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Quasibases for nonseparable p-groups

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Dedicated to László Fuchs on his 95th birthday

Abstract – This paper is an extension of the work developed in [4] on quasibases of abelian p-groups and based on the doctoral dissertation of Andrija Vodopivec [5]. We introduce the ideas of a δ-combination and height of an inductive quasibasis and show that the height of a quasibasis is invariant for related inductive quasibases. Moreover, an abelian p-group is separable if and only if the heights of all δ-combinations are zero. Finally, we show that an abelian p-group is not reduced if and only if there exists a δ-combination with infinite height.

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1. Introduction

We deal with abelian groups and we use all definitions and conventions in [3]. For some few classes of torsion-free groups there is a description by cardinals. For all other torsion-free groups there exists basically only a presentation by generators and relations, unavoidably. In view of the convenient description of (simply presented) torsion groups by Ulm–Kaplansky invariants, the use of generators and

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relations seems to be disadvantageous for torsion groups. But often groups are considered as extensions, and then things change. An explicit description of a mixed group as an extension of a torsion by a torsion-free group is impossible if the torsion group is given by Ulm–Kaplansky invariants. The torsion group has to be presented by generators and relations, the same way as the torsion-free group. Here the concept of a quasibasis [3, 33.5] comes into the game.

Investigating mixed groups we recognized that the concept of a quasibasis was not developed far enough for our needs. In [4] the concept of a quasibasis was reduced to that of an inductive quasibasis and p-groups are explicitly described by the corresponding diagonal relation arrays α. In particular, we showed that smallness of α is equivalent to splitting and independent diagonal relation arrays were shown to correspond uniquely to reduced, separable groups.

In this paper we determine a relation array of the generalized Prüfer group $\mathcal{H}_{2\omega+1}$, Theorem 4.3. We define a height of an inductive quasibasis and show that this is an invariant for related inductive quasibases, Theorem 5.8. Further, we define δ-combinations and characterize "separable" by the heights of δ-combinations, Theorem 6.3. Finally we establish a criterion for "nonreduced" in terms of heights, Theorem 6.6.

Our concept, for sure, needs additional development for promising applications in the theory of torsion groups. For more results see [5].

2. Preliminaries

We denote the ring of p-adic integers by $\mathbb{Z}_p$. As customary, define the p-adic norm of $\lambda \in \mathbb{Z}_p$ by $\|\lambda\| = p^{-n}$ if $\lambda \in p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p$. Moreover, $\lambda = \sum_{i\in\mathbb{N}_0} \lambda_i p^i$ will denote the standard representation of a p-adic integer $\lambda \in \mathbb{Z}_p$.

We consider subgroups of $\prod_{|I|} \mathbb{Z}_p$, the additive group of all tuples $(\lambda_k \mid k \in I)$ of p-adic integers, where $\lambda_k \in \mathbb{Z}_p$, over some index set I. A tuple $0 \neq (\lambda_k \mid k \in I) \in \prod_{|I|} \mathbb{Z}_p$ is called a *zero tuple* if for every natural number n the norm of almost all λ_k is less than p^{-n}. A zero tuple is called *normed*, if there is at least one unit among the entries λ_k. The zero tuples (together with the trivial tuple 0) form a subgroup $\left(\prod_{|I|} \mathbb{Z}_p\right)^*$ of $\prod_{|I|} \mathbb{Z}_p$, which clearly contains $\bigoplus_{|I|} \mathbb{Z}_p$. Moreover, $\left(\prod_{|I|} \mathbb{Z}_p\right)^* / \bigoplus_{|I|} \mathbb{Z}_p$ is the maximal divisible subgroup of $\prod_{|I|} \mathbb{Z}_p / \bigoplus_{|I|} \mathbb{Z}_p$.

Proposition 2.1. *Let* $G = \bigoplus_{k\in I} \langle g_i^k \mid i \in \mathbb{N}\rangle \cong \bigoplus_{|I|} \mathbb{Z}(p^\infty)$, *where* $pg_1^k = 0$, $pg_{i+1}^k = g_i^k$ *for all* $k \in I, i \in \mathbb{N}$. *Then* $D = \langle h_i \in G \mid i \in \mathbb{N}\rangle \cong \mathbb{Z}(p^\infty)$ *is a subgroup of* G, *where* $ph_1 = 0$, $ph_{i+1} = h_i$ *for all* $i \in \mathbb{N}$, *if and only if there is a normed zero tuple* $(\lambda_k \mid k \in I)$, *such that* $h_i = \sum_{k\in I} \lambda_k g_i^k$ *for all* $i \in \mathbb{N}$.

Proof. For each $i \in \mathbb{N}$, the element $0 \neq h_i \in \bigoplus_{k\in I}\langle g_i^k \mid i \in \mathbb{N}\rangle$ can be written in the form $h_i = \sum_{k\in I}\lambda_i^k g_i^k$, where $0 \leq \lambda_i^k < p^i$, $\lambda_i^k = 0$ for almost all $k \in I$, and $p \nmid \lambda_i^k$ for at least one $k \in I$, by order considerations. Furthermore, we have for each $i \in \mathbb{N}$,

$$0 = h_i - ph_{i+1} = \sum_{k\in I}\lambda_i^k g_i^k - \sum_{k\in I}\lambda_{i+1}^k p g_{i+1}^k = \sum_{k\in I}(\lambda_i^k - \lambda_{i+1}^k)g_i^k.$$

Hence, $(\lambda_i^k - \lambda_{i+1}^k)g_i^k = 0$, i.e., $p^i \mid (\lambda_i^k - \lambda_{i+1}^k)$ for all $k \in I$. For each $k \in I$ let

$$\lambda_k = \lambda_i^k + \sum_{j\geq i}(\lambda_{j+1}^k - \lambda_j^k) \in \mathbb{Z}_p,$$

where the equation holds for arbitrary $i \in \mathbb{N}$.

For a fixed $i \in \mathbb{N}$, $p^i \mid \lambda_k$ for almost all $k \in I$, because $\lambda_i^k = 0$ for almost all k. Therefore $(\lambda_k \mid k \in I)$ is a zero tuple. Moreover, $p \nmid \lambda_k$ for at least one $k \in I$, because $p \nmid \lambda_i^k$ for at least one $i \in \mathbb{N}$, i.e., $(\lambda_k \mid k \in I)$ is normed. In particular, $\lambda_k g_i^k = \lambda_i^k g_i^k$. Thus $h_i = \sum_{k\in I}\lambda_i^k g_i^k = \sum_{k\in I}\lambda_k g_i^k$ for all $i \in \mathbb{N}$.

Conversely, let $(\lambda_k \mid k \in I)$ be a normed zero tuple and $h_i = \sum_{k\in I}\lambda_k g_i^k$ for each $i \in \mathbb{N}$. Note $ph_1 = \sum_{k\in I} p\lambda_k g_1^k = 0$ and

$$ph_{i+1} = \sum_{k\in I}\lambda_k p g_{i+1}^k = \sum_{k\in I}\lambda_k g_i^k = h_i$$

for all $i \in \mathbb{N}$. In particular, the order $o(h_i) = p^i$, because $(\lambda_k \mid k \in I)$ is a normed zero tuple. Hence, $\langle h_i \in G \mid i \in \mathbb{N}\rangle \cong \mathbb{Z}(p^\infty)$. □

Following [4] the set

$$Q = \{a_i^k, x_j^u\} = \{a_i^k, x_j^u \mid i, j \in \mathbb{N}, k \in I, u \in I_j\} \subset G$$

is called a *quasibasis* of G, if

i. $\{x_j^u \mid j \in \mathbb{N}, u \in I_j\}$ is a basis of the basic subgroup $B = \bigoplus B_j$, where $o(x_j^u) = p^j$ for all $j \in \mathbb{N}, u \in I_j$;

ii. $G/B = \bigoplus_{k\in I} A^k$, where $A^k = \langle a_i^k + B \mid i \in \mathbb{N}\rangle \cong \mathbb{Z}(p^\infty)$, $k \in I$, and $pa_{i+1}^k + B = a_i^k + B$ for all $i \in \mathbb{N}, k \in I$, with $pa_1 + B = 0 + B$;

iii. $o(a_i^k) = p^i$ for all $i \in \mathbb{N}, k \in I$.

Note that

$$G = \langle a_i^k, x_j^u \mid i, j \in \mathbb{N}, k \in I, u \in I_j\rangle.$$

By [3, 33.5] every p-group has a quasibasis with corresponding relations

$$pa_{i+1}^k = a_i^k - \sum_{j\in\mathbb{N}} \sum_{u\in I_j} \alpha_{i,j}^{k,u} x_j^u \quad (i \in \mathbb{N}, k \in I, \alpha_{i,j}^{k,u} \in \mathbb{Z}).$$

Given a quasibasis $Q = \{a_i^k, x_j^u\}$ the array $\alpha = (\alpha_{i,j}^{k,u})$ is called a *corresponding relation array*, B the *corresponding* basic subgroup. Note that we may also assume $\alpha_{i,j}^{k,u} \in \mathbb{Z}_p$. For $n \in \mathbb{N}$ let $p^n Q = \{c_i^k, y_j^u \mid i, j \in \mathbb{N}, k \in I, u \in I_{j+n}\}$, where $y_j^u = p^n x_{j+n}^u$ and $c_i^k = p^n a_{i+n}^k$.

Lemma 2.2. *Let $Q = \{a_i^k, x_j^u\}$ be a quasibasis of G with relation array $\alpha = (\alpha_{i,j}^{k,u})$. Then for any $n \in \mathbb{N}$ the set $p^n Q$ is a quasibasis of $p^n G$ with corresponding array $(\alpha_{i+n,j+n}^{k,u})$.*

Proof. Since $p^n B = \bigoplus_{j\in\mathbb{N}} \bigoplus_{u\in I_{j+n}} \langle p^n x_{j+n}^u\rangle$ is a basic subgroup of $p^n G$ and $o(p^n x_{j+n}^u) = p^j$, $o(p^n a_{i+n}^k) = p^i$, the conditions (i) and (iii) hold. Since $p^n G/p^n B \cong G/B \cong \bigoplus_{|I|} \mathbb{Z}(p^\infty)$ condition (ii) follows. The relations

$$p^{n+1} a_{i+n+1}^k = p^n a_{i+n}^k - \sum_{j\in\mathbb{N}} \sum_{u\in I_{j+n}} \alpha_{i+n,j+n}^{k,u} p^n x_{j+n}^u$$

give rise to the indicated array. □

3. Inductive quasibases

A quasibasis $\{a_i^k, x_j^u\}$ is called an *inductive quasibasis*, see [4], if the corresponding relations are of the form $pa_{i+1}^k = a_i^k - b_i^k$ for $i \in \mathbb{N}, k \in I$, where $b_i^k \in B_i = \bigoplus_{u\in I_i} \langle x_i^u\rangle$, cf. also [1]. Furthermore, a relation array $\alpha = (\alpha_{i,j}^{k,u})$ is called *diagonal*, if $\alpha_{i,j}^{k,u} = 0$ for $i \neq j$. A diagonal array is denoted by $\alpha = (\alpha_i^{k,u}) = (\alpha_{i,i}^{k,u})$. By [4, Theorem 4 and Corollary 5], every p-group has an inductive quasibasis, and the corresponding relation array is diagonal. Note that an inductive quasibasis is based on a fixed decomposition $B = \bigoplus B_i$ of the basic subgroup, and we write $Q = \{a_i^k, \bigoplus B_i\}$ or $Q = \{a_i^k, B\}$ to suppress the generators of the basic subgroup.

Lemma 3.1. *Let $Q = \{a_i^k, \bigoplus B_i\}$ be an inductive quasibasis of G with corresponding relations $pa_{i+1}^k = a_i^k - b_i^k$, $i \in \mathbb{N}, k \in I$. Then $p^n a_{i+n}^k = a_i^k - \sum_{r=0}^{n-1} p^r b_{i+r}^k$ for all $n \in \mathbb{N}$.*

PROOF. We induct on n. Clearly, $pa^k_{i+1} = a^k_i - b^k_i$. By hypothesis

$$\begin{aligned} p^{n+1}a^k_{i+n+1} &= p^n a^k_{i+n} - p^n b^k_{i+n} \\ &= a^k_i - \sum_{r=0}^{n-1} p^r b^k_{i+r} - p^n b^k_{i+n} \\ &= a^k_i - \sum_{r=0}^{n} p^r b^k_{i+r}. \end{aligned}$$

□

Let G, H be groups with isomorphic basic subgroups $B = \bigoplus B_i \subset G$ and $C = \bigoplus C_i \subset H$, and $G/B \cong H/C$, i.e., in particular, for all i,$B_i \cong C_i$ are isomorphic homocyclic groups of exponent p^i. Let, assuming equal index sets, the corresponding quasibases be $Q = \{a^k_i, x^u_j\}$, $P = \{c^k_i, y^u_j\}$, and the corresponding relation arrays be $\alpha = (\alpha^{k,u}_{i,j})$, $\beta = (\beta^{k,u}_{i,j})$, respectively. Then the groups G, H, the quasibases P, Q and the relation arrays α, β are called *related*, respectively. In particular, if $G = H$ and $B = C$ we call the two quasibases $Q = \{a^k_i, x^u_j\}$, $P = \{c^k_i, y^u_j\}$ and the two corresponding relation arrays α, β of G *related*, respectively. The point for related relation arrays is that the respective index sets are equal. We tacitly assume this setting for those related pairs G, H, or for a single group G with fixed basic subgroup B.

Let $H = \varphi G$ with isomorphism φ, then by choice $C = \varphi B$ for some basic subgroup $B \subset G$ the groups G, H are related. In other words, related groups coincide in some invariants that are kept by isomorphism.

Otherwise, let G, H be related with related quasibases $Q = \{a^k_i, x^u_j\}$, $P = \{c^k_i, y^u_j\}$ and related relation arrays α, β. If there is another quasibasis P' of H such that the relation array β' corresponding to P' is equal to α, then $G \cong H$. This is a consequence of [4, Theorem 1], because the relations given by the relation array of a group are defining. Thus all results on changing the quasibasis, respectively changing the relation array, of a group include statements on isomorphism.

There is a strong relationship between two related inductive quasibases of a group.

LEMMA 3.2. *Let $Q = \{a^k_i, \bigoplus B_i\}$ and $P = \{c^k_i, \bigoplus B_i\}$ be related, inductive quasibases of G with corresponding relations $a^k_i - pa^k_{i+1} = b^k_i$ and $c^k_i - pc^k_{i+1} = d^k_i$. Then for each $k_0 \in I$ there is a normed zero tuple $(\lambda_k \mid k \in I)$ (depending on k_0), such that for all $n \in \mathbb{N}$,*

$$d^{k_0}_i - \sum_{k \in I} \lambda_k b^k_i \in p^n B_i$$

for almost all $i \in \mathbb{N}$.

Proof. Since

$$G/B = \bigoplus_{k\in I}\langle a_i^k + B \mid i \in \mathbb{N}\rangle = \bigoplus_{k\in I}\langle c_i^k + B \mid i \in \mathbb{N}\rangle \cong \bigoplus_{|I|}\mathbb{Z}(p^\infty)$$

by Proposition 2.1, there is, for a fixed $k_0 \in I$, a normed zero tuple $(\lambda_k \mid k \in I)$ (depending on k_0), such that $c_n^{k_0} = \sum_{k\in I}\lambda_k a_n^k + b_n, b_n \in B$ for all $n \in \mathbb{N}$. Hence for all $n \in \mathbb{N}$,

$$d_n^{k_0} - \sum_{k\in I}\lambda_k b_n^k = c_n^{k_0} - pc_{n+1}^{k_0} - \sum_{k\in I}\lambda_k(a_n^k - pa_{n+1}^k) = b_n - pb_{n+1} \in B_n,$$

because Q and P are inductive and related. The elements $b_n \in B$ are of the form $b_n = \sum_{i\in\mathbb{N}} b_{n,i}$, where $b_{n,i} \in B_i$. Thus for each $n \in \mathbb{N}$

$$b_n - pb_{n+1} = \sum_{i\in\mathbb{N}}(b_{n,i} - pb_{n+1,i}) \in B_n,$$

i.e., $b_{n,i} - pb_{n+1,i} = 0$ for all $i \in \mathbb{N}$ with $i \neq n$. Consequently, for all $n \in \mathbb{N}$,

$$\begin{aligned} b_{n,i} &= pb_{n+1,i} = p^2 b_{n+2,i} = \cdots = 0 && \text{if } i < n,\\ b_{n,i} &= pb_{n+1,i} = p^2 b_{n+2,i} = \cdots = p^{i-n}b_{i,i} && \text{if } i \geq n, \end{aligned}$$

and the first part of the following sum is 0, hence

$$\begin{aligned} b_n &= b_{n,1} + \cdots + b_{n,n-1} + b_{n,n} + b_{n,n+1} + b_{n,n+2} + \cdots\\ &= b_{n,n} + pb_{n+1,n+1} + p^2 b_{n+2,n+2} + \cdots + p^r b_{n+r,n+r} + \cdots \end{aligned}$$

This is a finite sum, thus we have the equality $p^r b_{n+r,n+r} = 0$ for all $n \in \mathbb{N}$, or $p^n \mid b_{n+r,n+r}$ for almost all $r \in \mathbb{N}_0$. This implies for all $n \in \mathbb{N}$,

$$\begin{aligned} d_{n+r}^{k_0} - \sum_{k\in I}\lambda_k b_{n+r}^k &= b_{n+r} - pb_{n+r+1}\\ &= \sum_{m\in\mathbb{N}_0} p^m b_{n+r+m,n+r+m} - \sum_{m\in\mathbb{N}_0} p^{m+1} b_{n+r+m+1,n+r+m+1}\\ &= b_{n+r,n+r} \in p^n B_{n+r} \end{aligned}$$

for almost all $r \in \mathbb{N}_0$, as claimed. □

Let $Q = \{a_i^k, x_j^u\}$ be an inductive quasibasis of G with corresponding relations

$$a_i^k - pa_{i+1}^k = \sum_{u\in I_i}\alpha_i^{k,u}x_i^u = b_i^k \in B_i. \tag{1}$$

We write the corresponding diagonal relation array $\alpha = (\alpha_i^{k,u})$ in the following form:

(2a) $$\alpha = (\alpha^k)_{k\in I},$$

(2b) $$\alpha^k = \operatorname{diag}(\alpha_1^k, \alpha_2^k, \ldots),$$

(2c) $$\alpha_i^k = (\alpha_i^{k,u})_{u\in I_i} \quad \text{with } i \in \mathbb{N},\ \alpha_i^{k,u} \in \mathbb{Z},$$

where $\alpha_i^k \in \mathbb{Z}^{|I_i|}$ is a tuple, with only finitely many nonzero entries, and α can be considered as a tuple of (infinite) diagonal matrices α^k.

Two related diagonal relation arrays $\alpha = (\alpha_i^{k,u})$ and $\beta = (\beta_i^{k,u})$, i.e., with equal index sets, are called *almost equal*, if for each k the equation $\alpha_i^k = \beta_i^k$ holds for almost all $i \in \mathbb{N}$.

The following proposition shows that a group allows a whole class of almost equal relation arrays, and, moreover, that almost equal relation arrays of groups imply isomorphism.

Proposition 3.3. *Let $Q = \{a_i^k, x_j^u\}$ be an inductive quasibasis of G with relation array $\alpha = (\alpha_i^{k,u})$ and let β be an array almost equal to α. Then there is an inductive quasibasis $P = \{c_i^k, x_j^u\}$ of G with relation array β and $c_i^k = a_i^k$ for each $k \in I$, and for almost all $i \in \mathbb{N}$.*

Related groups G, H with almost equal (related) relation arrays are isomorphic.

Proof. Since α and β are almost equal we need to show that we can make finitely many changes for each k. Thus it suffices to construct a new inductive quasibasis P of G which differs from Q only for one fixed k and a fixed i_k. Let $\beta = (\beta_i^k)$ be given by

$$\beta_i^k = \begin{cases} (\alpha_i^{k,u})_{u\in I_i} & \text{if } i \neq i_k, \\ (z^{k,u})_{u\in I_i} & \text{if } i = i_k, \end{cases}$$

where $(z^{k,u} \mid u \in I_{i_k}) \in \mathbb{Z}^{|I_{i_k}|}$ is an arbitrary tuple with only finitely many nonzero entries. We show that $P = \{c_i^k, x_j^u \mid i, j \in \mathbb{N}, k \in I, u \in I_j\} \subset G$ with

$$c_i^k = \begin{cases} a_i^k + p^{i_k - i} \displaystyle\sum_{u\in I_{i_k}} (z^{k,u} - \alpha_{i_k}^{k,u}) x_{i_k}^u & \text{if } i \leq i_k, \\ a_i^k & \text{if } i > i_k, \end{cases}$$

for all $k \in I$, is an inductive quasibasis of G. The conditions (i) and (iii) of the definition of a quasibasis are obviously satisfied. Since $c_i^k + B = a_i^k + B$ for all

$k \in I, i \in \mathbb{N}$, condition (ii) is also satisfied. Furthermore, for all $k \in I$,

$$\begin{aligned} pc_{i+1}^k &= p\Big(a_{i+1}^k + p^{i_k-i-1}\sum_{u\in I_{i_k}}(z^{k,u} - \alpha_{i_k}^{k,u})x_{i_k}^u\Big) \\ &= a_i^k - \sum_{u\in I_i}\alpha_i^{k,u}x_i^u + p^{i_k-i}\sum_{u\in I_{i_k}}(z^{k,u} - \alpha_{i_k}^{k,u})x_{i_k}^u \\ &= c_i^k - \sum_{u\in I_i}\alpha_i^{k,u}x_i^u \quad \text{if } i < i_k, \end{aligned}$$

$$\begin{aligned} pc_{i_k+1}^k &= pa_{i_k+1}^k \\ &= a_{i_k}^k - \sum_{u\in I_{i_k}}\alpha_{i_k}^{k,u}x_{i_k}^u \\ &= c_{i_k}^k - \sum_{u\in I_{i_k}}(z^{k,u} - \alpha_{i_k}^{k,u})x_{i_k}^u - \sum_{u\in I_{i_k}}\alpha_{i_k}^{k,u}x_{i_k}^u \\ &= c_{i_k}^k - \sum_{u\in I_{i_k}} z^{k,u}x_{i_k}^u, \end{aligned}$$

$$\begin{aligned} pc_{i+1}^k &= pa_{i+1}^k \\ &= a_i^k - \sum_{u\in I_i}\alpha_i^{k,u}x_i^u \\ &= c_i^k - \sum_{u\in I_i}\alpha_i^{k,u}x_i^u \quad \text{if } i > i_k. \end{aligned}$$

Hence β is the desired relation array.

By the argument above and because relation arrays provide defining relations, see [4, Theorem 1], the groups G, H are isomorphic. □

4. Construction of a quasibasis for $\mathcal{H}_{2\omega+1}$

Well known examples for nonseparable reduced p-groups are the generalized Prüfer groups H_σ for ordinals σ. For a definition see [3, Section 81]. By [3, 83.1] all generalized Prüfer groups are simply presented.

In [4] for the generalized Prüfer group $\mathcal{H}_{\omega+n}$ of length $\omega + n$ for natural n an inductive quasibasis was given with a corresponding relation array. Our next goal is to determine an inductive quasibasis of the generalized Prüfer group $\mathcal{H}_{2\omega+1}$ of length $2\omega + 1$. Note that the Ulm–Kaplansky-invariants of $\mathcal{H}_{2\omega+1}$ are

$$f_\sigma(\mathcal{H}_{2\omega+1}) = \begin{cases} \aleph_0 & \text{for } 0 \le \sigma < \omega, \\ 1 & \text{for } \omega \le \sigma \le 2\omega. \end{cases}$$

We construct a simply presented group $\mathcal{G}$ that has the same Ulm–Kaplansky-invariants as $\mathcal{H}_{2\omega+1}$. Hence $\mathcal{G} \cong \mathcal{H}_{2\omega+1}$, because simply presented groups with equal Ulm–Kaplansky-invariants are isomorphic, see [3, 83.3]. Further we use the presentation of this group $\mathcal{G}$ to obtain an inductive quasibasis.

We begin by developing some notation. Let $H = \langle h_0^0 \rangle \oplus \bigoplus_{k\in\mathbb{N}, i\in\mathbb{N}_0} \langle h_i^k \rangle$ be a free abelian group and $L = \langle p h_0^0, p^k h_0^k - h_0^0, p^i h_i^k - h_0^k \mid i, k \in \mathbb{N} \rangle$ a subgroup of H. We denote $\mathcal{G} = H/L = \langle g_0^0, g_i^k \mid k \in \mathbb{N}, i \in \mathbb{N}_0 \rangle$ with $g_0^0 = h_0^0 + L$ and $g_i^k = h_i^k + L$. The group $\mathcal{G}$ is given by the relations

$$p g_0^0 = 0, \quad p^k g_0^k = g_0^0$$

and

$$p^i g_i^k = g_0^k \quad \text{for } i, k \in \mathbb{N}.$$

In particular, $\mathcal{G}$ is simply presented. It is straightforward to show that the following hold for the groups L and $\mathcal{G}$ as described above.

i. Every $l \in L$ has the form $l = \lambda_0^0 h_0^0 + \sum_{k\in\mathbb{N}, i\in\mathbb{N}_0} \lambda_i^k h_i^k$, where $\lambda_i^k \in p^i\mathbb{Z}$ for $i, k \in \mathbb{N}$. Moreover, $h_0^0 \notin L$ and for $l = \sum_{k\in\mathbb{N}} \lambda_0^k h_0^k$, $\lambda_0^k \in p^k\mathbb{Z}$ for all $k \in \mathbb{N}$.

ii. For $r \in \mathbb{N}$ each $g \in p^r\mathcal{G}$ has the form $g = \sum_{k\in\mathbb{N}} \big(\mu_0^k g_0^k + \sum_{i>r} \mu_i^k g_i^k\big)$ with $\mu_0^k \in \mathbb{Z}$ and $\mu_i^k \in p^r\mathbb{Z}$ for $i > r$.

We determine a basic subgroup of $\mathcal{G}$ and construct an inductive quasibasis. Let $x_i^k = g_i^k - p g_{i+1}^k \in \mathcal{G}$ for all $i, k \in \mathbb{N}$ and let $\mathcal{B} = \langle x_i^k \mid k, i \in \mathbb{N} \rangle$.

Lemma 4.1. *The subgroup $\mathcal{B}$ of $\mathcal{G}$ defined above is a direct sum, $\mathcal{B} = \bigoplus_{i,k\in\mathbb{N}} \langle x_i^k \rangle$, with $o(x_i^k) = p^i$ for all $i, k \in \mathbb{N}$. Moreover, $\mathcal{B}$ is a basic subgroup of $\mathcal{G}$.*

Proof. Similar to the arguments for the generalized Prüfer group $\mathcal{H}_{\omega+1}$, see [3, Section 35, Example], it is easy to verify that $\{x_i^k \mid i, k \in \mathbb{N}\}$ is a p-independent system of $\mathcal{G}$ with each $b \in \mathcal{B}$ of the form $b = \sum_{i,k\in\mathbb{N}} \lambda_i^k x_i^k = \sum_{i,k\in\mathbb{N}} (\lambda_i^k - p\lambda_{i-1}^k) g_i^k$, where $0 \le \lambda_i^k < p^i$ and agreeing $\lambda_0^k = 0$. Moreover, $\mathcal{G}/\mathcal{B}$ is divisible with a decomposition into the $\mathbb{Z}(p^\infty)$ summands given by $\langle \bar{g}_i^1 \mid i \in \mathbb{N}_0 \rangle$ and $\langle p^{k-1} \bar{g}_i^k - p^k \bar{g}_i^{k+1} \mid i \in \mathbb{N}_0 \rangle$ for $k \in \mathbb{N}$ and where $\bar{g} = g + \mathcal{B}$. □

Lemma 4.2. $\mathcal{G} \cong \mathcal{H}_{2\omega+1}$.

Proof. Since $\mathcal{B} = \bigoplus_{i,k\in\mathbb{N}} \langle x_i^k \rangle$ and $p^\omega \mathcal{G} = \langle g_0^0, g_0^k \mid p g_0^0 = 0, p^k g_0^k = g_0^0 \text{ for } k \in \mathbb{N} \rangle$, we have $p^\omega \mathcal{G} = \mathcal{H}_{\omega+1}$, see [3, Section 83, Example 3]. So the the Ulm–Kaplansky invariants of the simply presented group $\mathcal{G}$ are equal to those of $\mathcal{H}_{2\omega+1}$. Thus $\mathcal{G} \cong \mathcal{H}_{2\omega+1}$ by the consideration above. □

Now we use the presentation of $\mathcal{G}$ to obtain an inductive quasibasis of $\mathcal{H}_{2\omega+1}$. In Lemma 4.1 we defined $\mathcal{B}$, the basic subgroup of $\mathcal{G}$, by $\mathcal{B} = \bigoplus_{i,k\in\mathbb{N}}\langle x_i^k\rangle$. This shows that condition (i) of an inductive quasibasis holds. Now we show conditions (ii) and (iii). Define the generators a_i^k as follows $a_i^0 = p^2 g_i^1$ and $a_i^k = -p^k g_i^k + p^{k+1} g_i^{k+1}$ for $i,k\in\mathbb{N}$. In particular, $a_i^0 - pa_{i+1}^0 = p^2 g_i^1 - p^3 g_{i+1}^1 = p^2 x_i^1 \in B_i$ and for all $i,k\in\mathbb{N}$

$$\begin{aligned} a_i^k - pa_{i+1}^k &= -p^k g_i^k + p^{k+1} g_i^{k+1} + p^{k+1} g_{i+1}^k - p^{k+2} g_{i+1}^{k+1} \\ &= -p^k x_i^k + p^{k+1} x_i^{k+1} \in B_i. \end{aligned}$$

Define A^k by $A^k = \langle a_i^k + \mathcal{B} \mid i \in \mathbb{N}\rangle \subset \mathcal{G}/\mathcal{B}$ for all $k\in\mathbb{N}_0$. Note that the subgroups A^k are precisely the $\mathbb{Z}(p^\infty)$ summands given in Lemma 4.1 by the generators $\langle \bar{g}_i^1 \mid i\in\mathbb{N}_0\rangle$ and $\langle p^{k-1}\bar{g}_i^k - p^k \bar{g}_i^{k+1} \mid i \in \mathbb{N}_0\rangle$. Hence condition (ii) holds. Finally we show condition (iii), that $o(a_i^k) = p^i$. This follows from $p^i a_i^0 = p^{i+2} g_i^1 = p^2 g_0^1 = 0$, $p^i a_i^k = p^i(-p^k g_i^k + p^{k+1} g_i^{k+1}) = -p^k g_0^k + p^{k+1} g_0^{k+1} = -g_0^0 + g_0^0 = 0$ and the defining relations for the groups $\mathcal{G}$ and L. We summarize the above results in the following theorem.

THEOREM 4.3. *$\mathcal{Q} = \{a_i^k, x_i^k \mid i\in\mathbb{N}, k\in\mathbb{N}_0\}$ with a_i^k and x_i^k as defined above is an inductive quasibasis of $\mathcal{G}\cong\mathcal{H}_{2\omega+1}$ with the corresponding relation array $\alpha^k = \operatorname{diag}(\alpha_1^k,\alpha_2^k,\ldots)$, where $\alpha_i^0 = (p^2,0,\ldots)$ and $\alpha_i^k = (\underbrace{0,\ldots,0}_{k-1},-p^k,p^{k+1},0,\ldots)$ for all $i,k\in\mathbb{N}$.*

5. Invariance of height for quasibases

Let $B = \bigoplus_{i\in\mathbb{N}} B_i$ be a basic subgroup of G, and let $B^\Pi = \prod_{i\in\mathbb{N}} B_i$. The elements $\delta\in B^\Pi$ are written in the form $\delta = (b_1,b_2,\ldots)$, where $b_i\in B_i$ for all $i\in\mathbb{N}$. Let $h(\bar\delta)$ denote the height of $\bar\delta = \delta + B$ in B^Π/B. If $h^B(b_i)$ denotes the height of b_i in B then it is easy to see, that $h(\bar\delta) = \liminf_{i\to\infty}\big(h^B(b_i)\big)$. Note that $h^B(b_i)=\infty$ if and only if $b_i = 0$ and $h^B(b_i)\in\{0,1,\ldots,i-1\}$ for $b_i\neq 0$.

Let $B_0^\Pi = \{\delta\in B^\Pi \mid h(\bar\delta)$ is not finite$\}$. Note that $B_0^\Pi = \widehat{B}$, the completion of B in the p-adic topology, cf. also [2]. Then $B\subset B_0^\Pi\subset B^\Pi$ and B_0^Π/B is the first Ulm subgroup of B^Π/B. Clearly, $\delta = (b_i\mid i)\in B_0^\Pi$ if and only if $\lim_{i\to\infty}(h^B(b_i)) = \infty$.

LEMMA 5.1. *$B_0^\Pi/B = p^\omega\big(B^\Pi/B\big)$ is the maximal divisible subgroup of B^Π/B. In particular, $h(\bar\delta)\in\mathbb{N}_0\cup\{\infty\}$ for $\bar\delta\in B^\Pi/B$. Moreover, $\operatorname{tor}\big(B^\Pi/B\big)\subset B_0^\Pi/B$.*

Proof. We show that B_0^Π/B is divisible. If $\delta = (b_i \mid i)$ and $\delta + B \in B_0^\Pi/B$, then there is a $j \in \mathbb{N}$ such that $h(b_i) \geq 1$ for all $i \geq j$, i.e., $\delta + B = (pc_i \mid i) + B = p(c_i \mid i) + B$, where $b_i = pc_i$ for all $i \geq j$. Consequently $B_0^\Pi/B = p\big(B_0^\Pi/B\big)$, as desired. In particular, $h(\bar{\delta}) \in \mathbb{N}_0 \cup \{\infty\}$ for $\bar{\delta} \in B^\Pi/B$. Moreover, it follows from $i = h(b_i) + n_i$, if the element $b_i \in B_i$ is of order p^{n_i}, that the torsion subgroup of B^Π/B is contained in B_0^Π/B. □

Recall the following rules for heights [3, Section 37].

Lemma 5.2. *The following properties hold for $\delta, \delta_1, \delta_2 \in B^\Pi$ and $\lambda \in \mathbb{Z}_p$ with $\|\lambda\| = p^{-n}$:*

i. $h(\bar{\delta}_1 + \bar{\delta}_2) \geq \min\{h(\bar{\delta}_1), h(\bar{\delta}_2)\}$,

ii. $h(\bar{\delta}_1 + \bar{\delta}_2) = h(\bar{\delta}_1)$, *if* $h(\bar{\delta}_1) < h(\bar{\delta}_2)$,

iii. $h(\lambda\bar{\delta}) = h(\bar{\delta}) + n$.

Proof. (i) and (ii) are obvious, cf. [3, Section 37]. Condition (iii) follows from $h(\lambda\bar{\delta}) = h(p^n\lambda'\bar{\delta}) = h(\lambda'\bar{\delta}) + n = h(\bar{\delta}) + n$, where $\lambda' \in \mathbb{Z}_p \setminus p\mathbb{Z}_p$ with $\lambda = p^n\lambda'$. □

Notation 5.3. Let $Q = \{a_i^k, \bigoplus B_i\}$ be an inductive quasibasis of G with corresponding relations $a_i^k - pa_{i+1}^k = b_i^k \in B_i$. Define $\delta^k = \delta^k(Q) = (b_1^k, b_2^k, \dots) \in B^\Pi$, then the Q*-tuple* $\Delta(Q) = (\delta^k(Q) \mid k \in I)$ describes the corresponding relations of G. An important property of these relations can be formulated by the height function h given by $h(Q) = \min\{h(\bar{\delta}^k(Q)) \mid k \in I\} \in \mathbb{N}_0 \cup \{\infty\}$. We will refer to it as the *height* of Q in G. For a zero tuple $(\lambda_k \mid k \in I) \neq 0$ and the Q-tuple $\Delta(Q)$ we define the sum $\delta = \sum_{k\in I} \lambda_k \delta^k(Q) = (\sum_{k\in I} \lambda_k b_i^k \mid i \in \mathbb{N}) \in B^\Pi$ and call it a Q*-combination*. Note that δ is a well defined element of B^Π, because the sum in each component is finite. A Q-combination is called *normed* if the zero tuple $(\lambda_k \mid k \in I)$ is normed.

If $\alpha = (\alpha_i^{k,u})$ is the diagonal relation array corresponding to the inductive quasibasis Q, then we write the relation array as in the Equations (1) and (2). Thus

$$\delta^k(Q) = (b_1^k, b_2^k, \dots) = \Big(\sum_{u\in I_i} \alpha_i^{k,u} x_i^u \;\Big|\; i \in \mathbb{N} \Big),$$

and $\Delta(Q) = (\delta^k \mid k)$ is the Q-tuple. In particular, the heights $h(\bar{\delta}^k(Q))$ are precisely determined by the p-powers dividing the entries $\alpha_i^{k,u}$. Hence also $h(Q) = \min\{h(\bar{\delta}^k(Q)) \mid k \in I\}$ can be read off the entries $\alpha_i^{k,u}$.

We now determine the heights of some quasibases that have been studied previously.

Example 5.4. Let $Q = \{a_i^k, x_i^u\}$ where $|I| = |I_i| = 1$ for all $i \in \mathbb{N}$, i.e., $B = \bigoplus_{i\in\mathbb{N}} \mathbb{Z}(p^i)$, and $G/B \cong \mathbb{Z}(p^\infty)$. Thus $\Delta(Q) = (b_1, b_2, \ldots)$, i.e., $h(Q) = \liminf_{i\to\infty}\big(h^B(b_i)\big)$. We now give the heights of three quasibases that appeared in [4, Section 2 and 5] together with their relation arrays:

$$\begin{aligned} H_{\omega+n}\colon \alpha &= \operatorname{diag}(p^n, p^n, \ldots),\\ \delta &= (p^n x_1, p^n x_2, \ldots),\\ h(\bar\delta) &= n,\end{aligned}$$

$$\begin{aligned} B\colon \alpha &= \operatorname{diag}(1, 0, 1, 0, \ldots)\\ \delta &= (x_1, 0, x_3, 0, \ldots),\\ h(\bar\delta) &= 1,\end{aligned}$$

$$\begin{aligned} \mathbb{Z}(p^\infty)\oplus B\colon \alpha &= \operatorname{diag}(1, p, p^2, p^3, \ldots),\\ \delta &= (x_1, px_2, p^2x_3, \ldots),\\ h(\bar\delta) &= \infty.\end{aligned}$$

In the next example we consider the generalized Prüfer group $\mathcal{G} \cong \mathcal{H}_{2\omega+1}$ and determine the Q-tuple $\Delta(\mathcal{Q})$, the heights $h(\bar\delta^k(\mathcal{Q}))$, and the height $h(\mathcal{Q})$ of the quasibasis $\mathcal{Q} = \{a_i^k, x_i^k\}$ for the generalized Prüfer group $\mathcal{H}_{2\omega+1}$.

Example 5.5. By Theorem 4.3 the generalized Prüfer group $\mathcal{H}_{2\omega+1}$ has the quasibasis $\mathcal{Q} = \{a_i^k, x_i^k \mid i \in \mathbb{N}, k \in \mathbb{N}_0\}$ and the corresponding relation array is $\alpha = (\alpha^k)_{k\in I}$ with $\alpha^k = \operatorname{diag}(\alpha_1^k, \alpha_2^k, \ldots)$, where

$$\alpha_i^0 = (p^2, 0, \ldots)$$

and

$$\alpha_i^k = (\underbrace{0, \ldots, 0}_{k-1}, -p^k, p^{k+1}, 0, \ldots) \quad \text{for all } i, k \in \mathbb{N}.$$

Thus

$$\Delta(\mathcal{Q}) = (\delta^k \mid k \in \mathbb{N}_0), \quad \text{where } \delta^0 = (p^2 x_i^1 \mid i),$$

and

$$\delta^k = (-p^k(x_i^k - p x_i^{k+1}) \mid i) \quad \text{for } k \in \mathbb{N}.$$

So the heights of the $\bar\delta^k$ are $h(\bar\delta^0) = 2$ and $h(\bar\delta^k) = k$ for $k \in \mathbb{N}$, hence $h(\mathcal{Q}) = h(\bar\delta^1) = 1$.

Our next objective is to show that the height is invariant for related inductive quasibases. Let Q be a quasibasis of a group G with Q-tuple $\Delta(Q) = (\delta^k(Q) \mid k)$. We begin with Proposition 5.6 by showing that for some fixed $k_0 \in I$ we may switch to a related quasibasis of G such that only the entry $\delta^{k_0}(Q)$ is changed and this in a quite arbitrary way.

Proposition 5.6. *Let $Q = \{a_i^k, \bigoplus B_i\}$ be an inductive quasibasis of G with Q-tuple $\Delta(Q) = (\delta^k(Q) \mid k)$. Let $k_0 \in I$ be fixed and let $\delta = \sum_{k\in I} \lambda_k \delta^k(Q)$ be a (normed) Q-combination with $p \nmid \lambda_{k_0}$. Then there is an inductive quasibasis $P = \{c_i^k, \bigoplus B_i\}$ of G, related to Q, with P-tuple $\Delta(P) = (\delta^k(P) \mid k)$ such that*

$$\delta^k(P) = \begin{cases} \delta & \text{if } k = k_0, \\ \delta^k(Q) & \text{if } k \neq k_0. \end{cases}$$

Proof. For

$$c_i^k = \begin{cases} \sum_{l\in I} \lambda_l a_i^l & \text{if } k = k_0, \\ a_i^k & \text{if } k \neq k_0, \end{cases}$$

we show that $P = \{c_i^k, \bigoplus B_i\}$ is an inductive quasibasis of G. The conditions (i) and (iii) in the definition are obvious and it remains to show (ii). Since

$$\lambda_{k_0} a_i^{k_0} = c_i^{k_0} - \sum_{k\in I\setminus\{k_0\}} \lambda_k c_i^k \in \langle c_i^k \mid k \in I, i \in \mathbb{N}\rangle$$

and $p \nmid \lambda_{k_0}$, we get

$$\langle a_i^k \mid k \in I, i \in \mathbb{N}\rangle = \langle c_i^k \mid k \in I, i \in \mathbb{N}\rangle.$$

Hence

$$G/B = \bigoplus_{k\in I} \langle a_i^k + B \mid i \in \mathbb{N}\rangle = \sum_{k\in I} \langle c_i^k + B \mid i \in \mathbb{N}\rangle.$$

Define $C^k = \langle c_i^k + B \mid i \in \mathbb{N}\rangle$. We now prove that $\sum_{k\in I} C^k$ is a direct sum. Since $C^k \cong \mathbb{Z}(p^\infty)$ for all $k \in I$, we may write an arbitrary element $c \in \sum_{k\in I} C^k$ in the form $c = \sum_{k\in I} \mu_k c_i^k + B$, $\mu_k \in \mathbb{Z}$ for some $i \in \mathbb{N}$. Then

$$\begin{aligned} c &= \sum_{l\in I} \mu_{k_0} \lambda_l a_i^l + \sum_{k\in I\setminus\{k_0\}} \mu_k a_i^k + B \\ &= \mu_{k_0} \lambda_{k_0} a_i^{k_0} + \sum_{k\in I\setminus\{k_0\}} (\mu_{k_0} \lambda_k + \mu_k) a_i^k + B. \end{aligned}$$

If $c = 0 \in G/B$, then $p^i \mid \mu_{k_0}\lambda_{k_0}$ and $p^i \mid (\mu_{k_0}\lambda_k + \mu_k)$ for all $k \in I \setminus \{k_0\}$, because Q is a quasibasis. Thus, $p^i \mid \mu_k$ for all $k \in I$, because $p \nmid \lambda_{k_0}$. This shows that $c = 0 \in G/B$ implies $\mu_k c_i^k \in B$, and the sum $\sum_{k \in I} C^k$ is direct. Consequently, P is a quasibasis of G. Moreover, it is inductive, because $c_i^k - pc_{i+1}^k \in B_i$ for all $k \in I$.

In particular, $\delta^k(P) = \delta^k(Q)$ for $k \neq k_0$, and

$$\begin{aligned}\delta^{k_0}(P) &= \Big(c_i^{k_0} - pc_{i+1}^{k_0} \;\Big|\; i \in \mathbb{N}\Big) = \Big(\sum_{l \in I} \lambda_l a_i^l - p \sum_{l \in I} \lambda_l a_{i+1}^l \;\Big|\; i \in \mathbb{N}\Big) \\ &= \Big(\sum_{l \in I} \lambda_l b_i^l \;\Big|\; i \in \mathbb{N}\Big) = \sum_{k \in I} \lambda_k \delta^k(Q) = \delta. \qquad \square\end{aligned}$$

An inductive quasibasis Q of G is called *normed*, if $h(\bar{\delta}^k) = h(Q)$ for every $k \in I$. Now we show that the group G has a normed, related inductive quasibasis P with $h(P) = h(Q)$.

Lemma 5.7. *For every inductive quasibasis Q of G there is a normed, related inductive quasibasis P of G with $h(P) = h(Q)$.*

Proof. Let $Q = \{a_i^k, \bigoplus B_i \mid k \in I\}$ be an inductive quasibasis of G. Recall the notation in Equation (1). Let $\delta^k = \delta^k(Q)$ for all $k \in I$. To construct P, we choose some $k_0 \in I$ with $h(\bar{\delta}^{k_0}) = h(Q)$ and define the subset $J = \{k \in I \mid h(\bar{\delta}^k) \neq h(Q)\} \subset I$. We use the idea of Proposition 5.6 and show that $P = \{c_i^k, \bigoplus B_i\}$ with

$$c_i^k = \begin{cases} a_i^k & \text{for } k \in I \setminus J, \\ a_i^k + a_i^{k_0} & \text{for } k \in J, \end{cases}$$

is a normed, related inductive quasibasis of G with $h(P) = h(Q)$. The set P clearly satisfies the conditions (i) and (iii) of the definition. Condition (ii) is also satisfied from the following. Note that for $k \in J$

$$c_i^k - pc_{i+1}^k = a_i^k - pa_{i+1}^k + a_i^{k_0} - pa_{i+1}^{k_0} = b_i^k + b_i^{k_0} \in B_i.$$

Thus

$$\begin{aligned}&\bigoplus_{k \in I} \langle c_i^k + B \mid i \in \mathbb{N}\rangle \\ &\quad = \Big(\bigoplus_{k \in J} \langle a_i^k + a_i^{k_0} + B \mid i \in \mathbb{N}\rangle\Big) \oplus \Big(\bigoplus_{k \in I \setminus J} \langle a_i^k + B \mid i \in \mathbb{N}\rangle\Big) \\ &\quad = \bigoplus_{k \in I} \langle a_i^k + B \mid i \in \mathbb{N}\rangle.\end{aligned}$$

Hence, P is an inductive quasibasis of G which is related to Q and given by

$$\delta^k(P) = \begin{cases} \delta^k & \text{for } k \in I \setminus J, \\ \delta^k + \delta^{k_0} & \text{for } k \in J. \end{cases}$$

By Lemma 5.2,

$$h(\bar{\delta}^k(P)) = h(\bar{\delta}^k + \bar{\delta}^{k_0}) = h(\bar{\delta}^{k_0}) = h(Q)$$

for all $k \in J$, because $h(\bar{\delta}^{k_0}) < h(\bar{\delta}^k)$. Thus $h(\bar{\delta}^k) = h(Q)$ for all $k \in I$ and hence, P is normed with $h(P) = h(Q)$. □

We are now ready to prove the main result in this section on the invariance of height for related inductive quasibases.

Theorem 5.8. *Related inductive quasibases of G have the same height.*

Proof. By Lemma 5.7 it suffices to show that $h(Q) = h(P)$ for two normed, related inductive quasibases $Q = \{a_i^k, \bigoplus B_i\}$ and $P = \{c_i^k, \bigoplus B_i\}$ of G. Assume $h(Q) > h(P)$ and let $h = h(P)$. Also, let $b_i^k = a_i^k - pa_{i+1}^k$ and $d_i^k = c_i^k - pc_{i+1}^k$ for all $i \in \mathbb{N}$, $k \in I$. By Lemma 3.2, let $k_0 \in I$ and let $(\lambda_k \mid k \in I)$ be a normed zero tuple such that

$$d_i^{k_0} - \sum_{k \in I} \lambda_k b_i^k \in p^n B_i, \tag{3}$$

for each $n \in \mathbb{N}$ and almost all $i \in \mathbb{N}$. Let $\delta^{k_0} = \delta^{k_0}(P) = (d_i^{k_0} \mid i \in \mathbb{N})$ and $\delta = \sum_{k \in I} \lambda_k \delta^k(Q) = (\sum_{k \in I} \lambda_k b_i^k \mid i \in \mathbb{N})$. From (3) it follows that $h(\bar{\delta}^{k_0} - \bar{\delta}) = \infty$. By Lemma 5.2 we get $h(\bar{\delta}) = h(\bar{\delta}^{k_0}) = h$. Thus the set $J = \{k \in I \mid p^{h+1} \nmid \lambda_k\}$ must be finite and nonempty. Write $\delta = \delta_1 + \delta_2$ where $\delta_1 = \sum_{k \in J} \lambda_k \delta^k(Q)$ and $\delta_2 = \sum_{k \in I \setminus J} \lambda_k \delta^k(Q)$. Thus,

$$h(\bar{\delta}_1) \geq \min\{h(\lambda_k \delta^k(Q)) \mid k \in J\} > h,$$

because of $h(\lambda_k \bar{\delta}^k(Q)) \geq h(\bar{\delta}^k(Q)) = h(Q) > h$ for all $k \in J$. On the other hand $h(\bar{\delta}_2) > h$, because $\delta_2 \in p^{h+1} B^{\Pi}$. Moreover, $h(\bar{\delta}_1), h(\bar{\delta}_2) > h$, and

$$h = h(\bar{\delta}) = h(\bar{\delta}_1 + \bar{\delta}_2) \geq \min\{h(\bar{\delta}_1), h(\bar{\delta}_2)\} > h,$$

a contradiction. Hence, $h(Q) = h(P)$. □

6. Quasibases of reduced groups

In this section we characterize separable and nonreduced groups in terms of the height of an inductive quasibasis. We begin with a lemma which describes the Ulm subgroup.

LEMMA 6.1. *Let $Q = \{a_i^k, \bigoplus B_i\}$ be an inductive quasibasis of G. Let $0 \neq g \in G$ be of order p^j. Then $g \in p^\omega G$ if and only if there is a normed zero tuple $(\lambda_k \mid k \in I)$, such that $g = \sum_{k\in I} \lambda_k a_j^k + b$, $b \in B$, and there is a natural number n such that*

$$g = p^n \sum_{k\in I} \lambda_k a_{j+n}^k = p^{n+1} \sum_{k\in I} \lambda_k a_{j+n+1}^k = p^{n+2} \sum_{k\in I} \lambda_k a_{j+n+2}^k = \cdots$$

In particular, $h\big(\sum_{k\in I} \lambda_k \bar{\delta}^k\big) \geq j$.

PROOF. Since $G/B = \bigoplus_{k\in I} \langle a_i^k + B \mid i \in \mathbb{N}\rangle \cong \bigoplus_{|I|} \mathbb{Z}(p^\infty)$, there is, by Proposition 2.1, a normed zero tuple $(\lambda_k \mid k \in I)$ such that $g \in \sum_{k\in I} \lambda_k a_j^k + B$ and the set $\{\sum_{k\in I} \lambda_k a_i^k + B \mid i \in \mathbb{N}\}$ generates a $\mathbb{Z}(p^\infty)$. Now let $g = \sum_{k\in I} \lambda_k a_j^k + b \in p^\omega G$ with $b \in \bigoplus_{i<l} B_i$ for some $l \in \mathbb{N}$. Then by Lemma 3.1

$$g = \sum_{k\in I} \lambda_k a_j^k + b = \sum_{k\in I} \lambda_k \Big(p^n a_{j+n}^k + \sum_{r=0}^{n-1} p^r b_{j+r}^k\Big) + b \in p^n G,$$

for all $n \in \mathbb{N}$. Hence for all $n \geq l$

$$\begin{aligned} g - p^n \sum_{k\in I} \lambda_k a_{j+n}^k &= \sum_{r=0}^{n-1} \sum_{k\in I} \lambda_k p^r b_{j+r}^k + b \\ &= \sum_{r=j}^{n+j-1} \sum_{k\in I} \lambda_k p^{r-j} b_r^k + b \\ &= \sum_{r=j}^{l-1} \sum_{k\in I} \lambda_k p^{r-j} b_r^k + b + \sum_{r=l}^{n+j-1} \sum_{k\in I} \lambda_k p^{r-j} b_r^k \in p^n B. \end{aligned}$$

Let $b_r = \sum_{k\in I} \lambda_k p^{r-j} b_r^k \in B_r$. In view of height and order considerations we conclude that $\sum_{r=j}^{l-1} b_r + b = 0$. Thus $\sum_{r=l}^{n+j-1} b_r \in p^n B$ for all $n \geq l$, i.e., $b_r = 0$ for all $r \geq l$. It follows that $g = p^n \sum_{k\in I} \lambda_k a_{j+n}^k$ for all $n \geq l$. Consequently $g \in p^\omega G$ has the indicated form. Moreover, $b_r = \sum_{k\in I} \lambda_k p^{r-j} b_r^k = 0$ implies that $p^j \mid \sum_{k\in I} \lambda_k b_r^k$, for all $r \geq l$, and $h(\sum_{k\in I} \lambda_k \bar{\delta}^k) \geq j$. □

Corollary 6.2. *Let* $Q = \{a_i^k, \bigoplus B_i\}$ *be an inductive quasibasis of* G. *If* $j > h(\bar{\delta}^{k_0})$, *then* $a_j^{k_0} \notin p^\omega G$.

Proof. If $a_j^{k_0} \in p^\omega G$, then by Lemma 6.1 there is a normed zero tuple $(\lambda_k \mid k \in I)$, such that $a_j^{k_0} = \sum_{k\in I} \lambda_k a_j^k + b$ and $a_j^{k_0} = p^n \sum_{k\in I} \lambda_k a_{j+n}^k$ for all n. Thus $b = 0$, $p^j|\lambda_k$ for all $k \neq k_0$, and consequently $a_j^{k_0} = p^n a_{j+n}^{k_0}$ for all n.

Now, let $j > h = h(\bar{\delta}^{k_0})$. So there are infinitely many $i \in \mathbb{N}$ where $p^{h+1} \nmid b_i^{k_0} = a_i^{k_0} - p a_{i+1}^{k_0} \in B_i$. Hence we may select an n with $p^{h+1} \nmid b_{j+n}^{k_0}$. We assume $a_j^{k_0} \in p^\omega G$ and apply Lemma 3.1, i.e.,

$$\sum_{r=0}^{n} p^r b_{j+r}^{k_0} = a_j^{k_0} - p^{n+1} a_{j+n+1}^{k_0} = 0.$$

Using $B = \bigoplus B_i$ we get $p^r b_{j+r}^{k_0} = 0$ for all $0 \leq r \leq n$. In particular, $p^n b_{j+n}^{k_0} = 0$ implies $p^j \mid b_{j+n}^{k_0}$. But $j > h$ further implies that $p^{h+1}|b_{j+n}^{k_0}$, a contradiction. Thus $a_j^{k_0} \notin p^\omega G$. □

Our next theorem describes separable groups in terms of the height of a quasibasis.

Theorem 6.3. *Let* $Q = \{a_i^k, \bigoplus B_i\}$ *be an inductive quasibasis of the reduced group* G. *Then* G *is separable if and only if* $h(\bar{\delta}) = 0$ *for all* Q*-combinations* δ.

Proof. Suppose G is separable and let $\delta = \sum_{k\in I} \lambda_k \delta^k(Q)$ be a Q-combination. If $h(\bar{\delta}) > 0$, then there is a $j \in \mathbb{N}$, such that $\sum_{k\in I} \lambda_k b_i^k \in pB_i$ for all $i \geq j$. By this fact combined with Lemma 3.1 we get for all $n \in \mathbb{N}$

$$\begin{aligned}\sum_{k\in I} \lambda_k p^{j-1} a_j^k &= \sum_{k\in I} \lambda_k p^{j-1}\Big(p^n a_{j+n}^k + \sum_{r=0}^{n-1} p^r b_{j+r}^k\Big)\\ &= p^{j+n-1} \sum_{k\in I} \lambda_k a_{j+n}^k + \sum_{r=0}^{n-1} p^{j+r-1} \sum_{k\in I} \lambda_k b_{j+r}^k\\ &= p^{j+n-1} \sum_{k\in I} \lambda_k a_{j+n}^k.\end{aligned}$$

This holds for all n, thus $\sum_{k\in I} \lambda_k p^{j-1} a_j^k \neq 0$ has infinite height, a contradiction, by [3, 65.1].

Conversely, assume that G is nonseparable, i.e., there is a $0 \neq g \in G$ of infinite height and order p^j. By Lemma 6.1 there is a normed zero tuple $(\lambda_k \mid k \in I)$ with $h(\sum_{k\in I} \lambda_k \bar{\delta}^k) \geq j > 0$, contradicting the hypothesis that $h(\bar{\delta}) = 0$ for all normed Q-combinations δ. □

The following lemma shows, that a Q-combination δ with $h(\bar{\delta}) = \infty$ allows to find a divisible subgroup of G.

Lemma 6.4. *Let $Q = \{a_i^k, \bigoplus B_i\}$ be an inductive quasibasis of G. If $\delta = \sum_{k\in I} \lambda_k \delta^k(Q)$ is a Q-combination with $h(\bar{\delta}) = \infty$, then there is a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ of natural numbers, such that $\langle d_i \in G \mid i \in \mathbb{N}\rangle \cong \mathbb{Z}(p^\infty)$ with $pd_1 = 0$ and $d_i = pd_{i+1} = p^{n_i} \sum_{k\in I} \lambda_k a_{i+n_i}^k$ for $i \in \mathbb{N}$.*

Proof. Since $h(\bar{\delta}) = \infty$, there is a strictly increasing sequence $(n_i)_{i\in\mathbb{N}}$ of natural numbers, such that $p^i \mid \sum_{k\in I} \lambda_k b_n^k$ for all $n \geq n_i$. Define $d_i = p^{n_i} \sum_{k\in I} \lambda_k a_{i+n_i}^k$ for all $i \in \mathbb{N}$. Clearly $pd_1 = p^{n_1+1} \sum_{k\in I} \lambda_k a_{n_1+1}^k = 0$. Moreover, for all $i \in \mathbb{N}$

$$\begin{aligned} pd_{i+1} &= p^{n_{i+1}+1} \sum_{k\in I} \lambda_k a_{i+n_{i+1}+1}^k \\ &= p^{n_i+(n_{i+1}-n_i+1)} \sum_{k\in I} \lambda_k a_{i+n_i+(n_{i+1}-n_i+1)}^k \\ &= p^{n_i} \sum_{k\in I} \lambda_k a_{i+n_i}^k - \sum_{r=0}^{n_{i+1}-n_i} p^{n_i+r} \sum_{k\in I} \lambda_k b_{i+n_i+r}^k \\ &= d_i, \end{aligned}$$

by Lemma 3.1 and the height condition above. Since $(\lambda_k \mid k \in I) \neq 0$, we have $d_i \neq 0$, for almost all $i \in \mathbb{N}$. Hence $\langle d_i \in G \mid i \in \mathbb{N}\rangle \cong \mathbb{Z}(p^\infty)$. □

Lemma 6.5. *Let $Q = \{a_i^k, \bigoplus B_i\}$ be an inductive quasibasis of G. If D is a divisible subgroup of G of rank 1, then there is a Q-combination $\delta = \sum_{k\in I} \lambda_k \delta^k(Q)$ such that $h(\bar{\delta}) = \infty$, and $D \subset \langle \sum_{k\in I} \lambda_k a_i^k \mid i \in \mathbb{N}\rangle$.*

Proof. Let $D = \langle g_i \in G \mid i \in \mathbb{N}\rangle \cong \mathbb{Z}(p^\infty)$ be a divisible subgroup of G with $pg_1 = 0$, $pg_{i+1} = g_i \neq 0$ for $i \in \mathbb{N}$. Since $\mathbb{Z}(p^\infty) \cong \langle g_i + B \mid i \in \mathbb{N}\rangle \subset G/B$, there is a normed zero tuple $(\lambda_k \mid k \in I)$, such that $g_i \in \sum_{k\in I} \lambda_k a_i^k + B$, for all $i \in \mathbb{N}$, by Proposition 2.1. Hence by Lemma 6.1

$$g_i = p^n \sum_{k\in I} \lambda_k a_{i+n}^k \quad \text{for almost all } n \in \mathbb{N} \text{ and all } i \in \mathbb{N}.$$

Thus for each $i \in \mathbb{N}$ and for almost all $n \in \mathbb{N}$

$$0 = g_i - pg_{i+1} = p^n \sum_{k \in I} \lambda_k (a^k_{i+n} - pa^k_{i+n+1}) = p^n \sum_{k \in I} \lambda_k b^k_{i+n},$$

i.e., $p^i \mid \sum_{k\in I} \lambda_k b^k_{i+n}$. Consequently, $h(\sum_{k\in I} \lambda_k \bar{\delta}^k) = \infty$, and, in particular, $D = \langle g_i \in G \mid i \in \mathbb{N}\rangle \subset \langle \sum_{k\in I} \lambda_k a^k_i \mid i \in \mathbb{N}\rangle$. □

The Lemmata 6.4 and 6.5 lead to the main result in this section.

THEOREM 6.6. *Let $Q = \{a^k_i, \bigoplus B_i\}$ be an inductive quasibasis of G. Then G is not reduced if and only if there is a Q-combination δ with $h(\bar{\delta}) = \infty$.*

The Theorems 6.3 and 6.6 characterize also the reduced nonseparable groups.

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