, let $G = \mathbb{Z} \oplus \mathbb{Q}$. Then $B_1 = \{(1,0),\ (0,1)\}$ is a splitting basis but $B_2 = \{(1,0),\ (1,1)\}$ is a basis that is not splitting. However, Proposition 3.1 shows that if G is indecomposable then all bases are indecomposable.

Proposition 3.1. *Let $G \in \mathcal{V}$ and $B \in \mathtt{Bases}(G)$. Then,*

1. *for any partition $B = C \dot\cup D$, $C_* \cap D_* = \{0\}$, and $C \dot\cup D$ is a splitting partition of B if and only if $C_* + D_*$ is pure in G;*
2. *$G = H \oplus K$ if and only if B has a splitting partition $B = C \dot\cup D$ with $H = C_*$ and $K = D_*$.*

Proof. (1) If $B = C \dot\cup D$ is any partition of B, then $[C] \cap [D] = 0$, so $C_* \cap D_* = 0$. Since $\operatorname{rank} C_* + \operatorname{rank} D_* = \operatorname{rank}(G)$, $C_* \oplus D_* = G$ if and only if $C_* + D_*$ is pure in G.

(2) ($\Longrightarrow$) Let $C \in \mathtt{Bases}(H)$ and $D \in \mathtt{Bases}(K)$. Then $H = C_*$ and $K = D_*$ so $B = C \dot\cup D$ is a splitting basis for G.

($\Longleftarrow$) Since $C_* \oplus D_* = G$ with $C \in \mathtt{Bases}(H)$ and $D \in \mathtt{Bases}(K)$, $G = H \oplus K$. □

Proposition 3.1 is most useful in the contrapositive, which we state for future reference.

Corollary 3.2. *Let $G \in \mathcal{V}$. Then G is indecomposable if and only if for all $B \in \mathtt{Bases}(G)$ and for all partitions $B = C \dot\cup D$, $C_* \oplus D_*$ is a subgroup of G with non-zero torsion quotient.* □

It is now routine to extend these results to complete decompositions of G. Let $B \in \mathtt{Bases}(G)$, and let $\mathcal{B} = \dot\bigcup_{i\in[t]} B_i$ be a splitting partition of B. Denote the corresponding decomposition $\bigoplus_{i\in[t]}(B_i)_*$ of G by $G(\mathcal{B})$.

Splitting partitions $\mathcal{B} = \dot\bigcup_{i\in[t]} B_i$ and $\mathcal{C} = \dot\bigcup_{j\in[s]} C_j$ of $B \in \mathtt{Bases}(G)$ are isomorphic, denoted $\mathcal{B} \cong \mathcal{C}$, if $t = s$ and there is a permutation π of t and isomorphisms α_i such that each $(C_{i*})\alpha_i = (B_{i\pi})_*$.

Let $G \in \mathcal{V}$ and $B \in \mathtt{Bases}(G)$. A decomposition $G = \bigoplus_{i\in[t]} A_i$ is *complete* if each A_i is indecomposable. A partition $\mathcal{B} = \dot{\bigcup}_{i\in[t]} B_i$ of B is a *complete splitting partition* if $G(\mathcal{B}) = \bigoplus_{i\in[t]} (B_i)_*$ is a complete decomposition of G.

Proposition 3.3. *Let* $G,\ A_i : i \in [t] \in \mathcal{V}$. *The following are equivalent*:

1. $G = \bigoplus_{i\in[t]} A_i$ *is a complete decomposition*;
2. *for each* $B_i \in \mathtt{Bases}(A_i),\ B = \dot{\bigcup}_{i\in[t]} B_i$ *is a complete splitting partition such that for all* $i \in [t],\ A_i = B_{i*}$;
3. G *has a basis* $B = \dot{\bigcup}_{i\in[t]} B_i$ *such that* $\bigoplus_{i\in[t]} (B_i)_*$ *is a complete decomposition which is pure in* G.

Proof. By Proposition 3.1, for all parts of the proposition, the statement holds if $t = 2$.

Assume that each statement holds if $t = k$ and let $t = k+1$. Let $H = \bigoplus_{i>1} A_i$, so $G = A_1 \oplus H$. Then all parts hold with G replaced by H, and hence by Proposition 3.1, all parts hold for G. □

4. Automorphisms of G

We first note without proof some well known properties of $\mathrm{Aut}(G)$. For any $\alpha \in \mathrm{Aut}(G)$ and any set $S \subseteq G$, $S\alpha$ denotes the set $\{s\alpha : s \in S\}$.

Lemma 4.1. *Let* $\alpha \in \mathrm{Aut}(G),\ a \in G,\ S \subseteq G,\ H$ *and* $K \in \mathtt{Subgroups}(G)$ *and* $B \in \mathtt{Bases}(G)$.

1. $S_*\alpha = (S\alpha)_*$.
2. $H \cap K = 0$ *if and only if* $H\alpha \cap K\alpha = 0$.
3. $\mathrm{type}(a) = \mathrm{type}(a\alpha)$.
4. *Let* $r \in \mathbb{Q},\ n_b \in \mathbb{Z}$ *for all* $b \in B$. *Whenever either side is defined, so is the other and* $\big(r \sum_{b\in B} n_b b\big)\alpha = r \sum_{b\in B} n_b (b\alpha)$. □

Proposition 4.2. $\mathrm{Aut}(G)$ *acts on* $\mathtt{Bases}(G)$. *This action preserves splitting partitions, indecomposable bases, and complete splitting partitions.*

Proof. Let $B \in \mathtt{Bases}(G)$, and $\alpha \in \mathrm{Aut}(G)$. Then Lemma 4.1 (4) implies that $B\alpha \in \mathtt{Bases}(G)$. Clearly, $B1_G = B$ and for all $\alpha,\ \beta \in \mathrm{Aut}(G),\ (B\alpha)\beta = B(\alpha\beta)$.

If $B = C \dot{\cup} D$ and $G = C_* \oplus D_*$, then $B\alpha = C\alpha \dot{\cup} D\alpha$ and $G = C_*\alpha \oplus D_*\alpha$. Conversely, if B has no splitting partition, then $B\alpha$ has no splitting partition.

Let $\mathcal{B} = \dot{\bigcup}_{i\in[t]} B_i$ be a complete splitting partition of B, so $G(\mathcal{B})$ is a complete decomposition. Then $G = \bigoplus_{i\in[t]}(B_i\alpha)_*$ is also a complete decomposition, so $\mathcal{B}\alpha = \dot{\bigcup}_{i\in[t]} B_i\alpha$ is a complete splitting partition of $B\alpha$. □

Corollary 4.3. *Let $G \in \mathcal{V}$. The following are equivalent*:

1. $\bigoplus_{i\in[t]} A_i$ *is a complete decomposition of* G;
2. G *has a basis with complete splitting partition* $B = \dot{\bigcup}_{i\in[t]} B_i$ *with* $B_i \in \mathtt{Bases}(A_i)$;
3. *for all* $\alpha \in \mathrm{Aut}(G)$, $\dot{\bigcup}_{i\in[t]} B_i\alpha$ *is a complete splitting partition of a basis of* G.

The action of $\mathrm{Aut}(G)$ on $\mathtt{Bases}(G)$, unlike that of $\mathrm{Aut}([G])$ on $\mathtt{Bases}([G])$, may be far from transitive; in fact its orbits determine the isomorphism classes of direct decompositions of G.

We say that complete decompositions $\mathcal{D}\colon \bigoplus_{i\in[t]} A_i$ and $\mathcal{E}\colon \bigoplus_{j\in[s]} C_j$ of G are *isomorphic*, denoted $\mathcal{D} \cong \mathcal{E}$, if their indecomposable summands are pairwise isomorphic, i.e. $t = s$ and there is a permutation π of $[t]$ such that for all $i \in [t]$, $C_{i\pi} \cong B_i$.

Proposition 4.4. *Let* $\mathcal{B} = \dot{\bigcup}_{i\in[t]} B_i$, $\mathcal{C} = \dot{\bigcup}_{j\in[s]} C_j$ *be complete partitions of* $B \in \mathtt{Bases}(G)$. *There exists* $\alpha \in \mathrm{Aut}(G)$ *such that* $\mathcal{B}\alpha = \mathcal{C}$ *if and only if* $G(\mathcal{B}) \cong G(\mathcal{C})$.

Proof. Assume there exists such $\alpha \in \mathrm{Aut}(G)$. Proposition 4.4 implies that for each $i \in [t]$, there is a $j \in [s]$ such that $(C_j)_* = (B_i\alpha)_*$ and this correspondence is 1–1. Hence $G(B) \cong G(C)$.

Conversely, if for all $i \in [t]$, $\alpha_i\colon (B_i)_* \to (C_{i\pi})_*$ are isomorphisms, then $\alpha = (\alpha_i\colon i \in [t]) \in \mathrm{Aut}(G)$. □

Theorem 4.5. *The orbits of* $\mathrm{Aut}(G)$ *acting on the complete decompositions of* G *are the isomorphic complete decompositions.*

Proof. Let $\mathcal{D}$, $\mathcal{E}$ be complete decompositions of G. Then by Proposition 4.4 $\mathcal{D} \cong \mathcal{E}$ if and only if there exists $\alpha \in \mathrm{Aut}(G)$ such that $\mathcal{D}\alpha = \mathcal{E}$.

Thus every orbit consists of isomorphism classes of complete decompositions, and every isomorphic pair of complete decompositions are in the same orbit. □

To clarify Theorem 4.5, note that in general, G may have several non-isomorphic complete decompositions, each of which determines several complete splitting partitions of bases of G. For each isomorphic pair $\mathcal{B}$, $\mathcal{C}$ of complete splitting partitions of bases, there may be several $\alpha \in \mathrm{Aut}([G])$ such that $\mathcal{B}\alpha = \mathcal{C}$.

5. The quasi category of $\mathcal{V}$

Let $G \in \mathcal{V}$. There is a class of subgroups of V which shed light on the structure of G.

DEFINITION 5.1. Let $H,\ G \in \mathcal{V}$. H is

- *quasi-equal to* G, written $H \doteq G$, if there exists $r \in \mathbb{Q}^*$ such that[2] $rH = G$;
- *quasi-isomorphic to* G, written $H \approx G$, if H and G are isomorphic to quasi-equal subgroups of V.

Quasi-equal groups may have very different structures. Fuchs [3, Example 2, §88] presents examples of groups $G \doteq H$ of arbitrary finite rank $n \geqslant 2$ such that G is completely decomposable and H is indecomposable.

The properties of these relations are summarised in the following proposition, whose proof is routine.

PROPOSITION 5.2. (1) *Quasi-equality and quasi-isomorphism are equivalences on* $\mathcal{V}$ *which extend equality and isomorphism respectively.*

(2) $(a/b)H = G$ *if and only if* $aH = bG \leqslant H \cap G \doteq G$.

(3) $H \doteq G$ *implies that* $[H] = [G]$ *and* $H \approx G$ *implies* $[H] \cong [G]$.

(4) *If* A *and* B *are pure subgroups of* G *with* $A \doteq B$, *then* $A = B$.

NOTATION 5.3. Let $G \in \mathcal{V}$.

- A *quasi-decomposition* G is a quasi-equality $G \doteq \bigoplus_i A_i$; the groups A_i are called *quasi-summands* of G.
- G is *strongly indecomposable* if it has no proper quasi-decompositions.
- A *strong decomposition* is a direct sum of strongly indecomposable groups, and a *strong quasi-decomposition of* G is a strong decomposition quasi-equal to G.

Completely decomposable groups are the type examples of strong decompositions, and almost completely decomposable (acd) groups of strong quasi-decompositions. In these cases, the direct summands of a strong (quasi-)decomposition are rank 1 groups.

[2] This definition, as well as the notation, differs from that in [3, §92]. However, it is more suited to our context.

Definition 5.4. The *quasi-automorphism group* of G

$$\mathbb{Q}^* \operatorname{Aut}(G) := \{r\alpha : r \in \mathbb{Q}^*,\ \alpha \in \operatorname{Aut}(G)\}.$$

The properties of $\mathbb{Q}^* \operatorname{Aut}(G)$ are summarised in the following proposition, whose proof follows immediately from the definitions:

Proposition 5.5. *For all $G \in \mathcal{V}$,*

1. $\operatorname{Aut}(G) \leqslant \mathbb{Q}^* \operatorname{Aut}(G) \leqslant \operatorname{Aut}([G])$;
2. $\operatorname{Aut}(G)$ *is a normal subgroup of* $\mathbb{Q}^* \operatorname{Aut}(G)$;
3. $H \doteq G$ *if and only if there exists* $r\alpha \in \mathbb{Q}^* \operatorname{Aut}([G])$ *such that* $rG\alpha = H$;
4. $\mathbb{Q}^* \operatorname{Aut}(G)$ *acts on the following sets*: *strongly indecomposable subgroups of G, quasi-decompositions of G, strong quasi-decompositions of G.* □

The notions of quasi-equality and quasi-isomorphism are due to [5] and [6] and were put into a categorical context by [9]. The properties of this category are outlined in [1, §7].

The most important such property is the existence and uniqueness of strong quasi-decompositions.

Theorem 5.6 (Jónsson's Theorem, [3, Theorem 92.5]). *Let $G \in \mathcal{V}$. Then G has strong quasi-decompositions.*

If $\bigoplus_{i\in[t]} A_i$ and $\bigoplus_{j\in[m]} C_j$ are strong quasi-decompositions of G, then $t = m$ and there is a permutation π of $[t]$ such that for all i, $A_i \approx C_{i\pi}$.

6. Jónsson bases

Throughout this section, $0 \neq G \in \mathcal{V}$. Let $\mathcal{J}$ be a set of representatives in $\mathcal{V}$ of the isomorphism classes of the strongly indecomposable quasi-summands of G. Jónsson's Theorem implies that $\mathcal{J}$ is finite. Let $\operatorname{Rep}(G)$ be the set of all such $\mathcal{J}$.

Recall that endomorphisms of G are identified with their extensions to $[G]$ and that for each $J \in \mathcal{J}$, $J_* = [J] \cap G$. Recall too that set partitions and group decompositions are assumed to be non-trivial.

Notation 6.1. (1) A *Jónsson basis* of G is a strong decomposition $\bigoplus_i A_i$ of finite index in G such that each A_i is pure in G.

(2) $\operatorname{Jon}(G)$ is the set of all Jónsson bases of G.

(3) If $A \in \operatorname{Jon}(G)$, the finite group G/A is a *Jónsson quotient* of G.

Remark 6.2. (1) If G is acd, $\mathrm{Jon}(G)$ consists of full completely decomposable subgroups.

(2) By Proposition 5.5 (4), $\mathrm{Aut}(G)$ acts on ((strongly) indecomposable) summands. The following proposition shows that $\mathrm{Aut}(G)$ acts on Jónsson bases.

Proposition 6.3. *Let* $\mathcal{J} \in \mathrm{Rep}(G)$.

1. *For all* $J \in \mathcal{J}$, *there exists a maximum* $n_J \in \mathbb{N}$ *such that* $\bigoplus_{J\in\mathcal{J}}(J_*)^{n_J} \in \mathrm{Jon}(G)$;
2. *for all* $A \in \mathrm{Jon}(G)$ *and all* $\alpha \in \mathrm{Aut}(G)$, $A\alpha \in \mathrm{Jon}(G)$;
3. *for all* $A \in \mathrm{Jon}(G)$, *there exists* $\alpha \in \mathrm{Aut}(G)$ *such that* $A\alpha = \bigoplus_{J\in\mathcal{J}}(J_*)^{n_J}$.

Proof. (1) Let n_J be the number of isomorphic copies of J which occur in some strong decomposition of finite index in G. By Jónsson's Theorem, n_J is uniquely determined. The group $\bigoplus_{J\in\mathcal{J}}(J_*)^{n_J}$ is a strong decomposition of finite index in G in which each summand J_* is a pure subgroup of G.

(2) If $A = \bigoplus_{i\in[t]} A_i$, then $A\alpha = \bigoplus_{i\in[t]}(A_i\alpha)$, a strong decomposition of finite index in G.

(3) For all $A = \bigoplus_{i\in[t]} A_i \in \mathrm{Jon}(G)$, there exists $\alpha_i \in \mathrm{Aut}(A_i)$, a partition $\mathcal{J} = \dot{\bigcup}\mathcal{J}_i$ and $m_J \in [0, n_J]$ with $\sum m_j = n_j$, such that $A_i\alpha_i = \bigoplus_{J\in\mathcal{J}_i}(J_*)^{m_J}$. Take $\alpha = \sum_i \alpha_i$. Then $A\alpha = \bigoplus_{J\in\mathcal{J}}(J_*)^{n_J}$ □

We now show the relation between decompositions of G and $\mathrm{Jon}(G)$. To clarify the notation, the decompositions of $A \in \mathrm{Jon}(G)$ in the following proposition, are not necessarily their decompositions into pure strongly indecomposable summands described in Notation 6.1(1).

Proposition 6.4. (1) *If* $G = \bigoplus_{i\in[t]} H_i$ *and* $A_i \in \mathrm{Jon}(H_i)$, *then* $\bigoplus_{i\in[t]} A_i \in \mathrm{Jon}(G)$.

(2) *Let* $A = \bigoplus_{i\in[t]} B_i \in \mathrm{Jon}(G)$, *then* $\bigoplus_{i\in[t]} B_{i*} \doteq G$ *and* $\bigoplus_{i\in[t]} B_{i*} = G$ *if and only if* $\sum_{i\in[t]} B_{i*}$ *is pure in* G.

(3) *If* $A = \bigoplus_{i\in[t]} B_i \in \mathrm{Jon}(G)$, *then for all* $J \in \mathcal{J}$, *there exists a partition* $n_J = \sum_{i\in[t]}(m_i)_J$ (*where we allow some terms to be* 0), *such that* $B_{i*} \cong \left(\bigoplus_{J\in\mathcal{J}}(J_*)^{(m_i)_J}\right)_*$.

Proof. (1) $\bigoplus_{i\in[t]} A_i$ is a strong decomposition of finite index in G whose summands are pure in H_i and hence in G.

(2) The groups B_{i*} are disjoint pure subgroups of G whose sum has finite index in G. Hence $\sum_{i\in[t]} B_{i*}$ is pure in G if and only if $G = \bigoplus_{i\in[t]} B_{i*}$.

(3) By Jónsson's Theorem, each B_i is isomorphic to a direct sum of elements of the set $\mathcal{J}$ and their sum accounts for all of them. Hence by and Proposition 5.2 (3), $B_{i*} \cong \left(\bigoplus_{J\in\mathcal{J}}(J_*)^{(m_i)_J}\right)_*$ for some $(m_i)_J \in [0, n_J]$ with $\sum_{i\in[t]}(m_i)_J = n_J$. □

Corollary 6.5 (Lady's Theorem, [4, Theorem 6.9]). *Let $G \in \mathcal{V}$. Then G has only finitely many non-isomorphic summands.*[3]

Proof. If H is a summand of G and $\mathcal{J} \in \mathrm{Rep}(G)$, then By Proposition 6.4, $H = A_*$ for some $A \cong \bigoplus_{J\in\mathcal{J}}(J_*)^{m_J}$, $m_j \in [0, n_j]$. There are only finitely many choices for J and m_J. □

Notation 6.6. Let $A \in \mathrm{Jon}(G)$.

- A decomposition $A = \bigoplus_{i\in[t]} A_i$ is a *splitting decomposition* of A if $G = \bigoplus_{i\in[t]} A_{i*}$;
- A is *non-split* if A has no splitting decomposition;
- $A = \bigoplus_{i\in[t]} A_i$ is a *complete splitting decomposition* if it is a splitting decomposition and each A_i is non-split.

Propositions 6.3 and 6.4 have the following immediate corollaries:

Corollary 6.7. (1) *G is indecomposable if and only if every $A \in \mathrm{Jon}(G)$ is non-split.*

(2) *If $G = \bigoplus_{i\in[t]} H_i$ then for all $A_i \in \mathrm{Jon}(H_i)$, $\bigoplus_{i\in[t]} A_i$ is a splitting decomposition of $A \in \mathrm{Jon}(A)$. The decomposition of G is complete if and only if the splitting decomposition of A is.*

Theorem 6.8. $\mathrm{Aut}(G)$ *acts on complete decompositions of G and on complete splitting Jónsson bases. In both cases, the orbits are the isomorphism classes of complete decompositions and complete splitting Jónsson bases.*

Proof. Let $\mathcal{D}\colon \bigoplus_{i\in[t]} H_i$ be an indecomposable decomposition of G and $\bigoplus_{i\in[t]} A_i$ a corresponding complete splitting Jónsson basis. Let $\alpha \in \mathrm{Aut}(G)$. Then $\bigoplus_{i\in[t]} H_i\alpha$ is also an indecomposable decomposition and $\bigoplus_{i\in[t]} A_i\alpha$ the corresponding complete splitting Jónsson basis.

[3] In his recent edition of *Abelian groups* [4, Lemma 6.8] Fuchs states that since Lady's Theorem is one of the most important results in torsion-free abelian group theory, a group-theoretical proof would be most welcome. Our proof replaces the Jordan–Zassenhaus Lemma in Lady's proof by Jónsson's Theorem, whose proof, while still ring theoretical, is rather more transparent.

On the other hand, let $\mathcal{E}: \bigoplus_{i\in[t]} K_i$ be an indecomposable decomposition of G with each K_i isomorphic to a unique H_i by some isomorphism α_i. Then $\alpha = (\alpha_i : i \in [t]) \in \mathrm{Aut}(G)$, so $\mathcal{E}$ is in the orbit of $\mathcal{D}$. Thus the orbit of $\mathcal{D}$ under $\mathrm{Aut}(G)$ consists of isomorphic indecomposable decomposition.

The argument extends to the corresponding Jónsson bases. □

Remark 6.9. It is clear that each decomposition of G refines to a complete decomposition, and each splitting decomposition of $A \in \mathrm{Jon}(G)$ refines to a complete splitting decomposition. In neither case is the refinement necessarily unique.

6.1 – *Jónsson quotients*

Let $G \in \mathcal{V}$ and $A \in \mathrm{Jon}(G)$. Let $\eta: G \to G/A$ be the natural surjection, and let $U \oplus W$ be a decomposition of G/A. We say $U \oplus W$ *lifts to* G if there exists a decomposition $G = K \oplus L$ such that $U = K\eta$ and $W = L\eta$.

Proposition 6.10. *$G/A = U \oplus W$ is a decomposition lifting to $G = K \oplus L$ if and only if A has a splitting decomposition $B \oplus C$ such that $U \cong B_*/B$ and $W = C_*/C$.*

Proof. ($\Longleftarrow$) If such a splitting decomposition exists, then $G = B_* \oplus C_*$ by Proposition 6.4(2), so $G/A = B_*/B \oplus C_*/C$.

($\Longrightarrow$) Suppose $G = K \oplus L$ with $U = K\eta$ and $W = L\eta$. Let $B = K \cap A$ and $C = L \cap A$, so $B \cap C = 0$. Let $a \in A$, say $a = k + \ell$ with $k \in K$ and $\ell \in L$. Since $a\eta = 0$, if $k\eta \neq 0$, then $0 \neq \ell\eta = -k\eta \in U \cap W$, a contradiction. Hence $B + C = A$. Thus $B \oplus C$ is a splitting decomposition of A and $G/A = U \oplus W$ lifts to $B_* \oplus C_* = G$. □

Remark 6.11. (1) If G is indecomposable, then [8, Example 16.8.11] shows that $G\eta$ need not be. However, if U is an indecomposable summand of $G\eta$ then $U\eta^{-1}$ is indecomposable in G.

(2) Since Jónsson quotients are finite, there is a least index $e \in \mathbb{N}$ and $A \in \mathrm{Jon}(G)$ such that $|G/A| = e$. Such A are called *regulating Jónsson bases* since in the case that G is almost completely decomposable, they are regulating subgroups as defined in a different way in [8, §4.2]. Thus regulating Jónsson bases are an extension to $\mathcal{V}$ of regulating subgroups of acd groups. Their properties remain to be investigated.

PROPOSITION 6.12. *Let $G \in \mathcal{V}$, $A \in \mathrm{Jon}(G)$ and $T = G/A$. Then $T = \bigoplus_i S_i$ is a decomposition which lifts to a complete decomposition $\bigoplus_i H_i$ of G if and only if $A = \bigoplus_i C_i$ is a complete splitting decomposition of A with $C_{i*} \cong H_i$ and each S_i has no decomposition which lifts to G.*

PROOF. The statement follows Proposition 6.10 and a routine induction on the number of summands. □

We turn now to the action of $\mathrm{Aut}(G)$ on the set of Jónsson quotients of G. The proof of the following lemma is a routine application of the definitions.

LEMMA 6.13. *Let $A \in \mathrm{Jon}(G)$ and $\alpha \in \mathrm{Aut}(G)$. Define $\bar{\alpha}: G/A \to G/(A\alpha)$ by $\bar{\alpha}: g + A \to g\alpha + A\alpha$. Then $\bar{\alpha}$ is an isomorphism and $(G/A)^{\alpha} = (G/A)\bar{\alpha}$ is a group action.* □

PROPOSITION 6.14. (1) *If $\eta: G \to G/A$ is the natural epimorphism, then $\alpha^{-1}\eta\alpha: G \to G/A\alpha$ is the natural epimorphism.*

(2) *If $G/A = S \oplus T$ and this decomposition lifts to G, then $G/A\alpha = S\bar{\alpha} \oplus T\bar{\alpha}$ and this decomposition also lifts to G.*

(3) *The map $\alpha \to \bar{\alpha}$ is a (non-abelian) group homomorphism. Its image is the group of automorphisms of G/A which lift to G, and its kernel is $\{\alpha \in \mathrm{Aut}(G): \alpha - 1 \in n\,\mathrm{End}(G)\}$ where $n = \exp(G/A)$.*

PROOF. (1) follows from the definition of the group action.

(2) Let $\mu: G/A \to S$ be the projection determined by the first decomposition. Then $\mu' = \bar{\alpha}^{-1}\mu\bar{\alpha}: G/A\alpha \to S\bar{\alpha}$ is the projection determined by the second decomposition. Since μ lifts to G, say $G = B_* \oplus C_*$ and $A = B \oplus C$, then $A\alpha = B\alpha \oplus C\alpha$ and $G = (B\alpha)_* \oplus (C\alpha)_*$, so μ' is the projection which determines the second decomposition.

(3) This is a routine calculation. □

Being finite, G/A has, up to isomorphism, a unique complete decomposition, which in general does not lift to G. We define a decomposition of G/A which lifts to G to be *unrefinable* if it has no refinement lifting to G.

THEOREM 6.15. *Let $G \in \mathcal{V}$ and let $\mathcal{A}$ be the set of unrefinable decompositions of Jónsson quotients of G. $\mathrm{Aut}(G)$ acts on $\mathcal{A}$ and the finitely many orbits of this action consist of isomorphic unrefinable decompositions.*

PROOF. Let $\mathcal{D} \in \mathcal{A}$, so $\mathcal{D} = \bigoplus_{I\in[t]} D_i$ an unrefinable decomposition of G/A for some $A \in \mathrm{Jon}(G)$. Then for all $\alpha \in \mathrm{Aut}(G)$, $\mathcal{D}$ is isomorphic to $\mathcal{D}\alpha := \bigoplus_{I\in[t]} D_i\bar{\alpha}$ as described in Lemma 6.13, so all elements of the orbit containing $\mathcal{D}$ are isomorphic. □

If $A' \in \mathrm{Jon}(G)$ it is possible that $G/A \cong G/A'$ even if A and A' are in different orbits of $\mathrm{Aut}(G)$. In any case, the number of isomorphism classes of unrefinable Jónsson quotients of G is no greater than the finite number of orbits of $\mathrm{Aut}(G)$ acting on complete decompositions of G

7. Strongly indecomposable groups

Strongly indecomposable groups play a crucial rôle in this paper. I am not aware of any published classification, but several important properties and examples can be found in [3, §92] and [4, §9]. In this section, I present a new characterisation of strongly indecomposable groups. Without loss of generality, we assume G is reduced and $\mathrm{rank}(G) > 1$. Recall that a strong decomposition in $\mathcal{V}$ is a direct sum of strongly indecomposable groups.

NOTATION 7.1. Let $R \leqslant G \in \mathcal{V}$.

- The pair (R, G) has *Property SI* if R is a strong decomposition such that G/R is an infinite torsion group with no decomposition lifting to G.
- If $B \in \mathtt{Bases}(G)$ then $(B)_* = \bigoplus_{b\in B} b_*$.

A routine calculation shows that Property SI is invariant under $\mathbb{Q}^*\,\mathrm{Aut}(G)$.

THEOREM 7.2. *Let $G \in \mathcal{V}$. Then G is strongly indecomposable if and only if for all $B \in \mathtt{Bases}(G)$, $((B)_*, G)$ has property SI.*

PROOF. ($\Longrightarrow$) Let $B \in \mathtt{Bases}(G)$, so $(B)_*$ is a strong decomposition and $G/(B)_*$ is torsion. If $G/(B)_*$ is finite, then for some $m \in \mathbb{N}$, $mG = (B)_*$, a contradiction, so $G/(B)_*$ is infinite. If some summand of $G/(B)_*$ lifts to G then G is decomposable, another contradiction. Hence $((B)_*, G)$ has Property SI.

($\Longleftarrow$) Suppose that for all $B \in \mathtt{Bases}(G)$, $((B)_*, G)$ has property SI, but that for some $m \in \mathbb{N}$, $mG = H \oplus K$. Let $C \in \mathtt{Bases}(H)$ and $D \in \mathtt{Bases}(K)$, so that by Proposition 3.1, $mG = C_* \oplus D_*$. Let $B = C \cup D$. Then $mG/(B)_*$ has a summand C_*/B_* lifting to the summand C_* of G, a contradiction. Hence no basis of mG has a splitting partition, so G is strongly indecomposable. □

A generalisation of Theorem 7.2, with essentially the same proof can be obtained by replacing the strong decomposition $(B)_*$ by an arbitrary full strong decomposition contained in G.

THEOREM 7.3. *Let $G \in \mathcal{V}$. Then G is strongly indecomposable if and only if for all full strong decompositions D contained in G, (D, G) has property SI.*

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